Exact semiclassical expansions for one-dimensional quantum oscillators

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A set of rules is given for dealing with WKB expansions in the one-dimensional analytic case, whereby such expansions are not considered as approximations but as exact encodings of wave functions, thus allowing for analytic continuation with respect to whichever parameters the potential function depends on, with an exact control of small exponential effects. These rules, which include also the case when there are double turning points, are illustrated on various examples, and applied to the study of bound state or resonance spectra. In the case of simple oscillators, it is thus shown that the Rayleigh–Schrödinger series is Borel resummable, yielding the exact energy levels. In the case of the symmetrical anharmonic oscillator, one gets a simple and rigorous justification of the Zinn-Justin quantization condition, and of its solution in terms of ''multi-instanton expansions.'' © *1997 American Institute of Physics.* [S0022-2488(97)02911-3]

I. INTRODUCTION

The time-independent one dimensional Schrödinger equation,

$$-\hbar^2 \frac{d^2\phi}{dq^2} + (V(q) - E)\phi = 0, \tag{0}$$

has always been a reservoir of useful models in quantum physics. In most such models the potential function V is analytic (e.g., a polynomial), enticing physicists into making analytic continuations in the complex q plane, a technique they are especially fond of. This is especially true for problems pertaining to semi-classical asymptotics, where analytic continuation of WKB expansions allows one to travel between ''classically allowed'' regions where the wave has an oscillatory behaviour [$E > V(q)$, so that the classical momentum $p = (E - V(q))^{1/2}$ is real] and ''classically forbidden'' regions ($E < V(q)$) where tunnelling takes place. But it is well known that analytic continuation of such divergent expansions as (1.1) should not be performed carelessly, because of Stokes phenomena.

Stokes phenomena have been much written about, and remain a controversial subject (cf. for instance the very lucid comments of Dingle in Ref. 1). One source of miscomprehension is the fact that most physicists work with WKB expansions as they would work with true functions (or almost so), whereas most modern mathematicians insist on considering them as asymptotic expansions in the sense of Poincaré, i.e., broad equivalence classes of functions (modulo all fastly decreasing functions of $\hbar$, for instance). The latter viewpoint makes it difficult to keep track of small exponential effects, i.e. tunnelling. The former leaves place to doubts as to the validity of operations to be performed on formal power series: consider for instance how Bender and Wu[2] guessed the singularity structure of the energy levels of the anharmonic oscillator, in the complex plane of the coupling constant; after presenting a nice zeroth-order semiclassical argument, they felt compelled to pursue in a completely different way, writing the following:

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> We did not use the above argument on grounds of rigor. It is not clear what is meant by an ''approximate zeroth-order analytic continuation.''

In this paper we provide tools for making such arguments rigorous, allowing for analytic continuations with respect to the variable q and whichever parameters equation (0) depends on, with an exact control of small exponentials to all orders. The underlying mathematics is an elaboration of ideas of Balian and Bloch,[3–5] pushed further in the one dimensional case by Voros,[6,7] and set on a firm mathematical basis using Ecalle's theory of resurgent functions.[8–12] Assuming V to be an arbitrary (possibly complex) polynomial function (in fact, most of what we shall say would still hold true when V is an entire function ''sufficiently well behaved at infinity''), it can be shown that WKB expansions, if ''well normalized'' as explained hereafter, are resurgent functions of the scale parameter $1/\hbar$. What this means precisely is explained in the introductory section of Ref. 12. For our present purpose all the reader needs to understand is the following practical implication of that statement: well normalized solutions of the Schrödinger equation can be exactly encoded by certain linear combinations of WKB expansions (which we call WKB symbols), thanks to resummation procedures of divergent series which generalize the well known Borel resummation procedure. But in contradistinction to the Borel case we do not have one resummation operation but two such, the so-called right and left resummation operations, which generally differ by small exponential terms (the case when they coincide is the Borel resummable case). A given wave function (i.e., a solution of the Schrödinger equation) can thus, if suitably normalized, be encoded by two different WKB symbols, its right (resp., left) symbol, from which the function is recovered by right (resp., left) resummation. These two symbols differ only by smaller exponential terms, and the former can be recovered from the latter by a formal operation which we call the Stokes automorphism, and which we denote by $\mathfrak{G}$. Analytic continuation of WKB symbols is possible along all paths of the complex q plane which avoid the so-called turning points, where $V(q)=E$. But our encoding (whether right or left) of wave functions by WKB symbols is discontinuous across special lines called Stokes lines. Stokes lines divide the complex q plane into simply connected regions called Stokes regions, and the way the right (resp., left) symbol of a wave function changes from one Stokes region R to another one R' is given by what we call the right (resp., left) connection isomorphism $\mathscr{C}^{+}_{R'R}$ (resp., $\mathscr{C}^{-}_{R'R}$). Decomposing the spaces of WKB symbols into their direct summands corresponding to the two possible determinations of the momentum, we can write $\mathscr{C}^{\pm}_{R'R}$ as a 2×2 matrix of operators, whose entries will be called the ''connection operators'' from R to R': more precisely the connection operator from (R,p) to (R',p') will be the entry corresponding to the initial determination p and the final determination p' of the momentum.

