ACTA ARITHMETICA
136.1 (2009)

Exact exponent in the remainder term of Gelfond's digit theorem in the binary case

by

VLADIMIR SHEVELEV (Beer-Sheva)

1. Introduction. For integers $m > 1$ and $a \in [0, m-1]$, define

$$T_{m,a}^{(j)}(x) = \sum_{0 \le n < x,\, n \equiv a \bmod m,\, s(n) \equiv j \bmod 2} 1, \qquad j = 1, 2, \tag{1}$$

where $s(n)$ is the number of 1's in the binary expansion of n. Gelfond [7] proved that

$$T_{m,a}^{(j)}(x) = \frac{x}{2m} + O(x^{\lambda}), \qquad j = 0, 1, \tag{2}$$

where

$$\lambda = \frac{\ln 3}{\ln 4} = 0.79248125\ldots. \tag{3}$$

This is the binary case of Gelfond's main digit theorem about the distribution of digit sums of arbitrary base in different residue classes. Gelfond's theorem initiated a whole line of research (see Notes on Chapter 3 in [1], as well as [10], [3], [9]). A related circle of works, dealing with the so-called Newman-like phenomena, started with the unexpected results of D. J. Newman [11] (see also [2], [5], [15]; again, an extensive bibliography may be found in [1]). In this paper, we shall be concerned only with the binary case of Gelfond's digit theorem. Recently, the author proved [13] that the exponent λ in the remainder term in (2) is the best possible when m is a multiple of 3 and is not the best possible otherwise. In this paper we give a simple formula for the exact exponent in the remainder term of (2) for an arbitrary m. Our method is based on constructing a recursion relation for the Newman-like sum corresponding to (1),

$$S_{m,a}(x) = \sum_{0 \le n < x,\, n \equiv a \bmod m} (-1)^{s(n)}. \tag{4}$$

2000 *Mathematics Subject Classification*: Primary 11A63.
Key words and phrases: cyclotomic cosets of 2 modulo m, order of 2 modulo m, difference equation, trigonometric sums.

DOI: 10.4064/aa136-1-7

It is sufficient for our purposes to deal with odd numbers m. Indeed, it is easy to see that, if m is even, then

$$S_{m,a}(2x) = (-1)^a S_{m/2,\lfloor a/2\rfloor}(x). \tag{5}$$

For $m > 1$ odd, consider the number $r = r(m)$ of distinct cyclotomic cosets of 2 modulo m [8, pp. 104–105]. E.g., $r(15) = 4$, since for $m = 15$ we have the following four cyclotomic cosets of 2: $\{1, 2, 4, 8\}$, $\{3, 6, 12, 9\}$, $\{5, 10\}$, $\{7, 14, 13, 11\}$.

Note that, if $C_1, \dots, C_r$ are all different cyclotomic cosets of 2 modulo m, then

$$\bigcup_{j=1}^{r} C_j = \{1, \dots, m-1\}, \qquad C_{j_1} \cap C_{j_2} = \emptyset, \quad j_1 \neq j_2. \tag{6}$$

Let h be the least common multiple of $|C_1|, \dots, |C_r|$,

$$h = [|C_1|, \dots, |C_r|]. \tag{7}$$

Note that h is of order 2 modulo m (this follows easily, e.g., from Exercise 3, p. 104 in [12]).

Definition 1. The *exact exponent* in the remainder term in (2) is $\alpha = \alpha(m)$ if

$$T^j_{m,a}(x) = \frac{x}{2m} + O(x^{\alpha+\varepsilon}), \qquad T^j_{m,a}(x) = \frac{x}{2m} + \Omega(x^{\alpha-\varepsilon}), \qquad \forall\varepsilon > 0.$$

Our main result is the following.

Theorem 1. *If $m \geq 3$ is odd, then the exact exponent in the remainder term in* (2) *is*

$$\alpha = \max_{1\leq l\leq m-1}\left(1 + \frac{1}{h\ln 2}\sum_{k=0}^{h-1}\left(\ln\left|\sin\frac{\pi l 2^k}{m}\right|\right)\right). \tag{8}$$

Note that, if 2 is a primitive root of an odd prime p, then $r = 1$, $h = p-1$. As a corollary of Theorem 1 we obtain the following result.

