Exact Free Vibration of Webs Moving Axially at High Speed

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Abstract: - Moving webs can be found in a wide range of industrial applications such as paper handling, textile manufacturing, and magnetic tape recording. The moving web in these applications, which is generally orthotropic, may experience speeds more than critical speed. Critical speed is defined as that axial speed where the system vibration has a vanishing eigenvalue and is subject to a buckling instability. At a supercritical speed, the web may experience types of instabilities and subsequently sever out of plane vibrations.
In this paper, based on thin plate theory, the equation of out-of-plane motion is derived for an orthotropic web. Then an exact method is employed to evaluate free vibration of the web in sub- and super-critical speeds. This method is in fact the Levy-type solution of the equation of motion in a stiffness matrix form. The exact vibration eigenvalues which are generally complex values, are the roots of stiffness matrix determinant. Since the terms of the determinant are complex transcendental functions of eigenvalues, classical eigenvalue solver can not be used. So a suitable algorithm is used here to extract eigenvalues in the two-dimensional plane of complex numbers.
Using a numerical example, the reliability of the formulation and the solution procedure is shown. The free vibration eigenvalues is extracted for a range of axial speeds. Based on the results, flutter and divergence instabilities of the moving web are studied at supercritical speeds.
Keywords: - Free vibration, Axially moving, Orthotropic web, Supercritical speed, Levy-type solution, Exact method, Flutter Instability, Divergence Instability.

1 Introduction

Mechanical systems comprising moving elastic continua can be found in numerous fields of engineering application e.g. band saw blades, power transmission belts, steel plates in galvanizing line, paper handling and textile manufacturing. Due to this prevalence, the examination of moving continua has a long tradition and produced a large amount of publications, mostly on axially moving slender structures like strings or beams. Comprehensive literature reviews can be found in Marynowski's book recently published by Springer [1].
Generally, axially moving continua in the form of thin, flat rectangular shape material with small flexural stiffness is called a web. Webs are moving at high speed, for example, in paper production the paper webs are transported with longitudinal speeds of up to 3000 m/min [2]. Because of their axially speed, moving materials experience a Coriolis acceleration component which renders such systems gyroscopic [3]. In a certain critical speed, first natural frequency of the gyroscopic system vanishes and the structure experience severe vibrations and bifurcation instability. In the speeds above the critical speed, termed supercritical speeds, the structure may experience divergence or flutter instability or become stable again.

In this paper using a two dimensional model, the free vibration and stability of webs are exactly solved at sub and supercritical axial speeds. The web which generally has orthotropic property is modeled by thin plate theory. The analytical results provide a benchmark for evaluating approximate methods applied to moving continua.

2 Theory

The differential equation of motion for a moving orthotropic web is:

$$D_{11}\frac{\partial^4 w}{\partial x^4}+2D_3\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}$$
$$-N_x\frac{\partial^2 w}{\partial x^2}-N_y\frac{\partial^2 w}{\partial y^2}-2N_{xy}\frac{\partial^2 w}{\partial x\partial y}$$
$$+\rho h\left(\frac{\partial^2 w}{\partial t^2}+2v\frac{\partial^2 w}{\partial x\partial t}+v^2\frac{\partial^2 w}{\partial x^2}\right)=0 \quad (1)$$

where

$$D_3 = D_{12} + 2D_{66}$$

The web is assumed to be subjected to a basic state of plane stress which is invariant in direction of web width and simply supported along transport direction (axial direction). Then in free vibration analysis, it can be postulated that, whatever type of vibration mode occurs, it is sinusoidal in the *y*-direction. This means that along any line in the web structure parallel to y-axis, transverse displacement w vary sinusoidally, so w can be written in the form

$$w(x,y,t) = W_n(x)[\exp(ik_n y) - \exp(-ik_n y)]e^{\lambda t} \quad (2)$$

where $k_n = n\pi/b$ is the wave number of the nth mode in y-direction (n =1,2,…), b is the web width, $W_n(x)$ is shape function along axial direction obtaining by solution the equation of motion for nth mode and values of λ extracting from an eigenvalue problem, are in general complex numbers as:

