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Exact inference for a simple step-stress model from the exponential distribution under time constraint

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Abstract In reliability and life-testing experiments, the researcher is often interested in the effects of extreme or varying stress factors such as temperature, voltage and load on the lifetimes of experimental units. Step-stress test, which is a special class of accelerated life-tests, allows the experimenter to increase the stress levels at fixed times during the experiment in order to obtain information on the parameters of the life distributions more quickly than under normal operating conditions. In this paper, we consider the simple step-stress model from the exponential distribution when there is time constraint on the duration of the experiment. We derive the maximum likelihood estimators (MLEs) of the parameters assuming a cumulative exposure model with lifetimes being exponentially distributed. The exact distributions of the MLEs of parameters are obtained through the use of conditional moment generating functions. We also derive confidence intervals for the parameters using these exact distributions, asymptotic distributions of the MLEs and the parametric bootstrap methods, and assess their performance through a Monte Carlo simulation study. Finally, we present two examples to illustrate all the methods of inference discussed here.

Keywords Accelerated testing · Bootstrap method · Conditional moment generating function · Coverage probability · Cumulative exposure model · Exponential distribution · Maximum likelihood estimation · Order statistics · Step-stress models · Tail probability · Type-I censoring

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1 Introduction

In many situations, it may be difficult to collect data on lift-time of a product under normal operating conditions as the product may have a high reliability under normal conditions. For this reason, accelerated life-testing (ALT) experiments can be used to force these products (systems or components) to fail more quickly than under normal operating condition. Some key references in the area of accelerated testing include Nelson (1990), Meeker and Escobar (1998), and Bagdonavicius and Nikulin (2002).

A special class of the ALT is called the *step-stress testing* which allows the experimenter to choose one or more stress factors in a life-testing experiment. Stress factors can include humidity, temperature, vibration, voltage, load or any other factor that directly affects the life of the products. In such a life-testing experiment, n identical units are placed on an initial stress level s_0 under a m-step-stress model, and only the successive failure times are recorded. The stress levels are changed to $s_1, \ldots, s_m$ at pre-fixed times $\tau_1 < \cdots < \tau_m$, respectively. The most common model used to analyse these times-to-failure data is the "cumulative damage" or "cumulative exposure" model. We consider here the situation when there is a time constraint on the duration of the experiment, say τ_{m+1}; that is, the experiment has to terminate before or at time τ_{m+1}. This is what is referred to as *Type-I Censoring* in reliability literature.

We consider here a simple step-stress model with only two stress levels. This model has been studied extensively in the literature. DeGroot and Goel (1979) proposed the tampered random variable model and discussed optimal tests under a Bayesian framework. Sedyakin (1966) introduced the cumulative exposure model in the simple step-stress case which has been further discussed and generalized by Bagdonavicius (1978) and Nelson (1980), while Miller and Nelson (1983) and Bai et al. (1989) discussed the determination of optimal time at which to change the stress level from s_0 and s_1. Bhattacharyya and Zanzawi (1989) proposed the tampered failure rate model which assumes that the effect of changing stress level is to multiply the initial failure rate function by a factor subsequent to the change times. Madi (1993) generalized this tampered failure rate model from the simple step-stress model (case $m = 1$) to the multiple step-stress model (case $m \geq 2$). Khamis and Higgins (1998) discussed the same generalization under the Weibull distribution. The tampered failure rate model as well as the model of Khamis and Higgins (1998) are special cases of Cox's proportional hazards model under step-stresses; see Bagdonavicius and Nikulin (2002). Xiong (1998) and Xiong and Milliken (1999) considered inference under the assumption of exponential life-time. They assumed that the mean life of an experimental unit is a log-linear function of the stress level, and developed inference for the two parameters of the corresponding log-linear link function. Watkins (2001) argued that it is preferable to work with the original exponential parameters eventhough the log-linear link function provides a simple reparametrization. It is important to mention here that the cumulative exposure model, under exponential lifetime distribution, becomes an accelerated life-testing model. Balakrishnan et al. (2007) derived the exact distributions of the MLEs (maximum likelihood estimators) under the exponential distribution when data are Type-II censored. Gouno and Balakrishnan (2001) reviewed the developments on step-stress accelerated life-testing. Gouno et al. (2004) discussed inference for step-stress models under exponential distribution when the available data are progressively Type-I censored.

In this article, we consider a simple step-stress model with two stress levels based on the exponential distribution when there is time constraint on the duration of the experiment. The model is described in detail in Sect. 2. Due to the form of time constraint, the MLEs of the unknown parameters do not always exist. We then discuss the conditional MLEs, and derive their conditional moment generating functions as well as the exact conditional distributions of the MLEs and their properties in Sect. 3. In Sect. 4, we discuss the exact method of constructing confidence intervals for the unknown parameters as well as the asymptotic method and the bootstrap methods. Monte Carlo simulation results and some illustrative examples are presented in Sects. 5 and 6, respectively. Finally, we make some concluding remarks in Sect. 7.

2 Model description and MLEs

Suppose that the time-to-failure data come from a cumulative exposure model, and we consider a simple step-stress model with only two stress levels s_0 and s_1 when there is a time constraint (say, τ_2) on the duration of the experiment. The lifetime distributions at s_0 and s_1 are assumed to be exponential with failure rates θ_1 and θ_2, respectively. The probability density function (PDF) and cumulative distribution function (CDF) are given by

$$f_k(t;\theta_k) = \frac{1}{\theta_k}\exp\left\{-t/\theta_k\right\}, \quad t \geq 0,\ \ \theta_k > 0,\ \ k = 1, 2 \tag{1}$$

and

$$F_k(t;\theta_k) = 1 - \exp\left\{-t/\theta_k\right\}, \quad t \geq 0,\ \ \theta_k > 0,\ \ k = 1, 2, \tag{2}$$

respectively. We then have the cumulative exposure distribution (CED) $G(t)$ as

$$G(t) = \begin{cases} G_1(t) = F_1(t;\theta_1) & \text{if } 0 < t < \tau_1 \\ G_2(t) = F_2\Big(t - \big(1 - \frac{\theta_2}{\theta_1}\big)\tau_1;\theta_2\Big) & \text{if } \tau_1 \leq t < \infty \end{cases}, \tag{3}$$

where $F_k(\cdot)$ is as given in (2). The corresponding PDF is

$$g(t) = \begin{cases} g_1(t) = \frac{1}{\theta_1}\exp\left\{-\frac{1}{\theta_1}t\right\} & \text{if } 0 < t < \tau_1 \\ g_2(t) = \frac{1}{\theta_2}\exp\left\{-\frac{1}{\theta_2}(t-\tau_1) - \frac{1}{\theta_1}\tau_1\right\} & \text{if } \tau_1 \leq t < \infty \end{cases}. \tag{4}$$

We have n identical units under an initial stress level s_0. The stress level is changed to s_1 at time τ_1, and the life-testing experiment is terminated at time τ_2, where $0 < \tau_1 < \tau_2 < \infty$ are fixed in advance. Let N_1 be the number of units that fail before τ_1, and N_2 be the number of units that fail before τ_2 at stress level s_1, then, we will observe the following observations:

$$\left\{t_{1:n} < \cdots < t_{N_1:n} \leq \tau_1 < t_{N_1+1:n} < \cdots < t_{N_1+N_2:n} \leq \tau_2\right\}, \tag{5}$$

where $t_{i:n}$ denotes the ith smallest failure time of the n units placed under test. Now, let $\mathbf{t}$ denote the vector of the $(N_1 + N_2)$ smallest failure times in (5).