The encoding of a wave function by its right or left symbol has the awkward feature of not preserving reality properties (the right and left symbol of a real wave function are complex conjugate to each other, so that except in the Borel resummable case none of them is real). When one is keen on keeping track of reality properties it is therefore convenient to replace right or left resummation by Ecalle's median resummation, kind of a ''geometrical mean'' between the two. Explicit computations of median symbols will be presented in this article, along with the corresponding computations of right and left symbols.

The rules for computing the Stokes automorphism and the connection isomorphisms are quite simple to state, without knowing anything of resurgence theory: we shall present them as kind of a ''do it yourself'' kit, which the reader can use in a great variety of situations.

In section II we deal with generic values of the energy, for which all turning points are simple: in that case we know from Ref. 11 that the connection isomorphisms are given by combinations of analytic continuations along suitable paths of the complex q plane, which we shall describe explicitly by simple pictograms.

In section III our pictographic rules are extended (using the results of Ref. 12) to the case when there are double turning points. The corresponding WKB expansions now involve special prefactors (which are not just power series of $\hbar$), and in section IV we show how to deal with these prefactors in the computations.

In the concluding section V we deal with the problem of solving the quantization condition for bound states (or resonances) with respect to the energy parameter: from the results of the previous sections we can thus obtain exact (resurgent) expressions for the energy levels, yielding rigorous justifications of such results as the Zinn–Justin ''multi-instanton expansions'' (see Refs. 13, 14 for instance), or the Bender and Wu complex branch point structure of the energy levels of the symmetrical quartic oscillator (cf. Refs. 2, 15–17). We study the latter problem in detail in Ref. 18.

One should emphasize that, in the spirit of Ecalle's resurgence theory,[19–23] a lot of interesting results can be obtained without ever computing explicitly the WKB expansions we are speaking about: they can be treated as implicit objects, the main interest of which lies in the small exponentials they generate through the ''resurgence'' process (which is completely described by our pictograms).

On the other hand our methods can also be used by computation lovers: examples of such formal computations are given in sections IV and V. The question immediately arises of how to deduce numerical information from these purely formal computations. The best answer we know is the ''hyper-asymptotic'' procedure of Berry *et al.*,[24–26] which gives wonderfully efficient ways of extracting very precise numerical information from divergent series, knowing the resurgence properties of these series.

A. Basic formal ingredients of complex WKB calculus

Following Voros,[6] we shall denote by $\dot{\mathbf{C}}$ the punctured complex q-plane (with the turning points deleted) and by $\dot{\mathbf{C}}_2$ its two-fold covering [i.e., the Riemann surface of $p(q)=(E-V(q))^{1/2}$]. Locally on that covering, complex WKB expansions read[27] as

$$\varphi(q)=(\varphi_0(q)+\varphi_1(q)\hbar+\varphi_2(q)\hbar^2+\cdots)e^{(i/\hbar)S(q)}, \tag{1.1}$$

where S (the complexified action function) is a primitive of p (i.e., $dS/dq=p$), and the expansion in front of the exponential is a formal power series in $\hbar$. The fact that (1.1) should be a formal solution of the Schrödinger equation characterizes this series up to an arbitrary normalization factor (an invertible formal power series in $\hbar$, with constant coefficients). A possible choice of normalization is

$$\varphi(q)=P(q,\hbar^2)^{-1/2}e^{(i/\hbar)\int_{q_0}^{q}P(q',\hbar^2)dq'} \tag*{$(1.2)_{q_0}$}$$

(Ref. 28) where the formal power series,

$$P(q,\hbar^2)=p(q)+p_1(q)\hbar^2+p_2(q)\hbar^4+\cdots ,$$

is defined as the even part (in $\hbar$) of the solution of the Riccati equation,

$$Y^2-i\hbar Y'=E-V(q)$$

[this Riccati equation is deduced from equation (0) by the change of unknown function $\phi=\exp(i/\hbar\int Y)$. Formally solving it, and separating even and odd parts (in $\hbar$), one checks that $Y_{\text{odd}}=i\hbar/2(Y'_{\text{even}}/Y_{\text{even}})$].

