Exact matrix product solution for the boundary-driven Lindblad XXZ chain

Gunter M. Schütz

Institute of Complex Systems II, Forschungszentrum Jülich, 52425 Jülich, Germany
and
Interdisziplinäres Zentrum für Komplexe Systeme, Universität Bonn

joint work with D. Karevski (Nancy) and V. Popkov, (MPI Dresden)

- Boundary driven Lindblad XXZ chain
- Matrix product ansatz for the stationary density matrix
- Isotropic Lindblad-Heisenberg chain
- Conclusions

1. Boundary-driven XXZ Lindblad chain

Non-equilibrium behaviour of open quantum system:

- Experimentally accessible (quasi one-dimensional spin chain materials, artificially assembled nanomagnets)
- Theoretically challenging:

- Interplay of magnon excitations, magnetization currents with twisted boundary fields (➔ non-equilibrium stationary state)

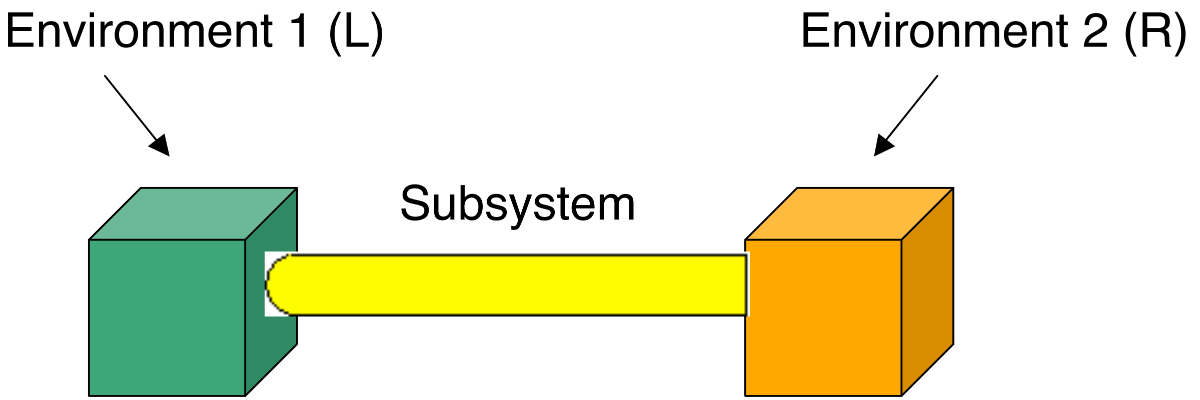
- Fundamental problems

  - ➢ No density matrix exp(-βH)
  - ➢ Non-linear response far from equilibrium
  - ➢ Interplay of bulk transport with boundary pumping

Lindblad equation for open quantum systems

Lindblad (1976): General time evolution equation of a quantum subsystem

E.g.



Total System: Hamiltonian $H_{tot} = H_L + H + H_R$

Subsystem: Quantum Hamiltonian H, reduced density matrix $\rho(t)$

Quantity of interest: Stationary density matrix $\rho^* = \lim_{t \to \infty} \rho(t) \neq \exp(-\beta H)$

Lindblad equation $$\frac{d}{dt}\rho = -i[H,\rho] + \mathcal{D}^L(\rho) + \mathcal{D}^R(\rho)$$

unitary part (subsystem), left dissipator, right dissipator

To preserve unitarity and normalization (Tr $\rho(t) = 1$):

$$\mathcal{D}^{L,R}(\rho) = D^{L,R}\rho D^{L,R\dagger} - 1/2\{\rho, D^{L,R\dagger}D^{L,R}\}$$

Boundary pumping: $\rho^* \neq \exp(-\beta H)$

Task: a) Choose H and $D^L(\rho) \neq D^R(\rho)$ appropriately for physical scenario

b) Find ρ^*

c) Compute observables

Lindblad equation for XXZ Heisenberg quantum chain

Anisotropic (XXZ) Heisenberg spin-1/2 quantum chain:

$$H = J\sum_k [\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta(\sigma_k^z \sigma_{k+1}^z - \varepsilon_0)] + g_1^L + g_N^R$$

Exchange constant: J=1/2

Bulk interaction: $\Delta = (q + q^{-1})/2$, $\varepsilon_0 = 1$

Boundary fields: $\vec{f}^L \cdot \vec{\sigma} = f^L \sigma_u^z$ (left) $\vec{f}^R \cdot \vec{\sigma} = f^R \sigma_v^z$ (right)

