УДК 621.647.23

Iakovliev A.
University of Southampton, Southampton, United Kingdom (ai1v14@soton.ac.uk)

EXACT ELASTIC STABILITY ANALYSIS BASED ON DYNAMIC STIFFNESS METHOD

Яковлев А.И.
Саусхемптонский университет, Саусхемптон, Великобритания

ТОЧНЫЙ МЕТОД РЕШЕНИЯ ЗАДАЧ НА УСТОЙЧИВОСТЬ С ПРИМЕНЕНИЕМ МЕТОДА ДИНАМИЧЕСКОЙ ЖЕСТКОСТИ

The article is dedicated to the discussion on the exact dynamic stiffness matrix method applied to the problems of elastic stability of engineering structures. The detailed formulation of the member dynamic stiffness matrix for beams is presented along with the general guidelines on automatisation of the assembly of member dynamic stiffness matrices into the global matrix that corresponds to the whole structure. The advantage of the dynamic stiffness matrix in case of parametric studies is explained. The problem of computing the eigenvalues of transcendental matrix is addressed. The straightforward approach as well as a powerful Witrick-Williams algorithm are discussed in details. The general guidelines on programming the DS matrix method are given as well.
Keywords: elastic stability, buckling, dynamic stiffness matrix, eigenvalues, Wittrick-Williams algorithm.

1. Introduction

Elastic stability problems faced by engineers these days feature extremely high levels of complexity. Usually, the structures of interest represent multilevel hierarchies of substructural assemblies. Finite element analysis has lately shown a considerable potential in solving such complex problems. However, the FE method has several disadvantages. The errors associated with the numerical approximations of the stresses and strains within a finite element using numerical shape functions are well known (Bathe 1996). Another weak point of the FE method is computational efficiency. Although, the effectiveness of programming of the FE procedures in commercial software has been led to perfection over the last thirty years the necessity to assemble large stiffness matrices and to perform costly operations like inversion is inevitable. These major drawbacks of the FE method give a strong background to look for any alternatives that can be used either in parallel or separately from the FE analysis.

This article is dedicated to the discussion of the exact Dynamic Stiffness method and its application to the problems of elastic stability. This method is not new and has been known for many decades. The exact date of the development of this method is difficult to trace. However, it is assumed to start from works by Professor John H. Argyris (Spalding 2014) who was the first to introduce the method of assembly structural components into a general stiffness matrix. The essence of the method is in writing exact stiffness relations for a single member of a structure in a general form and then assemble such equations into a matrix. Then the member matrices are assembled into a global structural stiffness matrix. The eigenvalues of the global matrix yield the critical buckling loads of a whole structure.

The member stiffness matrices derived from the exact stiffness relations are known as dynamic stiffness (DS) matrices (Williams et al. 2004). However, the name of such matrix was taken from the nomenclature widely accepted in the theory of vibrations, the approach is highly universal and can be readily applied to the problems of elastic stability. The member DS matrices as well as the assembled DS matrices are transcendental which means that their entries are transcendental functions of eigenvalues and define exact relations between nodal forces and nodal displacements of each member in a structure. In such a way the DS matrix method possesses considerable advantages compared to the finite element (FE) method in terms of computation efficiency and accuracy. However, the transcendental nature of a DS matrix significantly complicates the process of computing its eigenvalues. For this purpose an iterative algorithm capable of converging on any number of eigenvalues with any precision was developed by Wittrick and Williams and has their names (F. Williams and W. Wittrick 1970; Wittrick & Williams 1971).

This paper is dedicated to the detailed study of the DS matrix method in its application to the problems of elastic stability of structures. It contains numerous guidelines and tricks on automating and programming the method. We discuss here the formulation of the member DS matrix starting from member governing differential equation (Section 2). Then we proceed with the analysis of assembly of the global DS matrix and propose essential steps to program it. Finally, we discuss the methods of computing the eigenvalues of the DS matrix on a simple illustrative example. We study both the straightforward evaluation of the zeros of the determinant of the DS matrix and the Wittrick-Williams (W-W) algorithm. Although the W-W algorithm has been considerably studied in dynamics (Wittrick & Williams 1971;

DOI: http://dx.doi.org/10.20535/2305-9001.2016.77.78904

F. Williams and W. Wittrick 1970; Banerjee & Williams 1986) we believe that it requires a more in-depth analysis from the perspective of how it can be programmed for the case of elastic stability. We also believe that illustration of the application of the algorithm on a simple example is essential.

2. Exact dynamic stiffness matrix for buckling of a beam member

Consider a standard Sturm-Liouville differential equation in a form (Arfken et al. 2013):

$$\frac{d}{dx}\left[p(x)\frac{dW(x)}{dx}\right] + qW(x) = -\lambda rW(x), \tag{2.1}$$

Where $p(x), q(x), \lambda(x), r(x)$ are the coefficients that depend on x and defined in a real finite interval $x \in [a, b]$. Equation (2.1) along with boundary conditions define a classical Sturm-Liouville problem. The beam bending equation used in linear buckling analysis is derived from equation (2.1) by relating coefficients to corresponding physical quantities: $p(x) = EI(x)$ is related to the stiffness of a beam member, $q(x) = P$ - is the applied compressing loading and $\lambda(x), r(x)$ are taken to be zero. Thus, the simple bending equation reads as:

$$\frac{d}{dx}\left[EI(x)\frac{dW(x)}{dx}\right] + PW(x) = 0, \tag{2.2}$$

here $W(x)$ is the lateral deflection of a beam as shown in figure 2.1. Equation (2.2) can be generalised to account for the general case of beam bending. The generalised beam member equation is thus a fourth order differential equation with constant coefficients written as:

$$\frac{d^4}{dx^4}[W(x)] + \lambda^2 \frac{d^2}{dx^2}[W(x)] = 0. \tag{2.3}$$

Here $\lambda^2 = P/EI_i$ is the buckling load parameter.

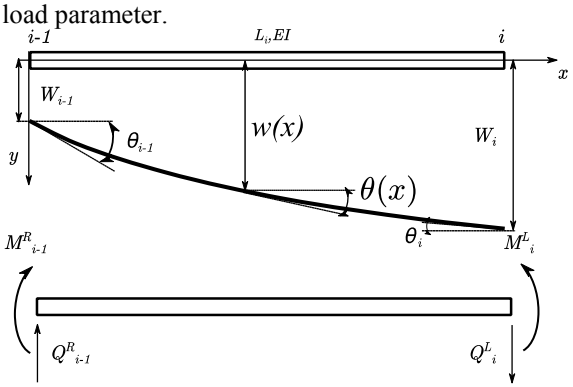

Fig. 2.1. The schematic representation of a beam as a structural member described with equation (*2.2*). The member has two nodes $i-1$ and i. The deflection and rotation at the nodes denoted as W_{i-1},W_i,$\theta_{i-1}$and θ_i respectively. The bending moments and lateral forces at the nodes are defined as $M_{i-1}, M_i, Q_{i-1}, Q_i$. The beam is compressed along the x-axis with a loading P

Equation (2.3) with appropriate boundary conditions defines the problem of elastic stability of a structural member represented by an elastic beam. It can be discretised according to the FE method and solved for eigenvalues of a generalised eigenvalue problem λ that define the critical buckling loads. Alternatively, this problem can be solved exactly employing the DS (Dynamic Stiffness) method. Here we proceed with the DS approach to the problem. We derive the dynamic stiffness matrix for the case of buckling of a beam member following the general guidelines published by Banerjee (Banerjee 1997). The author formulated a general algorithm for construction of the dynamic stiffness matrix that we adopt here for the case of buckling problems.

