Exact Minimum Eigenvalue Distribution of an Entangled Random Pure State

Satya N. Majumdar,[1] Oriol Bohigas,[1] and Arul Lakshminarayan *[2]

[1] *Laboratoire de Physique Théorique et Modèles Statistiques (UMR 8626 du CNRS), Université Paris-Sud, Bâtiment 100, 91405 Orsay Cedex, France.*

[2] *Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Straβe 38, D-01187 Dresden, Germany.*

A recent conjecture regarding the average of the minimum eigenvalue of the reduced density matrix of a random complex state is proved. In fact, the full distribution of the minimum eigenvalue is derived exactly for both the cases of a random real and a random complex state. Our results are relevant to the entanglement properties of eigenvectors of the orthogonal and unitary ensembles of random matrix theory and quantum chaotic systems. They also provide a rare exactly solvable case for the distribution of the minimum of a set of N *strongly correlated* random variables for all values of N (and not just for large N).

Key Words: entanglement, random pure state, extreme value statistics

I. INTRODUCTION

Entangelement has been studied extensively in the recent past due to its central role in quantum information and possible involvement in quantum computation [1, 2]. It is desirable in many instances to create states of large entanglement. Measures of entanglement have been studied mostly in the context of pure bipartite states, where the von-Neumann entropy of either subsystem is one of the measures of entanglement [2]. However there exist other measures of entanglement as well, e.g. the so called concurrence for two-qubit systems [3]. The entanglement of *random pure* quantum states is of interest as they have near maximal entanglement content, especially in the context of bipartite entanglement [4]. Apart from the issue of bipartite entanglement, statistical properties of such random states are relevant for quantum chaotic or non-integrable systems. The applicability of random matrix theory and hence of random states to systems with well-defined chaotic classical limits was pointed out long back [5]. They are also of relevance to other systems with no apparent classical limit [6, 7, 8].

In this paper, we focus on a bipartite quantum system. More precisely, we consider a bipartite partition of a NM-dimensional Hilbert space $\mathcal{H}^{(NM)}$ as $\mathcal{H}^{(NM)} = \mathcal{H}_A^{(N)} \otimes \mathcal{H}_B^{(M)}$. We can assume without loss of generality $N \leq M$. As an example of such a bipartite system, $\mathcal{A}$ may be considered a given subsystem (say a set of spins) and $\mathcal{B}$ may represent the environment (e.g., a heat bath). Any quantum state $|\psi\rangle$ of the composite system can be generally written as a linear combination, $|\psi\rangle = \sum_{i=1}^{N} \sum_{\alpha=1}^{M} x_{i,\alpha}\, |i^A\rangle \otimes |\alpha^B\rangle$ where $|i^A\rangle$ and $|\alpha^B\rangle$ denote two complete basis of $\mathcal{H}_A^{(N)}$ and $\mathcal{H}_B^{(M)}$ respectively and the coefficients $x_{i,\alpha}$'s form the entries of a rectangular $(N \times M)$ matrix X. Mutually nonexclusive properties of such a state are entanglement, randomness and statisical purity. Such a quantum state $|\psi\rangle$ is:

• **entangled:** if *not* expressible as a direct product of two states belonging to the two subsystems $\mathcal{A}$ and $\mathcal{B}$. Only in the special case when the coefficients have the product form, $x_{i,\alpha} = a_i b_\alpha$ for all i and α, the state $|\psi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle$ can be written as a direct product of two states $|\phi^A\rangle = \sum_{i=1}^{N} a_i |i^A\rangle$ and $|\phi^B\rangle = \sum_{\alpha=1}^{M} b_\alpha |\alpha^B\rangle$ belonging respectively to the two subsystems $\mathcal{A}$ and $\mathcal{B}$. In this case, the composite state $|\psi\rangle$ is *fully unentangled.* But otherwise, it is generically *entangled.*

• **random:** if the coefficients $x_{i,\alpha}$ are random variables drawn from an underlyting probability distribution. The simplest and the most common random state corresponds to choosing $x_{i,\alpha}$'s as independent and identically distributed Gaussian variables, real or complex.

• **pure:** if the density matrix of the composite system is simply given by, $\rho = |\psi\rangle\langle\psi|$ with the constraint $\mathrm{Tr}[\rho] = 1$, or equivalently $\langle\psi|\psi\rangle = 1$.

Given a *random, pure* and generically *entangled* composite state, important informations on the results of the measurement of any observable on the subsystem $\mathcal{A}$ can be derived from the reduced density matrix $\rho_A = \mathrm{Tr}_B[\rho]$, obtained upon tracing out the environmental degrees of freedom (i.e., those of subsystem $\mathcal{B}$). It is easy to show

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* Permanent address: Department of Physics, Indian Institute of Technology Madras, Chennai, 600036, India.

(see section II for details) that for a random pure state, $\rho_A = XX^\dagger$ is an $N \times N$ square matrix where X is the $N \times M$ rectangular coefficient matrix. The N unordered eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ of ρ_A carry important informations regarding the degree of the entanglement in the subsystem A. Given that the entries $x_{i,\alpha}$ of the coefficient matrix X are independent Gaussian variables (real or complex), the eigenvalues λ_i's of the matrix $\rho_A = XX^\dagger$ are also random variables and their joint probability density function (jpdf) is known [9, 10]

$$P(\lambda_1, \lambda_2, \cdots, \lambda_N) = B_{M,N}\delta\left(\sum_{i=1}^{N}\lambda_i - 1\right)\prod_{i=1}^{N}\lambda_i^{\frac{\beta}{2}(M-N+1)-1}\prod_{j<k}|\lambda_j - \lambda_k|^\beta. \tag{1}$$

Here $\beta = 1, 2$ corresponding to the real and complex entries of A and $B_{M,N}$ is the normalization constant that is known explicitly [10]. Several spectral properties associated with the jpdf in Eq. (1), in particular for the complex $\beta = 2$ case, have been studied extensively in the literature, for instance see the book [11] and references therein.

In principle, all informations about the spectral properties of the subsystem $\mathcal{A}$, including its degree of entanglement, are encoded in the jpdf (1). For example, one useful measure of entanglement is the von Neumann entropy $S = -\sum_{i=1}^{N}\lambda_i \ln(\lambda_i)$ which is a random variable. The average entropy $\langle S\rangle$ (where the average is performed with the measure in Eq. (1)) was computed for $\beta = 2$ by Page [12] and was found to be $\langle S\rangle \approx \ln(N) - \frac{N}{2M}$ for large $1 << N \leq M$. Noting that $\ln(N)$ is the maximal possible value of entropy of the subsystem $\mathcal{A}$, it follows that in the limit when $M >> N$, the average entropy, and hence the average entanglement, of a random pure state is near maximal. Later, the same result was shown to hold for the $\beta = 1$ case [13].