Choose $\sigma^z_u = \sin\theta_L\, \sigma^y + \cos\theta_L\, \sigma^z$ and $\sigma^z_v = -\sin\theta_R\, \sigma^x + \cos\theta_R\, \sigma^z$

y-z plane x-z plane

Boundary pumping:

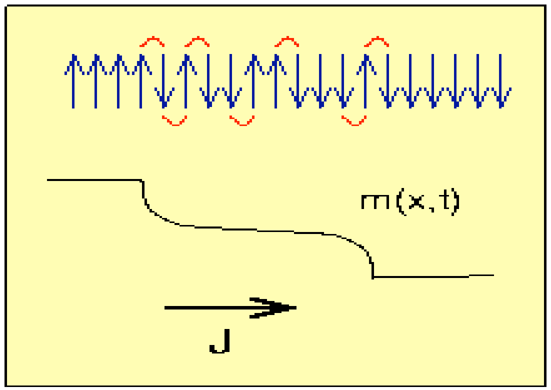
Consider Lindblad terms corresponding to complete polarization in the plane of the quantum boundary fields

$$D^L = \sqrt{\frac{\Gamma}{2}}(\sigma_1^x + i\cos\theta_L\sigma_1^y - i\sin\theta_L\sigma_1^z) \qquad \text{(y-z plane)}$$

$$D^R = \sqrt{\frac{\Gamma}{2}}(\cos\theta_R\sigma_N^x - i\sigma_N^y + \sin\theta_R\sigma_N^z) \qquad \text{(x-z plane)}$$

Stationary solution (without bulk dynamics): $\rho_{L,R} = (1 \pm \sigma^z_{u,v})/2)$

Bulk dynamics ==> Current, magnetization profile



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2. Matrix product ansatz for the stationary density matrix

- Determine ρ from stationary Lindblad equation $i[H, \rho] = \mathcal{D}^L(\rho) + \mathcal{D}^R(\rho)$
- Write $\rho = SS^\dagger / \mathrm{Tr}(SS^\dagger)$, $S \in C^{2N}$
- Matrix product ansatz $$S = \langle \phi | \Omega^{\otimes N} | \psi \rangle$$

with 2x2 matrix $$\Omega = \begin{pmatrix} A_1 & A_+ \\ A_- & A_2 \end{pmatrix}$$

where $\langle \phi |$ and $| \psi \rangle$ are vectors in some space and A_i are matrices

A_i, $\langle \phi |$ and $| \psi \rangle$ have to determined such that stationary LE is satisfied!

Two steps: (1) bulk part for A_i, (2) boundary part for $\langle \phi |$ and $| \psi \rangle$

Solution of LE (bulk part)

Step 1: Introduce local divergence condition (different from Prosen 2011)

- remember $H = \sum_{k=1}^{N-1} h_{k,k+1} + g_1^L + g_N^R$

  with 4x4 matrix $h = [\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \Delta (\sigma^z \otimes \sigma^z - 1)]/2$

  and 2x2 boundary matrices $g^L = f^L \sigma^z_u$, $g^R = f^R \sigma^z_v$

- introduce 2x2 matrix $$\Xi = \begin{pmatrix} E_1 & E_+ \\ E_- & E_2 \end{pmatrix}$$

  with non-commutative auxiliary matrices E_i

- require $$[h, \Omega \otimes \Omega] = \Xi \otimes \Omega - \Omega \otimes \Xi$$ (local divergence condition)

==> 16 quadratic equations for the 8 matrices A_i, E_i

$$\begin{pmatrix} 0 & \Delta A_1 A_+ - A_+ A_1 & \Delta A_+ A_1 - A_1 A_+ & 0 \\ -\Delta A_1 A_- + A_- A_1 & -[A_+\,,\,A_-] & -[A_1\,,\,A_2] & -\Delta A_+ A_2 + A_2 A_+ \\ -\Delta A_- A_1 + A_1 A_- & [A_1\,,\,A_2] & [A_+\,,\,A_-] & -\Delta A_2 A_+ + A_+ A_2 \\ 0 & \Delta A_- A_2 - A_2 A_- & \Delta A_2 A_- - A_- A_2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} E_1 A_1 - A_1 E_1 & E_1 A_+ - A_1 E_+ & E_+ A_1 - A_+ E_1 & E_+ A_+ - A_+ E_+ \\ E_1 A_- - A_1 E_- & E_1 A_2 - A_1 E_2 & E_+ A_- - A_+ E_- & E_+ A_2 - A_+ E_2 \\ E_- A_1 - A_- E_1 & E_- A_+ - A_- E_+ & E_2 A_1 - A_2 E_1 & E_2 A_+ - A_2 E_+ \\ E_- A_- - A_- E_- & E_- A_2 - A_- E_2 & E_2 A_- - A_2 E_- & E_2 A_2 - A_2 E_2 \end{pmatrix}$$