From the CED in (3) and the corresponding PDF in (4), we obtain the likelihood function of θ_1 and θ_2 based on the Type-I censored sample in (5) as follows:

1. If $N_1 = n$ and $N_2 = 0$ in (5), the likelihood function of θ_1 and θ_2 is

$$L(\theta_1, \theta_2|\mathbf{t}) = n! \prod_{k=1}^{n} g_1\left(t_{k:n}\right) = \frac{n!}{\theta_1^n} \exp\left\{-\frac{1}{\theta_1}\sum_{k=1}^{n} t_{k:n}\right\},$$
$$0 < t_{1:n} < \cdots < t_{n:n} < \tau_1; \tag{6}$$

2. In all other cases, the likelihood function of θ_1 and θ_2 is

$$\begin{aligned} L(\theta_1, \theta_2|\mathbf{t}) &= \frac{n!}{(n-N)!}\left\{\prod_{k=1}^{N_1} g_1\left(t_{k:n}\right)\right\}\left\{\prod_{k=N_1+1}^{N} g_2\left(t_{k:n}\right)\right\}\left\{1 - G_2(\tau_2)\right\}^{n-N} \\ &= \frac{n!}{(n-N)!\,\theta_1^{N_1}\theta_2^{N_2}} \exp\left\{-\frac{1}{\theta_1}D_1 - \frac{1}{\theta_2}D_2\right\}, \\ & 0 < t_{1:n} < \cdots < t_{N_1:n} < \tau_1 \le t_{N_1+1:n} < \cdots < t_{N:n} < \tau_2, \end{aligned} \tag{7}$$

where $N = N_1 + N_2$ $(2 \le N \le n)$, and

$$D_1 = \sum_{k=1}^{N_1} t_{k:n} + \left(n - N_1\right)\tau_1 \quad \text{and} \quad D_2 = \sum_{k=N_1+1}^{N} \left(t_{k:n} - \tau_1\right) + (n-N)(\tau_2 - \tau_1).$$

It is useful to note that D_i corresponds to the total time on test at stress s_{i-1} (for $i = 1, 2$). From the likelihood functions in (6) and (7), we observe some results listed in the following 2×2 table:

	$N_2 = 0$	$1 \le N_2 \le n - N_1$
$N_1 = 0$	$\hat{\theta}_1$ and $\hat{\theta}_2$ do not exist	$\hat{\theta}_1$ does not exist and D_2 is a complete sufficient statistic for θ_2
$N_1 \ge 1$	$\hat{\theta}_2$ does not exist and D_1 is a complete sufficient statistic for θ_1	$\hat{\theta}_1$ and $\hat{\theta}_2$ do exist and (D_1, D_2) is a joint complete sufficient statistic for (θ_1, θ_2)

From the table, we can see that the MLEs of θ_1 and θ_2 exist only when $N_1 \ge 1$ and $N_2 \ge 1$. In this situation, the log-likelihood function of θ_1 and θ_2 is given by

$$l(\theta_1, \theta_2|\mathbf{t}) = \log\frac{n!}{(n-N)!} - N_1 \log\theta_1 - N_2 \log\theta_2 - \frac{D_1}{\theta_1} - \frac{D_2}{\theta_2}. \tag{8}$$

From (8), the MLEs of θ_1 and θ_2 are readily obtained as

$$\hat{\theta}_1 = \frac{D_1}{N_1} \quad \text{and} \quad \hat{\theta}_2 = \frac{D_2}{N_2}, \tag{9}$$

respectively.

Remark 1 In the model considered above, we have not assumed any relationship between the mean failure times under the two stress levels.

Remark 2 In some situations, we may know the mean failure time $\theta_2 = \lambda\theta_1$ for a known λ. In this situation, the MLE of θ_1 exists when at least one failure occurs, and its exact distribution can be derived explicitly. One can also use the likelihood ratio test to test the hypothesis $H_0 : \theta_2 = \lambda\theta_1$ for a specified λ. Of course, the likelihood ratio test can only be carried out conditionally (i.e., under $N_i \geq 1, \ i = 1, 2$). The properties of such a test will require further investigation.

Let us now consider the following data sets to illustrate the method of estimation discussed here.

Example 1 We now consider the following data presented by Xiong (1998):

Stress Level	Failure Times					
$\theta_1 = e^{2.5}$	2.01	3.60	4.12	4.34		
$\theta_2 = e^{1.5}$	5.04	5.94	6.68	7.09	7.17	7.49
	7.60	8.23	8.24	8.25	8.69	12.05

The choices made by Xiong (1998) were

$$n = 20, \quad \theta_1 = e^{2.5} = 12.18249, \quad \theta_2 = e^{1.5} = 4.48169 \quad \text{and} \quad \tau_1 = 5.$$

In this case, had we fixed time $\tau_2 = 6, \ 7, \ 8, \ 9, \ 12$, we would obtain the MLEs of θ_1 and θ_2 from (9) to be

$$\hat{\theta}_1 = 23.5175 \quad \text{and} \quad \hat{\theta}_2 = 7.4900, \ 9.5533, \ 5.5729, \ 4.1291, \ 5.4927.$$

Example 2 Next, we consider the following data generated with $n = 35, \ \theta_1 = e^{3.5} = 33.11545, \ \theta_2 = e^{2.0} = 7.389056$ and $\tau_1 = 8$:

Stress Level	Times-to-Failure								
$\theta_1 = e^{3.5}$	1.46	2.22	3.92	4.24	5.47	5.60	6.12	6.56		
$\theta_2 = e^{2.0}$	8.19	8.30	8.74	8.98	9.43	9.87	11.14	11.76	11.85	12.14
	14.04	14.19	14.24	14.33	15.28	16.58	16.85	16.92	17.80	20.45
	13.05	13.49	20.98	21.09	22.01	26.34	28.66			

In this case, we chose the time $\tau_2 = 16,\ 20,\ 24$. The corresponding MLEs of θ_1 and θ_2 are found from (9) to be

$$\hat{\theta}_1 = 23.315467 \quad \text{and} \quad \hat{\theta}_2 = 8.971297,\ 9.316378,\ 9.646305.$$

3 Conditional distributions of the MLEs

To find the exact distributions of $\hat{\theta}_1$ and $\hat{\theta}_2$, we first derive the conditional moment generating functions (CMGF) of $\hat{\theta}_1$ and $\hat{\theta}_2$, conditioned on the event $A = \Big\{1 \le N_1 \le n-1$ and $1 \le N_2 \le n - N_1\Big\}$. For notational convenience, we denote $M_k\big(\omega\big)$ for the CMGF of $\hat{\theta}_k,\ k = 1, 2$. Then, we can write

$$\begin{aligned} M_{k|A}\big(\omega\big) &= \mathrm{E}\Big\{e^{\omega\hat{\theta}_k}\big|A\Big\} \\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-i} \mathrm{E}_{\theta_1,\theta_2}\Big\{e^{\omega\hat{\theta}_k}\big|N_1 = i, N_2 = j\Big\} \cdot \mathrm{P}_{\theta_1,\theta_2}\Big\{N_1 = i, N_2 = j|A\Big\}. \end{aligned} \quad (10)$$

Clearly, the numbers of failures occurring before τ_1 and between τ_1 and τ_2 has a trinomial distribution with probability mass function (pmf)

$$\mathrm{P}_{\theta_1,\theta_2}\{N_1 = i, N_2 = j\} = \binom{n}{i,\ j,\ n-i-j} p_1^i p_2^j p_3^{n-i-j}, \quad i = 0, 1, \ldots, n,\ j = 0, \ldots, n-i, \quad (11)$$

where

$$\begin{aligned} p_1 &= G_1(\tau_1) = 1 - e^{-\tau_1/\theta_1}, \\ p_2 &= G_2(\tau_2) - G_1(\tau_1) = \big(1 - p_1\big)\Big\{1 - e^{-(\tau_2-\tau_1)/\theta_2}\Big\}, \\ p_3 &= 1 - p_1 - p_2, \end{aligned}$$

and $\binom{n}{i,j,n-i-j}$ is the extended multinomial coefficient defined by

$$\binom{n}{i,\ j,\ n-i-j} = \frac{n!}{i!j!(n-i-j)!}.$$

Consequently, we can write

$$\mathrm{P}_{\theta_1,\theta_2}\Big\{N_1 = i, N_2 = j|A\Big\} = C_n \cdot \mathrm{P}_{\theta_1,\theta_2}\Big\{N_1 = i, N_2 = j\Big\}, \quad (12)$$

where

$$C_n = \frac{1}{1 - (1-p_1)^n - (1-p_2)^n + p_3^n}.$$

Now, to derive $E_{\theta_1,\theta_2}\left\{e^{\omega\hat{\theta}_k}\middle| N_1 = i, N_2 = j\right\}$, we need the following Lemma.

Lemma 1 *Let $T_{1:n} < \cdots < T_{n:n}$ denote the n order statistics from PDF $g(t)$ given in* (4)*. Then, the joint conditional PDF of $T_{1:n}, \ldots, T_{N_1+N_2:n}$, given $N_1 = i$ and $N_2 = j$, is* (*see* Arnold et al. 1992, David and Nagaraja 2003)

$$f\left(t_1, \ldots, t_{i+j} | N_1 = i, N_2 = j\right) = R_{ij} \cdot \exp\left\{-\frac{1}{\theta_1}\sum_{k=1}^{i} t_k - \frac{1}{\theta_2}\sum_{k=i+1}^{i+j}\left(t_k - \tau_1\right)\right\},$$
$$0 < t_1 < \cdots < t_i \le \tau_1 < t_{i+1} < \cdots < t_{i+j} \le \tau_2, \tag{13}$$

where

$$R_{ij} = \frac{n!\,p_3^{n-i-j}(1-p_1)^j}{\theta_1^i\theta_2^j(n-i-j)!\,P_{\theta_1,\theta_2}\left\{N_1 = i, N_2 = j\right\}}.$$

Proof The joint conditional PDF of $T_{1:n}, \ldots, T_{N_1+N_2:n}$, given $N_1 = i$ and $N_2 = j$, can be written as

$$f\left(t_1, \ldots, t_{i+j} | N_1 = i, N_2 = j\right) = \frac{n!}{(n-i-j)!\,P_{\theta_1,\theta_2}\left\{N_1 = i, N_2 = j\right\}}$$
$$\times\left\{\prod_{k=1}^{i} g_1(t_k)\right\}\left\{\prod_{k=i+1}^{i+j} g_2(t_k)\right\}\left\{1 - G_2(\tau_2)\right\}^{n-i-j}.$$

Upon substituting the expressions for g_1, g_2 and G_2, (13) follows.

