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Exact results for a two-dimensional one-component plasma on a sphere

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Résumé. — Pour étudier le modèle OCP du plasma classique à deux dimensions, nous proposons de placer les particules à la surface d'une sphère. Pour une valeur particulière de la température T_0 la fonction de partition ainsi que les fonctions de distribution peuvent être calculées explicitement. A cette température la limite thermodynamique de ce système coïncide avec celle du système plan.

Abstract. — In order to study the properties of the two-dimensional classical one-component plasma, we propose to confine the particles to the surface of a sphere. For a special value T_0 of the temperature, the partition and the distribution function can be calculated explicitly. The thermodynamic limit of this system coincides, for $T = T_0$, with the one obtained previously for a planar system.

In two recent articles Jancovici and Alastuey (JA) [1, 2] studied the equilibrium statistical mechanics of a classical two-dimensional one-component plasma (OCP) using methods of random matrices [3, 4]. For some special value T_0 of the temperature they calculated the free energy and the n-body distribution functions exactly.

In this letter we consider the case of a system of N particles confined to the surface of a sphere. We show that for the temperature T_0 this system has the same thermodynamic limit as the one considered by (JA).

The N particles of mass m and charge q are confined to the surface of a sphere centred in O and of radius R. The number density is

$$\rho = \frac{N}{4\,\pi R^2}\,. \tag{1}$$

The interaction potential between particles i and j located at M_i and M_j on the sphere is

$$v_{ij} = -\,q^2\,\text{Log}\left(\frac{r_{ij}}{L}\right) \tag{2}$$

where L is a length scale and r_{ij} is the length of the chord joining M_i and M_j i.e.

$$r_{ij} = 2\,R\sin\left(\frac{\psi_{ij}}{2}\right) \tag{3}$$

with

$$\psi_{ij} = \text{Arc}\cos\left(\frac{\mathbf{OM}_i\,.\,\mathbf{OM}_j}{R^2}\right). \tag{4}$$

The particles are embedded in a uniform neutralizing background of opposite charge.

The total potential energy, taking into account the influence of the background, is

$$V_N(1, 2, \ldots, N) = -\frac{q^2}{2}\sum_{i=1}^{N}\sum_{j>i}\text{Log}\frac{2\,R^2}{L^2}\times \\ \times(1-\cos\psi_{ij}) - \frac{N^2\,q^2}{4}\left(1-\text{Log}\frac{4\,R^2}{L^2}\right). \tag{5}$$

1. **Excess free energy.** — For the temperature $T_0 = q^2/2\,k_B$ (k_B Boltzmann constant) the excess canonical partition function is

$$Z_N(T_0) = e^{N^2/2}\cdot\left(\frac{L}{2}\right)^N\cdot R^N\int\prod_{i=1}^{N}d\Omega_i\,\times \\ \times\prod_{k=1}^{N}\prod_{l>k}\left(\frac{1-\cos\psi_{kl}}{2}\right) \tag{6}$$

where

$$d\Omega_i = \sin\theta_i\,d\theta_i\,d\varphi_i \tag{7}$$

(θ_i, φ_i spherical coordinates of M_i).

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Introducing the Cayley-Klein parameters defined by

$$\alpha_i = \cos \frac{\theta_i}{2} e^{i\varphi_i/2} \quad (8)$$

$$\beta_i = - i \sin \frac{\theta_i}{2} e^{-i\varphi_i/2} \quad (9)$$

we can write

$$1 - \cos \psi_{ij} = 2 \mid \alpha_i \beta_j - \alpha_j \beta_i \mid^2 . \quad (10)$$

The integrand of (6) now takes the form

$$\prod_{i=1}^{N} \prod_{j>i} \left(\frac{1 - \cos \psi_{ij}}{2} \right) = \prod_{k=1}^{N} \beta_k^{N-1} \prod_{i=1}^{N} \prod_{j>i} \left(\frac{\alpha_i}{\beta_i} - \frac{\alpha_j}{\beta_j} \right). \quad (11)$$

The second product in the r.h.s. of equation (11) is a Vandermonde [1] determinant. Expanding it and inserting into equation (6) we get

$$Z_N(T_0) = e^{N^2/2} (2 \pi L)^N R^N N ! \times \times \prod_{k=1}^{N} \frac{(k-1)!(N-k)!}{N!}. \quad (12)$$

This result has a structure similar to the expression derived by (JA) for a charged disk of radius R

$$Z_N(T_0) = K \pi^N N ! \prod_{k=1}^{N} \gamma(k, N) \quad (13)$$

where $\gamma(k, N)$ is the incomplete gamma function [2], and K is a constant.

The difference is that our calculation involves complete gamma functions.