While the average entropy is a useful measure of entanglement, it is not the unique one. In fact, important informations regarding the nature of entanglement of a random pure state can also be obtained (see Section II for a detailed discussion) by studying the probability distributions of the extreme eigenvalues $\lambda_{\max} = \max(\lambda_1, \lambda_2, \ldots, \lambda_N)$ and $\lambda_{\min} = \min(\lambda_1, \lambda_2, \ldots, \lambda_N)$. In particular, the probability distribution of the minimum eigenvalue $\lambda_{\min}$ provides, in addition to the nature of the entanglement, an important information about the degree to which the effective dimension of the Hilbert space of the subsystem $\mathcal{A}$ can be reduced.

In fact, the average value $\langle\lambda_{\min}\rangle$ (with respect to the measure in Eq. (1)) of the minimum eigenvalue was studied recently by Znidaric [14] for the case $N = M$ and based on the exact $\langle\lambda_{\min}\rangle$ for small values of N, Znidaric conjectured that $\langle\lambda_{\min}\rangle = 1/N^3$ for all N for the complex case ($\beta = 2$). The purpose of this paper is to provide *exact results for the full probability distribution* of $\lambda_{\min}$ for *all* N (for the case when $N = M$), both for the complex ($\beta = 2$) and the real ($\beta = 1$) cases. A byproduct of our general results is the proof of Znidaric's conjecture for $\beta = 2$. Our results are summarized as follows. Let $P_N(x)dx$ denote the probability that $x \leq \lambda_{\min} \leq x + dx$, i.e., $P_N(x)$ is the probability density function (pdf) of $\lambda_{\min}$. We show that

- **Complex case ($\beta = 2$):**

$$P_N(x) = N\,(N^2 - 1)\,(1 - Nx)^{N^2-2}\,\Theta(1 - Nx) \tag{2}$$

where $\Theta(x)$ is the standard Heaviside function, $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$. The k-th moment $\mu_k(N) = \langle\lambda_{\min}^k\rangle$ is given by

$$\mu_k(N) = \frac{\Gamma(k+1)\Gamma(N^2)}{N^k\,\Gamma(N^2 + k)}. \tag{3}$$

In particular, for $k = 1$, we get $\mu_1(N) = 1/N^3$ thus proving the recent conjecture in [14].

- **Real case ($\beta = 1$):** the result for the real case turns out to be a bit more complicated. For the pdf of $\lambda_{\min}$ we get

$$P_N(x) = A_N\,x^{-N/2}\,(1 - Nx)^{(N^2+N-4)/2}\,{}_2F_1\left(\frac{N+2}{2}, \frac{N-1}{2}, \frac{N^2+N-2}{2}, -\frac{1-Nx}{x}\right),\ 0 < x \leq 1/N \tag{4}$$

and $P_N(x) = 0$ for $x \geq 1/N$. The constant A_N is given by

$$A_N = \frac{N\,\Gamma(N)\,\Gamma(N^2/2)}{2^{N-1}\,\Gamma(N/2)\,\Gamma((N^2+N-2)/2)}, \tag{5}$$

and ${}_2F_1(\alpha, \beta, \gamma, z)$ is the standard Hypergeometric function defined as [15]

$${}_2F_1(\alpha, \beta, \gamma, z) = 1 + \frac{\alpha\,\beta}{\gamma}\,z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\,\frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\,\frac{z^3}{3!} + \ldots \tag{6}$$

The moments $\mu_k(N) = \langle \lambda_{\min}^k \rangle$ are also computed exactly and are given in Eq. (58). In particular, the average value ($k = 1$) decays for large N as

$$\mu_1(N) \approx \frac{c}{N^3} \tag{7}$$

where the prefactor c has a nontrivial value

$$c = 2\left[1 - \sqrt{\frac{\pi e}{2}}\,\mathrm{erfc}(1/\sqrt{2})\right] = 0.688641\cdots \tag{8}$$

The paper is organized as follows. In Section II, we provide a general introduction to the random pure states of a bipartite system and recapitulate some general facts leading to the jpdf (1). Section II and III provide the detailed calculations of the distribution of the minimum eigenvalue for the complex and the real cases respectively. Finally we conclude in Section IV with a summary and open questions. Some details of the calculations are presented in the two appendices.

II. A RANDOM PURE STATE OF A BIPARTITE SYSTEM

In this section we recall some general facts about a *random pure* (RP) state of a bipartite system, its entanglement properties and the associated random matrix ensemble. As mentioned in the introduction, let us consider a composite bipartite system $A \otimes B$ composed of two smaller subsystems A and B, whose respective Hilbert spaces $\mathcal{H}_A^{(N)}$ and $\mathcal{H}_B^{(M)}$ have dimensions N and M. The Hilbert space of the composite system $\mathcal{H}^{(NM)} = \mathcal{H}_A^{(N)} \otimes \mathcal{H}_B^{(M)}$ is thus NM-dimensional. Without loss of generality we will assume that $N \leq M$. Let $\{|i^A\rangle\}$ and $\{|\alpha^B\rangle\}$ represent two complete basis states for A and B respectively. Then, any arbitrary state $|\psi\rangle$ of the composite system can be most generally written as a linear combination

$$|\psi\rangle = \sum_{i=1}^{N}\sum_{\alpha=1}^{M} x_{i,\alpha}\,|i^A\rangle \otimes |\alpha^B\rangle \tag{9}$$

where the coefficients $x_{i,\alpha}$'s form the entries of a rectangular $(N \times M)$ matrix $X = [x_{i,\alpha}]$.

Now, the state $|\psi\rangle$ is a statistically *pure* state of the composite system if the density matrix of the composite system is given by

$$\rho = |\psi\rangle\,\langle\psi|. \tag{10}$$

Note that had the composite system been in a statistically *mixed* state, its density matrix would have been of the form

$$\rho = \sum_k p_k\,|\psi_k\rangle\,\langle\psi_k|, \tag{11}$$

where $|\psi_k\rangle$'s are the pure states of the composite system and $0 \leq p_k \leq 1$ denotes the probability that the composite system is in the k-th pure state, with $\sum_k p_k = 1$. In this paper, we will restrict ourselves to the case when the composite system is in a pure state denoted by $|\psi\rangle$. Then its density matrix in Eq. (10), upon using the decomposition in Eq. (9), can be expressed as

$$\rho = \sum_{i,\alpha}\sum_{j,\beta} x_{i,\alpha}\,x_{j,\beta}^*\,|i^A\rangle\langle j^A| \otimes |\alpha^B\rangle\langle\beta^B|, \tag{12}$$

where the Roman indices i and j run from 1 to N and the Greek indices α and β run from 1 to M. We also assume that the pure state $|\psi\rangle$ is normalized to unity so that $\mathrm{Tr}[\rho] = 1$. Hence the coefficients $x_{i,\alpha}$'s must be such that $\mathrm{Tr}[\rho] = 1$.