Theorem 2. *If p is an odd prime for which* 2 *is a primitive root, then the exact exponent in the remainder term in* (2) *is*

$$\alpha = \frac{\ln p}{(p-1)\ln 2}. \tag{9}$$

Theorem 2 generalizes the well-known result for $p = 3$ ([11], [2], [1]). Furthermore, we say that 2 is a *semiprimitive root* modulo p if 2 is of order $(p-1)/2$ modulo p and the congruence $2^x \equiv -1 \bmod p$ is not solvable. E.g., 2 is of order 8 modulo 17, but the congruence $2^x \equiv -1 \bmod 17$ has the solution $x = 4$. Therefore, 2 is not a semiprimitive root of 17. The first primes for which 2 is a semiprimitive root are (see [14, A 139035])

$$7, 23, 47, 71, 79, 103, 167, 191, 199, 239, 263, \dots . \tag{10}$$

For these primes we have $r = 2$ and $h = (p-1)/2$. As a second corollary of Theorem 1 we obtain the following result.

THEOREM 3. *If p is an odd prime for which 2 is a semiprimitive root, then the exact exponent α in the remainder term in* (2) *is also given by* (9).

We also prove the following lower estimate for $\alpha(m)$.

THEOREM 4. *For m odd,*

$$\alpha(m) \geq \frac{\ln m}{rh \ln 2}.$$

In particular, if $m = p$ is prime, then $rh = p - 1$ and

$$\alpha(p) \geq \frac{\ln p}{(p-1)\ln 2}.$$

Note that, if Artin's conjecture of the infinity of primes for which 2 is a primitive root is true, then by Theorem 2,

$$\liminf_{p\to\infty} \alpha(p) = 0.$$

In Section 2 we provide an explicit formula for $S_{m,a}(x)$, while in Sections 3–4 we prove Theorems 1–4.

2. Explicit formula for $S_{m,a}(x)$. Let $\lfloor x \rfloor = N$. We have

$$S_{m,a}(N) = \sum_{n=0,\, m|n-a}^{N-1} (-1)^{s(n)} = \frac{1}{m} \sum_{t=0}^{m-1} \sum_{n=0}^{N-1} (-1)^{s(n)} e^{2\pi i(n-a)t/m} \tag{11}$$
$$= \frac{1}{m} \sum_{t=0}^{m-1} \sum_{n=0}^{N-1} e^{2\pi i(\frac{t}{m}(n-a)+\frac{1}{2}s(n))}.$$

Note that the interior sum is of the form

$$\Phi_{a,\beta}(N) = \sum_{n=0}^{N-1} e^{2\pi i(\beta(n-a)+\frac{1}{2}s(n))}, \quad 0 \leq \beta < 1. \tag{12}$$

Putting

$$F_\beta(N) = e^{2\pi i\beta a} \Phi_{a,\beta}(N), \tag{13}$$

we note that $F_\beta(N)$ does not depend on a.

LEMMA 1. *If $N = 2^{\nu_0} + 2^{\nu_1} + \dots + 2^{\nu_\sigma}$ with $\nu_0 > \nu_1 > \dots > \nu_\sigma \geq 0$, then*

$$F_\beta(N) = \sum_{g=0}^{\sigma} e^{2\pi i(\beta \sum_{j=0}^{g-1} 2^{\nu_j} + g/2)} \prod_{k=0}^{\nu_g - 1} (1 + e^{2\pi i(\beta 2^k + 1/2)}). \tag{14}$$

Proof. Let $\sigma = 0$. Then by (12) and (13),

$$F_\beta(N) = \sum_{n=0}^{N-1} (-1)^{s(n)} e^{2\pi i\beta n} \tag{15}$$
$$= 1 - \sum_{j=0}^{\nu_0-1} e^{2\pi i\beta 2^j} + \sum_{0\le j_1<j_2\le \nu_0-1} e^{2\pi i\beta(2^{j_1}+2^{j_2})} - \cdots$$
$$= \prod_{k=0}^{\nu_0-1} (1 - e^{2\pi i\beta 2^k}),$$

which corresponds to (14) for $\sigma = 0$.