The general solution of the equation (2.3) is well known (Timoshenko & Gere 2012) and can be written in terms of the undetermined constants as:

$$w(x) = c_1 sin(\lambda x) + c_2 cos(\lambda x) + c_3 x + c_4. \tag{2.4}$$

Equation (2.4) defines the dependence of the member deflection on the unknown coefficients and the critical buckling load parameter λ. The equations for the cross-sectional rotation, bending moment and lateral force can be readily obtained from equation (2.4) (Timoshenko & Gere 2012) and are written as:

$$\theta(x) = -c_1\lambda sin(\lambda x) + c_2\lambda cos(\lambda x) + c_3 \tag{2.5}$$

$$M(x) = EI(-c_1\lambda^2 cos(\lambda x) - c_2\lambda^2 sin(\lambda x)) \tag{2.6}$$

$$F(x) = EI(c_1\lambda^3 sin(\lambda x) - c_2\lambda^3 cos(\lambda x)) + \lambda^2(-c_1\lambda sin(\lambda x) + c_2\lambda cos(\lambda x) + c_3) \tag{2.7}$$

Now we define nodal displacements of a member using equations (2.4)-(2.5) by substituting values of x that correspond to each node into each equation (2.4)-(2.5). Therefore, the system of four equations in terms of unknown coefficients is obtained. This system of equations relates displacements at the nodes of a member to the unknown constants. Written in a matrix form it reads:

$$\begin{bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & L & \sin\lambda L & \cos\lambda L \\ 0 & 1 & \lambda\cos\lambda L & -\lambda\sin\lambda L \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}. \tag{2.8}$$

Equation (2.8) can be represented in a compact form:

$$\mathbf{w} = \mathrm{T}_1(\lambda)\mathbf{c}. \tag{2.9}$$

Bold letters indicate vectors. Matrix $\mathrm{T}_1(\lambda)$ defines the relation between nodal displacements and unknown constants. This matrix is transcendental as it possesses an intrinsic dependence on its eigenvalues λ. The dependence is defined using the transcendental functions. Similar procedure is applied to the equations (2.6) and (2.7). The lateral loading shown in figure 2.1 is defined in terms of unknown coefficients. The corresponding system of equations reads as:

$$\begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} 1 & \lambda^2 & 0 & 0 \\ 0 & 0 & -EI\lambda^2 & 0 \\ 0 & \lambda^2 & -2\lambda^3\sin\lambda L & 0 \\ 0 & 0 & EI\lambda^2\cos\lambda L & EI\lambda^2\sin\lambda L \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}. \tag{2.10}$$

Equation (2.10) can also be represented in a compact form using vector notation:

$$\mathbf{F} = \mathrm{T}_2(\lambda)\mathbf{c}. \tag{2.11}$$

Matrix $\mathrm{T}_2(\lambda)$ relates nodal forces of a member with the vector of unknown coefficients. This matrix is also transcendental with respect to its eigenvalue λ.

Now we express the vector of unknown constants **c** from equation (2.9) (for this purpose we invert matrix $\mathrm{T}_1(\lambda)$):

$$\mathbf{c} = \mathrm{T}_1^{-1}(\lambda)\mathbf{w}, \tag{2.12}$$

and substitute it into equation (2.11):

$$\mathbf{F} = \mathrm{T}_2(\lambda)\mathrm{T}_1^{-1}(\lambda)\mathbf{w}. \tag{2.13}$$

Equation (2.13) gives an exact relation between the vector of nodal forces and the vector of nodal displacements employing a transcendental matrix:

$$\mathbf{K}(\lambda) = \mathrm{T}_2(\lambda)\mathrm{T}_1^{-1}(\lambda). \tag{2.14}$$

This matrix is called the exact Dynamic Stiffness Matrix. The word "Dynamic" came from the origins of the DS matrix method from the theory of vibrations. Since then the method has been employed in various fields but the name has been kept unchanged. The entries of the matrix were computed using equation (2.14) and is presented in the simplified form using the notation from the paper by Williams and Wittrick (Williams & Wittrick 1983) who in turn formalised the elastic stability functions computed and tabulated by Livesley and Chandler (Livesley & Chandler 1956):

$$\mathbf{K} = \frac{EI}{L}\begin{bmatrix} \frac{\gamma}{L^2} & \frac{\nu}{L} & -\frac{\gamma}{L^2} & \frac{\nu}{L} \\ \frac{\nu}{L} & \alpha & \frac{\nu}{L} & \beta \\ -\frac{\gamma}{L^2} & -\frac{\nu}{L} & \frac{\gamma}{L^2} & -\frac{\nu}{L} \\ \frac{\nu}{L} & \beta & -\frac{\nu}{L} & \alpha \end{bmatrix}, \tag{2.15}$$

Here α, β, ν and γ are the transcendental functions of eigenvalue λ:

$$\begin{cases} \alpha = \frac{1}{2}\left[\frac{L^2\lambda^2}{2\left(1-\frac{\lambda L}{2}\cot\frac{\lambda L}{2}\right)} + \lambda L\cot\frac{\lambda L}{2}\right], \quad \beta = \frac{1}{2}\left[\frac{L^2\lambda^2}{2\left(1-\frac{\lambda L}{2}\cot\frac{\lambda L}{2}\right)} - \lambda L\cot\frac{\lambda L}{2}\right], \\ \nu = \frac{L^2\lambda^2}{2\left(1-\frac{\lambda L}{2}\cot\frac{\lambda L}{2}\right)}, \quad \gamma = \frac{\lambda^3 L^3\cot\frac{\lambda L}{2}}{2\left(1-\frac{\lambda L}{2}\cot\frac{\lambda L}{2}\right)}. \end{cases} \tag{2.16}$$

The member DS matrix shows some similarities with a well-known numeric member stiffness matrix employed in FE analysis. The FE stiffness matrix of a member usually contains two different types of degrees of freedom (DOF). The first type of DOF corresponds to the nodes of each element and the second type of DOF corresponds to the nodes of a whole member. Here by member we understand a single complete beam that can be discretized with any arbitrary number of elements. According to Wittrick and Williams (Wittrick & Williams 1971) in the limit case when the number of finite elements of a structure tends to infinity the so-called condensed dynamic stiffness matrix can be located within the FE numerical stiffness matrix. The condensed stiffness matrix corresponds only to member nodal displacements and approaches the DS matrix. Thanks to this fact, the entries of the DS matrix can be regarded as limits to which the series of the infinite numerical stiffness matrix converges.

