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# EXACT FILL RATES FOR (*R, s, S*) INVENTORY CONTROL WITH GAMMA DISTRIBUTED DEMAND

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Discussion paper

# Exact fill rates for $(R; s; S)$ inventory control with gamma distributed demand

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Abstract

For the familiar $(R; s; S)$ inventory control system only approximate expressions exist for the fill rate, i.e. the fraction of demand that can be satisfied from stock. Best-known are the approximations derived from renewal theory by Tijms & Groenevelt (1984), holding under specific conditions; in particular, $S - s$ should be reasonably large. They considered, more specifically, the cases of normally and gamma distributed demand.

Here, an exact expression for the fill rate is derived, holding generally in the situation that demand has a gamma distribution with known integer-valued parameters, while lead time is constant. This formula is checked through extensive simulations; besides, detailed comparisons are made with Tijms & Groenevelt's approximation.

Key words: fill rate, gamma demand, inventory control, $(R; s; S)$-policy, simulation

Jel-code: C44

# 1 Introduction

One of the most frequently met inventory control methods is the $(R; s; S)$ system: inventory is checked at review moments, $R$ time-units apart; only if the inventory position is at or below $s$, an order up to level $S$ is placed. $R$ is called the review period, $s$ the reorder point. Orders are delivered with a fixed delay: the lead time $L$: Finally, backlogging of excess demand is assumed.

For evaluating inventory control methods, both cost-based and material performance measures can be found in the literature. Since cost factors are notoriously hard to determine, we will stick to the (European) tradition of material-based service measures: throughout this paper, our performance measure will be the fill rate $\beta$; i.e. the fraction of total demand that can be satisfied immediately from stock at hand.

So, demand is the only stochastic feature in our model. In earlier literature, demand often is assumed to be normally distributed; however, this distribution has the obvious restriction of being symmetric and the even more obvious disadvantage of taking negative values. Hence, following Burgin (1975) and Strijbosch & Moors (1999), demand will be assumed here to follow a (stationary) gamma-process $\Gamma(\lambda; \rho t)$; meaning that

- demand during any interval of length $t$ has distribution $\Gamma(\lambda; \rho t)$;
- demands during disjoint time intervals are independent.

Since $1/\lambda$ is just a scale parameter, for the moment it will be taken equal to 1; the shape parameter $\rho$ will be assumed to take only integer values, both for demand during review period $R$, and during lead time $L$: (So, in fact Erlang instead of gamma distributions are considered.)

Note the consequence of stochastic demand: not every review moment results in an order. Therefore, the number of review periods between subsequent orders is a random variable. Consequently, the length of a replenishment cycle - the interval between two deliveries - is stochastic too.

The remaining sections of this paper can be summarized as follows. In Section 2 our notation is introduced, describing the $(R; s; S)$ model in detail. Section 3 presents our main result: an exact expression for the fill rate $\beta$, attained under this model for given values of $R; L; s$ and $S$. Outcomes are presented in Section 4, together with extensive simulation results, while four special cases are considered in Section 5. Section 6 presents the comparison with the approximations of Tijms & Groenevelt (1984). The final Section 7 discusses a research plan involving important applications of these findings. From the function $\beta(R; L; s; S)$, it now is very easy to calculate the reorder point $s$ leading to a prescribed fill rate $\beta$. Recent experience (Strijbosch & Moors, 1999) has shown that it then is easy to find an approximate relation between $\beta$ and $s$ that is simple to use for practitioners. We plan to execute this follow up programme in the near future.

## 2 Notation

First, the $(R; s; S)$ system will be described in detail, introducing our notation. Review moments are denoted by $r_i$ $(i = 1; 2; \ldots)$; $r_{i+1} = r_i + R$. Some of these $r_i$ are order moments: only if the inventory position is smaller than $s$; an order (up to level $S$) is placed. (The inventory position is defined as the net stock plus all orders that have not yet been delivered.) Orders are delivered after a delay of length $L$; hence, the time between two subsequent order moment (and between two subsequent deliveries) is a multiple $kR$ of $R$. In other words, $kR$ is the length of this replenishment cycle (RC).

Per RC, net stock reaches its minimum just before delivery; if this minimum is negative, a shortage occurs. However, measuring shortage just by means of those minima leads to double counts: they occur if net stock remains negative after the subsequent delivery. Denoting the net stock at a specific delivery moment by $n$, and just before delivery by $n_-$, the shortage $t$ of this RC will be defined as