Corollary 1 *The CMGF of $\hat{\theta}_1$, given the event $A = \left\{1 \le N_1 \le n-1 \text{ and } 1 \le N_2 \le n - N_1\right\}$, is*

$$M_1(\omega|A) = C_n\sum_{i=1}^{n-1}\sum_{k=0}^{i} C_{ik} \cdot \frac{\exp\left\{\frac{\omega}{i}(n-i+k)\tau_1\right\}}{\left(1-\frac{\theta_1}{i}\omega\right)^i}, \quad \omega < \frac{1}{\theta_1}, \tag{14}$$

where

$$C_{ik} = (-1)^k\binom{n}{i}\binom{i}{k}\left\{\left(1-p_1\right)^{n-i} - p_3^{n-i}\right\}\left(1-p_1\right)^k$$

and C_n, p_1, p_2 and p_3 are as defined earlier.

Proof Using (9), (11), (12) and the result in Lemma 1 into Eq. (10), and simplifying the resulting expression, we obtain (14).

Corollary 2 *The CMGF of $\hat{\theta}_2$, given the event $A = \left\{1 \le N_1 \le n-1 \text{ and } 1 \le N_2 \le n - N_1\right\}$, is*

Table 1 Estimated coverage probabilities (in %) of confidence intervals for θ_1 and θ_2 based on 1,000 simulations with $n = 20,\ \theta_1 = e^{2.5},\ \theta_2 = e^{1.5}$ and $R = 1,000$

			90% CI			**95% CI**			**99% CI**		
	τ_1	τ_2	**BCa**	**App.**	**Exact**	**BCa**	**App.**	**Exact**	**BCa**	**App.**	**Exact**
θ_1	1	2	80.2	75.2	90.7	85.1	74.0	95.1	90.4	90.1	98.7
		3	81.8	74.4	90.9	86.4	76.9	95.5	91.9	90.2	98.7
		4	80.4	74.4	90.2	85.7	73.2	94.7	92.5	91.7	98.5
		5	80.5	76.8	89.4	85.2	73.7	94.4	92.1	93.2	99.4
		6	81.4	73.9	90.6	85.6	71.8	95.2	92.5	91.5	99.3
		7	81.3	75.8	89.7	86.4	75.8	95.6	90.4	92.6	98.7
		8	82.1	74.4	89.9	87.7	74.5	95.0	91.0	92.1	99.2
		9	81.9	73.5	90.7	86.0	75.4	94.1	93.3	92.4	99.5
		10	83.1	75.0	90.2	86.7	73.7	94.9	91.8	92.5	98.8
	2	3	87.2	84.1	90.6	88.6	81.8	94.9	92.0	92.3	99.4
		4	83.9	81.9	90.7	90.0	81.2	94.3	91.6	91.6	98.6
		5	82.5	81.9	89.4	89.1	81.3	94.6	93.8	93.2	98.8
		6	82.8	82.0	89.8	89.5	83.0	95.5	93.2	94.2	99.1
		7	81.8	84.3	90.7	89.8	81.3	95.8	92.9	92.3	98.5
		8	84.4	80.3	88.9	89.6	82.9	95.9	93.6	91.3	99.1
		9	83.9	81.8	90.3	89.0	81.1	95.3	93.1	91.8	98.9
		10	82.7	83.0	90.5	90.3	80.0	95.7	93.7	93.0	99.0
	3	4	84.9	82.3	90.8	86.4	87.0	94.6	91.3	94.3	99.0
		5	86.5	81.7	89.9	86.6	84.2	94.1	91.0	94.9	99.3
		6	85.4	85.6	90.9	89.4	88.1	95.8	92.1	93.3	99.3
		7	83.0	83.1	90.1	87.8	88.3	94.9	91.6	94.0	98.9
		8	85.1	85.7	90.4	89.2	88.4	94.7	92.8	94.3	99.3
		9	84.8	82.2	89.9	89.2	87.7	95.8	93.9	93.8	99.8
		10	84.0	85.0	90.8	88.6	87.5	94.9	91.4	94.4	99.1
	4	5	86.3	83.7	90.7	90.1	88.1	95.0	95.3	92.0	98.9
		6	87.0	84.1	90.8	88.4	89.6	95.4	95.8	94.6	99.1
		7	88.9	82.0	89.9	90.0	91.1	94.9	94.9	95.4	99.5
		8	87.4	83.3	90.9	90.5	87.1	94.9	95.4	93.9	98.7
		9	85.6	83.1	89.6	89.7	89.1	95.6	93.3	94.9	99.3
		10	86.8	82.5	89.4	90.2	88.6	94.6	95.9	94.3	99.1
θ_2	1	2	84.3	87.1	89.9	89.3	85.2	95.9	91.9	92.6	98.8
		3	90.4	84.5	90.7	92.8	90.7	95.8	94.4	93.9	99.5
		4	87.2	85.9	89.2	93.0	90.4	95.6	97.1	95.2	99.4
		5	90.5	87.6	90.5	95.2	90.5	94.6	98.1	95.7	98.5
		6	90.2	87.7	89.9	95.5	92.1	94.3	98.9	94.9	99.1
		7	90.0	86.1	89.5	94.3	92.1	95.5	98.8	95.0	99.4
		8	90.7	89.7	90.7	95.1	92.0	95.2	98.5	96.2	99.3
		9	90.4	88.4	90.4	95.6	92.4	95.9	98.0	96.7	98.8
		10	89.6	89.6	90.5	94.5	94.1	95.5	99.2	96.3	99.6

Table 1 continued

τ_1	τ_2	90% CI BCa	App.	Exact	95% CI BCa	App.	Exact	99% CI BCa	App.	Exact
2	3	86.1	88.7	89.9	89.1	84.9	95.7	91.2	91.1	99.3
	4	87.3	84.8	89.7	92.0	87.3	94.9	95.1	94.2	99.2
	5	88.4	86.0	90.2	93.6	89.0	94.2	97.1	95.7	98.8
	6	89.8	87.9	89.8	94.1	90.1	94.7	97.1	95.5	99.2
	7	90.7	86.9	90.3	95.2	90.6	95.3	98.2	96.1	99.1
	8	88.7	88.3	90.7	94.8	91.2	94.2	98.1	96.1	99.5
	9	91.3	86.9	89.9	94.5	91.8	94.5	98.7	95.6	99.2
	10	89.7	88.8	90.6	96.6	92.4	95.4	99.0	96.3	99.0
3	4	84.4	85.4	90.5	86.9	85.0	95.8	92.7	92.6	99.8
	5	87.3	85.1	90.9	91.7	89.4	94.8	95.4	93.9	98.4
	6	89.2	87.4	90.0	91.2	90.5	95.4	96.5	93.8	99.0
	7	89.8	86.7	89.0	93.6	90.4	95.8	97.6	95.8	98.8
	8	88.1	87.2	89.8	94.0	90.3	94.9	98.1	95.6	99.5
	9	90.7	88.0	90.0	92.5	91.3	94.3	97.9	95.5	98.9
	10	90.7	88.4	90.9	94.9	90.5	95.1	98.3	96.1	99.6
4	5	82.9	78.7	90.7	86.1	83.8	95.9	93.7	91.1	99.9
	6	86.8	82.7	90.7	90.9	87.3	94.3	93.7	94.6	99.2
	7	88.2	84.6	90.2	93.2	90.2	95.7	96.9	94.3	99.3
	8	88.8	87.7	89.8	93.4	89.6	94.3	97.5	94.3	98.9
	9	89.8	87.4	89.9	94.2	92.3	94.6	97.9	95.2	99.4
	10	88.7	86.5	89.8	95.3	91.7	95.2	98.5	94.9	99.1

$$M_2(\omega|A) = C_n \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{k=0}^{j} C_{ijk} \cdot \frac{\exp\left\{\frac{\omega}{j}(n-i-j+k)(\tau_2-\tau_1)\right\}}{\left(1-\frac{\theta_2}{j}\omega\right)^j}, \quad \omega < \frac{1}{\theta_2}, \tag{15}$$

where

$$C_{ijk} = (-1)^k \binom{n}{i,\ j,\ n-i-j} \binom{j}{k} p_1^i p_3^{n-i-j+k} (1-p_1)^{j-k},$$

and C_n, p_1, p_2 *and* p_3 *are as defined earlier.*

Proof Using (9), (11), (12) and the result in Lemma 1 into Eq. (10), and simplifying the resulting expression, we obtain (15).