Such expansions will be called well normalized at q_0 ($q_0\in\dot{\mathbf{C}}_2$); of course they are multivalued analytic on $\dot{\mathbf{C}}_2$, because the integral in the exponential depends on the homotopy class of the integration path. For our purposes it will be more convenient to work with slightly different normalization conventions, which read as

$$\varphi(q)=P(q,\hbar^2)^{-1/2}e^{(i/\hbar)\int_\infty^q(P(q',\hbar^2)-p(q'))dq'}e^{(i/\hbar)S(q)}, \tag{1.2$_\infty$}$$

where S is as in (1.1) (we have used the fact that $P-p$ is integrable at infinity); such expansions will be called well normalized at infinity (here again, of course, the precise meaning of this expression depends on the homotopy class of the integration path).

Whichever way WKB expansions are normalized, their multivaluedness can be described in the following way. Let γ be a cycle, i.e. a path in $\dot{\mathbf{C}}_2$ starting from some point q and getting back to the same point q with the same determination of the momentum p. Then it immediately follows from the above formulae that analytic continuation along γ transforms φ into $sgn(\gamma)a^\gamma\varphi$, where a^γ is the Voros multiplier of the cycle γ, defined by

$$a^\gamma=e^{(i/\hbar)\int_\gamma P(q,\hbar^2)dq}, \tag{1.3}$$

and $sgn(\gamma)=\pm 1$ is the ''signature'' of γ, given by $sgn(\gamma)=(-1)^{n(\gamma)/2}$, where $n(\gamma)$ is the ''index'' of γ, i.e. the number of times it encircles the turning points (counting them with their multiplicities, when they are not simple). Notice that $n(\gamma)$ is an even integer, because of our requirement that γ should not change the determination of the momentum.

Recalling that $P=p+O(\hbar^2)$, we can rewrite the Voros multiplier as

$$a^\gamma=a_\gamma e^{(i/\hbar)\omega_\gamma}, \tag{1.4}$$

with

$$\omega_\gamma=\int_\gamma p\,dq \tag{1.5}$$

and

$$a_\gamma=e^{i/\hbar\int_\gamma(P(q,\hbar^2)-p(q))dq}=1+O(\hbar)$$

(a formal series of integral powers of $\hbar$).

Of course a^γ depends only on the homology class of the cycle γ in the complex hyperelliptic curve $p^2=E-V(q)$. Furthermore

$$a^{\gamma+\gamma'}=a^\gamma a^{\gamma'}.$$

B. Wave interpretation of Stokes phenomena

It should be emphasized that in resurgence theory the meaning of such terms as Stokes lines, Stokes regions,... etc., is the same as in Dingle[1] (and as in Stokes' original article!): Stokes lines are not the places where two exponentials (corresponding to two opposite determinations of p) ''exchange their dominance,'' but the places where one of them is ''maximally dominant'' over the other (this is the natural point of view when WKB expansions are understood not as mere ''asymptotic expansions'' in the Poincaré sense, but as exact encodings of true functions through resummation procedures).

Locally in $\dot{\mathbf{C}}$ (the complex q plane minus the turning points) the space of WKB symbols splits into two subspaces, depending on which determination is chosen for the square root $p=(E-V(q))^{1/2}$ in formula (1.1).

Stokes lines can be characterized as those places in $\dot{\mathbf{C}}$ where the Stokes automorphism $\mathfrak{G}$ is not diagonal with respect to this splitting, so that speaking of ''a wave function with momentum p'' would be meaningless (do we mean a wave function with only components of momentum p in its right symbol, or in its left symbol?). This is why careful distinction should be made, in reading what follows, between those notions which concern symbols (i.e., formal objects: e.g., the notions of ''dominant'' and ''recessive'' symbols in Sec. I B 2) and those which concern wave functions (i.e., true solutions of the Schrödinger equation: e.g., the notion of an ''L-decaying'' wave function in Sec. I B 2).

