

Exact Results on Sinai's Diffusion

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Abstract: We study the continuum version of Sinai's problem of a random walker in a random force field in one dimension. A method of stochastic representations is used to represent various probability distributions in this problem (mean probability density function and first passage time distributions). This method reproduces already known rigorous results and also confirms directly some recent results derived using approximation schemes. We demonstrate clearly, in the Sinai scaling regime, that the disorder dominates the problem and that the thermal distributions tend to zero-one laws.

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1 Introduction

Since the introduction of the problem of a random walker in a random uncorrelated force field by Sinai [1] a great deal of activity has focused on this area. From the mathematical point of view, in one dimension, many results may be obtained, for example Kesten [4] and Golosov [5] were the first to derive the average value of the probability density function. Subsequently, results on the distribution of first passage times and consequently the probability distribution of the maximum displacement were obtained. The model is one of great interest as it is an analytically accessible test ground for the study of non equilibrium phenomena in a non mean field situation where activation processes play a clear role. Recently a real space renormalization group study has been carried out which leads to results on multitime properties of the process [8], demonstrating explicitly the phenomenon of aging in this model. Physically the model finds applications in the study of the random field Ising model (which at low temperature may be viewed as a system of annihilating random walkers). The logarithmic diffusion seen in the Sinai model serves as a paradigm for the dynamics of domain walls in droplet models for disordered spin systems [6]. A thorough discussion of the relevance of the Sinai model in condensed matter physics and a review of the early results obtained may be found in [11, 12].

In this paper we analyse the continuum version of the Sinai model, using techniques similar to those used in the theory of disordered quantum systems [14] and that have already been used in the study of the Sinai model [3, 10]. We rederive the results already known in a very compact fashion which in addition sheds more light on the nature of these solutions. We derive explicit representations for the distributions (and not just the disordered averaged values). We shall see for example that the first passage time distributions become zero-one laws, i.e. the thermal fluctuations become insignificant and everything is dominated by the disorder.

The version of the model we shall study is that of a diffusion X_t in a random force field. The equation of motion is the Langevin equation

$$\frac{dX_t}{dt} = \eta_t + \phi'(X_t), \quad (1)$$

where η is a thermal white noise in time, i.e. it is Gaussian of zero mean with correlation function $\langle \eta_t \eta_{t'} \rangle = 2T\delta(t - t')$. The force ϕ' is derived from a potential $\phi(x)$ which is a Brownian motion of zero mean such that $\overline{\phi'(x)\phi'(y)} = \sigma^2\delta(x - y)$. The result that the diffusion is logarithmic may be obtained through the following well known argument: The time to move from the origin to a point x is given by the Arrhenius law, i.e.

$$t \sim \exp((\phi(x) - \phi(0))/T) \sim \exp(\sigma\sqrt{x}/T), \quad (2)$$

by Brownian scaling, and hence $X(t) \sim \log^2(t)$. Exact results derived show that this, rather crude, argument in fact reflects very accurately the behaviour of the diffusion in the appropriate scaling regime. In this paper the logarithmic nature of the diffusion

is derived from first principles without recourse to the Arrhenius law or any scaling arguments.

2 Mathematical Formalism

We start with the model (1) of a particle moving in a quenched random white noise force field $\phi'(x)$ such that $\overline{\phi'(x)\phi'(y)} = 2\delta(x-y)$, here the overline indicates the average over the disorder (i.e. without loss of generality we have taken $\sigma = \sqrt{2}$). The particle is also subject to a random white noise in time such that the probability density function for a particle, started at the origin at time zero, to be at x at time t is given by the standard forward Fokker-Planck equation:

$$\frac{\partial}{\partial x} \left(\frac{\partial p}{\partial x} + \frac{d\phi}{dx} p \right) = \frac{\partial p}{\partial t}. \quad (3)$$

