# Exact reconstruction for cone-beam scanning along nonstandard spirals and other curves

Yangbo Ye[a,b*] , Shiying Zhao[b], Hengyong Yu[d] , and Ge Wang[a,b,c]
[a]Departments of Mathematics, [b]Department of Radiology, [c]Biochemical Engineering,
University of Iowa, Iowa City, Iowa 52242
[d]College of Communication Engineering,
Hangzhou Dianzi University, Hangzhou, Zhejiang 310018, China

## ABSTRACT

In this article we consider cone-beam CT projections along a nonstandard 3-D spiral with variable radius and variable pitch. Specifically, we generalize an exact image reconstruction formula by Zou and Pan (2004a) and (2004b) to the case of nonstandard spirals, by giving a new, analytic proof of the reconstruction formula. Our proof is independent of the shape of the spiral, as long as the object is contained in a region inside the spiral, where there is a PI line passing through any interior point. Our generalized reconstruction formula can also be applied to much more general situations, including cone-beam scanning along standard (Pack, *et al.* 2004) and nonstandard saddle curves, and any smooth curve from one endpoint of a line segment to the other endpoint, for image reconstruction of that line segment. In other words, our results can be regarded as a generalization of Orlov's classical papers (1975) to cone-beam scanning.

**Key words**: cone-beam CT, exact reconstruction, nonstandard spiral

## 1. INTRODUCTION

Cone-beam CT along nonstandard spirals is much more flexible than standard spiral CT in biomedical imaging applications. We anticipate major development along this direction for electron-beam CT/micro-CT (Wang and Ye, 2003), bolus-chasing CT angiography (Wang and Vannier, 2003), and other projects. Approximate reconstruction algorithms of Feldkamp-type for cone-beam scanning along nonstandard spirals were given in Wang, *et al.* (1991) and (1993). However, an exact counterpart in this general case has not been reported yet. An exact cone-beam reconstruction formula in the filtered backprojection format was proved for cone-beam scanning along standard spirals (Katsevich 2002, 2003, and 2004). A backprojected filtration counterpart of the Katsevich formula was proposed as well (Zou and Pan 2004a and (2004b). It is highly desirable to extend these important results to the general case of nonstandard spirals. Actually, the exact image reconstruction was recently studied in the case of cone-beam data along a spiral with variable pitch (Zou *et al.*, 2004c).

Let

$$\boldsymbol{\rho}(s) = (R(s)\cos s, R(s)\sin s, h(s)), \quad a \le s \le b, \tag{1.1}$$

be a nonstandard spiral with variable radius $R(s)$ and variable pitch $h(s)$. Assume that $R(s) > 0$ and $h'(s) > 0$ for any $s$, with $b - a > 4\pi$. Assume that there is a region $P$ inside the spiral such that for any point $\mathbf{r} = (x, y, z) \in P$, there is at least one PI line of the spiral passing through $\mathbf{r}$. Here a PI line is a line passing through two points $\boldsymbol{\rho}(s_1)$ and $\boldsymbol{\rho}(s_2)$ on the spiral, with $0 < s_2 - s_1 < 2\pi$ .

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* Yangbo Ye: yangbo-ye@uiowa.edu,
Shiying Zhao: shiying-zhao@uiowa.edu
Hengyong Yu: yuhengyong@hotmail.com
Ge Wang: ge-wang@uiowa.edu

Developments in X-Ray Tomography IV, edited by Ulrich Bonse, Proc. of SPIE Vol. 5535
(SPIE, Bellingham, WA, 2004) · 0277-786X/04/$15 · doi: 10.1117/12.559087

Consider an object function $f(\mathbf{r})$ whose support is contained in $P$, i.e., $f(\mathbf{r})=0$ for any $\mathbf{r}\notin P$. We will assume that $f(\mathbf{r})$ is continuous and smooth. For any unit vector $\boldsymbol{\beta}$, define the cone-beam projection of $f(\mathbf{r})$ from a point $\boldsymbol{\rho}(s)$ on the spiral by

$$D_f(\boldsymbol{\rho}(s),\boldsymbol{\beta})=\int_0^\infty f(\boldsymbol{\rho}(s)+t\boldsymbol{\beta})dt. \tag{1.2}$$