1. Canonical decomposition of a wave inside a Stokes region

Inside a Stokes region the Stokes automorphism $\mathfrak{S}$ is diagonal with respect to the above mentioned direct sum decomposition into opposite determinations of the momentum: if the left symbol of a wave in some Stokes region belongs to the subspace with given p, the same is true for its right symbol and conversely. It therefore follows that inside a Stokes region R the property of ''having a given determination of the momentum'' is not just a property of formal WKB expansions, it also has a meaning for true solutions of the Schrödinger equation. As an important example, let q belong to a classically allowed segment of the real axis (we consider here the ''real case,'' where V is real valued and E is real). Such a segment cannot coincide with a Stokes line. Let us restrict it so that it lies inside a Stokes region. By the above decomposition any wave function in such a segment can be written as a sum of two waves: one with only positive p terms in its symbol, which we can interpret as a wave propagating right; one with only negative p terms in its symbol, which we can interpret as a wave propagating left.

Scattering through or above a potential barrier can be analyzed by comparing this decomposition far right and far left along the real axis: let ϕ be a wave function which for q real $\ll 0$ propagates left in the above sense (this property defines it unambiguously, up to normalization); let $\phi=\phi_+ + \phi_-$ be for q real $\gg 0$ its decomposition into components of positive and negative p: for q real $\gg 0$ we can interpret ϕ_- as the incident wave and ϕ_+ as the reflected wave whereas ϕ, for q real $\ll 0$, is the transmitted wave.

2. Comparing decompositions across an unbounded Stokes line

In the real case the classically forbidden segments of the real axis do not lie inside Stokes regions: they are examples of Stokes lines.

Comparing the canonical decompositions (Sec. I B 1) in two Stokes regions R, R' separated by a Stokes line is especially interesting (and easier) along unbounded Stokes lines, which connect infinity to a turning point. Along such a Stokes line L the two possible determinations of the momentum p correspond, respectively, to waves which ''fade into the turning point'' (resp., ''fade away at infinity''): by this we mean that moving along L towards the turning point (resp., away from it) is the fastest way for the exponential $\exp(i/\hbar\int p dq)$ to decrease. WKB symbols with the corresponding determination of the momentum will be called dominant along L (resp., recessive along L); we denote by WKB^L (resp., WKB_L) the space of WKB symbols near L which are dominant (resp., recessive) along L.

Although the Stokes automorphism $\mathfrak{S}$ is not diagonal with respect to the decomposition $\mathrm{WKB}=\mathrm{WKB}_L \oplus \mathrm{WKB}^L$, the subspace WKB_L of recessive symbols turns out to be stable under $\mathfrak{S}$. In other words, the property for a wave of having a recessive symbol does not depend on whether the right or left symbol is considered. Such a wave will be called an L-decaying wave. In the two-dimensional vector space of solutions of the Schrödinger equation, L-decaying waves are the one-dimensional subspace of those solutions which decay exponentially at infinity along the Stokes line L. Such a solution can be normalized so that its symbol is Borel resummable, for every q on L (this is the case for WKB expansions which are ''well normalized at infinity'' along L, in the sense of Sec. I A).

Reflection against an infinite wall is a case where the present analysis applies near one end of the real axis (the classically forbidden one, which is a unbounded Stokes line L_-) and analysis (Sec. I B 1) near the other end (the classically allowed one): by computing the connection operator from WKB_{L_-} to either one of the two WKB components on the classically allowed end, we get the symbols of the incident and reflected wave, as compared to that of the (decaying) transmitted wave.

3. Search for bound states and resonances

Searching for bound states in a confining potential V (where both ends of the real axis are classically forbidden) amounts to finding those values of the energy E for which $\mathscr{W}_{-\infty}^{\mathrm{dec}}$, the subspace of wave functions decaying at $-\infty$, coincides with $\mathscr{W}_{+\infty}^{\mathrm{dec}}$, the subspace of wave functions decaying at $+\infty$.