Given the density matrix of the pure composite state in Eq. (12), one can then compute the reduced density matrix of, say, the subsystem $\mathcal{A}$ by tracing over the states of the subsystem $\mathcal{B}$

$$\rho_A = \mathrm{Tr}_B[\rho] = \sum_{\alpha=1}^{M}\langle\alpha^B|\rho|\alpha^B\rangle. \tag{13}$$

Using the expression in Eq. (12) one gets

$$\rho_A = \sum_{i,j=1}^{N} \sum_{\alpha=1}^{M} x_{i,\alpha}\, x_{j,\alpha}^{*}\, |i^A\rangle\langle j^A| = \sum_{i,j=1}^{N} W_{ij} |i^A\rangle\langle j^A| \tag{14}$$

where W_{ij}'s are the entries of the $N \times N$ square matrix $W = XX^{\dagger}$. In a similar way, one can express the reduced density matrix $\rho_B = \mathrm{Tr}_A[\rho]$ of the subsystem B in terms of the square $M \times M$ dimensional matrix $W' = X^{\dagger}X$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ denote the N eigenvalues of $W = XX^{\dagger}$. Note that these eigenvalues are nonnegative, $\lambda_i \geq 0$ for all $i = 1, 2, \ldots, N$. Now the matrix $W' = X^{\dagger}X$ has $M \geq N$ eigenvalues. It is easy to prove that $M - N$ of them are identically 0 and N nonzero eigenvalues of $W^{\dagger}$ are the same as those of W. Thus, in this diagonal representation, one can express ρ_A as

$$\rho_A = \sum_{i=1}^{N} \lambda_i\, |\lambda_i^A\rangle\, \langle \lambda_i^A| \tag{15}$$

where $|\lambda_i^A\rangle$'s are the eigenvectors of $W = XX^{\dagger}$. A similar representation holds for ρ_B. It then follows that one can represent the original composite state $|\psi\rangle$ in this diagonal representation as

$$|\psi\rangle = \sum_{i=1}^{N} \sqrt{\lambda_i}\, |\lambda_i^A\rangle \otimes |\lambda_i^B\rangle \tag{16}$$

where $|\lambda_i^A\rangle$ and $|\lambda_i^B\rangle$ represent the normalized eigenvectors (corresponding to nonzero eigenvalues) of $W = XX^{\dagger}$ and $W' = X^{\dagger}X$ respectively. This spectral decomposition in Eq. (16) is known as the Schimdt decomposition. The normalization condition $\langle\psi|\psi\rangle = 1$, or equivalently $\mathrm{Tr}[\rho] = 1$, imposes a constraint on the eigenvalues, $\sum_{i=1}^{N} \lambda_i = 1$.

Note that while each individual state $|\lambda_i^A\rangle \otimes |\lambda_i^B\rangle$ in the Schimdt decomposition in Eq. (16) is *unentangled*, their linear combination $|\psi\rangle$, in general, is *entangled*. This simply means that the composite state $|\psi\rangle$ can not, in general, be written as a direct product $|\psi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle$ of two states of the respective subsystems. The spectral properties of the matrix W, i.e., the knowledge of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$, in association with the Schimdt decomposition in Eq. (16), provide useful information about how entangled a pure state is. For example, as mentioned in the introduction, one useful measure of the entanglement is the von Neumann entropy, $S = -\sum_{i=1}^{N} \lambda_i \ln(\lambda_i)$.

In addition, the two extreme eigevalues, the largest $\lambda_{\max} = \max(\lambda_1, \lambda_2, \ldots, \lambda_N)$ and the smallest $\lambda_{\min} = \min(\lambda_1, \lambda_2, \ldots, \lambda_N)$ also provide useful information about the entanglement. Note that due to the constraint $\sum_{i=1}^{N} \lambda_i = 1$ and the fact that all eigenvalues are nonnegative, it follows that $1/N \leq \lambda_{\max} \leq 1$ and $0 \leq \lambda_{\min} \leq 1/N$. Consider, for instance, the following limiting situations. Suppose that the largest eigenvalue $\lambda_{\max} = \max(\lambda_1, \lambda_2, \ldots, \lambda_N)$ takes its maximum allowed value 1. Then due to the constraint $\sum_{i=1}^{N} \lambda_i = 1$ and the fact that $\lambda_i \geq 0$ for all i, it follows that all the rest $(N-1)$ eigenvalues must be identically 0. In that case, it follows from Eq. (16) that $|\psi\rangle$ is fully *unentangled*. On the other hand, if $\lambda_{\max} = 1/N$ (i.e., it takes its lowest allowed value), it follows that all the eigenvalues must have the same value, $\lambda_i = 1/N$ for all i, again due to the constraint $\sum_{i=1}^{N} \lambda_i = 1$. In this case, one can show that the pure state $|\psi\rangle$ is *maximally* entangled, as this state maximizes the von Neumann entropy $S = \ln(N)$.

In this paper, we will focus on the smallest eigenvalue $0 \leq \lambda_{\min} \leq 1/N$. As in the case of the largest eigenvalue above, let us consider the two limiting situations. When $\lambda_{\min}$ takes its maximal allowed value $\lambda_{\min} = 1/N$, it follows again from the constraint $\sum_{i=1}^{N} \lambda_i = 1$ that all the eigenvalues must have the same value $\lambda_i = 1/N$. This will thus make the state $|\psi\rangle$ *maximally* entangled. In the opposite case, when $\lambda_{\min} = 0$ takes its smallest allowed value, while it does not provide any information on the entanglement of the state $|\psi\rangle$, one sees from the Schmidt decomposition that the dimension of the effective Hilbert space of the subsystem A gets reduced from N to $N-1$. Indeed, if $\lambda_{\min}$ is very close to zero, one can effectively ignore the term containing $\lambda_{\min}$ in Eq. (16) and thus achieve a reduced Hilbert space, a process called 'dimensional reduction' that is often used in the compression of large data structures in computer vision [16, 17, 18]. Thus the knowledge of $\lambda_{\min}$ and in particular its proximity to its upper and lower limits provide informations on both the entanglement phenomenon as well as on the efficiency of the dimensional reduction process.

So far, our discussion is valid for an arbitrary pure state in Eq. (9) with any fixed coefficient matrix $X = [x_{i,\alpha}]$. Now, such a pure state will be called a *random pure* state if the coefficients $x_{i,\alpha}$'s are random variables, drawn from an underlying probability distribution. In particular, we will consider the case when the elements of X are independent and identically distributed random variables, real or complex, drawn from a Gaussian distribution: $\mathrm{Prob}[X] \propto \exp\left[-\frac{\beta}{2}\mathrm{Tr}(X^{\dagger}X)\right]$, where the Dyson index $\beta = 1, 2$ corresponds respectively to the real and complex

X matrices. The product $W = XX^\dagger$ is called the random Wishart matrix [19]. The joint distribution of the N nonnegative eigenvalues of W is known [20]

$$P^W(\lambda_1, \lambda_2, \ldots, \lambda_N) \propto e^{-\frac{\beta}{2}\sum_{i=1}^N \lambda_i} \prod_{i=1}^N \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j<k} |\lambda_j - \lambda_k|^\beta. \tag{17}$$

Note however, that in case of a *random pure* state $|\psi\rangle$ in Eq. (9), the eigenvalues of the matrix $W = XX^\dagger$ are not quite the same as that of the Wishart matrix, due to the additional constraint that $\mathrm{Tr}[\rho] = \mathrm{Tr}[W] = 1$. Thus, the eigenvalues of W that appear in the Schimdt decomposition in Eq. (16), are distributed according to the Wishart law in Eq. (17), but in addition have to satisfy the constraint $\sum_{i=1}^N \lambda_i = 1$. This constraint can be explicitly incorporated by multiplying a delta function $\delta(\sum_{i=1}^N \lambda_i - 1)$ to the Wishart measure in Eq. (17). With this additional delta function multiplying the Wishart measure, the exponential term in Eq. (17) just becomes a constant and can be absorbed into the overall normalization constant and one arrives at the jpdf of the eigenvalues of W mentioned in Eq (1) in the introduction.