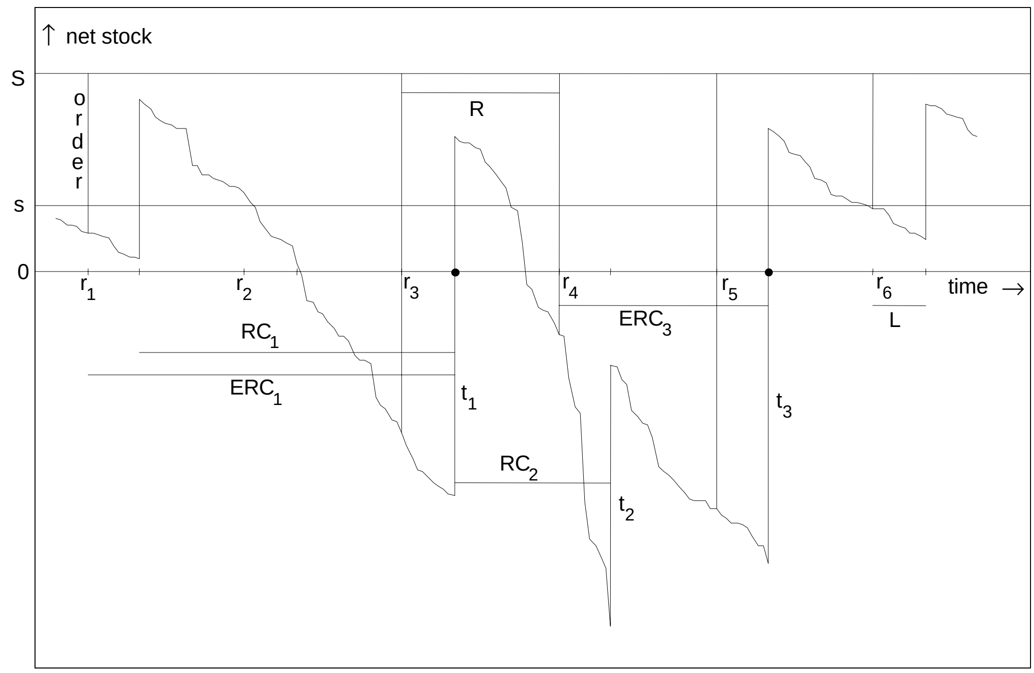
$$t = [-n_-]^+ - [-n]^+$$

where

$$x^+ = \begin{cases} x & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x \leq 0 \end{cases}$$

A shortage can occur only if demand between an order moment and the second next delivery exceeds $S$; this period of length $kR + L$ will be called an extended replishment cycle (ERC). Figure 2.1. shows these notations. Note that $t_1$ and $t_3$ are measured from the horizontal axis downwards; only for $t_2$ the correction term $[-n]^+$ is relevant.

Figure 2.1 (R; s; S)-system; L < R.




Note that our de…nition of t di¤ers just slightly from the usual one; although the same expression is used by both, standardly the second term $[¡n]^+$ refers to the start of the preceeding RC. Of course, in the addition process di¤erences cancel our; since we will be interested only in average shortage, both de…nitions might be used. We prefer ours, because it refers to a single moment in time.

In denoting random variables, corresponding capital letters will be used; e.g., for a random RC, K will denote its number of review periods. Hence, a possible shortage T at the end of a random RC is given by

$$T = [¡N_¡]^+ ¡ [¡N]^+ \qquad (2.1)$$

It should be stressed that this formula holds very generally: also for L ¸ R, and for any demand distribution. Since we assume stationary demand, the distribution of demand during any time-interval only depends on the length of the interval. The following notations will be used:

$X_k$ : demand during k review periods,
Z : demand during lead time

Of course $X_0 = 0$. Then, (2.1) can be rewritten as

$$T = [X_K + Z - S]^+ - [Z - S]^+$$

In the sequel, only the average shortage $E(T)$ will be needed. To evaluate that expectation, conditioning on the value of $K$ will be used. Defining

$$p_k = P(K = k), \; k \in \mathbb{N}$$

then gives the equally general expression

$$E(T) = \sum_{k=1} p_k E[X_k + Z - S]^+ - E[Z - S]^+ \tag{2.2}$$

The hardest nut to crack is the (conditional) distribution of $X_k + Z$ (demand in an ERC of given RC-length $k$). This distribution is derived in Section 3, leading to a formula for $E(T)$ that is suitable for computation. Then, the performance measure $\beta$ is found easily: denoting the expected demand during a review period by $\mu_R$, this fill rate equals

$$\beta = 1 - \frac{E(T)}{\mu_R E(K)} \tag{2.3}$$

The general expressions (2.2) and (2.3) will be evaluated now for stationary gamma demand.

# 3 The exact fill rate for gamma demand

The assumption that demand follows a stationary gamma process implies that in any period of length $t$ demand has distribution $\Gamma(\lambda, \rho t)$ with parameters $\lambda$ and $\rho$. Since $1/\lambda$ is simply a scale-parameter, the simplifying assumption $\lambda = 1$ will be used from here on. Normalizing $\rho t$ by equating it to $b$ for a single review period then leads to

$$\left.\begin{aligned} X_k &\sim \Gamma(1, kb) \\ Z &\sim \Gamma(1, d = bL/R) \end{aligned}\right\} \tag{3.1}$$

For disjoint time-intervals, these variables are independent.

The density and the cumulative distribution function of $Y$ will be denoted by $f_b$ and $F_b$ where

$$f_b(y) = \frac{y^{b-1}}{\Gamma(b)} e^{-y}, \; y \geq 0 \tag{3.2}$$

The probability distribution of $K$ follows. Introducing $q = S - s$ and using the convolution property of gamma distributions, the event

$$\{K = k\} = \{X_{k-1} \le q\} \cap \{X_k > q\}$$

has probability

$$p_k = P(K = k) = F_{(k-1)b}(q) - F_{kb}(q)$$

since $\{X_k \le q\}$ implies $\{X_{k-1} \le q\}$. For integer-valued $b$, this leads to

$$p_k = \sum_{j=1}^{b} \frac{q^{kb-j}}{(kb-j)!} e^{-q}; \; k \in \mathbb{N} \tag{3.3}$$

by use of the familiar property

$$F_{k+1}(q) = F_k(q) - \frac{q^k}{k!} e^{-q} = 1 - \sum_{i=0}^{k} \frac{q^i}{i!} e^{-q} \tag{3.4}$$

Next, the (conditional) distribution of demand $X_k + Z$ during an ERC with fixed length $k$ will be derived in five steps; consider first the case $k \ge 2$:

(i) Under the condition $A = \{X_{k-1} \le q\}$, the conditional density $f_A$ of $X_{k-1}$ satisfies

$$f_A(x) \propto f_{(k-1)b}(x); \; 0 \le x \le q$$

where $\propto$ denotes proportionality

(ii) Taking the convolution with $X_1$ gives the conditional density $g_A$ of $X_k$:

$$\begin{aligned} g_A(v) &= \int f_A(x) f_b(v-x)dx; \; 0 \le x \le q; v - x \ge 0 \\ &\propto \int_0^m f_{(k-1)b}(x) f_b(v-x)dx; v \ge x \end{aligned}$$

with $m = \min(q; v)$:

(iii) The additional condition $B = \{X_k > q\}$ implies $m = q$; hence, the conditional density $g_k$ of $X_k$ under $\{K = k\} = A \cap B$ becomes

$$g_k(v) = g_{A \cap B}(v) \propto \int_0^q f_{(k-1)b}(x) f_b(v-x)dx; \; v \ge q$$

(iv) Now using (3.2), repeated partial integration gives for integers $b$

$$\begin{aligned} g_k(v) &\propto e^{-v} \int_0^q x^{(k-1)b-1}(v-x)^{b-1}dx \\ &\propto \sum_{j=1}^{b} \frac{q^{kb-j}}{(kb-j)!} f_j(v-q); \; v \ge q \end{aligned}$$

(v) By taking convolutions once more, the conditional density $h_k$ of $U = X_k + Z$ under $\{K = k\}$ finally follows:

$$h_k(u) = \frac{1}{p_k} \sum_{j=1} \frac{q^{kb-j}}{(kb-j)!} f_{d+j}(u - q); \; u \geq q \tag{3.5}$$

It is easy to check that this expression holds for $k = 1$ as well.

Now, (2.2) implies

$$E(T) = \sum_{k=1} p_k \int_S (u - S) h_k(u) du - \int_S (u - S) f_d(u) du$$

Introducing for $a > 0$

$$v_a(x) = \int_x (u - x) f_a(u) du$$

and using (3.3), this leads to the final expression

$$E(T) = e^{-q} \sum_{j=1} \left( \sum_{k=1} \frac{q^{kb-j}}{(kb-j)!} \right) v_{d+j}(s) - v_d(S) \tag{3.6}$$

Since $\mu_R = b$, an exact expression for the fill rate $\beta$ follows at once. Note that the relation

$$v_a(x) = a[1 - F_{a+1}(x)] - x[1 - F_a(x)]$$

enables fast calculation of (3.6).

# 4 Outcomes and simulation results

A MATLAB program was written to calculate the expected loss from (3.6) and the corresponding fill rate from (2.3), for given values of the foursome $(b, d, s, q)$. Furthermore, an extremely fast Delphi program was developed to simulate our $(R, s, S)$ control system. The simulation results were used to check our derivations: besides, this simulation program is necessary in case of non-integer valued $b$ and $d$. For this reason, the core of our Delphi program is given in Appendix A. All our simulation experiments concerned 30,000 review periods; of course, the number of RC's depends on $E(K)$.

Table 4.1 shows detailed theoretical and simulated results (indicated by ^) for twelve selected values of $(b, d, s, q)$. (Note that $d/b \in \mathbb{N}$ implies that delivery coincides with a review moment; in that case, shortage, delivery and review are determined in this order.)

The last columns give the 95%- con…dence interval for the expected shortage, based on the variance of $T$, estimated from the simulation runs.

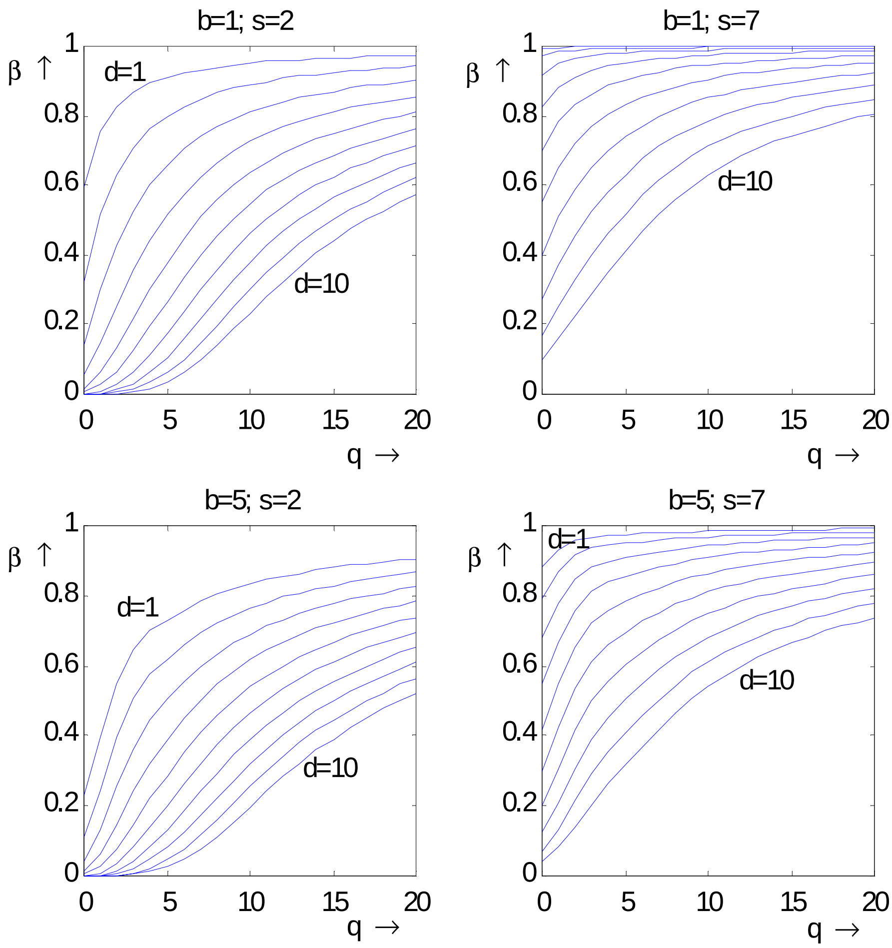
Table 4.1 Fill rate ¯ for $(R; s; S)$ control system.