Similarly, searching for resonances in a scattering problem of the kind considered in Sec. I B 1, or in a reflection problem of the kind considered in Sec. I B 2, amounts to finding those (complex) values of the energy for which the reflected component of the wave will vanish. Assuming that the incident wave comes from the right, this amounts to finding those values of the energy E for which $\mathscr{W}_{-\infty}$ coincides with $\mathscr{W}_{+\infty}^{-}$ (the subspace of waves with negative momentum near $+\infty$), where we have denoted by $\mathscr{W}_{-\infty}$ the following subspace:

$$\mathscr{W}_{-\infty}=\begin{cases}\mathscr{W}_{-\infty}^{\mathrm{dec}}, & \text{if } -\infty \text{ is classically forbidden,}\\ \mathscr{W}_{-\infty}^{-}, & \text{if } -\infty \text{ is classically allowed.}\end{cases}$$

Setting in similar fashion,

$$\mathscr{W}_{+\infty}=\begin{cases}\mathscr{W}_{+\infty}^{\mathrm{dec}}, & \text{if } +\infty \text{ is classically forbidden,}\\ \mathscr{W}_{+\infty}^{-}, & \text{if } +\infty \text{ is classically allowed,}\end{cases}$$

we thus see that the bound state or resonances energies can be defined in every case as those values of the energy for which $\mathscr{W}_{-\infty}=\mathscr{W}_{+\infty}$. This amounts of course to the vanishing of the obvious operator,

$$\mathscr{J}:\mathscr{W}_{-\infty}\rightarrow\mathscr{W}/\mathscr{W}_{+\infty},$$

which we shall call the Jost operator. To express this condition in a less abstract way, let us choose some basis $(\phi,\phi^\star)$ of the two dimensional vector space $\mathscr{W}$, depending analytically on E in the interval under concern, such that for every E in that interval,

(1) $\phi\in\mathscr{W}_{-\infty}$,
(2) $\phi^\star\notin\mathscr{W}_{+\infty}$ (so that $\phi^\star$ generates $\mathscr{W}$ mod. $\mathscr{W}_{+\infty}$).

Such a basis will be called a Jost basis. Then one can write

$$\phi=J(E)\phi^\star \mod \mathscr{W}_{+\infty},$$

where $J(E)$ is an analytic function of the energy, which we call the Jost function. The values of E we are looking for are just the zeros of this function.

II. SOLVING THE CONNECTION PROBLEM FOR GENERIC ENERGIES

We shall assume here that E is a non critical value of the energy, so that all turning points are simple.

For any choice R,R' of Stokes regions we shall give an explicit ''pictographic'' description of the connection isomorphisms $\mathscr{C}_{R'R}^{+}$ and $\mathscr{C}_{R'R}^{-}$ as finite sums of analytic continuation operators along suitable paths of $\mathbf{C}$.

In Subsection II A we will deal with the ''simple pattern'' case where all Stokes lines are unbounded. Then $\mathscr{C}_{R'R}^{+}=\mathscr{C}_{R'R}^{-}$ $(=:\mathscr{C}_{R'R})$.

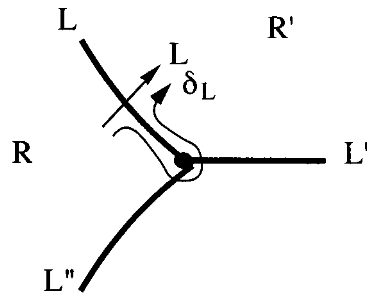


FIG. 1. Elementary connection operator across L.

In Subsection II B we will reduce the general case to the simple pattern case (a different reduction for $\mathscr{C}^{+}_{R'R}$ and $\mathscr{C}^{-}_{R'R}$).

In both cases we shall give explicit computations for explicit examples.

A. Connection isomorphisms for simple patterns

Each turning point is tied to three Stokes lines, and the ''simple pattern'' hypothesis means that these Stokes lines come from infinity.

In this case, as shown in Ref. 12, the property of being Borel resummable in some Stokes region R is preserved by analytic continuation to another Stokes region R'. Accordingly, the right and left connection isomorphisms are equal and we denote them simply by $\mathscr{C}_{R'R}$.

1. The elementary connection isomorphisms $\mathscr{C}_L$ and $\mathscr{C}_L^{-1}$

Let us consider a Stokes line L, the Stokes region R on the right side of L, and R' on its left side as shown in Fig. 1 (our convention is to orient unbounded Stokes lines toward the turning point). We shall call $\mathscr{C}_L := \mathscr{C}_{R'R}$ the elementary connection isomorphism across L.