Assuming that (14) is valid for every N with $s(N) = \sigma+1$, let us consider $N_1 = 2^{\nu_\sigma} b + 2^{\nu_{\sigma+1}}$ where b is odd, $s(b) = \sigma + 1$ and $\nu_{\sigma+1} < \nu_\sigma$. Let

$$N = 2^{\nu_\sigma} b = 2^{\nu_0} + \cdots + 2^{\nu_\sigma},$$
$$N_1 = 2^{\nu_0} + \cdots + 2^{\nu_\sigma} + 2^{\nu_{\sigma+1}}.$$

Notice that for $n \in [0, 2^{\nu_{\sigma+1}})$ we have

$$s(N + n) = s(N) + s(n).$$

Therefore,

$$F_\beta(N_1) = F_\beta(N) + \sum_{n=N}^{N_1-1} e^{2\pi i(\beta n+\frac{1}{2}s(n))}$$
$$= F_\beta(N) + \sum_{n=0}^{2^{\nu_{\sigma+1}}-1} e^{2\pi i(\beta n+\beta N+\frac{1}{2}(s(N)+s(n)))}$$
$$= F_\beta(N) + e^{2\pi i(\beta N+\frac{1}{2}s(N))} \sum_{n=0}^{2^{\nu_{\sigma+1}}-1} e^{2\pi i(\beta n+\frac{1}{2}s(n))}.$$

Thus, by (14) and (15),

$$F_\beta(N_1) = \sum_{g=0}^{\sigma} e^{2\pi i(\beta\sum_{j=0}^{g-1} 2^{\nu_j}+g/2)} \prod_{k=0}^{\nu_g-1} (1 + e^{2\pi i(\beta 2^k+1/2)})$$
$$+ e^{2\pi i(\beta\sum_{j=0}^{\sigma} 2^{\nu_j}+(\sigma+1)/2)} \prod_{k=0}^{\nu_{g+1}-1} (1 + e^{2\pi i(\beta 2^k+1/2)})$$
$$= \sum_{g=0}^{\sigma+1} e^{2\pi i(\beta\sum_{j=0}^{g-1} 2^{\nu_j}+g/2)} \prod_{k=0}^{\nu_g-1} (1 + e^{2\pi i(\beta 2^k+1/2)}). \ \blacksquare$$

Formulas (11)–(14) give an explicit expression for $S_m(N)$ as a linear combination of products of the form

$$\prod_{k=0}^{\nu_g-1}(1+e^{2\pi i(\beta 2^k+1/2)}), \qquad \beta=t/m,\ 0\le t\le m-1. \tag{16}$$

Remark 1. One may derive (14) from a very complicated general formula of Gelfond [7]. However, we preferred to give an independent proof.

In particular, if $N=2^\nu$, then from (11)–(13) and (15) for

$$\beta=t/m, \quad t=0,1,\ldots,m-1, \tag{17}$$

we obtain the known formula (cf. [4]):

$$S_{m,a}(2^\nu)=\frac{1}{m}\sum_{t=1}^{m-1}e^{-2\pi i\frac{t}{m}a}\prod_{k=0}^{\nu-1}(1-e^{2\pi i\frac{t}{m}2^k}). \tag{18}$$

3. Proof of Theorem 1. Consider the equation of order r

$$z^r+c_1z^{r-1}+\cdots+c_r=0 \tag{19}$$

with the roots

$$z_j=\prod_{t\in C_j}(1-e^{2\pi it/m}), \qquad j=1,\ldots,r. \tag{20}$$

Notice that for $t\in C_j$ we have

$$\prod_{k=n+1}^{n+h}(1-e^{2\pi it2^k/m})=\Big(\prod_{t\in C_j}(1-e^{2\pi it/m})\Big)^{h/h_j}=z_j^{h/h_j}, \tag{21}$$

where h is defined by (7). Therefore, for every $t\in\{1,\ldots,m-1\}$, according to (19) we have

$$\prod_{k=n+1}^{n+rh}(1-e^{2\pi it2^k/m}) + c_1\prod_{k=n+1}^{n+(r-1)h}(1-e^{2\pi it2^k/m})+\cdots+c_{r-1}\prod_{k=n+1}^{n+h}(1-e^{2\pi it2^k/m})+c_r=0. \tag{22}$$

After multiplication by $e^{-2\pi i\frac{t}{m}a}\prod_{k=0}^{n}(1-e^{2\pi it2^k/m})$ and summing over $t=1,\ldots,m-1$, by (18) we find

$$S_{m,a}(2^{n+rh+1})+c_1S_{m,a}(2^{n+(r-1)h+1}) +\cdots+c_{r-1}S_{m,a}(2^{n+h+1})+c_rS_{m,a}(2^{n+1})=0. \tag{23}$$