Now, in order to obtain the exact PDFs of $\hat{\theta}_1$ and $\hat{\theta}_2$ in (9), we need the following Lemma.

Lemma 2 *If* X *is a gamma random variable with parameters* α *and* β*, then the PDF of* $Y = X + \zeta$ *is of the form*

Table 2 Estimated coverage probabilities (in %) of confidence intervals for θ_1 and θ_2 based on 1,000 simulations with $n = 35,\ \theta_1 = e^{2.5},\ \theta_2 = e^{1.5}$ and $R = 1,000$

			90% CI			95% CI			99% CI		
	τ_1	τ_2	**BCa**	**App.**	**Exact**	**BCa**	**App.**	**Exact**	**BCa**	**App.**	**Exact**
θ_1	1	2	85.8	86.1	90.6	85.8	84.6	95.2	93.0	89.8	98.8
		3	86.9	83.2	90.8	86.9	94.0	95.4	92.5	90.0	98.9
		4	84.5	82.7	90.1	84.5	82.5	94.6	93.2	91.1	99.5
		5	85.7	82.1	89.3	85.7	86.9	94.5	93.1	91.1	99.6
		6	96.7	82.2	89.9	84.5	85.8	95.0	92.9	91.2	99.4
		7	84.0	84.5	89.8	87.2	84.7	95.3	92.8	91.1	98.9
		8	84.4	84.3	89.9	87.6	85.8	95.0	92.8	91.0	99.3
		9	87.0	83.7	90.6	88.4	83.7	94.9	91.7	92.2	99.7
		10	83.2	82.4	90.1	88.1	84.1	94.9	92.4	90.8	98.9
	2	3	83.7	85.5	90.2	88.0	86.1	94.8	92.5	92.6	99.9
		4	83.9	85.4	90.7	88.6	86.8	94.8	94.1	93.0	98.9
		5	84.3	85.3	89.8	89.4	87.1	94.9	93.4	93.3	99.8
		6	83.6	85.1	89.9	88.8	87.1	95.3	93.3	91.5	99.7
		7	84.5	85.8	90.5	86.8	86.0	95.1	92.8	93.6	99.5
		8	83.9	86.8	89.9	90.1	85.2	95.3	93.0	92.8	99.5
		9	83.1	83.3	90.2	88.7	86.8	95.2	93.6	93.7	98.9
		10	81.8	83.8	90.5	88.0	86.2	95.6	94.9	93.4	99.5
	3	4	86.6	85.1	90.4	91.8	89.6	94.9	95.7	94.8	99.8
		5	89.1	87.6	89.9	92.1	89.3	94.8	97.2	94.5	99.2
		6	87.3	86.0	90.3	91.9	89.8	95.1	96.9	94.9	99.8
		7	87.4	86.9	90.0	90.4	89.5	94.9	96.6	94.1	99.9
		8	87.9	84.6	90.4	92.0	88.6	94.8	96.6	93.4	100.0
		9	89.8	85.7	89.9	92.7	88.2	95.4	96.8	94.2	99.7
		10	89.1	85.9	90.5	91.3	88.8	94.9	96.3	94.3	99.2
	4	5	88.8	84.4	90.5	92.0	91.3	95.0	97.5	95.5	99.9
		6	89.9	86.8	90.3	94.5	90.3	95.1	97.0	94.3	99.6
		7	89.5	87.0	89.9	92.3	91.9	94.9	96.7	94.8	99.1
		8	89.2	87.8	90.4	93.8	91.9	94.7	96.5	94.9	99.7
		9	88.7	87.3	89.7	93.7	91.7	95.1	97.1	95.6	99.4
		10	89.6	86.7	89.5	92.2	91.5	94.8	96.6	95.7	99.5
θ_2	1	2	88.1	82.5	89.9	88.1	88.7	95.3	96.9	94.0	98.9
		3	87.9	88.2	90.2	87.9	90.5	95.2	97.4	96.5	99.9
		4	89.1	87.8	89.2	89.1	93.7	95.1	97.6	96.4	99.8
		5	89.3	89.4	90.2	89.3	93.2	94.9	98.5	96.1	98.9
		6	90.8	89.8	89.9	90.8	92.1	94.8	98.9	96.9	99.3
		7	90.9	90.0	89.8	95.8	92.5	95.5	99.0	97.3	99.5
		8	89.2	89.2	90.1	93.5	94.2	95.1	98.9	96.9	99.8
		9	89.6	91.2	90.4	95.0	93.5	95.4	98.2	98.0	98.8
		10	90.2	89.9	90.2	95.5	93.3	95.0	99.1	97.8	99.6

Table 2 continued

τ_1	τ_2	90% CI			95% CI			99% CI		
		BCa	**App.**	**Exact**	**BCa**	**App.**	**Exact**	**BCa**	**App.**	**Exact**
2	3	86.3	85.7	89.9	91.0	89.2	95.1	95.3	95.2	99.5
	4	87.6	87.6	89.8	93.0	90.8	94.9	97.4	95.0	99.1
	5	88.5	89.2	90.2	94.9	91.8	94.9	98.4	95.7	99.8
	6	89.4	89.9	89.8	93.5	92.7	94.7	98.3	96.9	99.6
	7	88.5	89.1	90.3	95.1	93.6	95.3	98.1	97.2	99.1
	8	88.5	88.7	90.3	92.5	93.2	94.7	99.2	97.4	99.4
	9	89.5	90.4	89.9	95.1	93.8	94.9	98.8	97.6	99.9
	10	89.7	91.9	90.1	94.6	93.0	95.3	98.8	97.2	99.1
3	4	86.8	84.3	90.2	90.5	89.4	95.2	94.7	94.5	99.8
	5	87.9	85.9	90.2	93.2	90.2	94.8	97.0	95.2	99.4
	6	89.0	89.2	90.0	93.4	92.6	95.0	97.5	96.2	99.0
	7	88.7	89.4	89.9	94.4	92.3	95.1	98.4	96.3	99.8
	8	90.4	91.2	89.9	94.9	92.6	94.9	97.7	96.3	99.5
	9	90.5	88.7	90.0	94.7	92.5	94.9	98.4	97.8	99.7
	10	91.1	90.9	90.3	94.8	92.9	95.1	98.5	96.9	99.6
4	5	86.5	81.9	90.4	88.3	86.7	95.2	93.2	93.5	99.9
	6	88.7	86.1	90.3	94.2	89.7	94.8	97.3	93.7	99.6
	7	89.0	89.4	90.1	94.1	90.5	95.3	97.7	96.7	99.6
	8	91.2	87.9	89.8	93.4	91.7	94.8	97.0	96.7	98.9
	9	89.9	89.7	89.9	95.1	92.7	94.9	98.5	96.5	99.2
	10	88.5	88.8	89.8	94.9	93.2	95.2	98.3	97.1	99.3

$$\gamma\left(x-\zeta;\alpha,\beta\right)=\begin{cases}\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\left(x-\zeta\right)^{\alpha-1}\mathrm{e}^{-(x-\zeta)/\beta} & \text{if } x>\zeta\\ 0 & \text{otherwise}\end{cases}, \tag{16}$$

and the MGF of $Y=X+\zeta$ *is of the form*

$$M_Y(\omega)=\frac{e^{\omega\zeta}}{\left(1-\beta\omega\right)^{\alpha}},\quad |\omega|<1/\beta. \tag{17}$$

Proof The proof follows from the well-known properties of the gamma distribution (see Johnson et al. 1994).

Theorem 1 *The conditional PDF of* $\hat{\theta}_1$*, given the event* $A=\Big\{1\le N_1\le n-1$ *and* $1\le N_2\le n-N_1\Big\}$*, is*

$$f_{\hat{\theta}_1}(x)=C_n\sum_{i=1}^{n-1}\sum_{k=0}^{i}C_{ik}\cdot\gamma\left(x-\tau_{ik};i,\frac{\theta_1}{i}\right), \tag{18}$$

where $\tau_{ik}=\frac{1}{i}(n-i+k)\tau_1$ *and* $\gamma(\cdot)$ *is as defined in* (16).