Note that this integral is actually taken over a finite interval, because the function $f(\mathbf{r})$ is compactly supported. When $\boldsymbol{\beta}$ points away from the support of $f(\mathbf{r})$, the integral vanishes. In the computation below, we will take the unit vector $\boldsymbol{\beta}$ as the one pointing to a point $\mathbf{r}\in\mathbf{R}^3$ from a point $\boldsymbol{\rho}(s)$ on the spiral:

$$\boldsymbol{\beta}(\mathbf{r},s)=\frac{\mathbf{r}-\boldsymbol{\rho}(s)}{|\mathbf{r}-\boldsymbol{\rho}(s)|}. \tag{1.3}$$

For a point $\mathbf{r}\in P$, denote the two end points of the PI line segment by $\boldsymbol{\rho}(s_1(\mathbf{r}))$ and $\boldsymbol{\rho}(s_2(\mathbf{r}))$, where $s_1=s_1(\mathbf{r})$ and $s_2=s_2(\mathbf{r})$ are the parameters of these two points on the spiral. We need to use a unit vector along the PI line:

$$\mathbf{e}_\pi(\mathbf{r})=\frac{\boldsymbol{\rho}(s_2(\mathbf{r}))-\boldsymbol{\rho}(s_1(\mathbf{r}))}{|\boldsymbol{\rho}(s_2(\mathbf{r}))-\boldsymbol{\rho}(s_1(\mathbf{r}))|}. \tag{1.4}$$

In Zou and Pan (2004a) and (2004b), a formula for exact image reconstruction for cone-beam CT along a standard spiral

$$\boldsymbol{\rho}(s)=(R\cos s, R\sin s, sh/(2\pi)),\quad a\le s\le b, \tag{1.5}$$

was proved. Here $R$ is a constant radius, and $h$ is a constant pitch. This standard spiral has a standard Tam-Danielsson window (Tam etc. (1998) and Danielsson etc. (1997)) on a detection plane. Zou and Pan's approach for the derivation of their formula ((3.4) below) uses geometry of the standard spiral and its Tam-Danielsson window, and its generalization to the general case of nonstandard spirals is not obvious.

In this article, we will not only give a new proof of the formula by Zou and Pan but also generalize it into a more general case. Our new proof will not be restricted to the standard geometry of the spiral and its associated Tam-Danielsson window, and rather supports a generalized version of (3.4) for exact image reconstruction with cone-beam scanning along nonstandard spirals with variable radii and pitches. Algorithm implementation and evaluation based on this generalization have been reported in Yu *et al.* (2004a) and (2004b).

As will be seen in the proof, our generalized reconstruction formula (3.4) will apply to a quite general class of scanning loci, including but not limited to nonstandard spirals defined by (1.1). Actually, as long as $\mathbf{r}$ is on a line segment and $\boldsymbol{\rho}(s)$ is a smooth curve running from one endpoint $\boldsymbol{\rho}(s_1(\mathbf{r}))$ of the line segment to the other endpoint $\boldsymbol{\rho}(s_2(\mathbf{r}))$, Eq. (3.4) would hold. This may be considered as a generalization of Orlov (1975) to cone-beam CT. In particular, our Theorem 3.1 can be applied to the cone-beam scanning along a saddle curve as in Pack *et al*. (2003), or more generally, to cone-beam scanning along a nonstandard saddle curve.

## 2. THE REGION OF PI LINES AND REGION OF UNIQUE PI LINES

Since the PI line concept is crucial in the Katsevich-type reconstruction, we continue making use of it for exact cone-beam reconstruction with nonstandard spiral loci. In this section, we analyze the existence and uniqueness of the PI lines in the nonstandard spiral case. Although the exact reconstruction does not necessarily depend on the uniqueness of the PI line, this uniqueness would minimize the redundancy in cone-beam data acquisition.

For a standard spiral (1.5) of a constant radius $R$ and pitch $h$, it was proved (Danielsson et al. 1997) that for any point $(x, y, z)$ inside the spiral, i.e., when $x^2 + y^2 < R^2$ and $a + 2\pi \le s \le b - 2\pi$, for $a$ and $b$ with $b - a > 4\pi$ as above, there is a unique PI line passing through $(x, y, z)$. When the spiral (1.1) is nonstandard, however, one cannot expect to have one and only one PI line passing through any given point inside the spiral in general. As pointed out in Ye et al. (2004a) and (2004b), one should seek two different kinds of regions inside the spiral (1.1). The region of PI lines as above is a region such that for any point in this region, there is one or more PI lines passing through it. The region of unique PI lines is a region such that passing through any point there is a unique PI line. We will denote the former by $P$ and the latter by $U$. Clearly $U \subset P$.