In what follows we shall find it useful to work with the Laplace transform of the density with respect to time which is given by

$$\frac{d}{dx} \left(\frac{dp}{dx} + \frac{d\phi}{dx} p \right) = Ep - \delta(x), \quad (4)$$

where E is the Laplace transform variable conjugate to t . The adjoint $\rho(x)$ of the probability density $p(x)$ (the probability to be at 0 given that the particle starts at x) is related to $p(x)$ via detailed balance

$$p(x) = \rho(x) \exp(-\phi(x) + \phi(0)) \quad (5)$$

and can be shown to satisfy the backward equation

$$\frac{d^2 \rho}{dx^2} - \frac{d\phi}{dx} \frac{d\rho}{dx} = E\rho - \delta(x). \quad (6)$$

We now proceed by making essentially a Hopf-Cole transformation of the problem

$$\begin{aligned} \rho(x) &= A \exp \left(- \int_0^x z_s^+ ds \right) \text{ for } x \geq 0 \\ \rho(x) &= A \exp \left(- \int_0^{-x} z_s^- ds \right) \text{ for } x \leq 0, \end{aligned} \quad (7)$$

where it is easy to see that

$$\frac{dz^\pm(s)}{ds} = -E + z^{\pm 2}(s) + \frac{d\phi(\pm s)}{ds} z^\pm(s) \quad s \geq 0. \quad (8)$$

The jump at $x = 0$ fixes the value of A to be

$$A = \frac{1}{z_0^+ + z_0^-}. \quad (9)$$

Hence we have found an explicit representation for ρ (and hence p) in terms of the two stochastic processes $z^\pm(s)$:

$$\rho(x) = \frac{1}{z_0^+ + z_0^-} \exp\left(-\int_0^x z_s^+ ds\right) \text{ for } x \geq 0 \quad (10)$$

$$\rho(x) = \frac{1}{z_0^+ + z_0^-} \exp\left(-\int_0^{-x} z_s^- ds\right) \text{ for } x \leq 0 \quad (11)$$

In general one requires boundary conditions for z^\pm , here we imagine that the diffusion is restricted to a finite segment of the real line $[-L, L]$ with reflecting boundaries at $x = \pm L$, hence

$$\frac{d\rho}{dx}|_{\pm L} = 0 \Rightarrow z_L^+ = z_L^- = 0. \quad (12)$$

On using the equation of detailed balance (5) and the integrated form of (8), one may show

$$\begin{aligned} p(x) &= \frac{z_0^+}{z_x^+(z_0^+ + z_0^-)} \exp\left(-\int_0^x \frac{E}{z_s^+} ds\right) \text{ for } x \geq 0 \\ p(x) &= \frac{z_0^-}{z_{-x}^-(z_0^+ + z_0^-)} \exp\left(-\int_0^{-x} \frac{E}{z_s^-} ds\right) \text{ for } x \leq 0. \end{aligned} \quad (13)$$

A useful property of p is that we may write

$$p(x) = -\frac{1}{E} \frac{z_0^\pm}{(z_0^+ + z_0^-)} \frac{d}{dx} \exp\left(-\int_0^{\pm x} \frac{E}{z_s^\pm} ds\right), \quad (14)$$

using this relation it is straightforward to explicitly verify that

$$\int_{-\infty}^{\infty} p(x) dx = \frac{1}{E}, \quad (15)$$

i.e. the conservation of probability. The observant reader will notice that the stochastic processes z^\pm have the pathology that they have no normalisable equilibrium measure and in addition that they may become infinite (finite time explosion). Let us remark here that, for probabilistic reasons, z must remain positive as $\rho(x)$ must decrease monotonically on increasing the distance from the origin. In fact the natural process to work with is $w_s^\pm = z_{L-s}^\pm$, which obeys the time reversed (with respect to z) stochastic differential equation

$$\frac{dw}{ds} = E - w^2 + \frac{d\phi}{ds} w. \quad (16)$$

Using the Stratonovich prescription the generator for the diffusion w is

$$\mathcal{G}_w = \frac{\partial}{\partial w} w \frac{\partial}{\partial w} w - (w^2 - E) \frac{\partial}{\partial w}. \quad (17)$$