$\lambda = \sigma + \mathrm{i}\omega$

where σ and ω are the real and imaginary parts of eigenvalues λ, respectively. While all eigenvalues λ are pure imaginary ($\lambda = \mathrm{i}\omega$), the traveling web is stable and the values of ω are the natural frequencies of the web. By increasing transport speed, the natural frequencies of the web decrease and at a certain speed, the first natural frequency of vibration vanishes (ω=0) and the web becomes unstable. This certain speed may be mentioned as *critical speed* v_{cr}. For axial speeds higher than the critical speed, the web can experience divergence or flutter instability that in the unstable conditions the real part of at least one of the eigenvalues λ is non-zero ($\sigma \neq 0$).

The critical speed can also be obtained by a static stability analysis based on the fact that the axially moving web becomes unstable if multiple equilibrium positions exist at any problem specification. The critical speed is the lowest speed at which multiple equilibrium positions exist [4]. From equation (1), the equilibrium position in static stability analysis satisfies following equation:

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2D_3\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} - N_x\frac{\partial^2 w}{\partial x^2} - N_y\frac{\partial^2 w}{\partial y^2} - 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + \rho h v^2\frac{\partial^2 w}{\partial x^2} = 0 \quad (3)$$

Also, the displacement function for buckling modes in the exact method can be written as:

$$w(x,y) = W_n(x)[\exp(ik_n y) - \exp(-ik_n y)] \quad (4)$$

In the present method, the web is divided to a small number of components that each component fills the domain between two rollers or a roller and an end boundary. Fig. 1b shows one of these components. The assumption of sinusoidally modes implies that the displacements (w_1, θ_1, w_2, *and* θ_2) and the additional forces (Q_1, M_1, Q_2, *and* M_2), that appear on the edge of every web component due to vibration or buckling, vary sinusoidally in the *y*-direction (Fig. 1b).

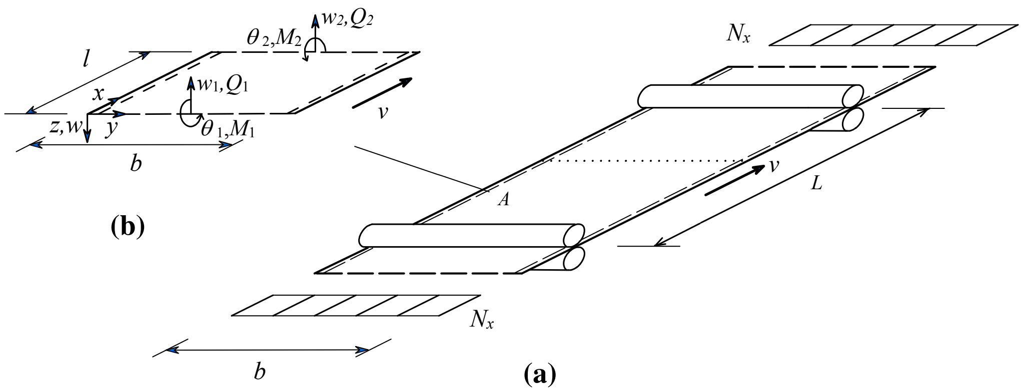

Fig. 1: (a) A web moving between some rollers
(b) Nodal displacements and forces for one of the web components.

The exact stiffness matrix is extracted for each component and by assembling stiffness matrices of all components, overall stiffness matrix of the structure will be obtained.

In the free vibration study, the determination of the exact stiffness matrix for the component shown in Fig. 1b requires solution of differential equation of motion presented in equation (1) as satisfy the boundary conditions. Substituting w from equation (2) into equation of motion (1) in the absence of N_y and N_{xy}, gives an ordinary differential equation with complex terms for nth mode as

$$D_{11}\frac{\partial^4 W_n}{\partial x^4}+[\rho h v^2 - N_x - 2D_3 k_n{}^2]\frac{\partial^2 W_n}{\partial x^2}$$
$$+2\rho h\lambda v\frac{\partial W_n}{\partial x}+(\rho h\lambda^2 + D_{22}k_n{}^4)W_n = 0 \qquad (5)$$

