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Exact Lattice Supersymmetry from Topological Field Theory

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Lattice Supersymmetry and Topological Field Theory

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We discuss the connection between supersymmetric field theories and topological field theories and show how this connection may be used to construct local lattice field theories which maintain an exact supersymmetry. It is shown how metric independence of the continuum topological field theory allows us to derive the lattice theory by blocking out of the continuum in a deformed geometry. This, in turn allows us to prove the cut-off independence of certain supersymmetric Ward identities.

1. Motivation

There have been many attempts to formulate supersymmetric theories on lattices. [1]. But, unfortunately, generic lattice models break supersymmetry explicitly allowing relevant SUSY violating operators to appear in the effective action. The couplings of such operators must then be fine tuned to approach a supersymmetric fixed point as the lattice spacing $a \to 0$.

Because of these problems one is motivated to try to preserve an element of SUSY on the lattice in the hope that this residual supersymmetry will protect the lattice theory from dangerous radiative corrections. In this talk I will discuss one way to achieve this – by exploiting a well-known connection between theories with extended supersymmetry and topological field theories [2]. The prototype example of this is supersymmetric quantum mechanics viewed as (0+1) dimensional lattice field theory. Another approach which also attempts to find lattice actions exhibiting an exact supersymmetry can be found in [3].

2. SUSY QM

Consider a model built from a discrete set of scalar fields ϕ_i with classical action $S_{\rm cl}(\phi) = 0$. Such a model is trivially invariant under the shift (topological) symmetry

$$\phi_i \to \phi_i + \epsilon_i$$

To quantize this theory we need to pick a gauge condition eg. $N_i(\phi) = 0$. The partition function of the gauge-fixed theory then takes the form

$$Z = \int D\phi e^{-\frac{1}{\alpha} N_i^2(\phi)} \det\left(\frac{\partial N_i}{\partial \phi_j}\right)$$

If the determinant is represented using anticommuting ghosts $\overline{\psi}_i$, ψ_i and an auxiliary field B_i is introduced we can rewrite this as

$$Z = \int D\phi D\psi D\overline{\psi} DB e^{-S_q}$$

where

$$S_q = -\frac{1}{2}\alpha B_i^2 + N_i B_i + \overline{\psi}_i \frac{\partial N_i}{\partial \phi_j}\psi_j$$

S_q is then invariant under the BRST symmetry:

$$\begin{aligned}
\delta\phi_i &= \psi_i \xi \\
\delta\overline{\psi}_i &= B_i \xi \\
\delta\psi_i &= 0 \\
\delta B_i &= 0
\end{aligned}$$

We see that indeed that δ is a nilpotent operator on these fields. Indeed, the action may be written

$$S_q = \delta\left(\overline{\psi}_i\left(N_i - \frac{\alpha}{2}B_i\right)\right)$$

We will show in the next section that this structure is rather special and allows us to write down expectation values *in the continuum version of this model* whose values are independent of the metric. These theories are thus termed topological quantum field theories TQFT.

Returning to our discrete model we can now make a specific choice of gauge function and an associated physical interpretation of the field indices. If the latter are now chosen to label a series of lattice sites on a circle and the field N_i chosen as

$$N_i = D_{ij}^+ \phi_j + P_i'(\phi)$$

where D^+ is the usual forward difference operator on the lattice and P' is some arbitrary polynomial function of the field ϕ, we can reinterpret the resulting theory as a lattice regulated version of supersymmetric quantum mechanics in Euclidean space [4]. Notice that this requires us also to reinterpret the ghosts of the topological theory as the physical fermions of the supersymmetric theory. This is one aspect of the way in which the TQFT and its SUSY parent differ. The physical state conditions arising from the gauge fixing of the shift symmetry ensure, in addition, that the true TQFT corresponds to the projection to the vacuum states of the parent SUSY theory. In our construction we drop this projection and treat the discrete TQFT action as a bona fide lattice action of the associated SUSY model possessing an exact supersymmetry. Our numerical results indicate that this is fine at least in low dimensions [5].

3. Continuum TQFT

The ingredients of a topological quantum field theory are a Riemannian manifold $g_{\mu\nu}$, a set of fields Φ and an action $S(\Phi)$ together with a special class of so-called topological observables $O(\Phi)$ which have the property

$$\frac{\delta}{\delta g^{\mu\nu}} \langle O \rangle = 0$$

This property can be guaranteed if the action is the variation of some other function with respect to a nilpotent symmetry $S = \delta \Lambda$. The proof requires that both the measure and the operator O be invariant under the symmetry (and furthermore the latter should not contain the metric explicitly) It is important for our purposes to notice that any such theory always contains a *trivial* set of topological observables – namely those functions which arise from the symmetry variation of

some other function $O = \delta O'$. In the language of supersymmetry these operators generate the supersymmetric Ward identities of the parent SUSY theory