Moreover, using the general formulas (11)–(14) for a positive integer u, we obtain the equality

$$(24)\qquad S_{m,a}(2^{rh+1}u)+c_1S_{m,a}(2^{(r-1)h+1}u)+\cdots+c_{r-1}S_{m,a}(2^{h+1}u)+c_rS_{m,a}(2u)=0.$$

Putting here

$$(25)\qquad S_{m,a}(2^u)=f_{m,a}(u),$$

we have

$$(26)\qquad f_{m,a}(y+rh+1)+c_1f_{m,a}(y+(r-1)h+1)+\cdots+c_{r-1}f_{m,a}(y+h+1)+c_rf_{m,a}(y+1)=0,$$

where

$$(27)\qquad y=\log_2 u.$$

The characteristic equation of (26) is

$$(28)\qquad v^{rh}+c_1v^{(r-1)h}+\cdots+c_{r-1}v^h+c_r=0.$$

A comparison of (28) and (20)–(21) shows that the roots of (28) are

$$(29)\quad v_{j,w}=e^{2\pi iw/h}\prod_{t\in C_j}(1-e^{2\pi it/m})^{1/h},\qquad w=0,\dots,h-1,\ j=1,\dots,r.$$

Thus,

$$(30)\qquad v=\max|v_{j,l}|=2\max_{1\le l\le m-1}\left(\prod_{k=0}^{h-1}\left|\sin\frac{\pi l2^k}{m}\right|\right)^{1/h}.$$

Generally speaking, some numbers in (20) could be equal. In view of (29), the $v_{j,w}$'s have the same multiplicities. If η is the maximal multiplicity, then according to (25) and (27),

$$(31)\qquad S_{m,a}(u)=f_{m,a}(\log_2 u)=O((\log_2 u)^{\eta-1}u^{\ln v/\ln 2}).$$

Nevertheless, at least

$$(32)\qquad S_{m,a}(u)=\Omega(u^{\ln v/\ln 2}).$$

Indeed, let, say, $v=|v_{1,w}|$ and suppose that in the solution of (26) with some natural initial conditions, all coefficients of $y^{j_1}v_{1,w}^y$, $j_1\le\eta-1$, $w=0,\dots,h-1$, are 0. Then $f_{m,a}(y)$ satisfies a difference equation with the characteristic equation not having roots $v_{1,w}$, and the corresponding relation for $S_{m,a}(2^n)$ (see (23)) has the characteristic equation (19) without the root z_1. This is impossible since by (18) and (21) we have

$$S_{m,a}(2^{h+1})=\frac{1}{m}\sum_{j=1}^{r}\sum_{t\in C_j}e^{-2\pi i\frac{t}{m}a}\prod_{k=1}^{h}(1-e^{2\pi i\frac{t}{m}2^k})=\frac{1}{m}\sum_{j=1}^{r}\sum_{t\in C_j}e^{-2\pi i\frac{t}{m}a}z_j^{h/h_j}.$$

Therefore, the coefficients considered do not all vanish, and (32) follows. Now from (30)–(32) we obtain (8). ∎

REMARK 2. In (8) it is sufficient to let l run over a system of distinct representatives of the cyclotomic cosets $C_1, \ldots, C_r$ of 2 modulo m.

REMARK 3. It is easy to see that there exists $l \geq 1$ such that $|C_l| = 2$ if and only if m is a multiple of 3. Moreover, for l we can take $m/3$. Now from (8), choosing $l = m/3$, we obtain $\alpha = \lambda = \ln 3/\ln 4$. This result was obtained in [13] together with estimates of the constants in $S_{m,0}(x) = O(x^\lambda)$ and $S_{m,0}(x) = \Omega(x^\lambda)$ which are based on the formula, proved in [13],

$$S_{m,0}(x) = \frac{3}{m} S_{3,0}(x) + O(x^{\lambda_1})$$

for $\lambda_1 = \lambda_1(m) < \lambda$ and Coquet's theorem [2].