Given the jpdf (1), we are interested here in the distribution of the minimum eigenvalue $\lambda_{\min}$. Let $Q_{N,M}(x) = \mathrm{Prob}[\lambda_{\min} \geq x]$ be the cumulative distribution of $\lambda_{\min}$. The pdf of $\lambda_{\min}$ is simply obtained by taking the derivative, $P_{N,M}(x) = -dQ_{N,M}(x)/dx$. Since the event $\lambda_{\min} \geq x$ necessarily implies that all the eigenvalues $\lambda_i \geq x$ (for all $i = 1, 2, \ldots, N$), it follows, upon using the explicit jpdf (1), that $Q_{N,M}(x)$ is precisely given by the multiple integral (with $N \leq M$)

$$Q_{N,M}(x) = B_{M,N} \int_x^\infty \cdots \int_x^\infty \delta\left(\sum_{i=1}^N \lambda_i - 1\right) \prod_{j<k} |\lambda_j - \lambda_k|^\beta \prod_{i=1}^N \lambda_i^{\frac{\beta}{2}(M-N+1)-1} d\lambda_i. \tag{18}$$

The real technical challenge is to evaluate this multiple integral. In the next two sections, we show how to compute this integral exactly respectively for $\beta = 2$ and $\beta = 1$, for all $M = N$, i.e., when the Hilbert spaces of the two subsystems have equal dimensions. In this case, i.e., when $M = N$, we will denote, for simplicity of notations, $Q_{N,N}(x) = Q_N(x)$ for the cumulative distribution of the minimum eigenvalue and the corresponding density by $P_{N,N}(x) = P_N(x) = -dQ_N(x)/dx$.

III. A COMPLEX RANDOM VECTOR

This section is devoted to finding exactly the distribution of the minimum eigenvalue $\lambda_{\min}$ or the minimum Schmidt coefficient for random complex states. Let

$$Q_N(x) = \mathrm{Prob}\left[\lambda_{\min} \geq x\right] = \mathrm{Prob}\left[\lambda_1 \geq x, \lambda_2 \geq x, \ldots, \lambda_N \geq x\right]. \tag{19}$$

Therefore

$$Q_N(x) = B_{N,N} \int_x^\infty \cdots \int_x^\infty \delta\left(\sum_{i=1}^N \lambda_i - 1\right) \prod_{j<k} (\lambda_j - \lambda_k)^2 \prod_{i=1}^N d\lambda_i \tag{20}$$

An evaluation of this multiple integral proceeds by introducing an auxiliary one defined by

$$I(x,t) = \int_x^\infty \cdots \int_x^\infty \delta\left(\sum_{i=1}^N \lambda_i - t\right) \prod_{j<k} (\lambda_j - \lambda_k)^2 \prod_{i=1}^N d\lambda_i, \tag{21}$$

so that $Q_N(x) = B_{N,N}\, I(x,1)$. Consider the following Laplace transform of $I(x,t)$:

$$\int_0^\infty I(x,t) e^{-st} dt = \int_x^\infty \cdots \int_x^\infty e^{-s\sum_{i=1}^N \lambda_i} \prod_{j<k} (\lambda_j - \lambda_k)^2 \prod_{i=1}^N d\lambda_i. \tag{22}$$

A linear shift and scaling $z_i = s(\lambda_i - x)$ results in

$$\int_0^\infty I(x,t) e^{-st} dt = \frac{e^{-sNx}}{s^{N^2}} \int_0^\infty \cdots \int_0^\infty e^{-\sum_{i=1}^N z_i} \prod_{j<k} (z_j - z_k)^2 \prod_{i=1}^N dz_i. \tag{23}$$

Thus the dependence on s and x just factors out of the integral. The integral happens to be one of the Selberg integrals which can be evaluated explicitly [21] and this gives

$$\int_0^\infty I(x,t)e^{-st}dt = \frac{e^{-sNx}}{s^{N^2}} \prod_{j=0}^{N-1} \Gamma(j+2)\Gamma(j+1). \tag{24}$$

An inverse Laplace transform yields

$$I(x,t) = \frac{\prod_{j=0}^{N-1}\Gamma(j+2)\Gamma(j+1)}{\Gamma(N^2)} \left(t - Nx\right)^{N^2-1} \Theta\left(t - Nx\right). \tag{25}$$

Using the known normalization constant [10]

$$B_{N,N} = \frac{\Gamma(N^2)}{\prod_{j=0}^{N-1}\Gamma(N-j)\Gamma(N-j+1)} \tag{26}$$

we finally arrive at

$$Q_N(x) = \text{Prob}\left[\lambda_{\min} \geq x\right] = B_{N,N}\, I(x,1) = \left(1 - Nx\right)^{N^2-1} \Theta\left(1 - Nx\right). \tag{27}$$

Subsequently, the pdf is given by

$$P_N(x) = -\frac{dQ_N(x)}{dx} = N(N^2-1)(1-Nx)^{N^2-2}\,\Theta(1-Nx). \tag{28}$$

A plot of this pdf can be found in Fig. 1 for $N = 4$. Thus $P_N(x)$ in $x \in [0, 1/N]$ has the limiting behavior

$$\begin{aligned} P_N(x) &\to N\,(N^2-1) \qquad \text{as} \quad x \to 0 \\ &= N\,(N^2-1)\,(1-Nx)^{N^2-2} \quad \text{as} \quad x \to 1/N \end{aligned} \tag{29}$$

Note that in the regime where $x << 1/N$, the pdf in Eq. (28) becomes exponential, $P_N(x) \approx N(N^2-1)\exp[-N(N^2-1)x]$. Let us also note that the distribution of the smallest eigenvalue in Eq. (27) is identical to that of the smallest intensity component of a complex random state derived recently [22], provided one replaces N^2 (in the exponent in Eq. (27)) by N.