Proof From (14), we have CMGF of $\hat{\theta}_1$, given the event A as

$$M_1(\omega|A) = C_n \sum_{i=1}^{n-1} \sum_{k=0}^{i} C_{ik} \cdot M_{Y_{ik}}(\omega),$$

where Y_{ik}'s are random variables distributed as $X_{ik} + \tau_{ik}$ with X_{ik} being distributed as gamma with shape parameter $\alpha = i$ and scale parameter $\beta = \frac{\theta_1}{i}$. Result in (18) is obtained by inverting the above relation between the moment generating functions.

Remark 3 It is important to note here that the conditional PDF of $\hat{\theta}_1$ derived in Theorem 1 will enable to develop exact conditional inference for the parameter θ_1 conditioned on the event $\left\{1 \le N_1 \le n-1 \text{ and } 1 \le N_2 \le n - N_1\right\}$. So, this should be considered as conditional inference for θ_1 during the joint estimation of the parameters θ_1 and θ_2. One can instead consider the conditional density of $\hat{\theta}_1$ conditioned on the event $N_1 \ge 1$ (which will then simply be the conditional density of the MLE $\hat{\theta}_1$ based on an ordinary Type-I censored sample from the Exponential(θ_1) distribution), but the corresponding inference in this case will not be in the framework of joint estimation of θ_1 and θ_2. For this reason, change in τ_2 will have an effect on the inference for θ_1, but it will be negligible (see, for example, Table 3).

Theorem 2 *The conditional PDF of $\hat{\theta}_2$, given $A = \left\{1 \le N_1 \le n-1 \text{ and } 1 \le N_2 \le n - N_1\right\}$, is*

$$f_{\hat{\theta}_2}(x) = C_n \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{k=0}^{j} C_{ijk} \cdot \gamma\left(x - \tau_{ijk}; j, \frac{\theta_2}{j}\right), \tag{19}$$

where $\tau_{ijk} = \frac{1}{j}(n-i-j+k)(\tau_2 - \tau_1)$ and $\gamma(\cdot)$ is as defined in (16).

Proof The result follows immediately from (15) upon using Lemma 2.

Corollary 3 *The first two raw moments of $\hat{\theta}_1$ are*

$$\mathrm{E}\left(\hat{\theta}_1\right) = \theta_1 + C_n \sum_{i=1}^{n-1} \sum_{k=0}^{i} C_{ik} \cdot \tau_{ik} \tag{20}$$

and

$$\mathrm{E}\left(\hat{\theta}_1^2\right) = \theta_1^2 + C_n \sum_{i=1}^{n-1} \sum_{k=0}^{i} C_{ik} \cdot \left(\tau_{ik}^2 + 2\tau_{ik}\theta_1 + \frac{1}{i}\,\theta_1^2\right), \tag{21}$$

respectively.

Proof These expressions follow readily from (18).

Corollary 4 *The first two raw moments of* $\hat{\theta}_2$ *are*

$$\mathrm{E}\big(\hat{\theta}_2\big) = \theta_2 + C_n \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{k=0}^{j} C_{ijk} \cdot \tau_{ijk} \tag{22}$$

and

$$\mathrm{E}\big(\hat{\theta}_2^2\big) = \theta_2^2 + C_n \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{k=0}^{j} C_{ijk} \cdot \left(\tau_{ijk}^2 + 2\tau_{ijk}\theta_2 + \frac{1}{j}\,\theta_2^2\right), \tag{23}$$

respectively.

Proof These expressions follow readily from (19).

Corollary 5 *The tail probabilities of* $\hat{\theta}_1$ *and* $\hat{\theta}_2$ *are*

$$\mathrm{P}_{\theta_1}\left\{\hat{\theta}_1 > \xi\right\} = C_n \sum_{i=1}^{n-1} \sum_{k=0}^{i} C_{ik} \cdot \Gamma\left(\frac{i}{\theta_1}\langle \xi - \tau_{ik}\rangle; i\right) \tag{24}$$

and

$$\mathrm{P}_{\theta_2}\left\{\hat{\theta}_2 > \xi\right\} = C_n \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{k=0}^{j} C_{ijk} \cdot \Gamma\left(\frac{j}{\theta_2}\langle \xi - \tau_{ijk}\rangle; j\right), \tag{25}$$

where $\langle w\rangle = \max\{0, w\}$ *and*

$$\Gamma(w;\alpha) = \int_w^\infty \gamma\big(x;\alpha,1\big)\mathrm{d}x = \int_w^\infty \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}\mathrm{d}x.$$

Proof The expressions in (24) and (25) follow by integration from (18) and (19), respectively.

Incidentally, proceeding exactly along the same lines as above, we can derive the joint CMGF and joint conditional density function of $\hat{\theta}_1$ and $\hat{\theta}_2$ as given in the following theorem.

Theorem 3 *The joint CMGF of* $\hat{\theta}_1$ *and* $\hat{\theta}_2$*, given the event* $A = \Big\{1 \le N_1 \le n-1$ *and* $1 \le N_2 \le n - N_1\Big\}$*, is*

$$M_{12}\big(\nu,\omega|A\big) = C_n \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{k=0}^{i} \sum_{l=0}^{j} A_{ijkl} \cdot \frac{e^{\frac{\nu}{i}(n-i+k)\tau_1}}{\big(1-\frac{\theta_1}{i}\nu\big)^i} \cdot \frac{e^{\frac{\omega}{j}(n-i-j+l)(\tau_2-\tau_1)}}{\big(1-\frac{\theta_2}{j}\omega\big)^j}, \tag{26}$$

and the joint conditional density of $\hat{\theta}_1$ *and* $\hat{\theta}_2$ *is*

$$f_{12}(x, y) = C_n \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{k=0}^{i} \sum_{l=0}^{j} C_{ijkl} \cdot \gamma\left(x - \tau_{ik}; i, \frac{\theta_1}{i}\right) \gamma\left(y - \tau_{ijl}; j, \frac{\theta_2}{j}\right), \tag{27}$$

where C_n, p_1, p_2, p_3, $\gamma(\cdot)$, *and* τ_{ik} *are all as defined earlier,* $\tau_{ijl} = \frac{1}{j}(n - i - j + l) \times (\tau_2 - \tau_1)$, *and*

$$C_{ijkl} = (-1)^{k+l} \binom{n}{i, j, n-i-j} \binom{i}{k} \binom{j}{l} p_3^{n-i-j+l} (1 - p_1)^{j+k-l}.$$

Corollary 6 *From* (26), *we readily obtain*

$$\mathrm{E}\big(\hat{\theta}_1 \hat{\theta}_2\big) = C_n \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{k=0}^{i} \sum_{l=0}^{j} C_{ijkl} \cdot (\theta_1 + \tau_{ik})(\theta_2 + \tau_{ijl}), \tag{28}$$

and

$$\begin{aligned} \mathrm{Cov}\big(\hat{\theta}_1, \hat{\theta}_2\big) = {} & C_n \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{k=0}^{i} \sum_{l=0}^{j} C_{ijkl} \cdot \tau_{ik} \tau_{ijl} \\ & - C_n^2 \left(\sum_{i=1}^{n-1} \sum_{k=0}^{i} C_{ik} \cdot \tau_{ik} \right) \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{k=0}^{j} C_{ijk} \cdot \tau_{ijk} \right), \end{aligned} \tag{29}$$

where C_{ik} *and* C_{ijk} *are as defined earlier. From* (29), *it can be seen that* $\hat{\theta}_1$ *and* $\hat{\theta}_2$ *are correlated in case of small samples eventhough they become asymptotically independent as shown in Sect.* 4.2.

4 Confidence intervals

In this section, we present different methods of constructing confidence intervals (CIs) for the unknown parameters θ_1 and θ_2. From Theorems 1 and 2, we can construct the exact CIs for θ_1 and θ_2, respectively. Since the exact conditional PDF of $\hat{\theta}_1$ and $\hat{\theta}_2$ are quite complicated, we also present the approximate CI for θ_1 and θ_2 for larger sample sizes. Finally, we use the parametric bootstrap method to construct CI for θ_1 and θ_2.

4.1 Exact confidence intervals

To guarantee the invertibility for the parameters θ_1 and θ_2, we assume that the tail probabilities of $\hat{\theta}_1$ and $\hat{\theta}_2$ presented in Corollary 5 are increasing functions of θ_1 and

θ_2, respectively. Several authors including Chen and Bhattacharyya (1988), Gupta and Kundu (1998), Kundu and Basu (2000), and Childs et al. (2003) have used this approach to construct exact CI in different contexts. Like all of them, we are also unable to establish the required monotonicity, but the extensive numerical computations we carried out seem to support this monotonicity assumption; see Fig. 1, for example.