The DS matrix is square, symmetric and positive definite similarly to the FE stiffness matrix because of the way how it is formulated from the self-adjoined operator equation. The most important property of this DS matrix is that it is

exact because it simply represents the governing differential equation (2.3) without any discretisation or approximations. Thus, the eigenvalues of the DS matrix correspond to the exact values of the critical buckling loads. Another important property of the DS matrix is that it can be assembled into a global DS matrix following the procedure similar to the assembly of numerical stiffness matrix implemented in FE analysis.

3. Assembly of the complete global dynamic stiffness matrix

Consider a two-span beam compressed by an axial loading P scathed in

Fig. 3.**1**. Each span has different stiffness and is represented by a member shown in figure 2.1. We write stiffness equations as relations between nodal forces and nodal displacements for each member and assemble them into a system of equations:

$$\begin{cases} \mathbf{F}_1 = \mathrm{K}_1(\lambda)\mathbf{w}_1, \\ \mathbf{F}_2 = \mathrm{K}_2(\lambda)\mathbf{w}_2. \end{cases} \tag{3.1}$$

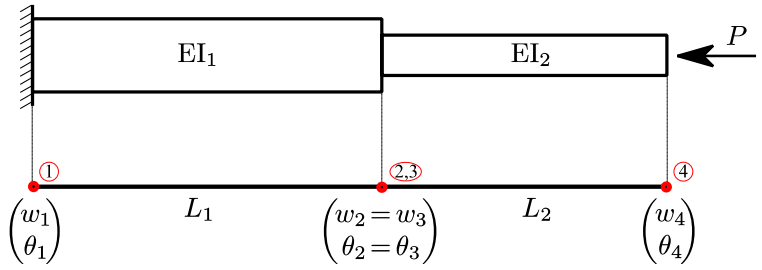

Fig. 3.1. Two-span beam compressed by the axial loading P. Each span has a different stiffness and length denoted by EI_1, EI_2, L_1 and L_2 respectively. The schematic representation of the same structure employed for assembly is scathed under the beam. Straight black lines denote members, while red dots represent nodes of each member. Each node is numbered. The nodal displacements are marked as w_i and θ_i where i is the node number

Here $\mathbf{F}_1$ and $\mathbf{F}_2$ are the vectors of nodal forces acting on each member and $\mathbf{w}_1$ and $\mathbf{w}_2$ are the vectors of nodal displacements respectively. The relations between nodal forces and nodal displacements for each member are defined with member DS matrices $\mathrm{K}_1(\lambda)$ and $\mathrm{K}_2(\lambda)$. Any system of equations can be assembled and presented in a matrix form:

$$\bar{F} = \bar{\mathbf{K}}\bar{w}. \tag{3.2}$$

Here $\bar{F}$ and $\bar{w}$ are the assembled force and displacement vectors and $\bar{\mathbf{K}}$ is the global DS matrix of the structure. Equation (3.2) can be rewritten:

$$\begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix}. \tag{3.3}$$

Here $\mathbf{K}_{11}$ and $\mathbf{K}_{22}$ are the submatrices of the global DS matrix that correspond to the member DS matrices that are overlapped. Overlapping is explained by the fact that degrees of freedom of the coincident nodes (nodes 2 and 3 on Fig. 3.**1**) coincide ensuring the continuity conditions are fulfilled.

Another important property of the global stiffness matrix is that it can be appended with other stiffness matrices that contain essential information about the structure. For instance, the two-span beam may be installed on the elastic foundation (see Fig. 3.**2**). In this case, the complete global DS matrix consists of two matrices:

$$\mathbb{K} = \bar{\mathbf{K}} + \mathbf{K}_s. \tag{3.4}$$

Here $\mathbb{K}$ is a complete global DS matrix and K_s is a matrix containing stiffness of elastic foundation as well as stiffness of artificial coupling between members. In this case, the K_{11} and K_{22} matrices do not overlap because they do not have any coinciding degrees of freedom. On the other hand, the coupling matrix K_s provides the information on a coupling between members and ensures continuity if displacements. The matrix relation between nodal forces and nodal displacements thus reads as:

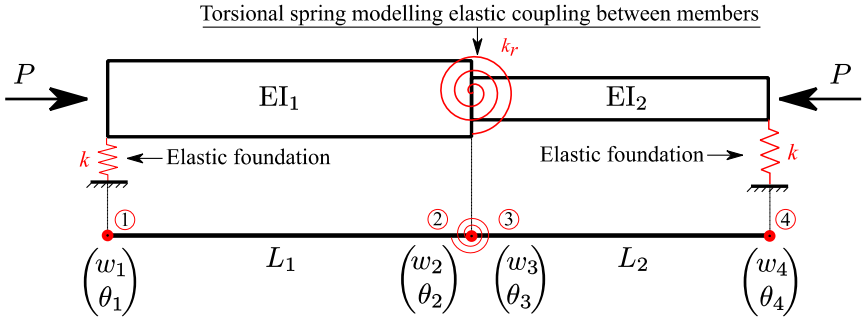

Fig. 3.2. The two-span beam on elastic foundation k with artificial elastic coupling between members introduced by a

$$\begin{matrix} & & \textbf{torsional spring } \mathbf{k_r} \\ \begin{bmatrix}\mathbf{F_1}\\ \mathbf{F_2}\end{bmatrix} = & \begin{bmatrix}\mathbf{K_{11}+K_{s11}} & \mathbf{K_{s12}}\\ \mathbf{K_{s21}} & \mathbf{K_{22}+K_{s22}}\end{bmatrix} & \begin{bmatrix}\mathbf{w_1}\\ \mathbf{w_2}\end{bmatrix}. \end{matrix} \tag{3.5}$$

Here, submatrices $K_{11} + K_{s11}$ and $K_{22} + K_{s22}$ correspond to the member stiffness plus the stiffness of the elastic foundation, while K_{s12} and K_{s21} (off-diagonal terms of the global DS matrix) define stiffness of coupling between members. Matrix $\mathbb{K}$ defined in equation (3.5) is a complete global DS matrix of a structure. In general, this matrix contains enough information to describe the majority of beam assemblies including plain frames with various clamping conditions.