The equilibrium distribution for w therefore satisfies

$$\mathcal{G}_w^\dagger P_0(w) = 0, \quad (18)$$

and is hence given by

$$P_0(w) = \frac{1}{K_0(2E^{\frac{1}{2}})} \frac{1}{w} \exp\left(-w - \frac{E}{w}\right), \quad (19)$$

where $K_0(x)$ is a Bessel function of the third kind [15]. It may be convenient, depending on the situation, to work with the alternative variable

$$w = E^{\frac{1}{2}} \exp(\gamma), \quad (20)$$

the fact that we are using the Stratonovich prescription means that we may apply the chain rule to the stochastic differential equation (16) and hence find:

$$\frac{d\gamma}{ds} = -2E^{\frac{1}{2}} \sinh(\gamma) - \frac{d\phi}{ds} \quad (21)$$

Slightly abusing notation, the equilibrium distribution for γ is given by:

$$P_0(\gamma) = \frac{1}{K_0(2E^{\frac{1}{2}})} \exp(-2E^{\frac{1}{2}} \cosh(\gamma)). \quad (22)$$

The boundary conditions induced on w are:

$$\left. \frac{d\rho}{dx} \right|_{\pm L} = 0 \rightarrow z_L^\pm = w_0^\pm = 0 \quad (23)$$

For example, for $x \geq 0$, ρ is given in terms of w as

$$\rho(x) = \frac{1}{w_L^+ + w_L^-} \exp\left(-\int_{L-x}^L w_s^+ ds\right). \quad (24)$$

In the limit $L \rightarrow \infty$ the boundary conditions at $\pm L$ should become unimportant. The following subtle argument demonstrates this fact. On sending L to infinity (whilst keeping x finite) the distribution of w_L is given by the equilibrium measure (19) (i.e. w_L is in equilibrium because L corresponds to a large *time* by which the process w is in equilibrium). In addition, as $L \rightarrow \infty$ the distribution of w_{L-x} is *also* given by the equilibrium distribution. Hence treating $L-x$ as the new origin for the process w^+ , L as a new origin for w^- , and using the independence of w^+ and w^- we may write the statistically identical expression:

$$\rho(x) = \frac{1}{w_x^+ + w_0^-} \exp\left(-\int_0^x w_s^+ ds\right), \quad (25)$$

where it is to be understood that w_0^\pm are in equilibrium.

3 Probability Density Functions

With the formalism elaborated in the previous section it is now immediate to calculate the probability density at the origin. This result has been calculated using the same technique as here in [3, 10] and we include it here for the sake of completeness as its derivation is now immediate. The same result may also be found in [13] where it is calculated using the replica method from disordered systems.

$$\overline{\rho(0)} = \int_0^\infty \frac{P_0(w)P_0(w')}{w+w'} dw dw', \quad (26)$$

and we start by expressing the denominator in the integral (26) as the integral over an exponential, i.e.

$$\overline{\rho(0)} = \frac{1}{E^{\frac{1}{2}} K_0(2E^{\frac{1}{2}})^2} \int_0^\infty \exp\left(-t(x+y) - E^{\frac{1}{2}}\left(x + \frac{1}{x}\right) - E^{\frac{1}{2}}\left(y + \frac{1}{y}\right)\right) \frac{dx dy}{xy} dt. \quad (27)$$

The integrals over x and y may now be performed yielding (in terms of Bessel functions of the third kind [15])

$$\overline{\rho(0)} = \frac{1}{E^{\frac{1}{2}} K_0(2E^{\frac{1}{2}})^2} \int_{2E^{\frac{1}{2}}}^\infty K_0^2(x) x dx. \quad (28)$$