(1) CI for θ_1

The exact CI for θ_1 can be constructed by solving the equations

$$P_{\theta_{1L}}\left\{\hat{\theta}_1 > \hat{\theta}_{obs}\right\} = \frac{\alpha}{2} \quad \text{and} \quad P_{\theta_{1U}}\left\{\hat{\theta}_1 > \hat{\theta}_{obs}\right\} = 1 - \frac{\alpha}{2}$$

for θ_{1L} (the lower bound of θ_1) and θ_{1U} (the upper bound of θ_1), respectively. A two-sided $100(1-\alpha)\%$ CI for θ_1, denoted by $(\theta_{1L}, \theta_{1U})$, can be obtained by solving the following two non-linear equations (either using the Newton–Raphson method or bisection method):

$$\frac{\alpha}{2} = C_n\left(\theta_{1L}, \hat{\theta}_2\right) \sum_{i=1}^{n-1} \sum_{k=0}^{i} C_{ik}\left(\theta_{1L}, \hat{\theta}_2\right) \cdot \Gamma\left(\frac{i}{\theta_{1L}}\langle \hat{\theta}_1 - \tau_{ik}\rangle; i\right)$$

and

$$1 - \frac{\alpha}{2} = C_n\left(\theta_{1U}, \hat{\theta}_2\right) \sum_{i=1}^{n-1} \sum_{k=0}^{i} C_{ik}\left(\theta_{1U}, \hat{\theta}_2\right) \cdot \Gamma\left(\frac{i}{\theta_{1U}}\langle \hat{\theta}_1 - \tau_{ik}\rangle; i\right),$$

where C_n, C_{ik}, τ_{ik} and $\Gamma(w; \alpha)$ are all as defined earlier.

(2) CI for θ_2

Similarly, a two-sided $100(1-\alpha)\%$ CI for θ_2, denoted by $(\theta_{2L}, \theta_{2U})$, can be obtained by solving the following two non-linear equations:

$$\frac{\alpha}{2} = C_n\left(\hat{\theta}_1, \theta_{2L}\right) \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{k=0}^{j} C_{ijk}\left(\hat{\theta}_1, \theta_{2L}\right) \cdot \Gamma\left(\frac{j}{\theta_{2L}}\langle \hat{\theta}_2 - \tau_{ijk}\rangle; j\right)$$

and

$$1 - \frac{\alpha}{2} = C_n\left(\hat{\theta}_1, \theta_{2U}\right) \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{k=0}^{j} C_{ijk}\left(\hat{\theta}_1, \theta_{2U}\right) \cdot \Gamma\left(\frac{j}{\theta_{2U}}\langle \hat{\theta}_2 - \tau_{ijk}\rangle; j\right),$$

where C_n, C_{ijk}, τ_{ijk} and $\Gamma(w; \alpha)$ are all as defined earlier.

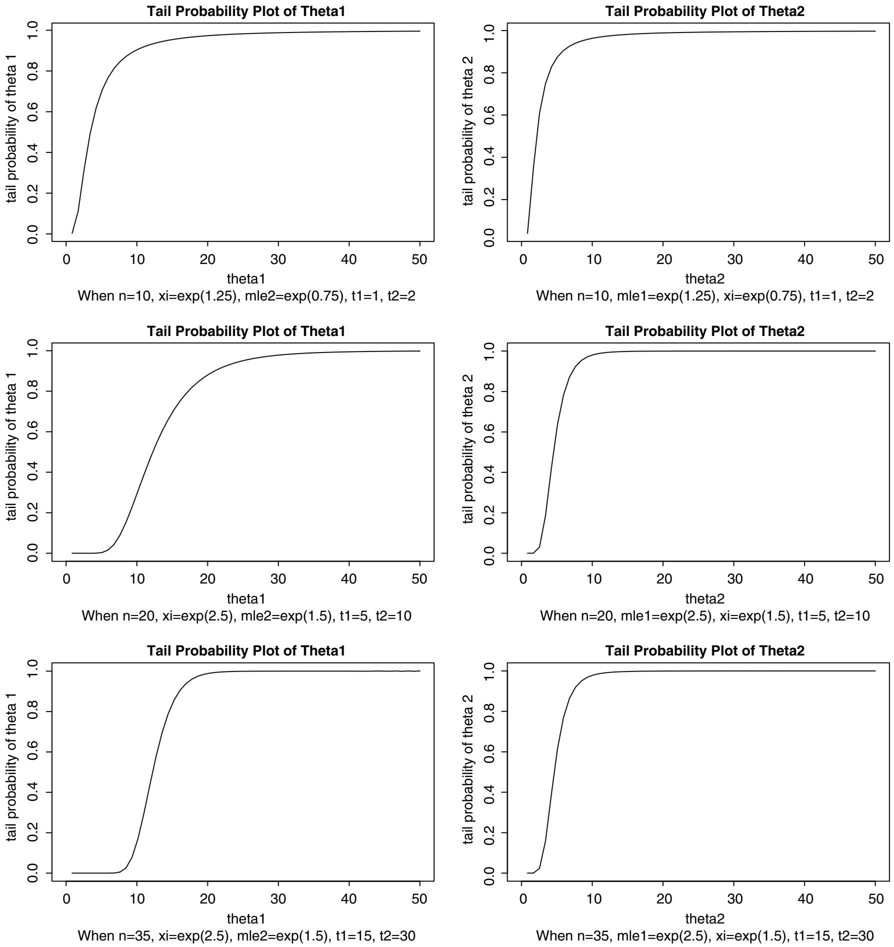

Fig. 1 Tail probability plot of $\hat{\theta}_1$ and $\hat{\theta}_2$

4.2 Approximate confidence intervals

For large N_1 and N_2, the observed Fisher information matrix of θ_1 and θ_2 is

$$\hat{I}(\theta_1, \theta_2) = \begin{bmatrix} \hat{I}_{11} & \hat{I}_{12} \\ \hat{I}_{21} & \hat{I}_{22} \end{bmatrix}_{\theta_1=\hat{\theta}_1, \theta_2=\hat{\theta}_2}, \tag{30}$$

where

$$\hat{I}_{ij} = -\mathrm{E}\left\{ \frac{\partial l(\theta_1, \theta_2|\mathbf{t})}{\partial\theta_i \partial\theta_j} \right\}\Bigg|_{\theta_1=\hat{\theta}_1, \theta_2=\hat{\theta}_2}, \quad i, j = 1, 2,$$

and $\hat{\theta}_1$ and $\hat{\theta}_2$ are as in (9). Therefore, we have

$$\hat{I}_{11} = -\mathrm{E}\left\{\frac{N_1}{\hat{\theta}_1^2} + \frac{2D_1}{\hat{\theta}_1^3}\right\} = \frac{N_1}{\hat{\theta}_1^2},$$
$$\hat{I}_{12} = 0 = \hat{I}_{21},$$
$$\hat{I}_{22} = -\mathrm{E}\left\{\frac{N_2}{\hat{\theta}_2^2} + \frac{2D_2}{\hat{\theta}_2^3}\right\} = \frac{N_2}{\hat{\theta}_2^2}.$$

The asymptotic variances of $\hat{\theta}_1$ and $\hat{\theta}_2$ can be obtained from (26) as

$$V_{11} = \mathrm{Var}(\hat{\theta}_1) = \frac{\hat{\theta}_1^2}{N_1} \quad \text{and} \quad V_{22} = \mathrm{Var}(\hat{\theta}_2) = \frac{\hat{\theta}_2^2}{N_2}.$$

We can then use the pivotal quantities for θ_1 and θ_2 as

$$\frac{\hat{\theta}_1 - \mathrm{E}(\hat{\theta}_1)}{\sqrt{V_{11}}} \quad \text{and} \quad \frac{\hat{\theta}_2 - \mathrm{E}(\hat{\theta}_2)}{\sqrt{V_{22}}},$$

where $\mathrm{E}(\hat{\theta}_1)$ and $\mathrm{E}(\hat{\theta}_2)$ are as given in (20) and (22), respectively. We can then express a two-sided $100(1-\alpha)\%$ approximate CI for θ_1 and θ_2 as

$$(\hat{\theta}_1 - W_1) \pm z_{1-\alpha/2}\sqrt{V_{11}}$$

and

$$(\hat{\theta}_2 - W_2) \pm z_{1-\alpha/2}\sqrt{V_{22}},$$

where

$$W_1 = C_n(\hat{\theta}_1, \hat{\theta}_2) \sum_{i=1}^{n-1} \sum_{k=0}^{i} C_{ik}(\hat{\theta}_1, \hat{\theta}_2) \cdot \tau_{ik},$$
$$W_2 = C_n(\hat{\theta}_1, \hat{\theta}_2) \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{k=0}^{j} C_{ijk}(\hat{\theta}_1, \hat{\theta}_2) \cdot \tau_{ijk},$$

and $z_{1-\alpha/2}$ is the upper $(\alpha/2)$ percentile of the standard normal distribution.