After the complete DS matrix of a structure is assembled it can be studied for eigenvalues that yield the critical buckling loads of the structure. It is known (F. Williams and W. Wittrick 1970) that buckling occurs when compressing load reaches its critical values that correspond to the eigenvalues of the following eigenvalue problem:

$$\begin{bmatrix}\mathbf{K_{11}+K_{s11}} & \mathbf{K_{s12}}\\ \mathbf{K_{s21}} & \mathbf{K_{22}+K_{s22}}\end{bmatrix}\begin{bmatrix}\mathbf{w_1}\\ \mathbf{w_2}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}. \tag{3.6}$$

In matrix notation:

$$\mathbb{K}\mathbf{w} = \mathbf{0}. \tag{3.7}$$

Concluding the analysis presented in this section several key points should be underlined. The DS matrix is a matrix that contains exact information about each structural member stiffness and is derived from the governing differential equation. The DS matrices of each member can be assembled in a global structural DS matrix which contains exact information about the stiffness of a whole structure. The complete DS matrix contains exact information about the stiffness of an arbitrary complex structure provided it is assembled from members which member DS matrices are known. The eigenvalues of the DS matrices directly correspond to the critical buckling loads. The DS matrix is transcendental and contains all the eigenvalue spectrum of the continuous structure.

It is now apparent that DS matrix method is a powerful and universal method that can be used as an alternative to the FE method as any frame structures or, generally speaking, any structure that consists of assemblies of members for which member DS matrices are known can be addressed by this method. This method is exact; thus the common errors associated with the numerical discretization of a continuum are avoided. It is also computationally efficient as the global DS matrices are usually much smaller than those assembled in the FE analysis. It can be explained by the fact that DS matrix is defined for an entire member rather than for a finite element. For example, in the case of a beam member, the DS matrix has only four degrees of freedom that correspond to each node. In FE analysis the corresponding numerical stiffness matrix is considerably larger than the DS one because the beam should be meshed with multiple elements to reach acceptable accuracy. Dependence of the computation accuracy of the FE analysis on the number of elements is a drawback compared to the DS matrix method.

The DS method, however, has its limitations. By virtue of the fact that the DS matrix is exact and possesses intrinsic transcendental dependence on its eigenvalues, the computation of the eigenvalues of equation (3.7) has been an almost unresolvable problem for a long time. This fact along with the difficulties in the assembly of the DS matrices for complex structures have been the major obstacles in development and application of the DS matrix method for decades since it was first introduced. These two issues are discussed in details in the next sections.

4. Eigenvalues of the exact dynamic stiffness matrix

Computation of the eigenvalues of a DS matrix is a known problem (Wittrick & Williams 1971; Williams 1981; Wanxie et al. 1997). Before the computationally efficient Wittrick-Williams algorithm was developed the standard procedure for computing eigenvalues of transcendental matrices was quite inefficient and unreliable (Williams et al. 2002). With the increase in computational capacities of modern computers, the straightforward calculations of the eigenvalues have become more reliable and can be automated to some extent. However, the danger of missing some eigenvalues still exists. In the first subsection of this section, we discuss the issues with the assembly of the DS matrices and the straightforward approach to computing its eigenvalues. The second subsection is dedicated to the Wittrick-Williams algorithm, its strong and weak points.

4.1. Computation of eigenvalues of the parameter dependent DS matrix:

Consider the member DS matrix given by equations (2.15) and (2.16). It is apparent that practical assembly of a considerable number of DS matrices into a global DS matrix is associated with mathematical difficulties as the entries are not float point numbers but some transcendental functions. One solution to this issue would be to use a numerical form of the DS matrix and apply Wittrick-Williams algorithm (W-W algorithm). This option is discussed in details in the next subsection. The second option is reasonable when numeric form of the DS matrix is not convenient. Such situation is possible when an exact analytical dependence of an eigenvalue on some parameter is required.

If the DS matrix is relatively small, i.e. from one to five members, the modern symbolic mathematic software is capable of assembling it in almost no time. Each member of the DS matrix is encapsulated within a symbolic function

that is then passed to a standard assembly process. This process is illustrated in Fig. 4.1.*1* for the case of a two-span beam.

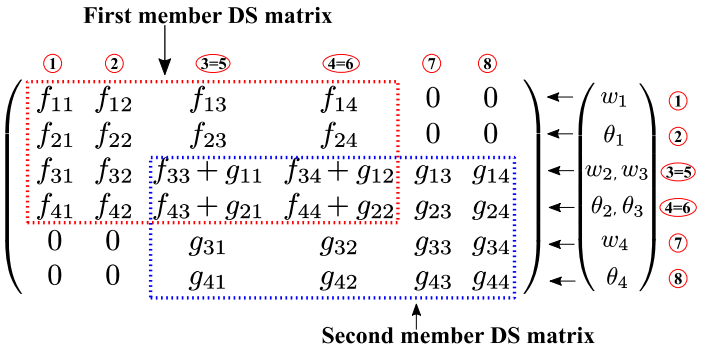

Fig. 4.1.1. The schematic representation of the global DS assembly process for the case of a two-span beam. Dashed rectangles show the member DS matrices. The overlap between two rectangles defines common degrees of freedom of both members. The overlap between matrices ensures that conditions of continuity of the structure are preserved. The corresponding degrees of freedom are numbered

The functions f and g contain expressions for the entries of a member DS matrices given by equation (2.16). Thus, the global DS matrix contains the same functions or their combinations (overlap region) with altered labels defining their positions in the global matrix. It is often required to consider some other means of coupling between members such as springs. Then the coupling matrix should be assembled separately following the same procedure. Thereafter, the complete global DS matrix is assembled according to equations (3.4) and (3.5).

One of the most important strong points of the DS matrix method is that any structural parameter such as member length, stiffness etc. can be regarded as a parameter of the system in general. Because the way how DS matrix is formulated using the exact symbolic functions it can be easy adapted to perform parametric studies. Each of functions f and g can contain an arbitrary number of meaningful physical parameters that describe the structure. Then, it is possible to derive exact analytical dependencies of the eigenvalues of the structure on these parameters. In the case of parametric studies, the straightforward approach to computing the eigenvalues of the DS matrix is desirable.