We now use the differential identity [15]

$$\frac{d}{dx} \left(\frac{x^2}{2} (K_0^2 - K_1^2) \right) = x K_0^2, \quad (29)$$

thus yielding the result:

$$\overline{\rho(0)} = \frac{4K_1(2E^{\frac{1}{2}})^2}{K_0(2E^{\frac{1}{2}})^2} - 4. \quad (30)$$

Using the asymptotic expansion for Bessel functions of the third kind [15] we find that for small E (corresponding to large time)

$$\overline{\rho(0)} \sim \frac{1}{E \log^2(E)}, \quad (31)$$

and hence using the Tauberian theorems for regularly varying functions we obtain the result

$$\overline{p(0, 0, t)} \sim \frac{1}{\log^2(t)}. \quad (32)$$

To examine the full (x, t) dependence we note that

$$\overline{\rho(x)} = \int \frac{P_0(\gamma_0^+)P_0(\gamma_0^-)}{E^{\frac{1}{2}} \exp(\gamma^+) + E^{\frac{1}{2}} \exp(\gamma_0^-)} G(\gamma_0^+, \gamma^+, x) d\gamma_0^+ d\gamma^+ d\gamma_0^-, \quad (33)$$

where we have used the Feynman-Kac Formula and G is the Greens function obeying

$$\frac{\partial G}{\partial x} = \frac{\partial^2 G}{\partial \gamma_0^2} - 2E^{\frac{1}{2}} \sinh(\gamma_0) \frac{\partial G}{\partial \gamma_0} - E^{\frac{1}{2}} \exp(\gamma_0) G, \quad (34)$$

subject to the initial condition

$$G(\gamma_0, \gamma, 0) = \delta(\gamma - \gamma_0). \quad (35)$$

In general the analysis of this Greens function is difficult and few analytic results are known. However if we restrict ourselves to the regime of Sinai scaling i.e we write $x = X \log^2(E)$ and $\gamma = -u \log(E)$, where u and X are of order one, we find a considerable simplification occurs which permits a rather thorough asymptotic analysis of this regime; this is the key remark that enables us to extend the method of [3] to obtain new results. In these new variables, the equation (34) becomes:

$$\frac{\partial G}{\partial X} = \frac{\partial^2 G}{\partial u_0^2} + (E^{\frac{1}{2}-u_0} - E^{\frac{1}{2}+u_0}) \log(E) \frac{\partial G}{\partial u_0} - \log^2(E) E^{\frac{1}{2}-u_0} G. \quad (36)$$

In the region where $u_0 > \frac{1}{2}$ we find the equation is dominated by the potential and drift terms i.e.

$$\frac{\partial G}{\partial X} = \log(E) E^{\frac{1}{2}-u_0} \frac{\partial G}{\partial u_0} - \log^2(E) E^{\frac{1}{2}-u_0} G, \quad (37)$$

which has the general solution

$$G(u_0, X) = \frac{E^{u_0-\frac{1}{2}}}{\log(E)} f\left(X + \frac{E^{u_0-\frac{1}{2}}}{\log^2(E)}\right), \quad (38)$$

and hence $G(u_0, X) \sim 0$ for $u_0 > \frac{1}{2}$. In the region $u_0 < -\frac{1}{2}$, the equation is dominated by the drift term, i.e.

$$\frac{\partial G}{\partial X} = -E^{\frac{1}{2}+u_0} \log(E) \frac{\partial G}{\partial u_0}, \quad (39)$$

the general solution to this equation being $G(u_0, X) = f\left(X + \frac{1}{E^{\frac{1}{2}+u_0} \log^2(E)}\right)$ hence $G(u_0, X) \sim f(X)$, i.e. it becomes independent of u_0 . In the interval $[-\frac{1}{2}, \frac{1}{2}]$ the equation is dominated by the diffusive term, i.e.

$$\frac{\partial G}{\partial X} = \frac{\partial^2 G}{\partial u_0^2}. \quad (40)$$