Moments of $\lambda_{\min}$: From the explicit expression of the pdf in Eq. (28) one can easily compute all the moments of $\lambda_{\min}$. For the k-th moment we get

$$\mu_k(N) = \langle \lambda_{\min}^k \rangle = \int_0^\infty x^k P_N(x)\,dx = \frac{\Gamma(k+1)\Gamma(N^2)}{N^k\,\Gamma(N^2+k)}. \tag{30}$$

In particular, for $k = 1$, we obtain for all N

$$\mu_1(N) = \langle \lambda_{\min} \rangle = \frac{1}{N^3}, \tag{31}$$

thus proving the recent conjecture by Znidaric [14] based on evaluations for small N. Putting $k = 2$ in Eq. (30), we get the second moment $\mu_2 = \frac{2}{N^4(N^2+1)}$. Thus the variance is given by

$$\sigma^2 = \mu_2(N) - [\mu_1(N)]^2 = \frac{1}{N^6}\left(\frac{N^2-1}{N^2+1}\right). \tag{32}$$

IV. A REAL RANDOM VECTOR

While complex random vectors are "generic", real vectors are important as well. For instance in the case when the system has a time-reversal symmetry or any anti-unitary symmetry the eigenfunctions can be in general chosen to be real and the relevant ensembles are the "orthogonal" ones (such as the Gaussian orthogonal ensemble and the circular orthogonal ensemble), wherein general orthogonal transformations leave the ensemble invariant [21, 23]. The entanglement properties of real and complex random states may, in general, differ. For instance for so called "single-particle" states or one-magnon states, real states have lower entanglement measured in terms of two-spin entanglement content than the case of the complex states [24]. In general, much less is known for random real states than the complex ones, although for instance several many-body Hamiltonians (say of spins) have natural time-reversal symmetry. In this section the distribution of the minimum eigenvalue of the real case is calculated exactly.

The jpdf of the eigenvalues λ_i in this case (we again restrict ourselves to the case $M = N$) is

$$P_N(\lambda_1, \cdots, \lambda_N) = C_{N,N}\delta\left(\sum_{i=1}^{N}\lambda_i - 1\right)\prod_{j<k}|\lambda_j - \lambda_k|\prod_{i=1}^{N}\frac{1}{\sqrt{\lambda_i}}, \tag{33}$$

where $C_{N,N}$ is the normalization constant and is known to be [10]

$$C_{N,N}^{-1} = \frac{2^N}{\pi^{N/2}\Gamma(N^2/2)}\prod_{j=0}^{N-1}\Gamma\left(\frac{j+1}{2}\right)\Gamma\left(\frac{j+3}{2}\right). \tag{34}$$

The cumulative distribution of the smallest eigenvalue, $Q_N(x) = \text{Prob}[\lambda_{\min} \geq x]$, is given by

$$Q_N(x) = C_{N,N}\int_x^\infty \cdots \int_x^\infty \delta\left(\sum_{i=1}^{N}\lambda_i - 1\right)\prod_{j<k}|\lambda_j - \lambda_k|\prod_{i=1}^{N}\frac{1}{\sqrt{\lambda_i}}\,d\lambda_i. \tag{35}$$

To evaluate this multiple integral, we proceed, as in the previous section, by defining an auxiliary integral $J(x,t)$ as

$$J(x,t) = \int_x^\infty \cdots \int_x^\infty \delta\left(\sum_{i=1}^{N}\lambda_i - t\right)\prod_{j<k}|\lambda_j - \lambda_k|\prod_{i=1}^{N}\frac{1}{\sqrt{\lambda_i}}\,d\lambda_i, \tag{36}$$

so that $Q_N(x) = C_{N,N}\, J(x, 1)$.

Taking the Laplace transform of Eq. (36) leads to

$$\int_0^\infty J(x,t)e^{-st}dt = \frac{1}{(2s)^{N^2/2}}\int_{2sx}^\infty \cdots \int_{2sx}^\infty e^{-\frac{1}{2}\sum_{i=1}^N y_i}\prod_{j<k}|y_j - y_k|\prod_{i=1}^{N}\frac{1}{\sqrt{y_i}}\,dy_i, \tag{37}$$

where the scaled variable $y_i = 2s\lambda_i$. We next use a result due to Edelman [25] for the Wishart orthogonal ensemble whose jpdf is given by

$$P_N^W(y_1, \cdots, y_N) = a_{N,N}e^{-\frac{1}{2}\sum_{i=1}^N y_i}\prod_{i=1}^{N}\frac{1}{\sqrt{y_i}}\prod_{j<k}|y_j - y_k| \tag{38}$$

where the normalization constant $a_{N,N}$ is

$$a_{N,N} = \frac{C_{N,N}}{2^{N^2/2}\Gamma(N^2/2)}. \tag{39}$$

For such an ensemble Edelman [25] showed that the distribution of the smallest eigenvalue $Q^W(z) = \text{Prob}\left[y_{\min} \geq z\right]$ is given explicitly by

$$Q^W(z) = \int_z^\infty \cdots \int_z^\infty P^W(y_1, \cdots, y_N)\prod_{i=1}^{N}dy_1\cdots dy_N = \frac{N\Gamma(N)}{2^{N-1/2}\Gamma(N/2)}\int_z^\infty \frac{e^{-Ny/2}}{\sqrt{y}}\,U\left(\frac{N-1}{2}, -\frac{1}{2}, \frac{y}{2}\right)dy. \tag{40}$$

where $U(a, b; z)$ is the confluent hypergeometric function [26] of the second kind that satisfies the differential equation

$$z\frac{d^2U}{dz^2} + (b - z)\frac{dU}{dz} - aU = 0 \tag{41}$$

with the boundary conditions

$$U(a, b, 0) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)}, \;\; U(a, b, z \to \infty) = 0. \tag{42}$$

Working back we therefore obtain

$$\int_0^\infty J(x,t)e^{-st}dt = \frac{1}{(2s)^{N^2/2}}\left[\frac{N\Gamma(N)}{2^{N-1/2}\Gamma(N/2)a_{N,N}}\right]\int_{2sx}^\infty \frac{e^{-Ny/2}}{\sqrt{y}}\,U\left(\frac{N-1}{2},-\frac{1}{2},\frac{y}{2}\right)dy. \tag{43}$$

To make further progress, it turns out to be easier to work with the probability density function rather than the cumulative distribution $Q_N(x)$,

$$P_N(x) = -\frac{dQ_N(x)}{dx} = -C_{N,N}\frac{d\,J(x,1)}{dx}. \tag{44}$$

Taking the derivative of Eq. (43) with respect to x leads to

$$-\int_0^\infty \frac{dJ(x,t)}{dx}e^{-st}dt = b_N\frac{1}{\sqrt{x}}\frac{e^{-Nsx}}{s^{(N^2-1)/2}}U\left(\frac{N-1}{2},-\frac{1}{2},sx\right), \tag{45}$$

where

$$b_N = \frac{N\Gamma(N)}{\Gamma(N/2)a_{N,N}2^{(N^2+2N-2)/2}}. \tag{46}$$

The task then is to find the Laplace inverse:

$$-\frac{dJ(x,t)}{dx} = \frac{b_N}{\sqrt{x}}\mathcal{L}_s^{-1}\left[\frac{e^{-Nsx}}{s^{(N^2-1)/2}}U\left(\frac{N-1}{2},-\frac{1}{2},sx\right)\right]. \tag{47}$$