| Parameters | | | | Theory | | | Simulation | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | 95% CI for $E(T)$ | |
| b | d | s | q | ¯ | $E(K)$ | $E(T)$ | b¯ | $E(\hat{K})$ | $E(\hat{T})$ | Lower | Upper |
| 1 | 1 | 2 | 0 | 0.5940 | 1.0000 | 0.4060 | 0.5996 | 1.0000 | 0.3980 | 0.3890 | 0.4070 |
| 1 | 2 | 2 | 0 | 0.3233 | 1.0000 | 0.6767 | 0.3243 | 1.0000 | 0.6729 | 0.6622 | 0.6835 |
| 2 | 1 | 2 | 0 | 0.4587 | 1.0000 | 1.0827 | 0.4564 | 1.0000 | 1.0901 | 1.0748 | 1.1054 |
| 2 | 2 | 2 | 0 | 0.2331 | 1.0000 | 1.5338 | 0.2323 | 1.0000 | 1.5368 | 1.5205 | 1.5531 |
| 1 | 1 | 2 | 1 | 0.7542 | 2.0000 | 0.4916 | 0.7585 | 2.0029 | 0.4807 | 0.4657 | 0.4957 |
| 1 | 2 | 2 | 1 | 0.5155 | 2.0000 | 0.9691 | 0.5165 | 2.0088 | 0.9672 | 0.9466 | 0.9877 |
| 2 | 1 | 2 | 1 | 0.6590 | 1.2838 | 0.8757 | 0.6572 | 1.2840 | 0.8826 | 0.8659 | 0.8992 |
| 2 | 2 | 2 | 1 | 0.4331 | 1.2838 | 1.4556 | 0.4311 | 1.2872 | 1.4658 | 1.4457 | 1.4859 |
| 1 | 1 | 2 | 2 | 0.8257 | 3.0000 | 0.5230 | 0.8277 | 3.0078 | 0.5151 | 0.4956 | 0.5346 |
| 1 | 2 | 2 | 2 | 0.6306 | 3.0000 | 1.1081 | 0.6325 | 3.0076 | 1.1008 | 1.0731 | 1.1286 |
| 2 | 1 | 2 | 2 | 0.7528 | 1.7546 | 0.8676 | 0.7515 | 1.7505 | 0.8723 | 0.8525 | 0.8921 |
| 2 | 2 | 2 | 2 | 0.5599 | 1.7546 | 1.5445 | 0.5590 | 1.7485 | 1.5434 | 1.5183 | 1.5685 |

The simulation results clearly con…rm our theoretical derivations; e.g., all twelve con…-dence intervals indeed contain $E(T)$:

The above calculations were repeated for all combinations of parameter values b; d; s 2 f1; 2; :::; 10g and q 2 f0; 1; :::; 20g: The maximum di¤erence between ¯ and ^ proved to be 1%; j¯ ¡ ^j < 0:4% held for 95% of all 21,000 combinations. Figure 4.1 summarizes some typical (exact) results.

Figure 4.1 Fill rate $\beta$ for various values of $(b; d; s; q)$:




The next section considers some special cases for formula (3.6), in particular the cases $R = 0; s = S; b = 1$ and $L = 0$:

# 5 Special cases

In case $R = 0$; the $(R; s; S)$ control system is simplified to the continous $(s; S)$ review system. Formula (3.6) now simplifies to the straightforward expression

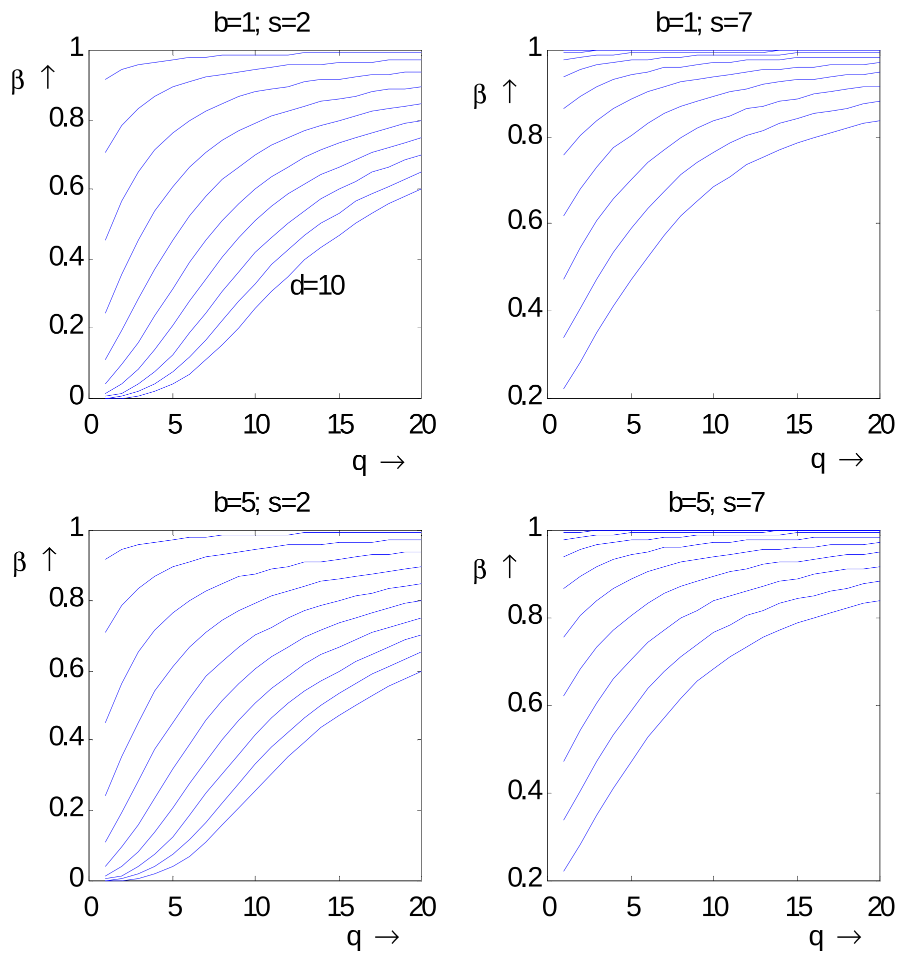
$$E(T) = v_d(s) - v_d(S) = E[Z - s]^+ - E[Z - S]^+$$

Since demand during any RC now equals $q$; (2.3) must be replaced by

$$\beta = 1 - \frac{E(T)}{q}$$

Figure 5.1 shows its typical behaviour.