With the convention that $\mathrm{WKB}^L(R)$ [resp., $\mathrm{WKB}_L(R)$] is the summand of $\mathrm{WKB}(R)$ consisting of all symbols which are dominant (resp., recessive) on L, $\mathscr{C}_L$ is given by its restrictions:

$$\mathscr{C}_L|\mathrm{WKB}^L(R) = L + \delta_L \quad \text{and} \quad \mathscr{C}_L|\mathrm{WKB}_L(R) = L,$$

where L means analytic continuation across L, and

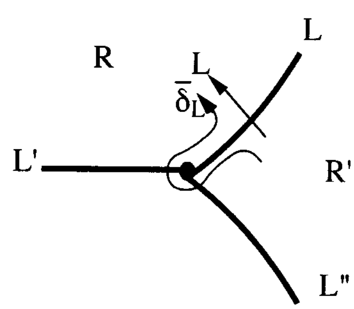


FIG. 2. Inverse elementary connection operator across L.

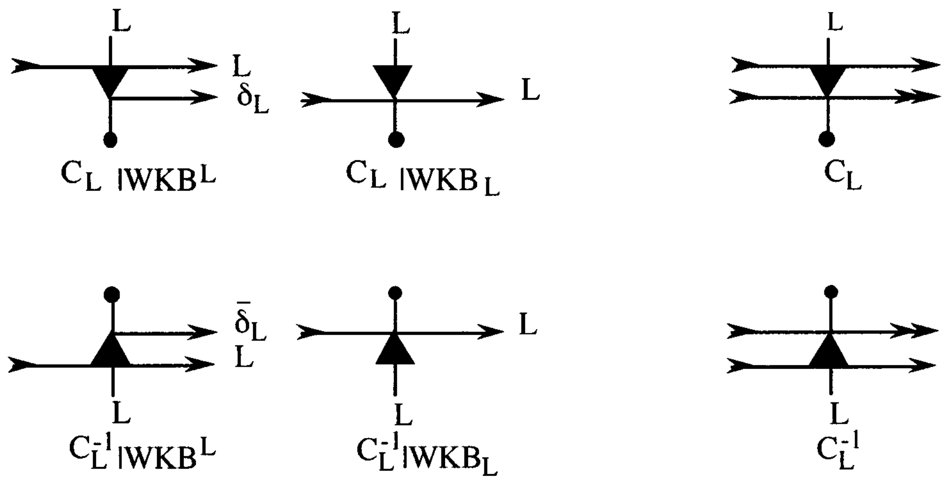


FIG. 3. Pictograms for elementary connection operators.

$$\delta_L:\mathrm{WKB}^L(R)\rightarrow\mathrm{WKB}_L(R')$$

is the so-called elementary connection operator across L, which we shall now compute.

Proposition II.1.1: The elementary connection operator δ_L is the analytic continuation operator around L from R to R' as shown in Fig. 1.

Proof (the idea of this proposition comes from Voros[6]): Just compare, for ψ in $\mathrm{WKB}^L(R)$, the two expressions of $\mathscr{C}_{R'R}\psi$ obtained first by crossing L clockwise and secondly by crossing L'' then L' anticlockwise (for the details, cf. Ref. 11; See also Refs. 29, 30). □

Consequence: In the same way one can describe $\mathscr{C}_L^{-1}:=\mathscr{C}_{RR'}$ as follows:

$$\mathscr{C}_L^{-1}{}_{|\mathrm{WKB}^L(R')}=L+\overline{\delta_L},\quad \mathscr{C}_L^{-1}{}_{|\mathrm{WKB}_L(R')}=L,$$

where $\overline{\delta_L}$ is the analytic continuation operator around L from R' to R as shown in Fig. 2.

Pictograms of $\mathscr{C}_L$ and $\mathscr{C}_L^{-1}$: It will be useful to describe the elementary connection isomorphisms $\mathscr{C}_L$ and $\mathscr{C}_L^{-1}$ by simple pictograms as shown on Fig. 3: the arrows on the base level of a triangle are carrying WKB^L symbols while those on the summit carry WKB_L ones.

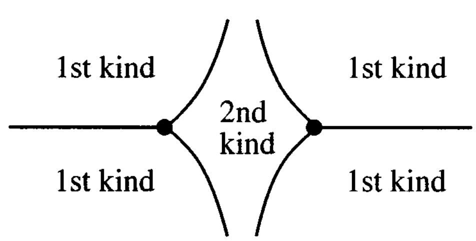


FIG. 4. The two kinds of Stokes regions.