The excess free energy for $T = T_0$ is readily derived from (12)

$$F_N(T_0) = - k_B T_0 \operatorname{Log} \frac{Z_N(T_0)}{(4 \pi R^2)^N}. \quad (14)$$

In the thermodynamic limit ($N \to \infty$, $R \to \infty$, ρ constant) one obtains

$$\lim_{N \to \infty} \frac{F_N(T_0)}{N} = q^2 \left[\frac{1}{2} - \frac{1}{4} \operatorname{Log} 2 \pi \right] - - \frac{q^2}{4} \operatorname{Log} (\rho \pi L^2) + O\left(\frac{\operatorname{Log} N}{N} \right) \quad (15)$$

which is identical to the result given in reference [1].

2. **Distribution functions.** — The n-particle distribution function for $T = T_0$ is given by

$$\rho_{N,T_0}^{(n)}(1, 2, \ldots, n) = \frac{N!}{(N-n)!} \frac{1}{Z_N(T_0)} \times \times \int \prod_{k=n+1}^{N} d\Omega_k \prod_{i=1}^{N} \prod_{j>i} \left(\frac{1 - \cos \psi_{ij}}{2} \right). \quad (16)$$

Using arguments parallel to those given in reference [2] one finds

$$\rho_{N,T_0}^{(n)}(1, 2, \ldots, n) = \rho^n \operatorname{Det} \{ [\alpha_k \alpha_l^* + \beta_k \beta_l^*]^{N-1} \} \quad (17)$$

with

$$1 \leqslant k \leqslant n$$
$$1 \leqslant l \leqslant n .$$

In particular

$$\rho_{N,T_0}^{(1)}(1) = \rho \quad (18)$$

$$\rho_{N,T_0}^{(2)}(1, 2) = \rho^2 g_{N,T_0}^{(2)}(1, 2) \quad (19)$$

with

$$g_{N,T_0}^{(2)}(1, 2) = 1 - \left(\frac{1 + \cos \psi_{1,2}}{2} \right)^{N-1} . \quad (20)$$

One can note that for $\psi_{12} = \pi$, $\rho_{N,T_0}(12) = 1$.

The system appears to be homogeneous for all N and the distribution functions are invariant by rotation of the sphere. Indeed, under a rotation R_0 characterized by its Cayley-Klein parameters, the coordinates (α_i, β_i) of particle i transform in the following way

$$\begin{pmatrix} \alpha_i' \\ \beta_i' \end{pmatrix} = \begin{pmatrix} \alpha_0 & \beta_0 \\ -\beta_0^* & \alpha_0^* \end{pmatrix} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} . \quad (21)$$

Using the fact that

$$\alpha_0 \alpha_0^* + \beta_0 \beta_0^* = 1 \quad (22a)$$

and

$$\alpha_i \alpha_i^* + \beta_i \beta_i^* = 1 \quad \text{for all } i \quad (22b)$$

we conclude that

$$\alpha_i' \alpha_j^{*\prime} + \beta_i' \beta_j^{*\prime} = \alpha_i \alpha_j^* + \beta_i \beta_j^* \quad (23)$$

and consequently $\rho_{N,T_0}^{(n)}(1, \ldots, n)$ is invariant under the rotation R_0. In particular expression (20) shows that $g_{N,T_0}^{(2)}(12)$ depends only on the relative distance between particles 1 and 2.

The thermodynamic limit of the functions $\rho_{N,T_0}^{(n)}(1, \ldots, n)$ is obtained in the following way : Define $\rho_k = R\theta_k$. The limit $N \to \infty$, $R \to \infty$, ρ constant, is taken by keeping ρ_k and φ_k constant for each particle k ($1 \leqslant k \leqslant n$). For an infinitely large sphere the particles will be situated in the tangent plane at the North Pole and there positions will be characterized by the polar coordinates (ρ_k, φ_k).

An expansion of (17) to second order in ρ_k/R gives

$$\rho_{T_0}^{(n)}(1, 2, \ldots, n) = \rho^n \exp\left[- \sum_{k=1}^{n} \mid Z_k \mid^2 \right] \times \times \operatorname{Det} [\exp(Z_k^* Z_l)] \quad (24)$$

where Z_k is defined by

$$Z_k = \sqrt{\pi\rho}\ \rho_k\ e^{i\varphi_k} . \tag{25}$$

The expression (24) coincides with the thermodynamic limit of the n-particle distribution function of a planar system.

3. **Conclusion.** — The thermodynamic limit of the system we have considered is identical to that of the planar two-dimensional OCP. It seems likely that this result remains valid for any temperature.

It is thus possible to advantageously simulate, by molecular dynamics or Monte-Carlo, a two-dimensional OCP by placing N-particles interacting *via* the potential (2) on the surface of a sphere. The system will be homogeneous, whatever the value of N, and the pair distribution function $g^{(2)}(r)$ can thus be obtained over a larger range of r values than would be possible, for a given value of N, from a simulation in a plane.

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