The region of PI lines can be determined for a given nonstandard spiral by the union of all PI line segments:

$$P = \bigcup_{0<s_2-s_1<2\pi} \left\{ t\boldsymbol{\rho}(s_1) + (1-t)\boldsymbol{\rho}(s_2) \,\middle|\, 0 < t < 1 \right\}.$$

Our new proof and generalization of the exact reconstruction formula are valid for any point $\mathbf{r} \in P$. To see the redundancy in cone-beam data acquisition, let us assume that there are two PI lines passing through a point $\mathbf{r} \in P$. Our generalized exact reconstruction formula (3.4) can be applied to the spiral arc from $\boldsymbol{\rho}(s_1)$ to $\boldsymbol{\rho}(s_3)$, as well as to the arc from $\boldsymbol{\rho}(s_2)$ to $\boldsymbol{\rho}(s_4)$. Consequently $f(\mathbf{r})$ will be computed twice. This redundancy has to be corrected by a weighted sum during the final image reconstruction.

To determine the region of unique PI lines, we observe that passing through a point there are more than one PI line if and only if there are four points $\boldsymbol{\rho}(s_j)$, $j = 1, \ldots, 4$, on the spiral with $s_1 < s_2 < s_3 < s_4$, $s_3 - s_1 < 2\pi$, and $s_4 - s_2 < 2\pi$, such that they are on the same plane. In fact, these four co-plane points are the end points of the PI lines segments on the spiral. One should exclude the degenerated case of three end points being on the same line, as the intersection point $\mathbf{r}$ of the two PI lines would then be on the spiral. This principle can be used to determine a region of unique PI lines for any given nonstandard spiral. Note that regions of unique PI lines have been determined when the spiral (1.1) has a variable pitch but a constant radius (Ye et al, 2004a), and when the spiral has a constant pitch but a variable radius (Ye et al, 2004b).

## 3. EXACT RECONSTRUCTION FORMULA

First let us define an integral kernel

$$K(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi i} \int_{\mathbf{R}^3} \operatorname{sgn}(\boldsymbol{v} \cdot \mathbf{e}_\pi(\mathbf{r})) e^{2\pi i \boldsymbol{v} \cdot (\mathbf{r} - \mathbf{r}')} d\boldsymbol{v}. \tag{3.1}$$

The integral in (3.1) diverges in the ordinary sense, and hence $K(\mathbf{r}, \mathbf{r}')$ is interpreted as a distribution. In this paper, we will avoid using distributions in our computation. Therefore the meaning of $K(\mathbf{r}, \mathbf{r}')$ is that it defines an integral transform from a function $g$ on $\mathbf{R}^3$ to

$$\int_{\mathbf{R}^3} K(\mathbf{r}, \mathbf{r}') g(\mathbf{r}') d\mathbf{r}' = \frac{1}{2\pi i} \int_{\mathbf{R}^3} \operatorname{sgn}(\boldsymbol{v} \cdot \mathbf{e}_\pi(\mathbf{r})) e^{2\pi i \boldsymbol{v} \cdot \mathbf{r}} d\boldsymbol{v} \int_{\mathbf{R}^3} g(\mathbf{r}') e^{-2\pi i \boldsymbol{v} \cdot \mathbf{r}'} d\mathbf{r}'. \tag{3.2}$$

In other words, the integral transform on the left side of (3.2) is by definition a twisted Fourier inverse transform of the Fourier transform of $g$. Note that the order of integration on the right side of (3.2) cannot be interchanged.