4.3 Bootstrap confidence intervals

The exact confidence intervals presented in Sect. 4.1 are computationally quite involved and become difficult to construct when sample size is large. Also, as we show later in Sect. 5, the approximate confidence intervals presented in Sect. 4.2 do not have satisfactory coverage probabilities unless the sample size is quite large. Hence, it is naturally of interest to check whether a bootstrap method provides a good alternate for this purpose. As will be seen later in Sect. 5, the adjusted percentile bootstrap method proposed here does perform well for large sample sizes and is also easy to implement. This suggests that one should therefore use the exact method in case of small to moderate sample sizes (up to 35) and the adjusted percentile bootstrap method for sample sizes larger than that as the exact method faces computational difficulties in this situation. In this subsection, we therefore describe a bootstrap method to construct CIs for θ_1 and θ_2, viz., the Adjusted percentile (BCa) interval; see Efron and Tibshirani (1988) for details. First, we describe the algorithm to obtain the Type-I censored sample. This algorithm will be utilized in the resampling needed for the bootstrap confidence interval in Sect. 4.3.2.

4.3.1 Bootstrap sample

Step 1. Given τ_1, τ_2 and the original Type-I censored sample, we obtain $\hat{\theta}_1$ and $\hat{\theta}_2$ from (9).

Step 2. Based on n, τ_1, $\hat{\theta}_1$ and $\hat{\theta}_2$, we generate a random sample of size n from Uniform (0, 1) distribution, and obtain the order statistics $(U_{1:n}, \ldots, U_{n:n})$.

Step 3. Find N_1 such that

$$U_{N_1:n} < 1 - e^{-\tau_1/\hat{\theta}_1} \leq U_{N_1+1:n}.$$

For $1 \leq i \leq N_1$, we set

$$t^*_{i:n} = -\hat{\theta}_1 \log\left(1 - U_{i:n}\right).$$

Step 4. Next, we generate a random sample of size $m = n - N_1$ from Uniform(0, 1) distribution, and obtain the order statistics $(V_{1:m}, \ldots, V_{m:m})$.

Step 5. Find N_2 such that

$$V_{N_2:m} < 1 - e^{-(\tau_2-\tau_1)/\hat{\theta}_2} \leq V_{N_2+1:m}.$$

For $1 \leq j \leq N_2$, we then set

$$t^*_{N_1+j:n} = \tau_1 - \hat{\theta}_2 \log\left(1 - V_{j:m}\right).$$

Step 6. Based on n, N_1, N_2, τ_1, τ_2, and ordered observations $\left\{t^*_{1:n}, \ldots, t^*_{N_1:n}, t^*_{N_1+1:n}, \ldots, t^*_{N_1+N_2:n}\right\}$, we obtain $\hat{\theta}^*_1$ and $\hat{\theta}^*_2$ from (9).

Step 7. Repeat Steps 2–6 R times and arrange all $\hat{\theta}^*_1$'s and $\hat{\theta}^*_2$'s in ascending order to obtain the bootstrap sample

$$\left\{\hat{\theta}^{*[1]}_k, \hat{\theta}^{*[2]}_k, \ldots, \hat{\theta}^{*[R]}_k\right\}, \quad k = 1, 2.$$

4.3.2 Adjusted percentile (BCa) interval

A two-sided $100(1-\alpha)\%$ BCa bootstrap confidence interval for θ_k is

$$\left(\hat{\theta}_k^{*[\alpha_{1k}R]},\ \hat{\theta}_k^{*[(1-\alpha_{2k})R]}\right),\quad k=1,2,$$

where

$$\alpha_{1k}=\Phi\left\{\hat{z}_{0k}+\frac{\hat{z}_{0k}+z_{\alpha/2}}{1-\hat{a}_k(\hat{z}_{0k}+z_{\alpha/2})}\right\}\ \text{ and }\ \alpha_{2k}=\Phi\left\{\hat{z}_{0k}+\frac{\hat{z}_{0k}+z_{1-\alpha/2}}{1-\hat{a}_k(\hat{z}_{0k}+z_{1-\alpha/2})}\right\}.$$

Here, $\Phi(\cdot)$ is the CDF of the standard normal distribution, and

$$\hat{z}_{0k}=\Phi^{-1}\left\{\frac{\#\text{ of }\hat{\theta}_k^{*[j]}<\hat{\theta}_k}{R}\right\},\quad j=1,\ldots,R,\ \ k=1,2.$$

A good estimate of the acceleration a_k is

$$\hat{a}_k=\frac{\sum_{i=1}^{N_k}\left[\hat{\theta}_k^{(\cdot)}-\hat{\theta}_k^{(i)}\right]^3}{6\left\{\sum_{i=1}^{N_k}\left[\hat{\theta}_k^{(\cdot)}-\hat{\theta}_k^{(i)}\right]^2\right\}^{3/2}},\quad i=1,\ldots,N_k,\ \ k=1,2,$$

where $\hat{\theta}_k^{(i)}$ is the MLE of θ_k based on the simulated Type-I censored sample with the ith observation deleted (i.e., the jackknife estimate), and

$$\hat{\theta}_k^{(\cdot)}=\frac{1}{N_k}\sum_{i=1}^{N_k}\hat{\theta}_k^{(i)},\quad i=1,\ldots,N_k,\ \ k=1,2.$$

In addition to this adjusted percentile bootstrap interval, the percentile bootstrap interval and the Studentized-t bootstrap interval were also included in the comparative study; but, since the adjusted percentile (BCa) interval yielded the best results overall among these three methods, only the results for this method are presented.

5 Simulation Study

In this section, we present the results of a Monte Carlo simulation study carried out in order to compare the performance of all the methods of inference described in Sect. 4. We chose the values of the parameters θ_1 and θ_2 to be $e^{2.5}$ and $e^{1.5}$, respectively; we also chose for n the values of 20 and 35, and several different choice for (τ_1,τ_2). We then determined the true coverage probabilities of the 90%, 95%, 99% confidence intervals for θ_1 and θ_2 by all the methods presented in Sect. 4. These values, based on

1,000 Monte Carlo simulations and $R = 1{,}000$ bootstrap replications, are presented in Tables 1 and 2.

From these tables, it is clear that the exact method of constructing confidence intervals (based on the exact conditional densities of $\hat{\theta}_1$ and $\hat{\theta}_2$ derived in Sect. 3) always maintains its coverage probability at the pre-fixed nominal level. The approximate method of constructing confidence intervals (based on asymptotic normality of $\hat{\theta}_1$ and $\hat{\theta}_2$) has its true coverage probability to be always less than the nominal level. Though the coverage probability improves for large sample size, we still find it to be unsatisfactory even for n as large as 35 particularly when τ_1 and τ_2 are not too large. Therefore, the approximate CI should not be used unless n is considerably large.

Among the three bootstrap methods of constructing confidence intervals mentioned in Sect. 4, it was observed that the Studentized-t interval seems to have considerably low coverage probabilities compared to the nominal level. The percentile interval and the adjusted percentile interval had their coverage probabilities to be better and somewhat closer to the nominal level. Eventhough the percentile method seems to be sensitive (for θ_1 when τ_1 is small), the method did improve a bit for larger sample size. Overall, the adjusted percentile method was observed to be the one (among the three bootstrap methods) with somewhat satisfactory coverage probabilities (not so for θ_1 when τ_1 is small), and so the results for this bootstrap method alone are included in the tables. Hence, this method may be used in case of large sample sizes when the computation of the exact CI becomes difficult.

6 Illustrative examples

In this section, we consider three examples. First, we revisit Examples 1 and 2 presented earlier in Sect. 2 in order to illustrate all the methods of inference developed in preceding sections. Example 3 gives some plots to show the monotonicity of the tail probabilities of $\hat{\theta}_1$ and $\hat{\theta}_2$ given in Corollary 5.

Example 1 (Revisited) From the formulas in Eqs. (20)–(23), we obtain the standard error of the MLEs of θ_1 and θ_2 to be

$\tau_1 = 5$	$\tau_2 = 6$	$\tau_2 = 7$	$\tau_2 = 8$	$\tau_2 = 9$	$\tau_2 = 12$
$\widehat{SE}(\hat{\theta}_1)$	21.44440	21.28597	21.18302	21.18202	21.18202
$\widehat{SE}(\hat{\theta}_2)$	4.79362	8.105016	3.604153	1.642129	1.880477

Further, the confidence intervals for θ_1 and θ_2 obtained by all three methods are presented in Tables 3 and 4, respectively. Note that the approximate confidence interval and the bootstrap confidence interval are both unsatisfactory upon comparing them with the exact confidence intervals. The problem is due to the small values of N_1 and N_2 in this case.