Consider transcendental eigenvalue problem 3.7. Let us assume that the complete global DS matrix is parameter dependent. It is easy to come up with some practical reasoning for such matrix to exist recalling Fig. 3.**1**. For instance, assume, that the coupling spring stiffness k_r is the governing parameter of the system. Thus, $\mathbb{K}$ matrix would have some functional dependence on two parameters, e.g., its eigenvalue λ and a structural parameter k_r. Then the eigenvalue problem 3.7 will be written as:

$$\mathbb{K}(\lambda, k_r)\mathbf{w} = \mathbf{0}. \tag{4.1.1}$$

This results in either vector of nodal displacements to be zero (trivial solution) or the DS matrix to be singular:

$$\begin{cases} \mathbf{w} = \mathbf{0} \\ \det[\mathbb{K}(\lambda, k_r)] = 0 \end{cases} \tag{4.1.2}$$

The last equation in the system defines the straightforward approach to computing the eigenvalues of the DS matrix. The equation for the determinant $F(\lambda, k_r)$ is formed from the symbolic functions f and g. Then it is set to be zero:

$$F(\lambda, k_r) = 0. \tag{4.1.3}$$

Equation (4.1.3) gives an exact relation between the eigenvalue of a structure in buckling composed of beam members and the structural parameter (spring stiffness in this case). Because it is difficult to study this equation in its symbolic form, it should be studied numerically. It is relatively cheap to transform the equation (4.1.3) into a numerical form. The grid of two vectors $\boldsymbol{\lambda_{1\times n}}$ and $\boldsymbol{k_{r_{1\times n}}}$ should be defined first. Secondly, the evaluation of equation (4.1.3) at each point of the grid results in the following numerical matrix $\mathbf{F_{n\times n}}$:

$$\begin{array}{c|} \\ \lambda_1 \\ \lambda_2 \\ \dots \\ \lambda_n \end{array} \begin{array}{c} \begin{array}{cccc} k_{r1} & k_{r2} & \dots & k_{\mathrm{rn}} \end{array} \\ \begin{pmatrix} F_{11} & F_{12} & \dots & F_{1n} \\ F_{21} & F_{22} & \dots & F_{2n} \\ \dots & \dots & \dots & \dots \\ F_{n1} & F_{n2} & \dots & F_{\mathrm{nn}} \end{pmatrix} \end{array} \tag{4.1.4}$$

This matrix numerically defines a surface of the determinant of the DS matrix with respect to its eigenvalue λ and the structural parameter k_r. This surface should be intersected with the zero plane to obtain parametric eigenvalue curves $\lambda = f(k_r)$. It can be done in a computationally efficient way according to the following steps. Firstly, some near-zero tolerance ε should be chosen $\varepsilon \approx 0$. Secondly, the selection algorithm is run on the entries $F_{ij}, i, j = [1, n]$ with the condition $|F_{km}| \leq \varepsilon$. Therefore, only the entries that are close to zero are selected. The values λ_k, k_{r_m} that correspond to F_{km} are selected from $\boldsymbol{\lambda}$ and $\boldsymbol{k_r}$ vectors. Therefore, a set of unique correspondences $\lambda_k \to k_{r_m}$ for which $|F_{km}| \approx 0$ is defined and can be plotted. Such plots exhibit parameter dependent eigenvalue curves of a structure. In such a way, various important properties of the structure with respect to the change of governing parameters can be studied.

4.2. Wittrick-Williams algorithm:

While the straightforward algorithm for computing the eigenvalues of the DS matrices has some advantages it is not suitable to compute eigenvalues of large DS matrices. The reason is that it is concerned with large symbolical operations that become extremely costly for large DS matrices. Another drawback of the straightforward approach is that it cannot guarantee that exactly all eigenvalues are computed within a chosen interval and none of them is missed. Therefore, this approach requires much of a human's assistance and is limited in automatisation.

On the other hand, the Wittrick-Williams algorithm is known for its ability to locate all the eigenvalues of the DS matrix in the given interval with arbitrary precision without missing a single one (F. Williams and W. Wittrick 1970; Williams & Wittrick 1983; Williams et al. 2004; Wittrick & Williams 1971). Along with the ability to be automated the unprecedented computation efficiency makes W-W algorithm a good option in the case when numerous eigenvalues of a large DS matrix need to be precisely computed (Williams et al. 2004). However efficient and powerful, the W-W algorithm has not received enough attention during the decades since its creation. It is still an option for researchers rather than for large industrial applications due to the complexity of its formulation and relatively small amount of studies focused on its application. However, we believe that this tendency can be changed. When properly studied and programmed along with the straightforward method discussed in the previous subsection, this method can hugely benefit the DS matrix approach and make it a strong competitor to the FE analysis. Thus, this discussion is dedicated to the analysis of the W-W algorithm along with a detailed explanation of the steps that need to be undertaken to program it for the case of buckling. The authors of the algorithm have published a paper dedicated to its application for the case of buckling (Williams & Wittrick 1983) in 1983, however, we believe that this algorithm needs a thorough explanation in light of modern programming techniques.

Consider equation (3.7). In general, it is valid only if either $det\ \mathbb{K} = 0$ or $\boldsymbol{w} = \boldsymbol{0}$ (recall the previous section). The authors stated (Wittrick & Williams 1971) that both conditions (4.1.2) should be considered to ensure that all eigenvalues have been computed. Originally the algorithm was defined to compute the vibration frequencies of the frame structures; thus all the notation and terminology is oriented toward the nomenclature widely accepted in dynamics. However, the algorithm is focused on computing the eigenvalues of various DS matrices that came from various problems. Thus, in general, it can be applied to any eigenvalue problem that is described by equation (3.7).

The main idea of the algorithm is based on Rayleigh's theorem on the frequencies of constrained structures and on the Sturm's theorem on the number of real distinct roots of a polynomial. Firstly, we quote here the Rayleigh's theorem reformulated by the authors (Wittrick & Williams 1971).

Theorem 1. *If one constraint is removed from a structure, the number of natural frequencies which lie below some fixed chosen frequency either remains unchanged or increases by one.*

We underline here that the same statement is true for the case of buckling in the same way as for vibration. The critical buckling loads are obtained from the eigenvalues of the DS matrix for buckling in a similar way that frequencies are computed from the eigenvalues of the DS matrix for vibration.

Secondly, we define necessary background and definitions before stating Sturm's theorem according to (Miller Thomas 1941). Sturm sequence or Sturm chain of polynomials is the sequence

$$f_0, f_1, f_2, \dots, f_n, \tag{4.2.1}$$

of polynomials obtained from a polynomial $f_0(x)$ and its derivative $f_1(x)$ applying Euclid's division algorithm.
The sequence 4.2.1 is proven to possess the following properties (Miller Thomas 1941):

1. The sequence can suffer a change in the number of variations of sign only at a root of $f_0(x)$.
2. The subsequence obtained by omitting the first term f_0 in 4.2.1 does not change the sign at a root of $f_0(x)$.
3. The product $f_0 f_1$ is negative close before the root of $f_0(x)$ and positive close after.

The third property along with the first two provides an essential feature of the Sturm sequence. Just before the root of $f_0(x)$ the product $f_0 f_1$ is negative, then at the root, the sign is lost since $f_0 = 0$ and shortly after the root the product has the same sign as f_1. The number of such lost signs in any specific semi-closed interval gives the exact number of roots of the polynomial $f_0(x)$. Finally, the Sturm's theorem can be stated.