Given an object function $f(\mathbf{r})$ of compact support in $P$, let us consider a special function $g$ which is given by following expression

$$g(\mathbf{r}') = \int_{s_1(\mathbf{r})}^{s_2(\mathbf{r})} \frac{\partial}{\partial q}\left(D_f(\boldsymbol{\rho}(q), \boldsymbol{\beta}(\mathbf{r}', s)) - D_f(\boldsymbol{\rho}(q), -\boldsymbol{\beta}(\mathbf{r}', s))\right)\Big|_{q=s} \frac{ds}{|\mathbf{r}' - \boldsymbol{\rho}(s)|}. \tag{3.3}$$

Note that, because of the upper and lower limits $s_2(\mathbf{r})$ and $s_1(\mathbf{r})$, our function $g(\mathbf{r}')$ also depends on $\mathbf{r} \in P$. As pointed out by Zou and Pan (2004a) and (2004b), $g(\mathbf{r}')$ is the weighted cone-beam backprojection of the derivative of the filtered data for the point $\mathbf{r} \in P$. It is the subject matter of Zou and Pan (2004a) and (2004b) that one can recover $f(\mathbf{r})$ by applying the integral transform (3.2) to $g(\mathbf{r}')$ in (3.3), for cone-beam projections along a standard spiral (1.5). In other words, they formulated (3.4) below for the standard spiral (1.5). What we will contribute in this paper is to prove (3.4) for a nonstandard spiral (1.1), as well as other smooth curves.

**Theorem 3.1.** *Consider a nonstandard spiral (1.1) with a region $P$ of PI lines. Let $f(\mathbf{r})$ be a function of compact support in $P$, whose 5th partial derivatives are absolutely integrable in $\mathbf{R}^3$. Then*

$$f(\mathbf{r}) = \int_{\mathbf{R}^3} K(\mathbf{r}, \mathbf{r}') g(\mathbf{r}') d\mathbf{r}', \tag{3.4}$$

*where the integral transform is defined in (3.2), $g(\mathbf{r}')$ is given by (3.3), $D_f(\boldsymbol{\rho}(q), \boldsymbol{\beta}(\mathbf{r}', s))$ by (1.2), and $\boldsymbol{\beta}(\mathbf{r}, s)$ by (1.3).*

We remark that in Theorem 3.1 we may simply assume that $f(\mathbf{r})$ is a smooth function of compact support in $P$, and hence all its partial derivatives exist and are continuous. The theorem is formulated in the present form because we will need this stronger version of the theorem in a subsequent work on exact reconstruction for discontinuous object functions.

## 4. PROOF OF THEOREM 3.1

With (3.2), the right side of (3.4) is

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\mathbf{R}^3} \operatorname{sgn}(\boldsymbol{v}' \cdot \mathbf{e}_\pi(\mathbf{r})) e^{2\pi i \boldsymbol{v}' \cdot \mathbf{r}} d\boldsymbol{v}' \int_{\mathbf{R}^3} g(\mathbf{r}') e^{-2\pi i \boldsymbol{v}' \cdot \mathbf{r}'} d\mathbf{r}' \\ &= \frac{1}{2\pi i} \int_{\mathbf{R}^3} \operatorname{sgn}(\boldsymbol{v}' \cdot \mathbf{e}_\pi(\mathbf{r})) e^{2\pi i \boldsymbol{v}' \cdot \mathbf{r}} d\boldsymbol{v}' \int_{\mathbf{R}^3} e^{-2\pi i \boldsymbol{v}' \cdot \mathbf{r}'} d\mathbf{r}' \\ &\quad \times \int_{s_1(\mathbf{r})}^{s_2(\mathbf{r})} \frac{\partial}{\partial q}\left(D_f(\boldsymbol{\rho}(q), \boldsymbol{\beta}(\mathbf{r}', s)) - D_f(\boldsymbol{\rho}(q), -\boldsymbol{\beta}(\mathbf{r}', s))\right)\Big|_{q=s} \frac{ds}{|\mathbf{r}' - \boldsymbol{\rho}(s)|}. \end{aligned} \tag{4.1}$$

We will first compute the expression on the right side of (4.1) contributed by $D_f(\boldsymbol{\rho}(q), \boldsymbol{\beta}(\mathbf{r}', s))$. Computation of the expression contributed by $-D_f(\boldsymbol{\rho}(q), -\boldsymbol{\beta}(\mathbf{r}', s))$ is similar.