If we apply detailed balance in (33) i.e. use

$$P_0(\gamma_0^+)G(\gamma_0^+, \gamma^+; x) = P_0(\gamma^+)G(\gamma^+, \gamma_0^+; x) \quad (41)$$

then we find

$$\overline{\rho(x)} = \int \frac{P_0(\gamma^+)P_0(\gamma_0^-)}{E^{\frac{1}{2}} \exp(\gamma^+) + E^{\frac{1}{2}} \exp(\gamma_0^-)} F(\gamma^+, x) d\gamma^+ d\gamma_0^- \quad (42)$$

where

$$F(\gamma^+, x) = \int_{-\infty}^{\infty} G(\gamma^+, \gamma_0^+, X) d\gamma_0^+. \quad (43)$$

By the Feynman-Kac formula $F(\gamma, x)$ obeys

$$\frac{\partial F}{\partial x} = \frac{\partial^2 F}{\partial \gamma^2} - 2E^{\frac{1}{2}} \sinh(\gamma) \frac{\partial F}{\partial \gamma} - E^{\frac{1}{2}} \exp(\gamma) F \quad (44)$$

with induced initial condition $F(\gamma, 0) = 1$. Returning to the variable w we finally obtain:

$$\overline{\rho(x)} = \frac{1}{K_0(2E^{\frac{1}{2}})^2} \int_0^{\infty} \frac{\exp(-w - w' - \frac{E}{w} - \frac{E}{w'})}{ww'(w + w')} F(\log(\frac{w}{E^{\frac{1}{2}}}), x) dw dw'. \quad (45)$$

We now exploit the fact that E is small

$$\begin{aligned} \overline{\rho(x)} &= \frac{1}{EK_0(2E^{\frac{1}{2}})^2} \int_0^{\infty} \frac{\exp(-Ez - Ez' - \frac{1}{z} - \frac{1}{z'})}{zz'(z + z')} F(\log(zE^{\frac{1}{2}}), x) dz dz' \\ &\sim \frac{1}{EK_0(2E^{\frac{1}{2}})^2} \int_0^{\infty} \frac{\exp(-\frac{1}{z} - \frac{1}{z'})}{zz'(z + z')} F(-\infty, x) dz dz' \\ &= \frac{F(-\infty, x)}{EK_0(2E^{\frac{1}{2}})^2} \int_0^{\infty} \exp(-u - u' - \alpha(u + u')) du du' d\alpha \\ &= \frac{F(-\infty, x)}{EK_0(2E^{\frac{1}{2}})^2}. \end{aligned} \quad (46)$$

From our previous analysis we find that for $u \in [-\frac{1}{2}, \frac{1}{2}]$

$$\frac{\partial F}{\partial X} = \frac{\partial^2 F}{\partial u^2}. \quad (47)$$

For $u_0 < -\frac{1}{2}$, F is constant and continuity at $u = -\frac{1}{2}$ implies that $\frac{\partial F}{\partial u}|_{u=-\frac{1}{2}} = 0$; hence $F(-\infty, x) = F(-\frac{1}{2}, x)$. Continuity at $u = \frac{1}{2}$ yields $F(\frac{1}{2}, x) = 0$. The equation (47) for F in $[-\frac{1}{2}, \frac{1}{2}]$ may be easily solved in terms of Fourier modes to yield

$$F(u, X) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left(\left(\frac{2n+1}{2}\right)\left(\frac{1}{2} - u\right)\pi\right) \exp\left(-\frac{(2n+1)^2}{4}\pi^2 X\right). \quad (48)$$

Note that if one had a Brownian motion in $[-\frac{1}{2}, \frac{1}{2}]$ started at a position u with an absorbing boundary at $u = \frac{1}{2}$ and a reflecting boundary at $u = -\frac{1}{2}$, then $F(u, X)$ is simply the probability that the particle has not been absorbed before time X , indeed the function F appears directly in the solution of Kesten [4] with this very interpretation [16].