Figure 5.1 Fill rate $\beta$ for $(s, S)$ control system $(R = 0)$.




In case $s = S$, or $q = 0$, the simpler $(R, S)$ control system is obtained. Since now $P(k = 1) = 1$, (3.6) reduces to

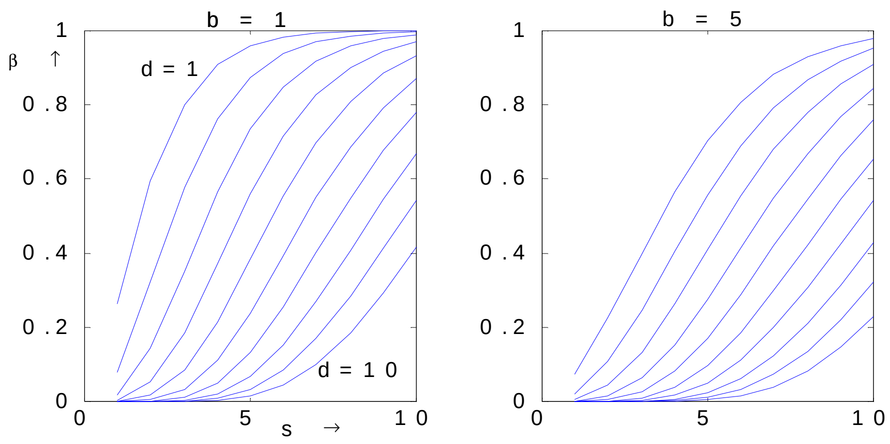
$$E(T) = v_{d+b}(S) - v_d(S) = E[X_1 + Z - S]^+ - E[Z - S]^+$$

which is obvious again; compare de Kok (1990) or Strijbosch & Moors (1999). The behaviour of

$$\beta = 1 - [v_{d+b}(S) - v_d(S)]/b$$

is shown in Figure 5.2.

Figure 5.2 Fill rate $\beta$ for $(R, S)$ control system ($q = 0$).




In case $b = 1$, (3.6) is reduced to

$$E(T) = v_{d+1}(s) - v_d(S) \tag{5.1}$$

It can be derived directly from the general starting formula

$$E(T) = E[X_k + Z - S]^+ - E[Z - S]^+ \tag{5.2}$$

as follows. If $r_i$ denotes an order moment, net stock at the preceeding review moment $r_{i-1}$ can be written as

$$S - X_{K-1} = Y + s$$

with $Y \geq 0$. Denoting demand between $r_{i-1}$ and $r_i$ by $X_1$ then gives

$$E[X_K + Z - S]^+ = E[X_1 - Y + Z - s | X_1 \geq Y]^+ \tag{5.3}$$

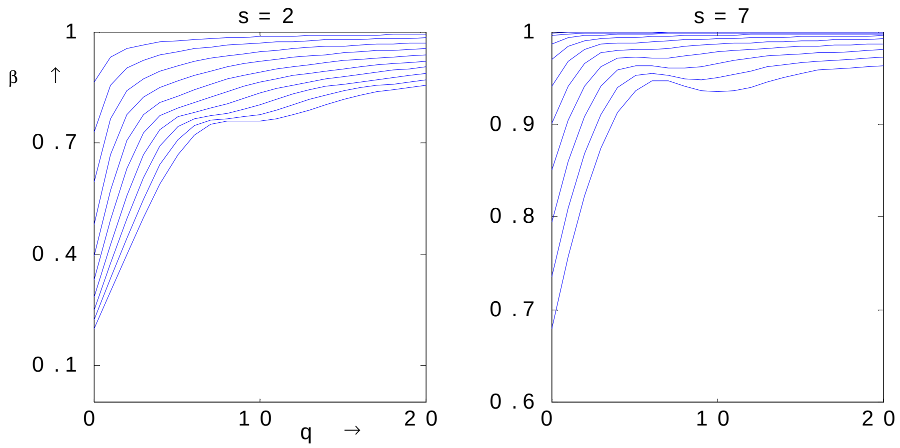
Since $b = 1$, $X_1$ has the (standard) exponential distribution $Ne(1)$, with the characteristic property that its conditional distribution under condition $\{X_1 \geq c\}$ is $Ne(1)$ again. Consequently, under condition $\{X_1 \geq Y\}$, the undershoot $U = X_1 - Y$ is a $Ne(1)$-distributed variable as well. Combining (5.2) and (5.3) leads to

$$E(T) = E[U + Z - s]^+ - E[Z - S]^+$$

which is (5.1).

In case $L = 0$, finally, the parameter $d$ disappears from (3.6). Figure 5.3 shows the behaviour of $\beta$ for this situation.

Figure 5.3 Fill rate $\beta$ for $(R;s;S)$ control system with $L = 0$.




Note that $\beta$ is not always increasing in $q$. This is probably due to the discreteness of RC length $K$: an increase in $q$ may have as consequence that ordering is postponed for another review period.