Denote by

$$F(\boldsymbol{v}) = \int_{\mathbf{R}^3} f(\mathbf{r}) e^{-2\pi i \boldsymbol{v} \cdot \mathbf{r}} d\mathbf{r} \tag{4.2}$$

the Fourier transform of $f(\mathbf{r})$. Then

$$\left(\frac{\partial}{\partial q} D_f(\boldsymbol{\rho}(q), \boldsymbol{\beta}(\mathbf{r}', s))\right)\Bigg|_{q=s} = \int_0^\infty \frac{\partial}{\partial q} f(\boldsymbol{\rho}(q) + t\boldsymbol{\beta}(\mathbf{r}', s))\Bigg|_{q=s} dt$$
$$= \int_0^\infty \frac{\partial}{\partial q} \int_{\mathbf{R}^3} F(\boldsymbol{\nu}) e^{2\pi i \boldsymbol{\nu}\cdot(\rho(q)+t\beta(\mathbf{r}',s))} d\boldsymbol{\nu} \Bigg|_{q=s} dt \tag{4.3}$$

by Fourier's inversion formula. We take the derivative under the inner integral to get

$$\left(\frac{\partial}{\partial q} D_f(\boldsymbol{\rho}(q), \boldsymbol{\beta}(\mathbf{r}', s))\right)\Bigg|_{q=s} = 2\pi i \int_0^\infty dt \int_{\mathbf{R}^3} \left(\boldsymbol{\nu} \cdot \frac{d\boldsymbol{\rho}(s)}{ds}\right) F(\boldsymbol{\nu}) e^{2\pi i \boldsymbol{\nu}\cdot(\rho(s)+t\beta(\mathbf{r}',s))} d\boldsymbol{\nu}. \tag{4.4}$$

As we assumed that the 5th partial derivatives of $f(\mathbf{r})$ are absolutely integrable in $\mathbf{R}^3$, its Fourier transform $F(\boldsymbol{\nu})$ is bounded by $O((1+|\boldsymbol{\nu}|)^{-5})$, by the Riemann-Lebesgue theorem. Consequently, the inner integral on the right side of (4.3) is dominated by $\int_{\mathbf{R}^3}(1+|\boldsymbol{\nu}|)^{-5} d\boldsymbol{\nu}$, while the inner integral on the right side of (4.4) is dominated by a convergent integral $\int_{\mathbf{R}^3}(1+|\boldsymbol{\nu}|)^{-4} d\boldsymbol{\nu}$. This proved that it is legitimate to interchange the order of differentiation and the inner integral in (4.3), and hence (4.4) is valid.

Substituting (4.4) to the right side of (4.1), the expression on the right side of (4.1) contributed by $D_f(\boldsymbol{\rho}(q), \boldsymbol{\beta}(\mathbf{r}', s))$ becomes

$$\int_{\mathbf{R}^3} \operatorname{sgn}(\boldsymbol{\nu}' \cdot \mathbf{e}_\pi(\mathbf{r})) e^{2\pi i \boldsymbol{\nu}'\cdot\mathbf{r}} d\boldsymbol{\nu}' \int_{\mathbf{R}^3} e^{-2\pi i \boldsymbol{\nu}'\cdot\mathbf{r}'} d\mathbf{r}'$$
$$\times \int_{s_1(\mathbf{r})}^{s_2(\mathbf{r})} ds \int_0^\infty dt' \int_{\mathbf{R}^3} \left(\boldsymbol{\nu} \cdot \frac{d\boldsymbol{\rho}(s)}{ds}\right) F(\boldsymbol{\nu}) e^{2\pi i \boldsymbol{\nu}\cdot(\rho(s)+t'(\mathbf{r}'-\rho(s)))} d\boldsymbol{\nu}, \tag{4.5}$$

where we changed variables to $t' = t/|\mathbf{r}' - \boldsymbol{\rho}(s)|$ using (1.3).

As $f(\mathbf{r})$ is of compact support, we can differentiate the right side of (4.2) under the integration. This way one can see that $F(\boldsymbol{\nu})$ is a smooth function, and its partial derivatives are also bounded by $O((1+|\boldsymbol{\nu}|)^{-5})$. Therefore, the function $(\boldsymbol{\nu} \cdot d\boldsymbol{\rho}(s)/ds)F(\boldsymbol{\nu})$ and its partial derivatives are smooth and bounded by $O((1+|\boldsymbol{\nu}|)^{-4})$. Using this fact, we may apply integration by parts many times to the innermost integral in (4.5) by differentiating $(\boldsymbol{\nu} \cdot d\boldsymbol{\rho}(s)/ds)F(\boldsymbol{\nu})$ and integrating the exponential function. Each application of integration by parts will yield a linear function of $t'$ in the denominator. This consequently proves that the innermost integral in (4.5) is a rapidly decreasing function of $t'$. Consequently we can legitimately interchange the integrals with respect to $s$ and $t'$ in (3.5). Since the innermost integral in (4.5) is dominated by $\int_{\mathbf{R}^3}(1+|\boldsymbol{\nu}|)^{-4} d\boldsymbol{\nu}$, we can further interchange the integrals with respect to $s$ and $\boldsymbol{\nu}$ in (4.5).