In the Laplace transformed variables one finds that

$$\overline{p(x, E)} \sim \frac{4}{\pi E \log^2(E)} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \exp\left(-\frac{(2n+1)^2}{4} \pi^2 \frac{x}{\log^2(E)}\right). \quad (49)$$

Putting in the value of F in equation (46) and asymptotically inverting the Laplace transform using the Tauberian theorems we obtain the result of Kesten [4]:

$$\overline{p(x, t)} \sim \frac{4}{\pi \log^2(t)} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \exp\left(-\frac{(2n+1)^2}{4} \pi^2 \frac{x}{\log^2(t)}\right) \quad (50)$$

4 First Passage Time Distributions

In this section we investigate the distribution of first passage times for the Sinai diffusion, these results of course give information of the distribution of the maximum of the process.

Let us define $P(x, t)$ to be the probability starting at x that the diffusion reaches the origin before time t . It is convenient to work with P evaluated at a random exponential time T_E of rate E , i.e.

$$P(x, E) = \int_0^{\infty} P(x, t) E \exp(-Et) dt = \langle \exp(-ET(x)) \rangle, \quad (51)$$

where the angled brackets indicate the thermal average (i.e. the average over the white noise η) and $T(x) = \inf\{t : X_t = 0 \mid X_0 = x\}$, i.e. the first passage time to 0 starting from x . A recurrence equation may now be found for $P(x, E)$ as follows, starting the process at x evolve the particle for a time dt , then

$$P(x, E) = \langle (1 - E dt) P(x + dX_t, E) \rangle. \quad (52)$$

Here the first term is simply the probability that the time T_E does not occur during the first step and the second term is P evaluated at the new position $x + dX_t$. Expanding $P(x + dX_t, E)$ to order dt and averaging over dX_t , one obtains the first passage time equation

$$\frac{d^2 P}{dx^2} - \frac{d\phi}{dx} \frac{dP}{dx} = EP \quad (53)$$

Using the boundary condition $P(0, E) = 1$ and making the Hopf-Cole transformation of the previous section we find:

$$P(x, E) = \exp\left(-\int_0^x z_s ds\right). \quad (54)$$

where z_s obeys the stochastic differential equation (8). Re-expressing the above in terms of the process w we find

$$P(x, E) = \exp\left(-\int_{L-x}^L w_s ds\right). \quad (55)$$

Consider $x = L$, i.e. we consider the time taken to cross the sample with reflecting initial conditions at the starting point (hence $z_L = w_0 = 0$). If we consider average value of the q^{th} power of $P(L, E)$ i.e. $\overline{P^q(L, E)}$ for fixed L and E , using the Feynman-Kac formula one finds that $\overline{P^q(L, E)} = F_q(L, E)$ where

$$\frac{\partial F_q}{\partial x} = \frac{\partial^2 F_q}{\partial \gamma_0^2} - 2E^{\frac{1}{2}} \sinh(\gamma_0) \frac{\partial F_q}{\partial \gamma_0} - qE^{\frac{1}{2}} \exp(\gamma_0) F_q. \quad (56)$$

It is now important to note that for $q > 0$, in the Sinai scaling regime, the prefactor q does not alter the solution for F_q , hence one has that $\overline{P^q(L, E)} = \overline{P(L, E)^q}$, that is to say that P tends to either zero or one independently of the thermal noise i.e. the distribution of $P(L, E)$ for fixed L and E , \mathcal{P} tends to the form $\mathcal{P}(P) = \alpha\delta(P - 1) + (1 - \alpha)\delta(P)$. Using the calculation of the previous section for F we find therefore, in the regime of large times and in the Sinai scaling regime where $L/\log^2(E) \sim O(1)$, that

$$\alpha = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \exp\left(-\frac{(2n+1)^2}{4} \pi^2 \frac{L}{\log^2(E)}\right). \quad (57)$$