4 Commutation relations: $0 = [E_i, A_i]$

8 relations with q-commutators, e.g., $\Delta A_1 A_+ - A_+ A_1 = E_1 A_+ - A_1 E_+$

4 relations with commutators, e.g. $[A_+ , A_-] = E_2 A_1 - A_2 E_1$

- Solution of all 16 equations in terms of only three matrices $A_{\pm}$, Q with relations

$$[A_+ , A_-] = - (q-q^{-1}) (b\bar{b}\, Q - c\bar{c}\, Q^{-1})$$

$$QA_{\pm} = q^{\pm 1} A_{\pm} Q$$

$$Q\, Q^{-1} = Q^{-1} Q = 1$$

by setting ($b,\bar{b},c,\bar{c}$ arbitrary)

$A_1 = b\, Q + c\, Q^{-1}$, $A_2 = \bar{b}\, Q + \bar{c}\, Q^{-1}$ (diagonal part of Ω)

$E_{\pm} = 0$

$E_1 = (q-q^{-1})/2\, (b\, Q - c\, Q^{-1})$, $E_2 = -(q-q^{-1})/2\, (\bar{b}\, Q - \bar{c}\, Q^{-1})$ (diagonal part of Ξ)

- Relations define Ω and Ξ in terms of $A_{\pm}$, Q
  - Proof by straightforward computation
  - $\Xi + \kappa\Omega$ is also a solution

Relation to quantum algebra $U_q[SU(2)]$

Use parametrization

$$b = \frac{\alpha}{q - q^{-1}} \frac{\nu}{\lambda}, \qquad \bar{b} = \frac{\alpha}{q - q^{-1}} \frac{1}{\lambda \nu},$$

$$c = -\frac{\alpha}{q - q^{-1}} \mu \lambda, \qquad \bar{c} = -\frac{\alpha}{q - q^{-1}} \frac{\lambda}{\mu}$$

and define $A_\pm = i\alpha S_\pm$, $Q = \lambda\, q^{Sz}$

==> Defining relations for $U_q[SU(2)]$

$$[S_+, S_-] = \frac{q^{2S_z} - q^{-2S_z}}{q - q^{-1}}$$

$$q^{S_z} S_\pm = q^{\pm 1} S_\pm q^{S_z}.$$

- Matrix product ansatz with $U_q[SU(2)]$ generators!
- Symmetry of bulk Hamiltonian (without boundary fields)

Representation theory

Define $[x]_q = (q^x - q^{-x})/ (q - q^{-1})$

- Finite-dimensional irreps not of interest

-Infinite-dimensional representation (with complex parameter p)

$$S_z = \sum_{k=0}^{\infty} (p - k)|k\rangle\langle k|,$$

$$S_+ = \sum_{k=0}^{\infty} [k + 1]_q |k\rangle\langle k + 1|,$$

$$S_- = \sum_{k=0}^{\infty} [2p - k]_q |k + 1\rangle\langle k|$$

==> Explicit form of Ω!

Solution of LE (boundary part)

Step 2: Condition on boundary vectors

- remember $H = \sum_{k=1}^{N-1} h_{k,k+1} + g_1^L + g_N^R$

  with 4x4 matrix $h = [\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \Delta (\sigma^z \otimes \sigma^z - 1)]/2$

  and 2x2 boundary matrices $g^L = f^L \sigma^z_u$, $g^R = f^R \sigma^z_v$

- define 2x2 matrix $\Phi = [g,\Omega]$ and introduce $\Upsilon_k := \Omega^{\otimes(k-1)} \otimes \Upsilon \otimes \Omega^{\otimes(N-k)}$

==> local divergence condition implies

$$[H, \Omega^{\otimes N}] = \Phi_1^L + \Xi_1 + \Phi_N^R - \Xi_N$$

(reduction of infinitesimal unitary part of evolution to boundary terms)