in which

$$W_n(x)=\sum_{m=1}^{4} A_{mn}\exp(r_{mn}x) \qquad (6)$$

where A_{mn} are coefficients deriving from boundary conditions at $x=0$ and $x=l$ and r_{mn}, the wave numbers in x-direction, can be obtained by substituting equation (6) into equation (5), that yields an polynomial equation for nth mode as

$$D_{11}r_{mn}{}^4 + (\rho h v^2 - N_x - 2D_3 k_n{}^2)r_{mn}{}^2$$
$$+2\rho h\lambda v\, r_{mn} + (\rho h\lambda^2 + D_{22}k_n{}^4) = 0 \qquad (7)$$

Equation (7) has four roots ($m=1$ to 4) corresponding to each mode ($n=1, 2, 3\ldots$), which are complex in general.

In the stability study, by using equations (3), (4) and (6), equation (7) is eliminated as the following form

$$r_{mn}{}^4 + \alpha_1 r_{mn}{}^2 + \alpha_2 = 0 \qquad (8)$$

where

$$\alpha_1 = (\rho h v^2 - N_x - 2D_3 k_n{}^2)/D_{11},$$
$$\alpha_2 = D_{22}k_n{}^4/D_{11},$$

and four series of roots are

$$r_{mn} = \pm\tfrac{1}{\sqrt{2}}\sqrt{\alpha_1 \pm \sqrt{\alpha_1{}^2 - 4\alpha_2}} \qquad (9)$$

For a component of the moving web shown in Fig. 2b, displacement function of equations (6) or (8) satisfies the boundary conditions at $y=0$ and $y=b$ as at these two sides the transverse displacement w and resultant bending moment per unit length of x-direction M_y vanish. By satisfying the boundary conditions on two other sides, stiffness matrix of the component can be obtained. These edges conditions, as shown in Fig. 2b, are defined by

at $x=0$: $\quad Q_x = Q_1 \quad , \quad w = -w_1$

$$M_x = -M_1 \ , \quad \frac{\partial w}{\partial x} = -\theta_1 \qquad (10)$$

at $x=l$: $\quad Q_x = -Q_2 \quad , \quad w = -w_2$

$$M_x = M_2 \ , \quad \frac{\partial w}{\partial x} = -\theta_2 \qquad (11)$$

where, the resultant shear force and bending moment per unit length of y-direction (Q_x and M_x respectively) are related to w by the following equations.

$$Q_x = -D_{11}\frac{\partial^3 w}{\partial x^3} - (D_{12}+4D_{66})\frac{\partial^3 w}{\partial x \partial y^2}$$
$$-(N_x - \rho h v^2)\frac{\partial w}{\partial x} + \rho h v\frac{\partial w}{\partial t} \qquad (12)$$

$$M_x = -[D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2}]. \qquad (13)$$

In the equation of shearing force Q_x, allowance has been made for the Kirchhoff edge effect and for the components of the in-plane loads arising from distortion of the web. The effect of the axial velocity on the edge shear has also been considered. Edge displacements and edge forces vectors can be defined by

$$\{d\} = \{\theta_1, w_1, \theta_2, w_2\}^T \qquad (14)$$
$$\{p\} = \{M_1, Q_1, M_2, Q_2\}^T \qquad (15)$$

Using expressions (2), (6) and (10) through (15) edge displacements and edge forces vectors in free vibration may be written as:

$$\{d\} = \{\bar{d}\}[\exp(ik_n y) - \exp(-ik_n y)]e^{\lambda t} \qquad (16)$$
$$\{p\} = \{\bar{p}\}[\exp(ik_n y) - \exp(-ik_n y)]e^{\lambda t} \qquad (17)$$

where

$$\{\bar{d}_n\} = -\sum_{m=1}^{4}\begin{Bmatrix} r_{mn} \\ 1 \\ r_{mn}\exp(r_{mn}l) \\ \exp(r_{mn}l) \end{Bmatrix} A_{mn}$$
$$\Rightarrow \{\bar{d}_n\}_{4\times1} = [X_n]_{4\times4}\{A_n\}_{4\times1} \qquad (18)$$