Now the innermost integral becomes

$$\int_{s_1(\mathbf{r})}^{s_2(\mathbf{r})} \left(\boldsymbol{\nu} \cdot \frac{d\boldsymbol{\rho}(s)}{ds}\right) e^{2\pi i(1-t')\boldsymbol{\nu}\cdot\boldsymbol{\rho}(s)} ds. \tag{4.6}$$

Changing variables from $s$ to $u = \boldsymbol{\nu} \cdot \boldsymbol{\rho}(s)$, we get

$$du = \left( \boldsymbol{v} \cdot \frac{d\boldsymbol{\rho}(s)}{ds} \right) ds ,$$

and (4.6) becomes

$$\int_{\boldsymbol{v}\cdot\boldsymbol{\rho}(s_1(\mathbf{r}))}^{\boldsymbol{v}\cdot\boldsymbol{\rho}(s_2(\mathbf{r}))} e^{2\pi i(1-t')u}\, du = \frac{e^{2\pi i(1-t')\boldsymbol{v}\cdot\boldsymbol{\rho}(s_2(\mathbf{r}))} - e^{2\pi i(1-t')\boldsymbol{v}\cdot\boldsymbol{\rho}(s_1(\mathbf{r}))}}{2\pi i(1-t')} . \tag{4.7}$$

Using (4.7) and (4.5), we now get the expression on the right side of (4.1) contributed by $D_f\left(\boldsymbol{\rho}(q), \boldsymbol{\beta}(\mathbf{r}', s)\right)$ as

$$\begin{aligned} &\int_{\mathbf{R}^3} \operatorname{sgn}(\boldsymbol{v}'\cdot\mathbf{e}_\pi(\mathbf{r})) e^{2\pi i\boldsymbol{v}'\cdot\mathbf{r}}\, d\boldsymbol{v}' \int_{\mathbf{R}^3} e^{-2\pi i\boldsymbol{v}'\cdot\mathbf{r}'}\, d\mathbf{r}' \\ &\times \int_{-\infty}^{1} \frac{dt}{2\pi i t} \int_{\mathbf{R}^3} F(\boldsymbol{v}) e^{2\pi i(1-t)\boldsymbol{v}\cdot\mathbf{r}'} \cdot \left( e^{2\pi i t\boldsymbol{v}\cdot\boldsymbol{\rho}(s_2(\mathbf{r}))} - e^{2\pi i t\boldsymbol{v}\cdot\boldsymbol{\rho}(s_1(\mathbf{r}))} \right) d\boldsymbol{v}, \end{aligned} \tag{4.8}$$

by setting $t = 1 - t'$.

From $D_f\left(\boldsymbol{\rho}(s), -\boldsymbol{\beta}\right) = \int_{-\infty}^{0} f\left(\boldsymbol{\rho}(s) + t\boldsymbol{\beta}\right) dt$, we can see that the contribution of $-D_f\left(\boldsymbol{\rho}(q), -\boldsymbol{\beta}(\mathbf{r}', s)\right)$ to the right side of (4.1) equals

$$\begin{aligned} &-\int_{\mathbf{R}^3} \operatorname{sgn}(\boldsymbol{v}'\cdot\mathbf{e}_\pi(\mathbf{r})) e^{2\pi i\boldsymbol{v}'\cdot\mathbf{r}}\, d\boldsymbol{v}' \int_{\mathbf{R}^3} e^{-2\pi i\boldsymbol{v}'\cdot\mathbf{r}'}\, d\mathbf{r}' \\ &\times \int_{1}^{\infty} \frac{dt}{2\pi i t} \int_{\mathbf{R}^3} F(\boldsymbol{v}) e^{2\pi i(1-t)\boldsymbol{v}\cdot\mathbf{r}'} \cdot \left( e^{2\pi i t\boldsymbol{v}\cdot\boldsymbol{\rho}(s_2(\mathbf{r}))} - e^{2\pi i t\boldsymbol{v}\cdot\boldsymbol{\rho}(s_1(\mathbf{r}))} \right) d\boldsymbol{v} \end{aligned} \tag{4.9}$$