Also Lindblad operator has only boundary parts:

==> split stationary LE into two boundary equations

$$\mathcal{D}^L(SS^\dagger) = i(\Phi_1^L + \Xi_1)S^\dagger - iS(\Phi_1^{L^\dagger} + \Xi_1^\dagger),$$

$$\mathcal{D}^R(SS^\dagger) = i(\Phi_N^R - \Xi_N)S^\dagger - iS(\Phi_N^{R\dagger} - \Xi_N^\dagger),$$

Define $A_0 = (A_1+A_2)/2$, $A_z = (A_1-A_2)/2$, Make decomposition

- left boundary: $S = \langle\phi| [A_0 + A_z\sigma^z + A_+\sigma^+ + A_-\sigma^-] \otimes \Omega^{\otimes(N-1)} |\psi\rangle$

- right boundary: $S = \langle\phi| \Omega^{\otimes(N-1)} \otimes [A_0 + A_z\sigma^z + A_+\sigma^+ + A_-\sigma^-] |\psi\rangle$

(likewise $S^\dagger$)

==> Two separate sets of equations for action of A_i on boundary vectors

==> Complete construction of ρ with some constraints on parameters

3. Isotropic Lindblad-Heisenberg chain

- Isotropic Heisenberg chain: Δ=1 (q=1)
- SU(2) symmetric (only bulk Hamiltonian, not boundary fields, not Lindblad terms)
- For convenience: $\alpha = \lambda = 1$, $\mu = \nu = i$
- $[x]_1 = x$, limits $q \to 1$ in representation well-defined

$$\Omega = i \begin{pmatrix} S^z & S_+ \\ S_- & -S^z \end{pmatrix}, \quad \Xi = i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or in vector notation $\vec{S} = (S_x, S_y, S_z), \quad \vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$

$$\Omega = i\vec{S} \cdot \vec{\sigma}, \quad \Xi = i\mathbb{1}$$

Solution of boundary equations:

- Key idea: Introduce coherent states

$$\langle\phi| := \sum_{n=0}^{\infty} \frac{\phi^n}{n!}\langle 0|(S_+)^n = \sum_{n=0}^{\infty} \phi^n|\langle n|,$$

$$|\psi\rangle := \sum_{n=0}^{\infty} \frac{\psi^n}{n!}(S_-)^n|0\rangle = \sum_{n=0}^{\infty} \psi^n \binom{2p}{n}|n\rangle$$

- SU(2) commutation relations:

$$\langle\phi|S_z = \langle\phi|(p - \phi S_+), \qquad S_z|\psi\rangle = (p - \psi S_-)|\psi\rangle,$$

$$\langle\phi|S_- = \phi\langle\phi|(2p - \phi S_+) \qquad S_+|\psi\rangle = \psi(2p - \psi S_-)|\psi\rangle$$

==> Action of S_z, S_- reduced to action of S_+! (right boundary: S_z, S_+ to S_+)

- Left Lindblad operator can be obtained from complete polarization along z-axis by unitary transformation $U = \exp(i\theta_L \sigma^x/2)$

==> new basis $\Omega(\theta_L) = i[S_z(\theta_L)\sigma_u^z + S_+(\theta_L)\sigma_u^+ + S_-(\theta_L)\sigma_u^-]$

with

$$S_z(\theta_L) = S_z \cos\theta_L + i\sin\theta_L \frac{S_+ - S_-}{2},$$

$$S_+(\theta_L) = \frac{S_+ + S_-}{2} + \cos\theta_L \frac{S_+ - S_-}{2} + iS_z \sin\theta_L$$

$$S_-(\theta_L) = \frac{S_+ + S_-}{2} - \cos\theta_L \frac{S_+ - S_-}{2} - iS_z \sin\theta_L$$

- new left boundary equations require: $\langle\phi|S_-(\theta_L) = 0, \qquad \langle\phi|S_z(\theta_L) = p\langle\phi|$

- Solution:

$\phi = \tan(\theta_L/2), \qquad p = \frac{i}{\Gamma - 2if^L}$

Proof: Coherent state relations and solution for ϕ lead to

$$S = \langle \phi | \Omega^{\otimes N} | \psi \rangle = i p \sigma_u^z \otimes \tilde{S} + \sigma_u^+ \otimes W$$

where $\tilde{S} = \langle \phi | \, \Omega^{\otimes (N-1)} \, | \psi \rangle$, $W = i \, \langle \phi | \, [S_+ + S^-] \, \Omega^{\otimes (N-1)} \, | \psi \rangle$

live on space for N-1 sites

$$\Longrightarrow \quad SS^\dagger = |p|^2 \mathbb{1} \otimes \tilde{S}S^\dagger - ip\sigma_u^- \otimes \tilde{S}W^\dagger - (ip)^* \sigma_u^+ \otimes W\tilde{S}^\dagger + \sigma_u^+ \sigma_u^- \otimes WW^\dagger .$$

Left Lindblad:

$$\mathcal{D}^L(SS^\dagger) = 2\Gamma |p|^2 \sigma_u^z \otimes \tilde{S}\tilde{S}^\dagger + \Gamma ip\sigma_u^- \otimes \tilde{S}W^\dagger + \Gamma (ip)^* \sigma_u^+ \otimes W\tilde{S}^\dagger .$$

Left Hamiltonian:

$$i[H, SS^\dagger] \,\|_{\mathrm{Left}} = -[ip + (ip)^*]\sigma_u^z \otimes \tilde{S}\tilde{S}^\dagger - \sigma_u^- \otimes [1 - 2if^L(ip)]\tilde{S}W^\dagger - \sigma_u^+ \otimes [1 + 2if^L(ip)^*]W\tilde{S}^\dagger$$

} equal with condition on p

Treatment of right boundary similar:

- Lindblad operator can be obtained from complete polarization along (-z)-axis by unitary transformation $U = \exp(i\theta_R \sigma^y/2)$

- new right boundary equations require: $S_+(\theta_R)|\psi\rangle = 0$

- Solution: $\psi = -\tan(\theta_R/2)$, $f^L = -f^R$

==> Complete explicit construction of ρ for isotropic case

Remark: For anisotropic case and no quantum boundary fields relation between representation parameter p and Lindblad coupling strength Γ reads

$$2\Gamma = i\,(q^p + q^{-p}) / [p]_q$$

Currents and magnetization profiles:

Local conservation law for local magnetization:

$$\frac{d}{dt}\, \sigma_n^\alpha = j_{n-1}^\alpha - j_n^\alpha \qquad \text{for } \alpha = x,y,z$$

with currents $j_n^\alpha = 2 \sum_{\beta,\gamma} \varepsilon_{\alpha\beta\gamma} \; \sigma_n^\beta \sigma_{n+1}^\gamma =$

==> Stationary case: $< j_n^\alpha > = j^\alpha \;\; \forall\, n$

- Untwisted model $\theta_L = \theta_R = 0$ [Prosen 2011]:

- $< \sigma_n^x > = < \sigma_n^y > = 0 \; \forall n$ (flat magnetization profiles for x and y component
- $j^x = j^y = 0$

Proof: z-Parity symmetry $U_z = (\sigma^z)^{\otimes N}$ of density matrix: $U_z \rho U_z = \rho$

==> $< \sigma_n^b > = \mathrm{Tr}\,(\sigma_n^b \rho) = \mathrm{Tr}\,(\sigma_n^b U_z \rho U_z) = \mathrm{Tr}\,(U_z \sigma_n^b U_z\, \rho) = - < \sigma_n^b >$ for b=x,y

and similar for j^x, j^y

- <u>Twisted case:</u> $< \sigma_n^\alpha > \neq 0, \quad j^\alpha \neq 0 \ \forall \ \alpha$

All components have non-zero expectation!

- $< \sigma_n^z > = - < \sigma_{N+1-n}^z > \ \forall \ n$
- $j^x = - j^y$

Proof: Key idea: Consider instead of parity another symmetry U of ρ

Specifically, for $\theta_R = - \theta_L = \pi/2$

$U = U_x V R$

with Space reflection R: $n \rightarrow N + 1 - n$,

x-y Rotation of spins $V = \mathrm{diag}(1,i)^{\otimes N}$

4. Conclusions

- Matrix product construction of stationary density matrix for boundary driven XXZ-Lindblad-chain using local-divergence condition

- Quadratic matrix algebra

- Relation with bulk symmetry, but not boundary terms

- Non-trivial magnetization profiles and non-vanishing magnetization current for all spin components even in isotropic case with general boundary twist

<u>Open problems:</u>

- Extension to other quantum systems with nearest-neighbour interaction

- Relationship with bulk symmetry and (possibly) full integrability

- Dynamical matrix product ansatz