First, an application of the convolution theorem leads to

$$\mathcal{L}_s^{-1}\left[\frac{e^{-Nsx}}{s^{(N^2-1)/2}}\right] = \frac{1}{\Gamma\left(\frac{N^2-1}{2}\right)}(t-Nx)^{(N^2-3)/2}\,\Theta\left(t-Nx\right). \tag{48}$$

Second, using an integral representation of the hypergeometric function $U(a,b,z)$ [26] namely

$$U(a,b,z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}dt \tag{49}$$

one obtains the following inverse:

$$\mathcal{L}_s^{-1}\left[U\left(\frac{N-1}{2},-\frac{1}{2},sx\right)\right] = \frac{x^{3/2}}{\Gamma(\frac{N-1}{2})}t^{(N-3)/2}(x+t)^{-(N+2)/2}. \tag{50}$$

Using the two inverses in Eqs. (48) and (50) and the convolution theorem, we get upon simplifying

$$\begin{aligned}\mathcal{L}_s^{-1}\left[\frac{e^{-Nsx}}{s^{(N^2-1)/2}}U\left(\frac{N-1}{2},-\frac{1}{2},sx\right)\right] &= \frac{x^{3/2}}{\Gamma\left(\frac{N^2-1}{2}\right)\Gamma\left(\frac{N-1}{2}\right)}\int_0^{t-Nx} t'^{\frac{N-2}{2}}\left(x+t'\right)^{-\frac{N+2}{2}}\left(t-t'-Nx\right)^{\frac{N^2-3}{2}}dt' \\ &= \frac{x^{-(N-1)/2}}{\Gamma\left(\frac{N^2+N-2}{2}\right)}(t-Nx)^{\frac{N^2+N-4}{2}}\,{}_2F_1\left(\frac{N+2}{2},\frac{N-1}{2},\frac{N^2+N-2}{2},-\frac{t-Nx}{x}\right).\end{aligned} \tag{51}$$

Here ${}_2F_1(a,b,c,z)$ is the standard hypergeometric function [26], and the integral can be found in [15]. Using this along with Eqs. (44,47) and substituting $t=1$, we finally get the p.d.f. of the minimum eigenvalue $\lambda_{\min}$ as

$$P_N(x) = A_N\,x^{-N/2}\,(1-Nx)^{(N^2+N-4)/2}\,{}_2F_1\left(\frac{N+2}{2},\frac{N-1}{2},\frac{N^2+N-2}{2},-\frac{1-Nx}{x}\right),\; 0<x\leq 1/N \tag{52}$$

and $P_N(x)=0$ for $x\geq 1/N$. The constant A_N is given by

$$A_N = \frac{N\,\Gamma(N)\,\Gamma(N^2/2)}{2^{N-1}\,\Gamma(N/2)\,\Gamma((N^2+N-2)/2)}. \tag{53}$$

This solves exactly for the distribution of the minimum eigenvalue of the reduced density matrices of bipartite random real states when the dimensions of the subspaces are equal. In the simplest possible case of real states of two qubits, $N = 2$, the distribution is simply

$$P_2(x) = \frac{1-2x}{\sqrt{x(1-x)}},\ 0 < x \le 1/2; \quad P_2(x) = 0,\ x \ge 1/2. \tag{54}$$

This follows from Eq. (52) as ${}_2F_1(2, 1/2, 2, x) = 1/\sqrt{1-x}$. Alternatively it almost immediately follows from the jpdf in Eq. (33) as there are only two eigenvalues that sum to unity in this case, and the distribution of the one which is less than one-half is precisely $P_2(x)$. In Fig. 1, we plot the pdf $P_N(x)$ of $\lambda_{\min}$ for $N = 4$, both for the complex case given in Eq. (28) and the real case given in Eq. (52)

In appendix-A, we work out the limiting behavior of $P_N(x)$ as $x \to 0$ and $x \to 1/N$. For general N, one finds

$$\begin{aligned} P_N(x) &\approx \left[\frac{\sqrt{\pi}\,\Gamma(N)\,\Gamma(N^2/2)}{2^{N-1}\,\Gamma^2(N/2)\,\Gamma((N-1)/2)}\right] x^{-1/2} \quad \text{as} \quad x \to 0 \\ &\approx A_N\, N^{-N/2}\,(1-Nx)^{(N^2+N-4)/2} \quad \text{as} \quad x \to 1/N \end{aligned} \tag{55}$$

Comparing this limiting behavior in the real case in Eq. (55) with that of the complex case in Eq. (29) one finds that while in the former $P_N(x)$ diverges as $x^{-1/2}$ as $x \to 0$, in the latter it approaches a constant. In the other limit $x \to 1/N$, both the densities approach zero as a power law $(1-Nx)^{\nu}$, but with different exponents $\nu = N^2 - 2$ (for the complex case) and $\nu = (N^2 + N - 4)/2$ for the real case.

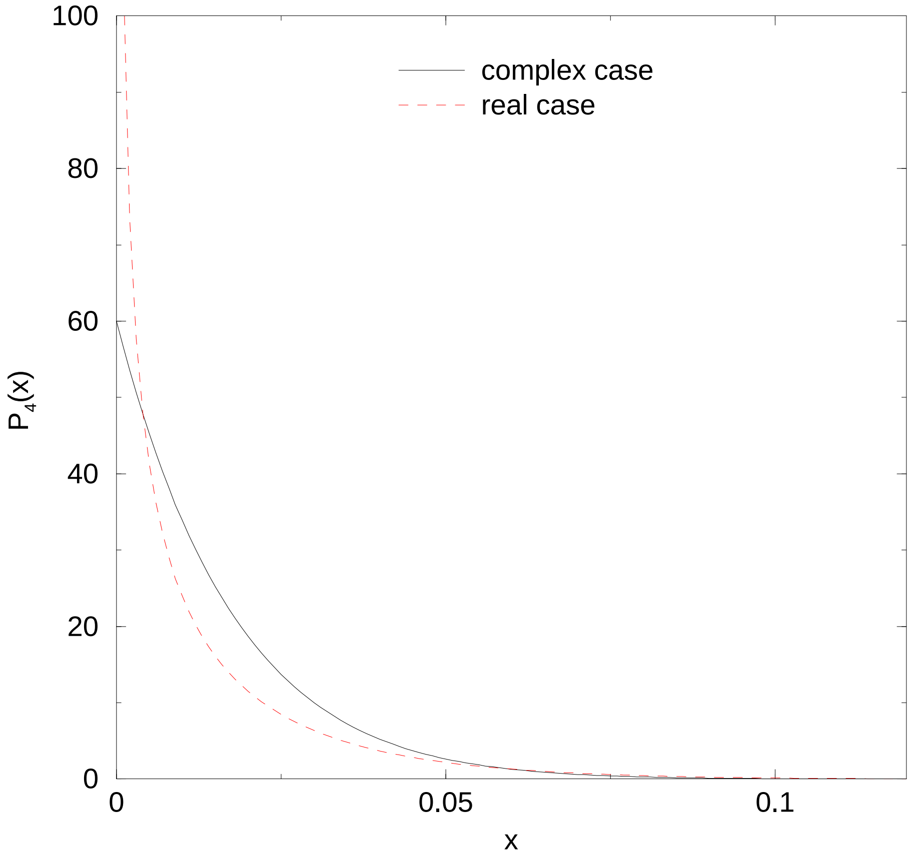

FIG. 1: The p.d.f $P_N(x)$ of the minimum eigenvalue $\lambda_{\min}$ vs. x for $N = 4$, for the complex and the real cases (Eqs. (28) and (52) respecively). In the complex case, the density approaches a constant as $x \to 0$, whereas for the real case, it diverges as $x^{-1/2}$ as $x \to 0$.