EXAMPLE 1. Let $m = 17$, $a = 0$. Then $r = 2$, $h = 8$,

$$C_1 = \{1, 2, 4, 8, 16, 15, 13, 9\}, \qquad C_2 = \{3, 6, 12, 7, 14, 11, 5, 10\}.$$

The calculation of

$$\alpha_l = 1 + \frac{1}{8 \ln 2} \sum_{k=0}^{17} \left(\ln \left| \sin \frac{\pi l 2^k}{17} \right| \right)$$

for $l = 1$ and $l = 3$ gives

$$\alpha_1 = -0.12228749\ldots, \qquad \alpha_3 = 0.63322035\ldots.$$

Therefore by Theorem 1, $\alpha = 0.63322035\ldots$. Moreover, we will prove that

$$\alpha = \frac{\ln(17 + 4\sqrt{17})}{\ln 256}.$$

Indeed, according to (23), for $n = 0$ and $n = 1$ we obtain the system ($S_{17,0} = S_{17}$):

$$\begin{cases} c_1 S_{17}(2^9) + c_2 S_{17}(2) = -S_{17}(2^{17}), \\ c_1 S_{17}(2^{10}) + c_2 S_{17}(2^2) = -S_{17}(2^{18}). \end{cases} \tag{33}$$

By direct calculations we find

$$S_{17}(2) = 1, \qquad S_{17}(2^2) = 1, \qquad S_{17}(2^9) = 21,$$
$$S_{17}(2^{10}) = 29, \quad S_{17}(2^{17}) = 697, \quad S_{17}(2^{18}) = 969.$$

Solving (33) we obtain

$$c_1 = -34, \qquad c_2 = 17.$$

Thus, by (23) and (24),

$$S_{17}(2^{n+17}) = 34 S_{17}(2^{n+9}) - 17 S_{17}(2^{n+1}), \qquad n \geq 0, \tag{34}$$

$$S_{17}(2^{17}x) = 34 S_{17}(2^9 x) - 17 S_{17}(2x), \qquad x \in \mathbb{N}. \tag{35}$$

Putting furthermore

$$S_{17}(2^x) = f(x), \tag{36}$$

we have

$$f(y+17) = 34f(y+9) - 17(y+1),$$

where $y = \log_2 x$. Hence,

$$f(x) = O((17 + 4\sqrt{17})^{x/8}),$$

that is

$$S_{17}(x) = O((17 + 4\sqrt{17})^{\frac{1}{8}\log_2 x}) = O(x^{\alpha}), \tag{37}$$

where

$$\alpha = \frac{\ln(17 + 4\sqrt{17})}{\ln 256} = 0.633220353\ldots.$$

4. Proofs of Theorems 2–4

Proof of Theorem 2. By the assumptions of Theorem 2 we have $r = 1$ and $h = p - 1$. Using (8) we have

$$\alpha = 1 + \frac{1}{(p-1)\ln 2} \ln \prod_{k=0}^{p-2} \left| \sin \frac{\pi 2^k}{p} \right|$$
$$= 1 + \frac{1}{(p-1)\ln 2} \ln \prod_{l=1}^{p-1} \sin \frac{\pi l}{p}.$$

Furthermore, using the identity

$$\prod_{l=1}^{p-1} \sin \frac{l\pi}{p} = \frac{p}{2^{p-1}} \tag{38}$$

([6, p. 378] for example), we find

$$\alpha = 1 + \frac{1}{(p-1)\ln 2}(\ln p - (p-1)\ln 2) = \frac{\ln p}{(p-1)\ln 2}. \ \blacksquare$$

Remark 4. In this case, (24) has the simple form

$$S_{p,a}(2^p u) + c_1 S_{p,a}(2u) = 0.$$

Since in the case of $a = 0$ or 1 we have

$$S_{p,a}(2) = (-1)^{s(a)},$$

while in the case of $a \geq 2$,

$$S_{p,a}(2a) = (-1)^{s(a)},$$

putting

$$u = \begin{cases} 1, & a = 0, 1, \\ a, & a \geq 2, \end{cases}$$

we find

$$c_1 = (-1)^{s(a)+1} \begin{cases} S_{p,a}(2^p), & a = 0, 1, \\ S_{p,a}(a2^p), & a \ge 2. \end{cases}$$

In particular, if $p = 3$ and $a = 2$ we have $c_1 = S_{3,2}(16) = -3$ and

$$S_{3,2}(8u) = 3S_{3,2}(2u).$$

Proof of Theorem 3. By the assumptions of Theorem 3 we have $r = 2$ and $h = (p-1)/2$, so that cyclotomic cosets of 2 modulo p satisfy

$$C_1 = -C_2.$$

Therefore, in (8) we obtain the same values for $l_1 = 1$ and $l_2 = p - 1$. Thus,

$$\alpha = 1 + \frac{2}{(p-1)\ln 2} \ln\biggl(\prod_{l=1}^{p-1} \sin\frac{\pi l}{p}\biggr)^{1/2} = \frac{\ln p}{(p-1)\ln 2}. \ \blacksquare$$