Therefore the right side of (4.1) is now equal to the sum of (4.8) and (4.9):

$$\begin{aligned} &\int_{\mathbf{R}^3} \operatorname{sgn}(\boldsymbol{v}'\cdot\mathbf{e}_\pi(\mathbf{r})) e^{2\pi i\boldsymbol{v}'\cdot\mathbf{r}}\, d\boldsymbol{v}' \int_{\mathbf{R}^3} e^{-2\pi i\boldsymbol{v}'\cdot\mathbf{r}'}\, d\mathbf{r}' \\ &\times \int_{-\infty}^{\infty} \operatorname{sgn}(1-t) \frac{dt}{2\pi i t} \int_{\mathbf{R}^3} F(\boldsymbol{v}) e^{2\pi i(1-t)\boldsymbol{v}\cdot\mathbf{r}'} \cdot \left( e^{2\pi i t\boldsymbol{v}\cdot\boldsymbol{\rho}(s_2(\mathbf{r}))} - e^{2\pi i t\boldsymbol{v}\cdot\boldsymbol{\rho}(s_1(\mathbf{r}))} \right) d\boldsymbol{v}. \end{aligned} \tag{4.10}$$

By similar arguments as before, we can interchange order of integration in (4.10). This way we can change variables from $\mathbf{r}'$ to $\mathbf{s} = (1-t)\mathbf{r}'$ with $d\mathbf{r}' = d\mathbf{s}/|1-t|^3$, and then from $\boldsymbol{v}'$ to $\boldsymbol{\mu} = \boldsymbol{v}'/(1-t)$ with $d\boldsymbol{v}' = |1-t|^3\, d\boldsymbol{\mu}$ and $\operatorname{sgn}(\boldsymbol{v}'\cdot\mathbf{e}_\pi(\mathbf{r})) = \operatorname{sgn}(\boldsymbol{\mu}\cdot\mathbf{e}_\pi(\mathbf{r}))\operatorname{sgn}(1-t)$. Consequently, (4.10) becomes

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{dt}{2\pi i t} \int_{\mathbf{R}^3} \operatorname{sgn}(\boldsymbol{\mu}\cdot\mathbf{e}_\pi(\mathbf{r})) e^{2\pi i(1-t)\boldsymbol{\mu}\cdot\mathbf{r}}\, d\boldsymbol{\mu} \\ &\times \int_{\mathbf{R}^3} e^{-2\pi i\boldsymbol{\mu}\cdot\mathbf{s}}\, d\mathbf{s} \int_{\mathbf{R}^3} F(\boldsymbol{v}) e^{2\pi i\boldsymbol{v}\cdot\mathbf{s}} \cdot \left( e^{2\pi i t\boldsymbol{v}\cdot\boldsymbol{\rho}(s_2(\mathbf{r}))} - e^{2\pi i t\boldsymbol{v}\cdot\boldsymbol{\rho}(s_1(\mathbf{r}))} \right) d\boldsymbol{v} \\ &= \int_{-\infty}^{\infty} \frac{dt}{2\pi i t} \int_{\mathbf{R}^3} \operatorname{sgn}(\boldsymbol{\mu}\cdot\mathbf{e}_\pi(\mathbf{r})) e^{2\pi i(1-t)\boldsymbol{\mu}\cdot\mathbf{r}}\, d\boldsymbol{\mu} \\ &\times \int_{\mathbf{R}^3} e^{-2\pi i\boldsymbol{\mu}\cdot\mathbf{s}} \left( f(\mathbf{s} + t\boldsymbol{\rho}(s_2(\mathbf{r}))) - f(\mathbf{s} + t\boldsymbol{\rho}(s_1(\mathbf{r}))) \right) d\mathbf{s}, \end{aligned} \tag{4.11}$$

by the Fourier inversion formula. The innermost integral on the right side of (4.11) can be written as

$$\int_{\mathbf{R}^3} f(\mathbf{s}) \left( e^{-2\pi i\boldsymbol{\mu}\cdot(\mathbf{s} - t\boldsymbol{\rho}(s_2(\mathbf{r})))} - e^{-2\pi i\boldsymbol{\mu}\cdot(\mathbf{s} - t\boldsymbol{\rho}(s_1(\mathbf{r})))} \right) d\mathbf{s} = F(\boldsymbol{\mu}) \left( e^{2\pi i t\boldsymbol{\mu}\cdot\boldsymbol{\rho}(s_2(\mathbf{r}))} - e^{2\pi i t\boldsymbol{\mu}\cdot\boldsymbol{\rho}(s_1(\mathbf{r}))} \right).$$