Theorem 2. *Let $f_0, f_1, f_2, \dots, f_n$ be the Sturm chain of a polynomial f(x) and let S(x) denote the number of sign changes (ignoring zeros) in the sequence*

$$f_0(x), f_1(x), f_2(x), \dots, f_n(x). \tag{4.2.2}$$

Then, for two real numbers: $a < b$, $x \in (a, b]$ the number N of distinct rots of f(x) is equal:

$$N = S(a) - S(b). \tag{4.2.3}$$

The idea of the W-W algorithm is to define an arbitrary interval $(a, b]$ and to count the number of all eigenvalues of the DS matrix within this range to multiplicity using two counters based on the Theorem 1 and Theorem 2 respectively. However, this procedure does not give the values for the eigenvalue, only the number of them in the interval. This issue is addressed by performing the iterative reduction of the initial range to a set of intervals each of

which contains only one eigenvalue. Then, the eigenvalues can be easily computed up to any required precision within each interval using numerical root finding techniques such as Newton's method.

Consider a transcendental DS matrix $\mathbb{K}(\lambda)$ and select any arbitrary trial magnitude of the eigenvalue λ_t. We will count the total number of eigenvalues that lie in the range $(0, \lambda_t]$. It will be performed by counting how many times the determinant of DS matrix changes its sign in this interval. Such procedure is called *sign count* of a matrix (Wittrick & Williams 1971).

In order to perform the sign count the matrix $\mathbb{K}(\lambda)$ should be reduced to its upper triangular form $\mathbb{K}(\lambda)^{\Delta}$ using Gauss elimination process without pivoting. It is an essential procedure as it transforms the matrix to a form that satisfies requirements of the Theorem 2. Wittrick and Williams have proven (Wittrick & Williams 1971) that if the sequence of minors of DS matrix $\mathbb{K}(\lambda)$ is constructed in a form:

$$\{K_0, K_1, K_2, \dots, K_n\}, \tag{4.2.4}$$

where K_0 is a zero-order minor $K_0 = +1$ and K_r are the leading principal minors of order r, where $r = \overline{1,n}$ and n is a size of $\mathbb{K}(\lambda)$. And if the matrix is reduced to the upper triangular form without pivoting, then the elements on the main diagonal of $\mathbb{K}(\lambda)^{\Delta}$ form the following sequence:

$$\left\{\frac{K_1}{K_0}, \frac{K_2}{K_1}, \frac{K_3}{K_2}, \dots, \frac{K_n}{K_{n-1}}\right\}. \tag{4.2.5}$$

This sequence possesses the property of the Sturm sequence from Theorem 2. It is essential to ensure that the reduction to the upper triangular form is performed *without pivoting* because otherwise the signs of the sequence will be altered leading to the erroneous results. Thus, the standard LU decomposition that is widely used instead of Gaussian elimination due to its efficiency is not recommended unless it is insured that the decomposition does not alter the signs of the sequence 4.2.5.

After the upper triangular form of the DS matrix $\mathbb{K}(\lambda)^{\Delta}$ has been obtained, the number of roots of its determinant can be computed in a straightforward way according to the Theorem 2. Substitution of $\lambda = \lambda_t$ into $\mathbb{K}(\lambda)^{\Delta}$ yields the numerical form of the matrix $\mathbb{K}(\lambda_t)^{\Delta}$. Count the number of negative elements on the main diagonal to obtain the initial estimate of the number of the eigenvalues of the DS matrix in the chosen interval.

The sign count of a matrix does not give the number of all eigenvalues in the interval if the interval is sufficiently large. This can be easily demonstrated using member DS matrix presented in equation (2.15). It is a 4×4 matrix which becomes a 2×2 matrix if fixed-free boundary conditions are applied. Such matrix reads as:

$$\mathbb{K} = EI \begin{bmatrix} -\dfrac{L\lambda^3 \cot\left(\frac{\lambda L}{2}\right)}{2\left(\frac{\lambda L \cot\left(\frac{\lambda L}{2}\right)}{2} - 1\right)} & \dfrac{L\lambda^2}{2\left(\frac{\lambda L \cot\left(\frac{\lambda L}{2}\right)}{2} - 1\right)} \\ \dfrac{L\lambda^2}{2\left(\frac{\lambda L \cot\left(\frac{\lambda L}{2}\right)}{2} - 1\right)} & -\dfrac{L^2\lambda^2}{4\left(\frac{\lambda L \cot\left(\frac{\lambda L}{2}\right)}{2} - 1\right)} - \dfrac{\lambda L \cot\left(\frac{\lambda L}{2}\right)}{2} \end{bmatrix}. \tag{4.2.6}$$

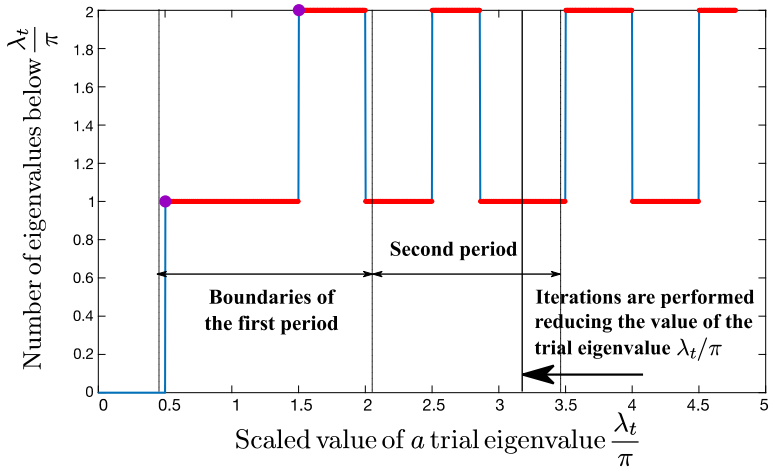

Fig. 4.2.1. The sign count of a DS in the range $\lambda_t = [0, 5\pi]$. The red lines contain multiple dots corresponding to the number of eigenvalues below the current trial one. Purple dots denote small intervals that contain true eigenvalues of the structure. The first and the second periods of the DS matrix are marked with vertical lines. Black arrow shows the direction of iterations performed by reducing the trial eigenvalue by small increments starting from the initial trial value

Thus, the maximum number of negative elements on the main diagonal of such matrix is 2. However, the matrix contains an infinite number of eigenvalues. It is apparent that the sign count can only show the exact number of eigenvalues and only in a small interval that contains no more than two eigenvalues. The sign count of a $n \times n$ matrix is capable of counting no more than n eigenvalues as no more than n negative elements exist on the main diagonal of its upper triangular form. We propose calling the interval $\lambda_t \in [\lambda_1, \lambda_2]$ in which each element on the main diagonal of $\mathbb{K}(\lambda_t)^{\Delta}$ becomes negative only once a *period* of the DS matrix. Each period contains no more than n eigenvalues and the periods are repeated up to infinity. To illustrate this we plotted the sign count of a

matrix given by equation (4.2.6) in an interval $\lambda_t = [0,5\pi]$ against the magnitude of the eigenvalue in Fig. 4.2.***1***.