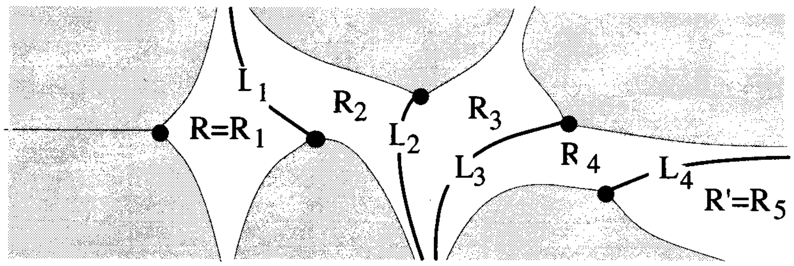


FIG. 5. The canonical sequence of Stokes regions and Stokes lines.

The diagrams on the right of the picture can be read in the following way: following a horizontal arrow without changing level means analytic continuation across the Stokes line involved (L operator). Climbing down (resp., up) a triangle is allowed only from base to summit and represents an operator δ_L (resp., $\overline{\delta_L}$); it means analytic continuation around the Stokes line involved. The number of arrows on each segment helps us to remember how many elementary WKB symbols are carried by this segment.

2. The connection isomorphism $\mathscr{C}_{R'R}$

For any two Stokes regions R and R' the connection isomorphism $\mathscr{C}_{R'R}$ can be computed as the product of successive elementary connection isomorphisms along any chain of adjacent Stokes regions. Our aim here is to exhibit such a chain which is canonical.

Global topological properties:

(1) In the simple case we are interested in here, each turning point is tied to three unbounded Stokes lines coming from infinity.

(2) One can distinguish only two kinds of Stokes regions as shown in Fig. 4.

The first property is a simple remark. The second one is due to the following result, whose proof is given in Ref. 31 (see also Ref. 32):

Lemma II.1.1: The action function S maps conformally each Stokes region either onto a half-plane (first kind) or onto a strip (second kind).

Vocabulary: Two Stokes regions will be called adjacent when they have a common Stokes line in their boundary; conversely, two Stokes lines will be called successive when they bound a common Stokes region.

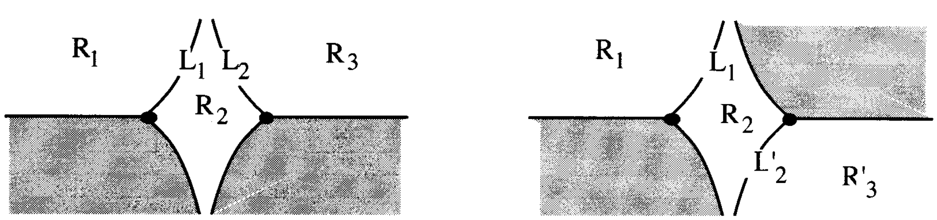


FIG. 6. Two canonical sequences for the harmonic oscillator.

FIG. 7. Pictograms for the connection isomorphisms of Fig. 6.

First elementary property of successive Stokes lines: For two successive Stokes lines L and L' bounding a common Stokes region R of the first kind one has $\mathrm{WKB}^L(R)=\mathrm{WKB}_{L'}(R)$ or, equivalently $\mathrm{WKB}_L(R)=\mathrm{WKB}^{L'}(R)$.

A canonical way of connecting two given Stokes regions R and R′: The three Stokes lines tied to a turning point divide $\mathbf{C}$ into three connected components, among which one or two contains neither R nor R'. After shading all such components for every turning point, we are left with a finite sequence $R_1=R,_2,\ldots,R_{n+1}=R'$ of successively adjacent Stokes regions separated by a finite sequence $L_1,\ L_2,\ldots,\ L_n$ of successive Stokes lines, as illustrated by Fig. 5.*Remark:* All Stokes regions R_i between R and R' in the above sequence are of the second kind.

The canonical sequence thus yields a canonical way to go from R to R'.

Proposition II.1.2: The canonical way is the only way to go from R to R' across Stokes lines all tied to different turning points.

Proof of the proposition: Let λ be a connecting R to R' as described in the proposition. One easily checks that the shaded regions in the above construction are exactly those which λ does not cross.

Corollary II.1.1: For any choice R,R' of Stokes regions, the connection isomorphism $\mathscr{C}_{R'R}$ is canonically given by the product

$$\mathscr{C}_{R'R}=\mathscr{C}_{L_n}^{\epsilon_n}\mathscr{C}_{L_{(n-1)}}^{\epsilon_{(n-1)}}\cdots\mathscr{C}_{L_1}^{\epsilon_1},$$

where $L_1,L_2,\ldots,L_n$ is the canonical sequence and $\epsilon_i=+1$ (resp., -1) if the path λ of the proposition above crosses L_i from right to left (resp., from left to right).