Therefore (4.11) and hence the right side of (4.1) becomes

$$\int_{\mathbf{R}^3} F(\boldsymbol{\mu})\,\mathrm{sgn}(\boldsymbol{\mu}\cdot\mathbf{e}_\pi(\mathbf{r}))e^{2\pi i\boldsymbol{\mu}\cdot\mathbf{r}}\,d\boldsymbol{\mu}\int_{-\infty}^{\infty}\left(e^{2\pi it\boldsymbol{\mu}\cdot(\boldsymbol{\rho}(s_2(\mathbf{r}))-\mathbf{r})}-e^{2\pi it\boldsymbol{\mu}\cdot(\boldsymbol{\rho}(s_1(\mathbf{r}))-\mathbf{r})}\right)\frac{dt}{2\pi it}. \tag{4.12}$$

Now we need a classical integral formula to compute the inner integral in (4.12):

$$\int_{-\infty}^{\infty}\left(e^{2\pi ita}-e^{2\pi itb}\right)\frac{dt}{2\pi it}=1, \tag{4.13}$$

for $a>0>b$. Since $\mathbf{r}$ is on the PI line between $\boldsymbol{\rho}(s_2(\mathbf{r}))$ and $\boldsymbol{\rho}(s_1(\mathbf{r}))$, $\boldsymbol{\mu}\cdot(\boldsymbol{\rho}(s_2(\mathbf{r}))-\mathbf{r})$ and $\boldsymbol{\mu}\cdot(\boldsymbol{\rho}(s_1(\mathbf{r}))-\mathbf{r})$ are of opposite signs, while the sign of the former is indeed $\mathrm{sgn}(\boldsymbol{\mu}\cdot\mathbf{e}_\pi(\mathbf{r}))$. Using (4.13), (4.12) is immediately simplified to

$$\int_{\mathbf{R}^3} F(\boldsymbol{\mu})\,e^{2\pi i\boldsymbol{\mu}\cdot\mathbf{r}}\,d\boldsymbol{\mu}=f(\mathbf{r}),$$

which completes the proof of Theorem 3.1.

## 5. DISCUSSION AND CONCLUSION

Our analytic proof of the extended Katsevich-type reconstruction formula has not only confirmed the exactness of the backprojected-filtration methodology for cone-beam reconstruction from a standard cone-beam spiral scan but also revealed the feasibility of backprojected-filtration-based exact cone-beam reconstruction from a nonstandard cone-beam spiral scan. As argued by Zou and Pan (2004a) and (2004b), exact Katsevich-type reconstruction from the data in the Tam-Danielson window is doable using a backprojected filtration algorithm. According to our above findings, exact image reconstruction from the cone-beam data within the generalized Tam-Danielson window in the nonstandard spiral case should be equally doable. We are working along this direction to study the numerical stability and optimize our algorithms (Yu *et al.* 2004a and 2004b). Furthermore, in the case of other types of scanning loci such as saddle curves (Pack, *et al.* 2003), the reconstruction formula as given in Theorem 3.1 should hold as well, as long as the generalized Tam-Danielson window can be appropriately adapted to allow the minimum data are included. Relevant results will be reported in other publications.

In conclusion, we have studied exact cone-beam CT reconstruction from projections along a nonstandard 3D spiral with a variable radius and variable pitch by generalizing a Katsevich-type formula by Zou and Pan (2004a) and (2004b), from the standard spiral cone-beam scanning to the cone-beam scanning along nonstandard spirals, standard and nonstandard saddle curves, and virtually any smooth curves. Our proof has avoided explicit handling of the signum function, and is independent of the specific shape of the scanning locus under the condition that the object is contained in a region of PI lines, and the data are contained in an appropriately extended Tam-Danielson window.

## ACKNOWLEDGMENTS

This work was partially supported by a Carver Scientific Research Initiative grant and NIH/NIBIB grants (EB002667 and EB004287).

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