Moments of $\lambda_{\min}$: One can use the explicit result for the p.d.f. $P_N(x)$ of $\lambda_{\min}$ in Eq. (52) to calculate its k-th moment

$$\mu_k(N) = \langle {\lambda_{\min}}^k \rangle = A_N \int_0^{1/N} x^{k-N/2} (1-Nx)^{(N^2+N-4)/2} \, {}_2F_1\left(\frac{N+2}{2}, \frac{N-1}{2}, \frac{N^2+N-2}{2}, -\frac{1-Nx}{x}\right) dx$$
$$= A_N \int_0^{\infty} y^{(N^2+N-4)/2} (N+y)^{-(N^2+2k)/2} \, {}_2F_1\left(\frac{N+2}{2}, \frac{N-1}{2}, \frac{N^2+N-2}{2}, -y\right) \quad (56)$$

where we made a change of variable $y = -N + 1/x$ in the first line. We next use the following known integral [15]

$$\int_0^{\infty} x^{\gamma-1} (x+z)^{-\sigma} \, {}_2F_1(\alpha, \beta, \gamma, -x) \, dx = \frac{\Gamma(\gamma)\Gamma(\alpha-\gamma+\sigma)\Gamma(\beta-\gamma+\sigma)}{\Gamma(\sigma)\Gamma(\alpha+\beta-\gamma+\sigma)} \, {}_2F_1(\alpha-\gamma+\sigma, \beta-\gamma+\sigma, \alpha+\beta-\gamma+\sigma, 1-z) \quad (57)$$

in Eq. (56) and also the value of A_N from Eq. (53) to arrive at an explicit expression for the k-th moment (valid for all N),

$$\mu_k(N) = \frac{\Gamma(N+1)\Gamma(N^2/2)\Gamma(k+2)\Gamma(k+1/2)}{\Gamma(N/2)\Gamma(k+N^2/2)\Gamma(k+(N+3)/2)2^{N-1}} \, {}_2F_1\left(k+2, k+1/2, k+\frac{N+3}{2}, 1-N\right). \quad (58)$$

One can verify that $\mu_0(N) = 1$, thus ensuring the correct normalization. For the average value of $\lambda_{\min}$ we use $k = 1$ and get

$$\mu_1(N) = \langle \lambda_{\min} \rangle = \frac{\sqrt{\pi}\,\Gamma(N)}{N\,\Gamma(N/2)\,\Gamma((N+5)/2)2^{N-1}} \, {}_2F_1\left(3, \frac{3}{2}, \frac{N+5}{2}, 1-N\right). \quad (59)$$

Thus the expression for $\langle \lambda_{\min} \rangle$ for arbitrary N in the real case is considerably more complicated than its counterpart in Eq. (31) for the complex case. One finds, from Eq. (59), that $\mu_1(N)$ decreases with increasing N, e.g., $\mu_1(1) = 1$, $\mu_1(2) = (4-\pi)/8$, $\mu_1(3) = (2-\sqrt{3})/9$ etc. In appendix-B, we show that asymptotically for large N, $\mu_1(N)$ decays as

$$\mu_1(N) \approx \frac{c}{N^3}; \quad \text{where} \quad c = 2\left[1 - \sqrt{\frac{\pi e}{2}}\,\mathrm{erfc}(1/\sqrt{2})\right] = 0.688641\cdots \quad (60)$$

where $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} du$ is the complementary error function. The large N result in Eq. (60) for the real case should be compared to that of the complex case where $\mu_1(N) = 1/N^3$. One sees that the average value of the minimum eigenvalue in the former case is less by a constant factor $c = 0.688641\cdots$ compared to the later case. In Fig. 2, we plot both the exact formula for $\mu_1(N)$ in Eq. (59) and the asymptotic form in Eq. (60) against N.

V. CONCLUSION

In this paper we have computed the exact probability distribution of the minimum eigenvalue $\lambda_{\min}$ of an entangled random (both real and complex) pure state of a bipartite system composed of two subsystems whose respective Hilbert spaces have equal dimensions $M = N$. We have also computed exactly all the moments of $\lambda_{\min}$ for all N. As a byproduct, we prove that $\langle \lambda_{\min} \rangle = 1/N^3$ for all N for complex matrices, a result recently conjectured [14]. The pdf of the minimum eigenvalue in the real case differs significantly from its complex counterpart.

Apart from providing important informations on the nature of the entanglement of a random pure state as well as on the degree to which the dimension of the Hilbert space of a subsystem can be reduced, our result for the distribution of the minimum eigenvalue has some relevance in the general context of extreme value statistics. This subject has been around for a long time [27], but has seen a recent resurgence due to its many applications in diverse areas such as engineering, economics and physical sciences [28]. If the underlying random variables are *independent and identically distributed* then there are three possible limiting universal distributions for the extreme events, the Fréchet, the Gumbel and the Weibull distributions. However, much less is known when the underlying random variables are *strongly correlated*. In such cases, the limiting distribution (for large N) of the maximum is known exactly only in very few cases. For example, the limiting distribution of the largest eigenvalue of a $N \times N$ Gaussian unitary random matrix (GUE) is given by the celebrated Tracy-Widom law [29], which has found many recent applications [30]. Similarly, the Tracy-Widom law also describes the limiting distribution of the largest eigenvalue of Wishart matrices [31, 32], for random matrices with certain non-Gaussian entries [33] and the scaled height of a $(1+1)$-dimensional growth models [30, 34]. The probabilities of large deviations of $\lambda_{\max}$, outside the regime of the validity of the Tracy-Widom law, have also been computed recently both for Gaussian [35] and Wishart matrices [18, 31]. Other examples for