# 6 Comparison with the approximations of Tijms & Groenevelt (1984)

Various computational methods for determining approximately optimal $(s;S)$ control rules exist in the literature, with both periodic and continuous review. As was already pointed out by Bashyam and Fu (1998), it has been widely recognized that penalty costs, and in particular, the cost of losing customer goodwill, are difficult to assess. Therefore, many papers deal with the problem to determine an $(s;S)$ pair that minimizes total setup and holding costs under the constraint that the solution satisfies a desired customer service level. Service level may be defined as the probability ($\alpha$) of not being out of stock in a given period, or as the fraction ($\beta$) of demand statisfied directly from the shelf, or as the fraction $(1 - \gamma)$ of demand being on backorder each period. Two papers on service level constraints are most relevant for the analysis in the present paper. First, Schneider and Rinquest (1990) develop a Service Level Power Approximation, using a $\gamma$-service level constraint and assuming fixed lead times. Further, Tijms and Groenevelt (1984) (TG) develop tractible approximations for the periodic and continuous review $(s;S)$ system, using a $\beta$-level constraint and allowing stochastic lead times. Both papers are important contributions for the practitioner. Due to the use of asymptotic results from renewal theory in order to approximate the undershoot distribution, an important limitation of both approaches is that the difference $q = S - s$ should be sufficiently large compared to the average demand during a review period; in our notation they demand:

$$2q \geq 3b \tag{6.1}$$

Our analysis in Section 3 does not need the undershoot distribution, thanks to the conditioning on the length of the RC. Consequently, we were able to find exact expressions for the expected shortage and the fill rate. That makes it interesting to compare the approximations of TG with our exact results.

The key result in TG is their formula (7), giving an approximation for the fill rate, holding for general demand patterns and stochastic lead times, provided condition (6.1) is satisfied and the required service level is high. (Note that for deterministic lead times, this formula was already derived by Schneider (1978, 1981) by means of asymptotic results of Roberts (1962).) Denoting this approximate value by $\beta_T$ and adopting our notation, their results can be rewritten for our stationary gamma demand as

$$\beta_T = 1 - \left[\int_s^\infty (x-s)^2 f_{b+d}(x)dx - \int_s^\infty (x-s)^2 f_d(x)dx\right] / [(2q+b+1)b] \tag{6.2}$$

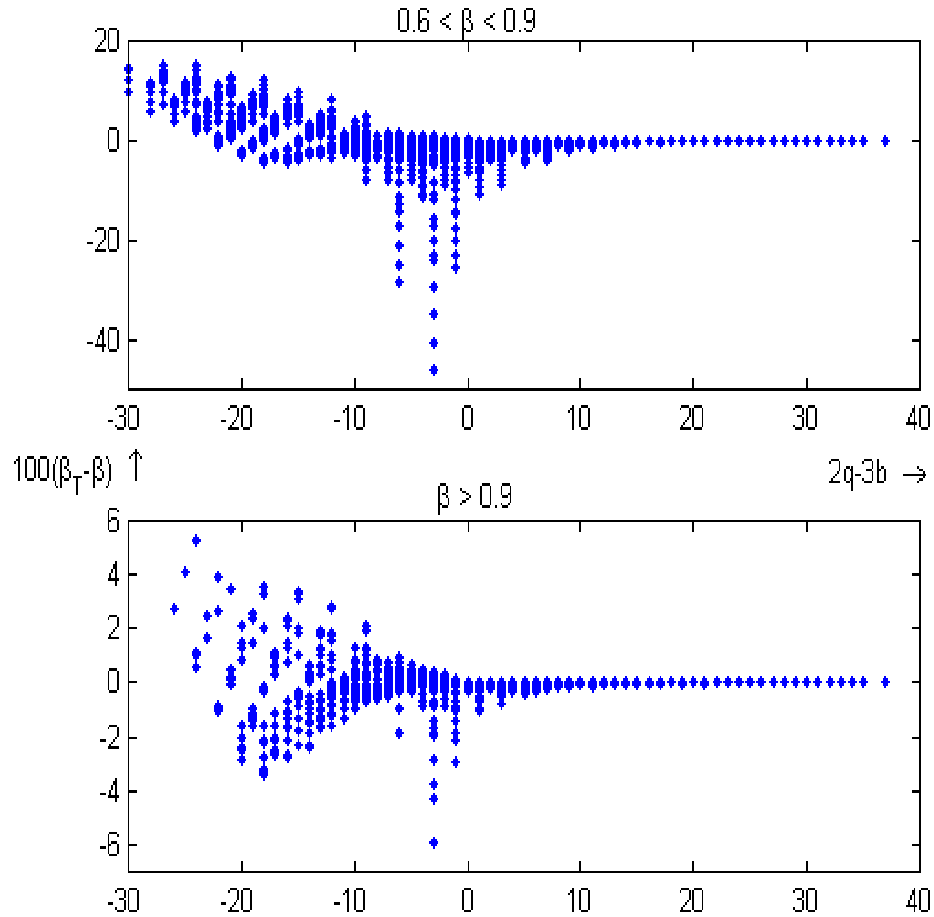
Using

$$\int_s^\infty (x-s)^2 f_\rho(x)dx = \rho(\rho+1)[1-F_{\rho+2}(s)] - 2\rho s[1-F_{\rho+1}(s)] + s^2[1-F_\rho(s)] \tag{6.3}$$

(6.2) can easily be calculated.

Figure 6.1 shows the errors (in percentages) in the TG approximations, ordered according to $2q - 3b$, the crucial quantity for the applicability of $\beta_T$. Of the previously used 21,000 combinations of values of $(b, d, s, q)$, only those leading to $\beta > 0.6$ are presented. Since high values of $\beta$ are important in practice, separate pictures for $\beta < 0.9$ and $\beta > 0.9$ are given.