One may also consider the case where $L \rightarrow \infty$, in this limit we see that

$$P(x, E) = \exp\left(-\int_0^x w_s ds\right). \quad (58)$$

but where the boundary condition on w_0 is that it is in equilibrium with the measure (19), here we therefore find

$$\overline{P^q(x, E)} = \overline{P(x, E)^q} = \int_{-\infty}^{\infty} P_0(\gamma_0) F(\gamma_0, \frac{x}{\log^2(E)}). \quad (59)$$

In the Sinai scaling regime $x/\log^2(E) \sim O(1)$ we may therefore write:

$$\overline{P(x, E)} = \frac{1}{K_0(2E^{\frac{1}{2}})} \int_{-\infty}^{\infty} \exp(-E^{\frac{1}{2}-u_0} - E^{\frac{1}{2}+u_0}) F(u_0) |\log(E)| du_0, \quad (60)$$

which becomes, in the limit of small E ,

$$\overline{P(x, E)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(u_0) du_0 = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \exp\left(-\frac{(2n+1)^2}{4} \frac{\pi^2 x}{\log^2(E)}\right). \quad (61)$$

Here it is interesting to note that one may calculate explicitly $\overline{\log(P(x, E))}$ which is given as

$$\overline{\log(P(x, E))} = -x \int_0^\infty P_0(w) w dw , \quad (62)$$

which in the limit of small E (or large time), but not dependent on being in the Sinai scaling regime, yields

$$\overline{\log(P(x, E))} \sim -\frac{x}{\log(E)}, \quad (63)$$

hence, whereas the scaling of the integer moment of $P(x, E)$ was as $\frac{x}{\log^2(E)}$, the scaling of the average of the logarithm is as $\frac{x}{\log(E)}$, suggesting that in a typical sample the scaling should be as $\frac{x}{\log(E)}$. If we consider the case where x is infinitesimally close to the origin we see that the probability to return to the origin (simply developing the exponential in equation (58) to first order in x) is

$$\overline{P(x, E)} = 1 - \frac{x}{\log(E)}, \quad (64)$$

and hence following the definition of [8, 9] the probability of not arriving at the origin behaves as $p_{pr}(L) \sim L^{-\theta}$ where L is the length scale $L = \log^2(E)$ and θ is therefore the persistence exponent for the diffusion and is given by $\theta = \frac{1}{2}$, thus confirming the results of [8, 9].

5 The Mean First Passage Time

In the Sinai model we recall that there are two kinds of average, a thermal average over the realisations of the temporal white noise and also the average over the realisations of the disorder. There has been much interest in the distribution of the mean first passage time (MFPT). Various authors have remarked upon the multifractality of its distribution, in this section we present an extremely simple argument to demonstrate this fact. Consider the case where there is a reflecting boundary at L , using the formalism of the previous sections or using the formula for MFPTs in [17] we have

$$\frac{d^2}{dx^2} \langle T_L(x) \rangle - \frac{d\phi}{dx} \frac{d}{dx} \langle T_L(x) \rangle = -1, \quad (65)$$

where the subscript L indicates the reflecting boundary conditions at L and $T_L(x)$ is the first time the particle arrives at 0 given that it starts at x (the angled brackets indicating the thermal average). The solution to this equation for a particle starting at L is

$$\langle T_L(L) \rangle = \int_0^L dx \exp(\phi(x)) \int_x^L dy \exp(-\phi(y)). \quad (66)$$

The moments averaged over the disorder are

$$\begin{aligned}
\overline{\langle T_L(L) \rangle^q} &= \int_{\{x_n < y_n\}} \prod_1^q dx_n dy_n \exp \left(\sum_{ij} (\min(x_i, x_j) - \min(y_i, y_j) - 2 \min(x_i, y_j)) \right) \\
&= \int_{\{x_n < y_n\}} \prod_1^q dx_n dy_n \exp \left(-\frac{1}{2} \sum_{ij} (|x_i - x_j| + |y_i - y_j| - 2|x_i - y_j|) \right). \quad (67)
\end{aligned}$$