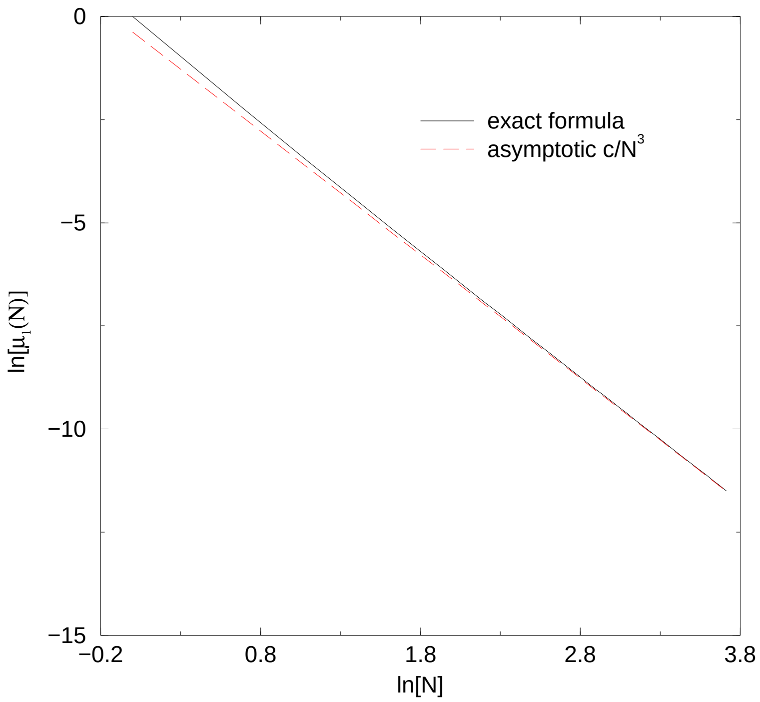

FIG. 2: A log-log plot of the exact formula of $\mu_1(N)$ in Eq. (59) vs. N compared with the asymptotic formula $\mu_1(N) \approx c/N^3$ vs. N with $c = 0.688641$ for the real case.

which the limiting distribution is known exactly include the maximum relative height of a class of one dimensional fluctuating interfaces in their steady states in a finite system [36, 37] and $1/f^\alpha$ noise signals [38]. In contrast, much less is known about the distribution of the extreme eigenvalues for finite N, a notable exception being the minimum eigenvalue for $N \times N$ Wishart matrices whose distribution was computed exactly by Edelman for all N [25]. In our present context, the eigenvalues of a random pure state are also strongly correlated due to the presence of the Vandermonde term $\prod_{j<k} |\lambda_j - \lambda_k|^\beta$ in the jpdf (1). So our results provide another rare exactly solvable case for the distribution of the minimum of a set of N *strongly correlated* random variables, and this is not just for large N but for *any finite* N.

Computing the distribution of $\lambda_{\min}$ for unequal dimensions ($M \neq N$) of the Hilbert spaces of the subsystems remains a challenging open problem.

APPENDIX A: LIMITING BEHAVIOR OF $P_N(x)$ FOR THE REAL CASE

In this appendix, we derive the behavior of $P_N(x)$ in Eq. (55), starting from the exact expression of $P_N(x)$ in Eqs. (52) and (53). The behavior near the upper limit $x \to 1/N$ is simple to derive. Using ${}_2F_1(\alpha, \beta, \gamma, 0) = 1$, one immediately finds from Eq. (52) that as $x \to 1/N$

$$P_N(x) \to A_N\, N^{-N/2}\, (1 - Nx)^{(N^2+N-4)/2} \tag{A1}$$

In contrast, deriving the behavior of $P_N(x)$ as $x \to 0$ is slightly more tricky. To derive this, we first use the following identity of the hypergeometric function [15]

$${}_2F_1(\alpha, \beta, \gamma, z) = (1-z)^{-\beta}\, {}_2F_1(\beta, \gamma - \alpha, \gamma, z/(z-1)) \tag{A2}$$

to rewrite

$$P_N(x) = \frac{A_N}{\sqrt{x}}\, (1 - Nx)^{(N^2+N-4)/2}\, [1 - (N-1)x]^{-(N-1)/2}\, {}_2F_1\left(\frac{N-1}{2}, \frac{N^2-4}{2}, \frac{N^2+N-2}{2}, \frac{1-Nx}{1-(N-1)x}\right) \tag{A3}$$

Now, in this form, it is easy to take the limit $x \to 0$. One gets, as $x \to 0$,

$$P_N(x) \to \frac{A_N}{\sqrt{x}}\, {}_2F_1\left(\frac{N-1}{2}, \frac{N^2-4}{2}, \frac{N^2+N-2}{2}, 1\right). \tag{A4}$$

Using further the following identity [15]

$$ {}_2F_1(\alpha,\beta,\gamma,1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} \tag{A5} $$

and the expression for A_N in Eq. (53) we get, as $x \to 0$

$$ P_N(x) \to \left[\frac{\sqrt{\pi}\,\Gamma(N)\,\Gamma(N^2/2)}{2^{N-1}\,\Gamma^2(N/2)\,\Gamma((N-1)/2)}\right] x^{-1/2}. \tag{A6} $$

APPENDIX B: ASYMPTOTIC BEHAVIOR OF $\mu_1(N)$ FOR LARGE N FOR THE REAL CASE

In this appendix we derive the asymptotic behavior for large N of $\mu_1(N)$ for the real case given in Eq. (59). We first use the following integral representation of the hypergeometric function [15]

$$ {}_2F_1(\alpha,\beta,\gamma,z) = \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 t^{\beta-1}\,(1-t)^{\gamma-\beta-1}\,(1-tz)^{-\alpha}\,dt, \tag{B1} $$

where $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the standard Beta function. Using this representation, we can express $\mu_1(N)$ in Eq. (59) as an integral

$$ \mu_1(N) = \frac{2^{4-N}\Gamma(N)}{N^2\Gamma^2(N/2)} \int_0^1 t^{1/2}\,(1-t)^{N/2}\,[1+(N-1)t]^{-3}\,dt, \tag{B2} $$

which is still exact for all N. Next we consider the integral above, rescale $t = x/N$ and then take the large N limit as follows,

$$ \begin{aligned} \int_0^1 t^{1/2}\,(1-t)^{N/2}\,[1+(N-1)t]^{-3}\,dt &= \frac{1}{N^{3/2}} \int_0^N x^{1/2}\,(1-x/N)^{N/2} \left[1+\frac{N-1}{N}x\right]^{-3} dx \\ &\approx \frac{1}{N^{3/2}} \int_0^\infty \frac{x^{1/2}e^{-x/2}}{(1+x)^3}\,dx. \end{aligned} \tag{B3} $$

Also, by Stirling's formula, $\Gamma(N) \approx \sqrt{2\pi}\,N^{N-1/2}\,e^{-N}$ for large N. Using these results in Eq. (B2) we get, to leading order for large N,

$$ \mu_1(N) \approx \frac{c}{N^3} \tag{B4} $$

where the prefactor c is given by the expression

$$ c = \frac{4\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \frac{x^{1/2}e^{-x/2}}{(1+x)^3}\,dx = 2\left[1-\sqrt{\frac{\pi e}{2}}\,\mathrm{erfc}(1/\sqrt{2})\right] = 0.688641\ldots \tag{B5} $$

where $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,du$ is the complementary error function.

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