Figure 6.1 Deviations of approximate $\beta_T$ from exact $\beta$:




The top picture reveals that for intermediate $\beta$-values $\beta_T$ may be 46% too low and 15% too high, especially for negative $2q - 3b$: Even if (6.1) is not violated, $\beta_T$ may be up to 10% too low. For $\beta > 0.9$; deviations of 6% in both directions may occur; however, if (6.1) is not violated, the deviation is at most 0.6%.

A more theoretical comparison is enabled by noting that (6.3) may be rewritten as

$$\int_s^\infty (x - s)^2 f_\nu(x)dx = \nu v_{\nu+1}(s) - s v_\nu(s) \qquad (6.4)$$

It can be checked directly that the functions $v_a$ satisfy the recursive relation

$$(a + 1)v_{a+2}(x) = (a + x + 2)v_{a+1}(x) - x v_a(x) \qquad (6.5)$$

Repetitive use then leads to

$$\int_s^\infty (x - s)^2 f_{b+d}(x)dx = 2\sum_{j=1}^{b} v_{d+j}(s) + dv_{d+1}(s) - sv_d(s)$$

whence (6.2) can be rewritten for Erlang distributed demand as

$$\beta_T = 1 - \frac{\frac{1}{b}\sum_{j=1}^{b} v_{d+j}(s)}{q + (b+1)/2} \tag{6.6}$$

On the other hand, introducing for $j = 1, 2, \ldots, b$

$$\alpha_j = e^{-q}\sum_{k=1}^{\infty} \frac{q^{kb-j}}{(kb-j)!}$$

for $q > 0$ (and $\alpha_b = 1$ for $q = 0$) leads to

$$bE(K) = q + \sum_{j=1}^{b} j\alpha_j$$

so that (2.3) now may be written as

$$\beta = 1 - \frac{\sum_{j=1}^{b} \alpha_j v_{d+j}(s) - v_d(S)}{q + \sum_{j=1}^{b} j\alpha_j} \tag{6.7}$$

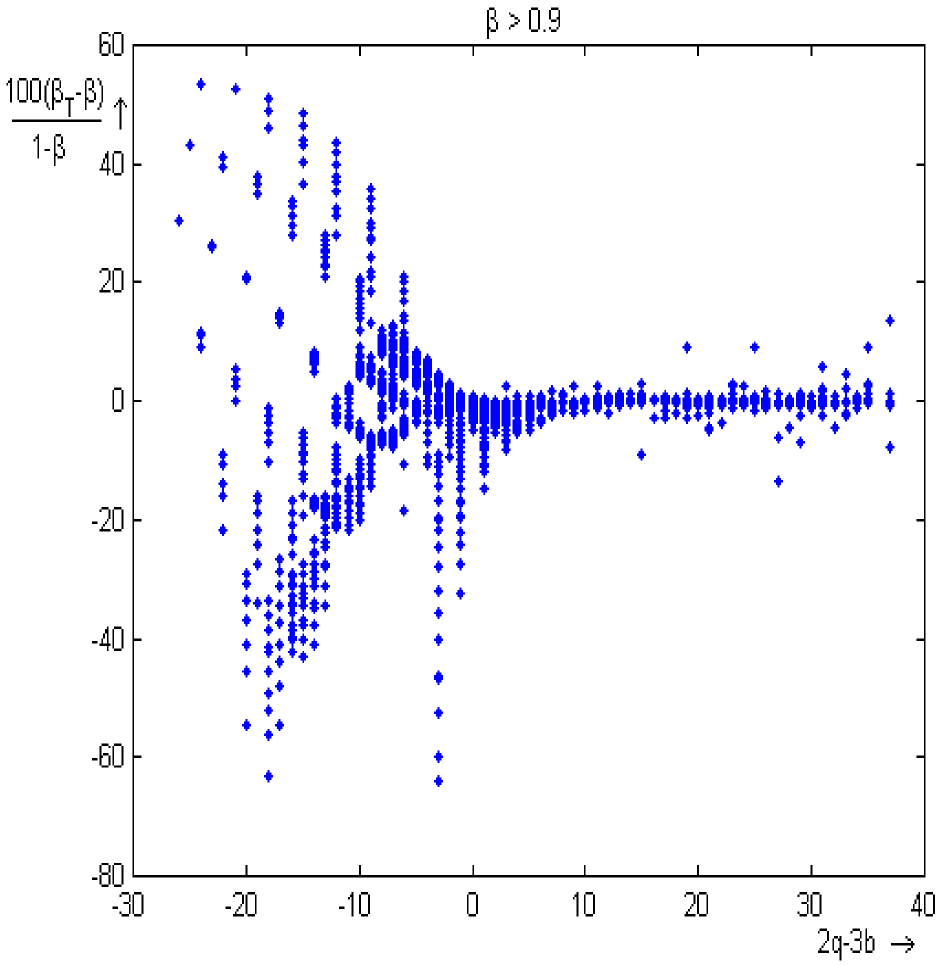
So, TG approximates the weighted mean of the $v_{d+j}(s)$ by their simple average. Note that substituting $\alpha_j = 1/b$ in (6.7) gives (6.6) - apart from the term $v_d(S)$, but of course, the $\alpha_j$ are strictly increasing.

# 7 Summary and further research

In this paper an exact formula has been derived for the average shortage in a replenishment cycle of an $(R, s, S)$ inventory control system where demand follows a stationary gamma process. It is assumed that lead time is deterministic and that demand during a review period and during the lead time have integer valued shape parameters. Extensive Monte Carlo experiments confirmed these theoretical findings.