The figure presents a graph on which the quantity of eigenvalues with magnitude smaller than the magnitude of a chosen trial eigenvalue λ_t/π is plotted against the magnitude of the trial eigenvalue. The computation was performed in the following manner. The interval $\lambda_t = [0,5\pi]$ was split into 1500 small intervals $d\lambda = 0.01$. The vector of trial values was constructed $\boldsymbol{\lambda_t} = [0, d\lambda, 2d\lambda, \dots, id\lambda, \dots, 15]$. Then, each λ_{t_i} was substituted into equation (4.2.6). and the corresponding numerical matrix was reduced to the upper triangular form $\mathbb{K}_i^{\Delta}(\lambda_{t_i})$. A simple count of the number of the negative elements on the main diagonal of each matrix $\mathbb{K}_i^{\Delta}(\lambda_{t_i})$ yielded the vector of sign counts of the DS matrix $\boldsymbol{S}\{\mathbb{K}\} = [S_1, S_2, \dots, S_i, \dots]$. Obviously, each element of $\boldsymbol{S}\{\mathbb{K}\}$ directly corresponds to the element of $\boldsymbol{\lambda_t}$ vector showing how many times the determinant of the DS matrix changed sign from zero to λ_{t_i}.

The map of $\boldsymbol{\lambda_t}$ and $\boldsymbol{S}$ vectors is plotted in Fig. 4.2.***1*** in red dots illustrating the process of converging on eigenvalues of DS matrix from above. Blue lines are drawn solely for the convenience of representation. It is apparent from the figure that the sign count "jumps" from 1 to 2 in a periodic manner. The only difference to this periodicity is exhibited in the first period $\lambda_t = [0.5\pi, 2\pi]$. In this period the sign count changes from 0 to 1 for $\lambda_t = 0.5\pi$ stating that a single eigenvalue exists in the interval $[0.5\pi - d\lambda, 0.5\pi + d\lambda]$. Then, the sign count increases from 1 to 2 stating that another eigenvalue exists in the interval $[1.5\pi - d\lambda, 1.5\pi + d\lambda]$. It is known from the textbook (Timoshenko & Gere 2012) that the exact magnitudes of the lowest two eigenvalues for the fixed-free beam are $\lambda_1 = 0.5\pi$ and $\lambda_2 = 1.5\pi$. These points are shown in the figure with purple dots. Thus, we see that the sign count of the matrix easily defines small intervals that enclose the eigenvalues of the DS matrix in the first period. However, the sign count does not produce any reliable information in the second period and so forth.

In cases when higher eigenvalues of the buckling problem are required the second counter of the W-W algorithm should be applied along with the sign count. If the sign count gives the precise number of the eigenvalues within each period of the DS matrix the second counter should obviously be able to account for the number of periods and for the number of the eigenvalues located within each period. The formulation of such counter is based on the Theorem 1. According to the original paper by Wittrick and Williams (Wittrick & Williams 1971), the counter $J_0(\lambda_{t_i})$ is equal to the number of eigenvalues that would lie below λ_{t_i} if the structure has been completely constrained (all possible degrees of freedom suppressed). In our case, $J_0(\lambda_{t_i})$ equals to the number of eigenvalues of a fixed-fixed beam that are smaller in magnitude than λ_{t_i}. All the eigenvalues of a fixed-fixed beam are known (Timoshenko & Gere 2012) thus $J_0(\lambda_{t_i})$ can be easily computed. In a case when there are many members assembled in one structure the $J_0(\lambda_{t_i})$ counter should be simply multiplied by the number of members. If the members are of different kinds the $J_0(\lambda_{t_i})$ counter for each kind of members should be computed and then all the counters summed up. The simplicity with which the multiple substructures are encountered defines the power of the W-W algorithm for the case of complex hierarchical structures with multiple substructures.

The W-W algorithm can now be stated (Wittrick & Williams 1971). When converging from above, i.e. performing iterative reduction of the trial eigenvalue $\lambda_{t_i} = \lambda_t - id\lambda_t$ the total number of the eigenvalues that lie below the chosen one is equal:

$$J(\lambda_{t_i}) = J_0(\lambda_{t_i}) + s\{\mathbb{K}(\lambda_{t_i})\}, \tag{4.2.7}$$

where $J_0(\lambda_{t_i})$ is the number of eigenvalues smaller than λ_{t_i} if the structure were fully constrained and $s\{\mathbb{K}(\lambda_{t_i})\}$ is the sign count of the DS matrix $\mathbb{K}$.

The application of the algorithm is illustrated for the case of a single fixed-free member defined by equation (4.2.6) in Fig. 4.2.***2***. It is apparent from the second graph in Fig. 4.2.***2*** that the W-W algorithm has located the small intervals $[\lambda_{t_i} - d\lambda, \lambda_{t_i} + d\lambda]$ that contains eigenvalues of the DS matrix. By changing initial interval and running the algorithm one obtains any desired number of eigenvalue estimations. If the precision of the magnitudes of eigenvalues required is higher than the chosen step, then either the step should be reduced or a root finding technique (Newton's method) applied within each small interval that contains eigenvalues.

Summarizing everything stated above the following set of steps to calculate eigenvalues of the assembled dynamic stiffness $\mathbb{K}$ matrix is proposed:

- Chose initial eigenvalue interval $(0, \lambda_t]$.
- Substitute the value λ_t into functions in equation (2.16) that define entries of $\mathbb{K}$ matrix. This operation will transform the DS matrix into a numerical matrix $\mathbb{K}(\lambda_t)$.
- Reduce matrix $\mathbb{K}(\lambda_t)$ to the upper triangular form $\mathbb{K}(\lambda_t)^{\Delta}$ using the Gauss technique without pivoting.
- Count the number of the negative elements on the main diagonal of $\mathbb{K}(\lambda_t)^{\Delta}$. This number $s\{\mathbb{K}(\lambda_t)\}$ is the initial estimation of the number of eigenvalues in the chosen interval.
- Compute $J_0(\lambda)$ correction to the initial count by evaluating the number of the fixed-fixed solutions of each member with magnitude smaller than λ_t. If the number of substructures is n, then the correction for the whole structure is defined as $nJ_0(\lambda)$.