Hence we see that the moment $\overline{\langle T_L(x) \rangle^q}$ is the partition function for a one dimensional anti-Coulomb gas (in the sense that like charges attract and opposite charges repel), where each particle has its antiparticle which is conditioned to be on its right. Rescaling the spatial variables we obtain

$$\overline{\langle T_L(L) \rangle^q} = L^{2n} \int_{\{x_n < y_n\}} \prod_1^q dx_n dy_n \exp \left(-\frac{L}{2} \sum_{ij} (|x_i - x_j| + |y_i - y_j| - 2|x_i - y_j|) \right), \quad (68)$$

where all the integrations are now in $[0, 1]$. When L becomes large this corresponds to the limit of low temperature and the partition function is dominated by the lowest energy state which is clearly when $x_n = 0$ and $y_n = 1$ for all n , i.e. all the like charges condense together to form two macrocharges which repel and hence lie on the two boundary points of the unit interval. This energy is simply $q^2 L$. Hence

$$\log \left(\overline{\langle T_L(L) \rangle^q} \right) \sim q^2 L, \quad (69)$$

clearly demonstrating the multifractality. The distribution is infact approximately log-normal.

6 Probabilistic Interpretation of Results

Here we shall interpret the results derived in this paper in a probabilistic fashion which gives us some intuition for the structure of the Sinai problem. In terms of stochastic processes we shall consider the process $u_x = -\gamma_x / \log(E)$; if we define the rescaled time variable $x = X \log^2(E)$ then the stochastic equation obeyed by u is

$$\frac{du}{dx} = \frac{1}{\log^2(E)} \frac{du}{dX} = -\frac{1}{\log(E)} \left(E^{\frac{1}{2}+u} - E^{\frac{1}{2}-u} - \frac{1}{\log^2(E)} \frac{d}{dX} \phi(X \log^2(E)) \right). \quad (70)$$

Using Brownian scaling we may write $\phi(X \log^2(E)) \equiv \phi(X) |\log(E)|$ and hence the induced stochastic differential equation on u as a function of the rescaled time X is

$$\frac{du}{dX} = -\log(E) (E^{\frac{1}{2}+u} - E^{\frac{1}{2}-u}) + \frac{d}{dX} \phi(X) \quad (71)$$

In the region outside the interval $[-\frac{1}{2}, \frac{1}{2}]$ the process is dominated by a drift which returns the particle to the interval. In the interval $[-\frac{1}{2}, \frac{1}{2}]$ the process is simply a Brownian motion. If we now consider the functional critical to our study of this problem

$$H(X) = \exp\left(-\int_0^x w_s ds\right) \equiv \exp\left(-\log^2(E) \int_0^X E^{\frac{1}{2}-u_s} ds\right), \quad (72)$$

we see that as $E \rightarrow 0$

$$\begin{aligned} H(X) &\rightarrow 1 \text{ if } \max(u_s) < \frac{1}{2} \\ &\rightarrow 0 \text{ if } \max(u_s) > \frac{1}{2}, \end{aligned} \quad (73)$$

thus clarifying at an intuitive level the zero-one type results obtained in the previous sections.

7 Conclusions and Perspectives

We have shown that the method presented in [3, 10] (and similar to those used in the Quantum Mechanics of one dimensional disordered systems [14]) may be extended to produce explicit results on spatio-temporal probability distribution functions in the continuum version of the Sinai model; notably the result of Kesten for the averaged probability density function and also for various first passage time probabilities. The method demonstrates the preponderance of zero-one type laws where the disorder dominates. The continuum formulation of the problem also allows a very simple demonstration of the multi-fractal nature of the mean first passage time.

The great advantage of the technique is that it allows one to represent transition probability densities as functionals of a well defined stochastic process and therefore quantities depending on several times may in principle be analysed, thus permitting the study of aging in these systems. Work on these extensions is currently under progress [18].

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