To our knowledge the best solution up to now for this problem was the approximation derived by Schneider (1978, 1981) and Tijms & Groenevelt (1984). For gamma distributed demand, we confirmed that their approximations are satisfactory, in particular for high service levels, and provided that the condition $q > 1.5b$ is satisfied: for $\beta > 0.9$, we found deviations $|\beta_T - \beta|$ of at most 0.6%. Note however, that even such small deviations may be of importance when very high service levels are required. To illustrate this, the lower half of Figure 6.1 is presented in a slighty di¤erent way: Figure 7.1 shows relative deviations 100 (¯$_T$ ¡ ¯) = (1 ¡ ¯) up to §14% even if (6.1) is satis…ed.

Figure 7.1 Deviations of approximate ¯$_T$ from exact ¯, relative to 1 ¡ ¯:




In fact, we derived in this paper the function ¯(b; d; s; q): From that, numerical calculation of s(b; d; ¯; q) is easy and straightforward; e.g. by means of the MATLAB procedure fzero: As an illustration, Table 7.1 shows some results.

Table 7.1 Exact reorder point $s$ in $(R; s; S)$ control system for $\beta = 0.95$.

| $b$ | $d$ | $q = 1$ | $q = 5$ | $q = 9$ |
|---|---|---|---|---|
| 1 | 1 | 4.0378 | 2.7636 | 2.1054 |
| 2 | 1 | 4.8566 | 3.5058 | 2.8046 |
| 1 | 2 | 5.5833 | 4.2100 | 3.4596 |
| 2 | 2 | 6.3248 | 4.8941 | 4.1220 |

Results like this are useful in practice, when a given service level $\beta$ is wanted, given $b, d$ and $q$. Standardizing $s$ immediately gives the safety factor $c$:

$$c = (s - \mu_{R+L})/\sigma_{R+L} = (s - b - d)/\sqrt{b + d}$$

where $\mu_{R+L}$ and $\sigma_{R+L}$ denote the mean and standard deviation, respectively, of demand during review plus lead time. It is dimensionless and hence independent of the scale parameter $1/\lambda$; consequently, safety factors are applicable if demand follows a general gamma process $\Gamma(\lambda; \rho t)$.

Although the necessary calculations are indeed reasonably simple, they may be forbidding for large-scale application. Hence, at the moment we are looking for an even simpler (approximative) numerical procedure. This will be done in the spirit of Strijbosch & Moors (1999), where highly accurate approximate safety factors for the $(R; S)$ control system were developed, using regression techniques. More precisely, from suitable sets of values of $(\beta; b; d; s; q)$ a regression function

$$\hat{c} = f(b; d; \beta; q)$$

will be derived for broad ranges of the regressors.

This approach is hampered by an important limitation of our paper: our results only hold for integer valued shape parameters. There, our simulation program comes in handy: it gives simulated values $\hat{\beta}(b; d; s; q)$; for $b, d \notin \mathbb{N}$ too. (If necessary the present precision may be improved by using runs longer than 30,000 review periods.) Including these simulated $\hat{\beta}$ in the regression analysis then leads to approximations

$$\tilde{c} = f(b; d; \beta(\hat{\beta}); q)$$

holding for intervals of $b$ and $d$ values.

The resulting approximation will have three important properties: firstly, programming the calculation is reduced to a few simple lines of code; secondly, calculation time is reduced to a (very small) fraction of the time necessary to solve s from (5.2), which is crucial when large numbers of stock keeping units are involved; thirdly, desired precision can be adapted by narrowing or broadening the ranges of the inputparameters. By approximating the dimensionless safety factors, the approximations are appropriate for gamma distributions with scale parameter unequal to one. As is described in Strijbosch & Moors (1999), the loss of precision can easily be kept lower than the loss of precision due to the necessary estimations of demand parameters in practice.

# Appendix A

Simulation of periodic review $(R, s, S)$-system using gamma demand distribution and discreet event simulation.

Additonal notation:

$g \;=\; \frac{d}{b} = \frac{L}{R}$

$x_i$ : demand during $[r_i, r_i + R(g - \lfloor g \rfloor))$;
$x_i$ is a realisation of $X_{L - \lfloor g \rfloor R} \sim \Gamma(1, d - b\lfloor g \rfloor)$

$y_i$ : demand during $[r_i + R(g - \lfloor g \rfloor), r_{i+1})$;
$y_i$ is a realisation of $X_{R - L + \lfloor g \rfloor R} \sim \Gamma(1, b - d + b\lfloor g \rfloor)$

$z_i \;=\; x_i + y_i$ demand during $[r_i, r_{i+1})$

$i_{i-}$ : inventory position immediately before $r_i$

$i_i$ : inventory position on $r_i$

$n_{i-}$ : net stock immediately before $r_i + L$

$n_i$ : net stock on $r_i + L$ (immediately after delivery, if any)

$$o_i = \begin{cases} 1 & \text{if } i_{i-} < s \quad \text{do order at } r_i \\ 0 & \text{if } i_{i-} \geq s \quad \text{don't order at } r_i \end{cases}$$

$w_i \;=\; \sum_{j=1}^{\lfloor g \rfloor} z_{i-1+j} + x_{i+\lfloor g \rfloor}$ : demand in $[r_i, r_i + L)$

Calculation scheme:

$$i_{1-} = s + q = S$$

$$i_{i-} = i_{i-1} - z_{i-1}, \ (i > 1)$$

$$i_i = \begin{cases} s + q & \text{if } o_i = 1 \\ i_{i-} & \text{if } o_i = 0 \end{cases}$$

$$n_{i-} = i_{i-} - w_i$$

$$n_i = n_{i-} + (s + q - i_{i-})\, o_i$$

$$\bar{b} = 1 - \sum_i \left[ (-n_{i-})^+ - (-n_i)^+ \right] \Big/ \sum_i z_i$$

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