$$\{\bar{p}_n\} = \sum_{m=1}^{4}\begin{Bmatrix} D_{11}r_{mn}{}^2 - D_{12}k_n{}^2 \\ -D_{11}r_{mn}{}^3 + (D_{12}k_n{}^2 + 4D_{66}k_n{}^2 + N_x - \rho h v^2)r_{mn} - \lambda\rho h v \\ -(D_{11}r_{mn}{}^2 - D_{12}k_n{}^2)\exp(r_{mn}l) \\ [D_{11}r_{mn}{}^3 - (D_{12}k_n{}^2 + 4D_{66}k_n{}^2 + N_x - \rho h v^2)r_{mn} + \lambda\rho h v]\exp(r_{mn}l) \end{Bmatrix} A_{mn}$$
$$\Rightarrow \{\bar{p}_n\}_{4\times1} = [Y_n]_{4\times4}\{A_n\}_{4\times1} \qquad (19)$$

Combining equations (18) and (19) by elimination $\{A_n\}$ yields

$$\{\overline{p}_n\} = [s_n]\{\overline{d}_n\} \tag{20}$$

where

$$[s_n] = [Y_n][X_n]^{-1} \tag{21}$$

Here $[s_n]$ is the exact stiffness matrix of a component of the web that in general contains complex elements associated with the transport speed v; however, it is Hermitian in form. In the free vibration analysis, the elements of the stiffness matrix are transcendental functions of the natural frequencies, axial velocity, in-plane force, and of the wave numbers in x and y-directions. By assembling stiffness matrix of all elements of the moving web and eliminating constrained degrees of freedom, the overall exact stiffness matrix of the multi-span orthotropic web moving over an elastic foundation $[S_n]$ is obtained. By the present method, precise results can be obtained by only a few components, leading to small order of overall stiffness matrix. The natural frequencies can be determined by vanishing the determinant of $[S_n]$, i.e.

$$\det[S_n(\lambda)] = 0$$

The terms in the determinant are transcendental functions of ω and therefore the conventional techniques of eigensolving cannot be used. At supercritical speeds, the eigenvalues are complex numbers and therefore an algorithm is required to search for the eigenvalues in the two-dimensional plane of complex numbers. In the current study the use is made of such an algorithm as is presented later.

As mentioned before, with a change of axial speed value, stability of the moving web can change to instability. This happens when one or several eigenvalues λ cross the imaginary axis [5]. The case when a pair of complex conjugate eigenvalues crosses the imaginary axis with a frequency $\omega = \mathrm{Im}\lambda \neq 0$ is known in technical literature as flutter instability, and the case when a real negative eigenvalue λ crosses the zero and becomes positive is called divergence instability. Flutter and divergence are dynamic and static forms of instability, respectively.

Equations (18) through (21) can be used for static stability analysis too, only by replacement λ with zero. Here, an eigenvalue problem is also produced, but its eigenvalue is the axial speed. Thus the critical speed v_{cr} is obtained by the following expression.

$$\det[S_n(v_{\mathrm{cr}})] = 0 \tag{22}$$

3 Numerical Results

Non-dimensional variables used in the results are introduced as

$$\Omega = \omega \frac{b^2}{2\pi^2}\sqrt{\frac{\rho h}{D_{11}}}, \quad \Gamma = \sigma \frac{b^2}{2\pi^2}\sqrt{\frac{\rho h}{D_{11}}}, \quad r = \frac{L}{b},$$

$$(c, c_{cr}) = (v, v_{cr})\frac{b}{\pi}\sqrt{\frac{\rho h}{D_{11}}}, \quad k_x = N_x \frac{b^2}{\pi^2 D_{11}} \tag{23}$$

where Ω and Γ are dimensionless imaginary and real parts of vibration eigenvalues. r is aspect ratio of the web. c and c_{cr} denote non-dimensional axial speed and critical axial speed of the web, respectively. k_x is in-web load parameter along x (positive when tensile). D_{11} in the relations (23) is defined as:

$$D_{11} = \frac{E_x}{12(1-\nu_{xy}\nu_{yx})} \tag{24}$$

In this section, the free vibration and stability of an orthotropic web is studied at low and high speeds. It is assumed that the web deflection and rotation at x=0, L are zero (Fig. 1a). The web has aspect ratio of r=3 and subjected to a uniform axial load with k_x=1.0. Dimensionless material properties are as follows:

E_y/E_x=0.5, G_{xy}/E_x=0.25, ν_{xy}=0.2, ν_{xy}/ν_{yx}=E_x/E_y

where E_x and E_y are major and minor elastic modulus of the web, respectively and G_{xy} is shear modulus.