4. Lattice Action as a Perfect Action

This metric independence of the continuum theory may be exploited so as to derive the lattice model directly from the continuum. The lattice theory can be obtained by a process of blocking out of the continuum in a deformed metric. We illustrate this in the 1D model we have introduced. In the continuum the bosonic action takes the form

$$S_B = \int dt\, e(t) \left[\frac{1}{e(t)} \frac{d\phi}{dt} + P'(\phi) \right]^2$$

where $e(t)$ is the einbein Change variables to *block* fields:

$$\phi^B(t) = \int dt'\, e(t') B_\beta^- \left(t - t' \right) \phi(t')$$

where

$$B_\beta^-(t) = \frac{1}{2a} \left[L_\beta\left(t + \delta \right) - L_\beta\left(t - a + \delta \right) \right]$$

with

$$\lim_{\beta \to \infty} L_\beta(t) = \theta(t)$$

Consider also an associated deformed metric

$$e_\beta(t) = \sum_{n=1}^{N} \frac{a}{A(\beta)} L_\beta'(t - na)$$

We can easily show that in $\lim_{\beta \to \infty}$

$$\phi^B(t) = \sum_{n=1}^{N} \phi(na) \frac{1}{2A_L} \left[\theta(t - na) - \theta(t - (n+1)a) \right]$$

Thus the field ϕ^B approaches a constant value within cells of a 1D lattice – changing only on moving from one cell to another. Furthermore, its derivative takes the form

$$\frac{1}{e_\beta(t)} \frac{d\phi^B}{dt} = \sum_{n=1}^{N} \frac{1}{2aA_L} D_{nm}^- \phi_m \times$$
$$\left[\theta(t - (n-1/2)a) - \theta(t - (n+1/2)a) \right]$$

which is constant now within cells of a *dual* lattice. Furthermore, the continuum bosonic action evaluated on such a continuum field resembles the lattice bosonic action

$$\lim_{\beta \to \infty} S_B(\phi^B) = \sum_{n=1}^{N} a \left[\frac{1}{2A_L a} D_{nm}^{-} \phi_m^B + P_n'(\phi^B) \right]^2$$

Similar arguments apply to the full action evaluated on both scalar and fermion block fields. As $\beta \to \infty$ one can show that Z is dominated by such block fields (the form of the scalar kinetic term ensures this) Thus the lattice theory, from the point of view of topological observables, is merely the continuum theory in the limit of a singular background and evaluated using a new set of block variables. We would then expect all the BRST symmetry of continuum theory to be manifested at the quantum level on the lattice. Our simple model affords an example of the latter feature.

SUSY QM actually has 2 topological symmetries in the continuum which correspond to the flip $\psi \to \overline{\psi}$ – the original BRST symmetry (now called δ_1) and a second BRST symmetry δ_2. Now the lattice action is, by construction, classically invariant under δ_1 but *not* under the second symmetry $\delta_2 S \neq 0$. The breaking term would be a total derivative in the continuum but is non-vanishing in general on the lattice. Nevertheless, as our previous argument would have us believe, the Ward identities following from δ_2 are accurately satisfied on the lattice. The tables below show numerical data for the bosonic and fermionic contributions to Ward identities for both δ_1 and δ_2 for a simple model with $P' = g\phi^3$ on a $N = 4$ site lattice with large lattice spacing.

Table 1

$\delta_{(1)} \left(\overline{\psi}_t x_0 \right)$

t	$< x(0)N^{(1)}(t) >$	$< \psi(0)\overline{\psi}(t) >$
0	0.8895(11)	-0.8898(3)
1	0.6152(10)	-0.6155(3)
2	0.4294(11)	-0.4295(3)
3	0.3024(11)	-0.3028(3)

Table 2

$\delta_{(2)} \left(\psi_t x_0 \right)$

t	$< x(0)N^{(2)}(t) >$	$< \overline{\psi}(0)\psi(t) >$
0	-0.8895(11)	0.8898(3)
1	-0.3016(11)	0.3028(3)
2	-0.4294(11)	0.4295(3)
3	-0.6160(10)	0.6155(3)

5. Conclusions

Certain models with extended SUSY can be related to topological quantum field theories. The BRST charge(s) that appear in the latter theories are formed from linear combinations of supercharges (a procedure termed *twisting* in the literature) The topological symmetries may often be extended to the lattice and lead to models which maintain some exact residual supersymmetry. Indeed, the algebra $Q^2 = 0$ evades all the usual no-go theorems associated with lattice SUSY. Furthermore, we have showed how the lattice theories may be obtained by blocking out of continuum in carefully chosen background geometry. This procedure naturally generates a $r = 1$ Wilson mass term for the fermions. Finally these ideas may be extended to 2D sigma models and Yang-Mills models in four dimensions which is work currently underway.

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