Proof of Theorem 4. According to (19)–(20),

$$c_r = (-1)^r \prod_{j=1}^{r} \prod_{t\in C_j} (1 - e^{2\pi it/m}) = (-1)^r \prod_{t=1}^{m-1} (1 - e^{2\pi it/m}).$$

Thus, using (38) we have

$$|c_r| = 2^m \prod_{t=1}^{m-1} \sin\frac{\pi t}{m} = m.$$

Consequently, by (29),

$$\prod_{j=1}^{r} |v_{j,w}| = m^{1/h}, \qquad w = 0, 1, \ldots, h-1.$$

Therefore,

$$v = \max|v_{j,w}| \ge m^{1/rh}$$

and Theorem 4 follows. ■

Using Theorems 1–3, in particular, we find

$$\alpha(3) = 0.7924\ldots, \quad \alpha(5) = 0.5804\ldots, \quad \alpha(7) = 0.4678\ldots,$$
$$\alpha(11) = 0.3459\ldots, \quad \alpha(13) = 0.3083\ldots, \quad \alpha(17) = 0.6332\ldots,$$
$$\alpha(19) = 0.2359\ldots, \quad \alpha(23) = 0.2056\ldots, \quad \alpha(29) = 0.1734\ldots,$$
$$\alpha(31) = 0.6358\ldots, \quad \alpha(37) = 0.1447\ldots, \quad \alpha(41) = 0.4339\ldots,$$
$$\alpha(43) = 0.6337\ldots, \quad \alpha(47) = 0.1207\ldots.$$

Acknowledgments. The author is grateful to Professor Daniel Berend (Department of Mathematics and of Computer Science, Ben-Gurion University) for his significant influence on the final version of this paper.

The article is partly supported by a grant of the Kamea Foundation of Israeli Ministry of Absorption.

References

[1] J.-P. Allouche and J. Shallit, *Automatic Sequences: Theory, Applications, Generalizations*, Cambridge Univ. Press, 2003.

[2] J. Coquet, *A summation formula related to the binary digits*, Invent. Math. 73 (1983), 107–115.

[3] C. Dartige et G. Tenenbaum, *Sommes des chiffres de multiples d'entiers*, Ann. Inst. Fourier (Grenoble) 55 (2005), 2423–2474.

[4] M. Drmota and M. Skałba, *Rarified sums of the Thue–Morse sequence*, Trans. Amer. Math. Soc. 352 (1999), 609–642.

[5] M. Drmota and T. Stoll, *Newman's phenomenon for generalized Thue–Morse sequences*, Discrete Math. 308 (2008), 1191–1208.

[6] G. Freiman and H. Halberstam, *On a product of sines*, Acta Arith. 49 (1988), 377–385.

[7] A. O. Gelfond, *Sur les nombres qui ont des propriétés additives et multiplicatives données*, ibid. 13 (1968), 259–265.

[8] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, Elsevier/North-Holland, 1977.

[9] C. Mauduit et J. Rivat, *Sur un problème de Gelfond : la somme des chiffres des nombres premiers*, Ann. of Math., to appear.

[10] C. Mauduit and A. Sárközy, *On the arithmetic structure of sets characterized by sum of digits properties*, J. Number Theory 61 (1996), 25–38.

[11] D. J. Newman, *On the number of binary digits in a multiple of three*, Proc. Amer. Math. Soc. 21 (1969), 719–721.

[12] D. Redmond, *Number Theory: an Introduction*, Dekker, New York, 1996.

[13] V. Shevelev, *Generalized Newman phenomena and digit conjectures on primes*, Int. J. Math. Math. Sci. 2008, ID 908045.

[14] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, http://www.research.att.com.

[15] T. Stoll, *Multi-parametric extensions of Newman's phenomenon*, Integers 5 (3) (2005), #14.

Department of Mathematics
Ben-Gurion University of the Negev
Beer-Sheva 84105, Israel
E-mail: shevelev@bgu.ac.il

Received on 10.6.2008
and in revised form on 1.8.2008 (5734)