It is adequate to divide the web into two components, derive the exact stiffness matrix of each component and assemble them. Finally we have to equalize the determinant of the overall stiffness matrix to zero and compute its roots. The determinant has so a complicated form forces us to utilize a numerical code to determine the roots. Also, since these roots may generally have both real and imaginary parts, the iterations must be done over the complex plane rather than an axis in simpler ones. Exact values of dimensionless real and imaginary parts of eigenvalues at different dimensionless speed are presented in Table 1. This values are relevant the first mode of free vibration derived with n=1. The normalized critical speed of the web derived by Equation (22) is c_{cr}=3.60008.

To show how the numerical algorithm is employed to extract the eigenvalues, the variation of the absolute value of the $\det[S_n(\lambda)]$ in logarithmic scale over the complex plane for the cases in which the dimensionless speed are equal to 4 and 5 is drawn in Fig. 2. Also to have a prospect about the web vibration, the qualitative behaviors (but not the exact response) of the web at dimensionless speeds are drawn in Fig. 3. According to this figure, between c=3.6 which is completely oscillatory and c=3.8 which has divergence instability, we must have a critical speed, in which; the system begins to show instability. Fig. 5 shows also that between c=4 and c=5 we have another instability since the behavior of system switches from pure oscillation to a flutter instability one.

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4 Conclusions

The Levy-type solusion has been employed to analyze free vibration of axially moving orthotropic webs in sub- and super-critical speeds. The exact stiffness matrix of each component of the web has been obtained, so precise results can be derived for a multi-span moving web, by only a few numbers of elements. Terms of this matrix are implicit functions of the eigenvalues of free vibration, in-plane forces act on the web, axial speed and the geometry of the web. The vibration eigenvalues are extracted within the domain of complex numbers using a suitable algorithm.

The free vibration eigenvalues of such webs obtained by the method can be served as a benchmark for checking the accuracy of other numerical methods. Nevertheless the method has some restrictions about boundaries and in-plane loads.

An example has been presented to examine the ability of the method for modeling of problems. In the example, different types of web instability (divergence or flutter) were shown in different axial speeds.

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Table 1: Exact free vibration eigenvalues for the first vibration mode of the moving orthotropic web

Normalized Axial Speed	**Normalized Eigenvalue**	
c	**Real Part (Γ)**	**Imaginary Part (Ω)**
0.0	0	0.694714
1.0	0	0.644181
2.0	0	0.491919
3.0	0	0.453442
3.2	0	0.165807
3.4	0	0.0914237
3.6	0	0.000646352
3.8	0.02085	0
4.0	0	0.0490121
5.0	0.080648	0.302775
6.0	0.182754	0