Example 2 (Revisited) From the formulas in Eqs. (20)–(23), we obtain the standard error of the MLEs of θ_1 and θ_2 to be

$\tau_1 = 8$	$\tau_2 = 16$	$\tau_2 = 20$	$\tau_2 = 24$
$\widehat{SE}(\hat{\theta}_1)$	10.27773	10.28401	10.28193
$\widehat{SE}(\hat{\theta}_2)$	2.718509	2.411893	2.303771

Further, the confidence intervals for θ_1 and θ_2 obtained by all three methods are presented in Table 5. Note that all the intervals for θ_2 are close to the exact confidence interval, and the approximate and bootstrap confidence intervals for θ_1 are highly unsatisfactory compared to the exact confidence interval. This is so because N_1 is small and N_2 is large in this case.

Example 3 Although we can not prove the monotonic increasing property of the tail probability functions given in Corollary 5, we present some plots of $P\left\{\hat{\theta} > \xi\right\}$ for different choices of n, and τ_1 and τ_2, in Fig. 1. These plots all display the monotonicity of the probabilities of interest.

7 Conclusions

In this paper, we have considered a simple step-stress model with two stress levels from exponential distributions when there is time constraint on the duration of the experiment. We have derived the MLEs of the unknown parameters θ_1 and θ_2, and their exact conditional distributions. We have also proposed several different procedures for constructing confidence intervals for θ_1 and θ_2. We have carried out a simulation

Table 3 Interval estimation for θ_1 based on the data in Example 1 with $\tau_1 = 5$ and different τ_2

τ_2	Method	90%	95%	99%
6	Bootstrap BCa	(11.8452, 97.0986)	(10.4813, 97.9780)	(7.9084, 98.7994)
	Approximation	(0.0000, 35.1448)	(0.0000, 38.8501)	(0.0000, 46.0919)
	Exact	(11.4823, 71.8781)	(10.1474, 93.3925)	(8.0940, 166.5306)
7	Bootstrap BCa	(11.6452, 97.1715)	(10.0295, 98.2370)	(8.0759, 99.1133)
	Approximation	(0.0000, 35.4373)	(0.0000, 39.1426)	(0.0000, 46.3844)
	Exact	(11.5931, 72.5194)	(10.2461, 94.2236)	(8.1736, 168.0092)
8	Bootstrap BCa	(11.5395, 95.6438)	(10.2748, 97.3843)	(8.6848, 98.9594)
	Approximation	(0.0000, 35.6525)	(0.0000, 39.3578)	(0.0000, 46.5997)
	Exact	(11.6965, 72.9479)	(10.3429, 94.7722)	(8.2602, 168.9658)
9	Bootstrap BCa	(11.8333, 96.1042)	(10.1246, 98.1306)	(7.4839, 98.6661)
	Approximation	(0.0000, 35.6561)	(0.0000, 39.3614)	(0.0000, 46.6032)
	Exact	(11.7003, 72.9524)	(10.3471, 94.7774)	(8.2656, 168.9753)
12	Bootstrap BCa	(11.4474, 96.4329)	(10.3128, 98.5669)	(7.4530, 99.8978)
	Approximation	(0.0000, 35.6561)	(0.0000, 39.3614)	(0.0000, 46.6032)
	Exact	(11.7006, 72.9580)	(10.3467, 94.7793)	(8.2639, 168.9228)

Table 4 Interval estimation for θ_2 based on the data in Example 1 with $\tau_1 = 5$ and different τ_2

τ_2	**Method**	**90%**	**95%**	**99%**
6	Bootstrap BCa	(2.8799, 16.6801)	(2.4932, 17.3774)	(1.7947, 18.0875)
	Approximation	(0.0000, 14.9771)	(0.0000, 16.6460)	(0.0000, 19.9077)
	Exact	(2.7403, 61.6015)	(2.3523, 117.4822)	(1.7900, 561.5936)
7	Bootstrap BCa	(4.0308, 26.5112)	(2.9103, 29.9906)	(3.1983, 33.2359)
	Approximation	(0.0000, 15.8027)	(0.0000, 17.5407)	(0.0000, 20.9376)
	Exact	(4.1066, 32.9363)	(3.5998, 45.9218)	(2.8281, 99.5966)
8	Bootstrap BCa	(2.5731, 8.2473)	(2.2987, 9.8267)	(2.0628, 12.0612)
	Approximation	(1.2354, 8.1647)	(0.5717, 8.8284)	(0.0000, 10.1256)
	Exact	(3.1190, 11.2912)	(2.8251, 13.2468)	(2.3466, 18.6546)
9	Bootstrap BCa	(2.1930, 5.2337)	(2.3852, 5.8879)	(2.3917, 7.1737)
	Approximation	(1.7884, 5.8839)	(1.3961, 6.2762)	(0.6293, 7.0430)
	Exact	(2.5643, 7.3382)	(2.3566, 8.3046)	(2.0086, 10.7583)
12	Bootstrap BCa	(3.4681, 6.4423)	(2.9746, 7.4671)	(2.1930, 9.0009)
	Approximation	(2.4996, 7.9478)	(1.9778, 8.4697)	(0.9578, 9.4896)
	Exact	(3.5333, 9.3778)	(3.2633, 10.5022)	(2.8071, 13.2944)

Table 5 Interval estimation for θ_1 and θ_2 based on the data in Example 2 with $\tau_1 = 8$ and different τ_2

	τ_2	**Method**	**90%**	**95%**	**99%**
θ_1	16	Bootstrap BCa	(15.4715, 44.0562)	(14.0231, 53.2283)	(12.6611, 110.4923)
		Approximation	(8.8162, 33.0712)	(6.4928, 35.3945)	(1.9521, 39.9352)
		Exact	(14.3235, 39.5828)	(13.1967, 45.3996)	(11.3027, 56.2873)
	20	Bootstrap BCa	(14.9623, 43.4934)	(13.5602, 53.1624)	(12.6110, 66.9014)
		Approximation	(8.8184, 33.0734)	(6.4951, 35.3967)	(1.9543, 39.9375)
		Exact	(14.3235, 39.4201)	(13.1967, 45.1434)	(11.3027, 56.3097)
	24	Bootstrap BCa	(14.7173, 43.3530)	(13.9817, 51.7492)	(13.0987, 88.3893)
		Approximation	(8.8171, 33.0721)	(6.4938, 35.3954)	(1.9530, 39.9362)
		Exact	(14.3235, 40.0600)	(13.1967, 45.2894)	(11.3027, 56.2863)
θ_2	16	Bootstrap BCa	(6.1549, 14.7354)	(5.8507, 16.9423)	(5.1816, 22.5316)
		Approximation	(4.7326, 12.3528)	(4.0027, 13.0827)	(2.5761, 14.5093)
		Exact	(6.0103, 14.2983)	(5.5908, 15.7817)	(4.8775, 19.3389)
	20	Bootstrap BCa	(6.3969, 13.9389)	(6.1930, 16.0229)	(5.6165, 22.1152)
		Approximation	(5.4263, 12.6502)	(4.7344, 13.3421)	(3.3820, 14.6945)
		Exact	(6.4928, 14.2075)	(6.0793, 15.5093)	(5.3570, 18.5517)
	24	Bootstrap BCa	(6.9175, 14.6662)	(6.6156, 17.3868)	(5.6610, 23.8502)
		Approximation	(5.8964, 12.9923)	(5.2167, 13.6719)	(3.8883, 15.0004)
		Exact	(6.8614, 14.3892)	(6.4449, 15.6304)	(5.7166, 18.4912)

study to compare the performance of all these procedures. We have observed that the approximate method of constructing confidence intervals (based on the asymptotic normality of the MLEs $\hat{\theta}_1$ and $\hat{\theta}_2$) and the Studentized-t bootstrap conference interval are both unsatisfactory in terms of the coverage probabilities. Eventhough the percentile bootstrap method seems to be sensitive for small values of τ_1 and τ_2, the method does improve for larger sample size. Overall, the adjusted percentile method seems to be the one (among the three bootstrap methods) with somewhat satisfactory coverage probabilities (not so for θ_1 when τ_1 is small). Hence, our recommendation is to use the exact method whenever possible, and the adjusted percentile method in case of large sample size when the computation of the exact confidence interval becomes difficult. We have also presented some examples to illustrate all the methods of inference discussed here as well as to support the conclusions drawn.

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