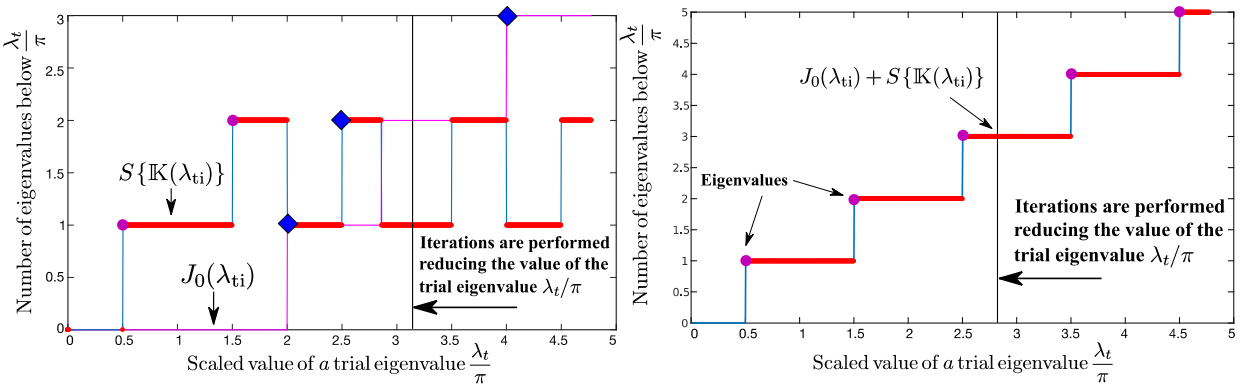

Fig. 4.2.2. The first graph represents the sign count of a matrix as in Fig. 4.2.1 along with the graph of $J_0(\lambda_{t_i})$. The purple dots represent the eigenvalues located by the sign count of a matrix in the first period. The blue squares correspond to changes in the $J_0(\lambda_{t_i})$ counter. The second graph shows the total counts of the eigenvalues according to W-W algorithm. The lines on the second graph were obtained by summing two lines on the first graph. The purple dots indicate the eigenvalues of the DS matrix located by W-W algorithm

- The sum of the initial estimation and the correction $J_1(\lambda_t) = J_{0_1}(\lambda_t) + s\{\mathbb{K}(\lambda_t)\}$ gives the exact number of eigenvalues with magnitudes smaller than λ_t. Here sub-subscript 1 denotes the first iteration.
- Start the second iteration by reducing λ_t by a small increment $d\lambda_t$. Thus, the new interval is written as $(0, \lambda_t - d\lambda_t] = (0, \lambda_{t_2}]$.

- Repeat the procedure for the new eigenvalue interval. The new number of eigenvalues $J_2(\lambda_t) = J_{0_2}(\lambda_{t_2}) + s\{\mathbb{K}(\lambda_{t_2})\}$ that lie within the new interval is computed. Then, the two numbers J_1 and J_2 are compared. If these two numbers coincide, then no eigenvalue of the dynamic stiffness matrix $\mathbb{K}$ lie in the small interval $d\lambda_t$. If, however, these two numbers are different, i.e. $J_1 - J_2 = z$, then z eigenvalues lie in the small interval $d\lambda_t$.
- Start the third iteration from reducing the initial interval by the increment $2d\lambda_t$ and repeat the procedure described in the previous steps. Following iterations are performed in the same way until either $\lambda_t - nd\lambda_t = 0$, or $J_n(\lambda) = 0$. These conditions ensure that the whole initial interval was analysed.
- In such a way, the whole initial interval is divided into n small intervals $d\lambda_t$. The number of eigenvalues within each interval is known. The intervals for which $J_i - J_{i+1} = z$ contain z eigenvalues while those for which $J_i - J_{i+1} = 0$ do not contain any eigenvalues. Iterative root finding operation can be performed in each small interval that contains eigenvalues to locate eigenvalues up to any required precision.

5. Conclusions

The DS matrix method has been shown to be a powerful approach to study elastic stability of structures. The method is exact and provides analytical dependences between the eigenvalues or critical buckling loads of structures and the physical parameters of these structures. The approach was discussed in details at all stages. Firstly, we have shown how member DS matrices are derived from member governing differential equations. Secondly, we have explained the process of assembly of the complete DS matrix of a structure from the member DS matrices along with stiffness matrices that contain information on coupling between members or on elastic supports. Finally, we discussed the approaches to compute the arbitrary number of eigenvalues of DS matrices with any required precision.

We believe that the DS matrix approach can be a good substitution for the FE analysis in many cases. In buckling it can be successfully applied to exactly compute the critical buckling loads of complex hierarchical structures that consist of large number of members much faster than by FE method. The DS matrix method is also capable of giving the exact analytical dependences of the critical buckling loads on any number of physical parameters which is essential for parametric studies. There is no known engineering software that operates with the DS matrix method for buckling. However, we have shown that this method can be automated and made very universal. Member DS matrices are known and the assembly can be easily programmed following the guidelines given here. The computation of the eigenvalues of the complete global DS matrix according to the W-W algorithm can also be readily automated by the appropriate computer routine developed following the steps proposed in section 4.2.

The DS method can also be extended beyond the problems of vibrations or buckling. As long as the member DS matrix is known it can be used along with appropriately modified W-W algorithm to address various physical problems. This fact makes the DS matrix method as universal as the FE analysis.

Аннотация. *Статья посвящена обсуждению применения точного метода динамической матрицы жесткости к проблемам упругой устойчивости инженерных конструкций. В статье представлен вывод динамической матрицы жесткости для структурных балочных компонентов с последующим руководством к автоматизации сборки глобальной динамической матрицы жесткости для всей конструкции из матриц для каждого компонента. Также обсуждаются преимущества метода динамической матрицы жесткости в случае параметрических расчетов. Проанализирована проблема расчета собственных значений трансцендентной матрицы жесткости. В деталях анализируются как метод прямого расчета собственных значений, так и мощный алгоритм Виттрика-Вильямса. В статье также приводится общее руководство к программированию метода динамической матрицы жесткости.*
Ключевые слова: Потеря устойчивости, динамическая матрица жесткости, собственные значения, алгоритм Виттрика-Вильямса.

Анотація. *Стаття присвячена обговоренню застосування точного методу динамічної матриці жорсткості до проблем пружної стійкості інженерних конструкцій. В статті наведено вивід динамічної матриці жорсткості для структурних балочних компонентів з наступним керівництвом щодо автоматизації зборки глобальної динамічної матриці жорсткості для всієї конструкції з матриць для кожного компонента. Також обговорюються переваги метода динамічної матриці жорсткості у випадку параметричних розрахунків. Проаналізована проблема розрахунку власних значень трансцендентної матриці жорсткості. В деталях аналізується як метод прямого розрахунку власних значень, так і потужний метод Віттріка-Вільямса. В статті також наводиться загальна інструкція до програмування методу динамічної матриці жорсткості.*
Ключові слова: Втрата стійкості, динамічна матриця жорсткості, власні значення, алгоритм Віттріка-Вільямса.

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Подана до редакції 03.09.2016