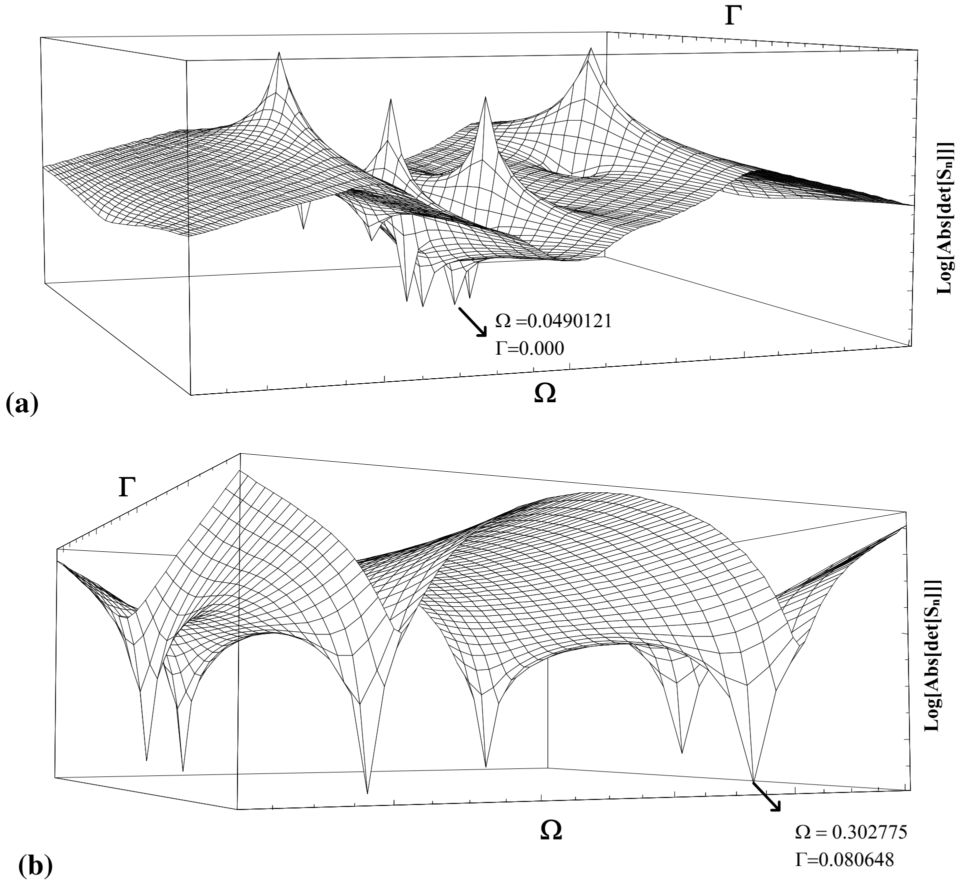

Fig. 2. Overall stiffness matrix determinant respect to the real and imaginary parts of eigenvalues. (a) c=4, (b) c=5

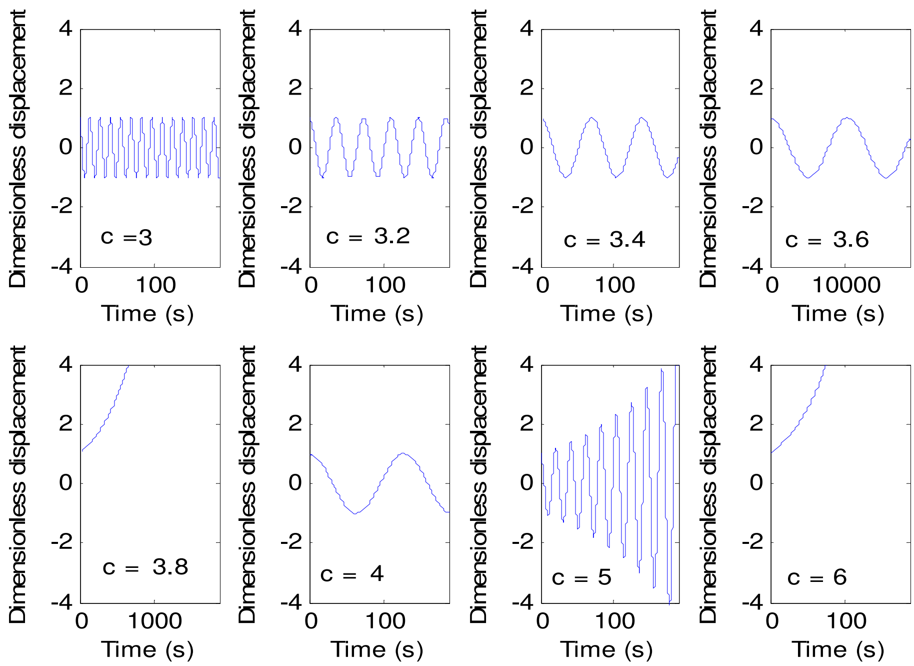

Fig. 3. Lateral deformation of the web center respect to time