Definition II.1.1: Two successive Stokes lines in the canonical sequence are called parallel (resp., antiparallel) if the canonical way to cross them is the same (resp., opposite).

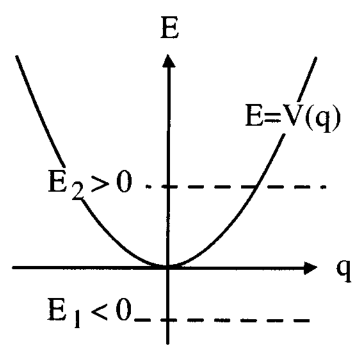


FIG. 8. The harmonic oscillator.

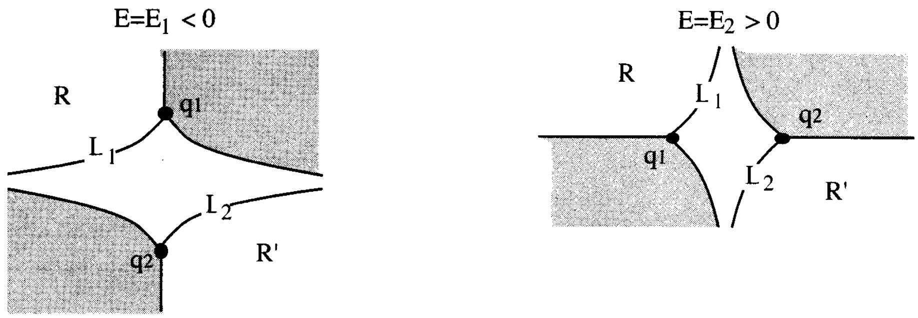


FIG. 9. Stokes patterns for the harmonic oscillator.

For instance, in Fig. 5 above, L_1, L_2 and L_3, L_4 are antiparallel while L_2, L_3 are parallel.

Second elementary property of successive Stokes lines: If two successive Stokes lines L and L' bound a common Stokes region R of the second kind and are not tied to the same turning point then the following properties are equivalent:

(1) L and L' are parallel.
(2) L and L' have the same asymptotic direction.
(3) $\mathrm{WKB}^L(R) = \mathrm{WKB}^{L'}(R)$.

A direct consequence of these properties is the following:

Proposition II.1.3: In the simple pattern case, the composition of our canonical sequence of elementary connection isomorphisms can be described pictographically by pasting side by side the successive pictograms of Sec. II A 1, Fig. 3.

For instance, the connection isomorphisms corresponding to the two Stokes patterns of Fig. 6 are described by the two pictograms on Fig. 7.

Proof of the proposition: The only thing to understand here is the fact that the conventions of Fig. 3, whereby dominant and recessive WKB symbols are distinguished according to the level of the corresponding horizontal arrows, are compatible from one elementary pictogram to the next one. Since all Stokes lines in the canonical sequence are tied to different turning points, this compatibility property follows from the second elementary property of the successive Stokes lines above. □

3. The harmonic oscillator

Taking $V(q) = q^2$ for our potential function (Fig. 8), we get the Stokes patterns shown on Fig. 9. The corresponding connexion isomorphisms are shown on Fig. 10.

FIG. 10. Pictograms of the connection isomorphisms of Fig. 8.

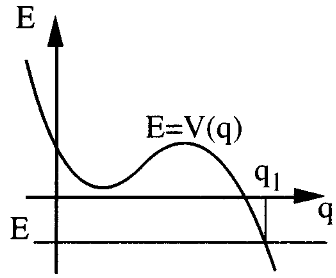


FIG. 11. The cubic barrier.

4. The cubic barrier (subcritical case)

Let us consider a cubic barrier and a value E of the energy as shown on Fig. 11.

The corresponding Stokes pattern is thus given by Fig. 12.

Following the scheme announced in Sec. I B, we shall analyze our wave function ϕ in terms of incoming, reflected and transmitted waves.

Since the transmitted wave must decay at $q=-\infty$, we can take $\phi(q)\in\mathscr{W}_{-\infty}^{\mathrm{dec}}$ to be the Borel sum for $q<q_1$ of some recessive WKB expansion φ, well normalized at $-\infty$. The connection isomorphism $\mathscr{C}_{R_+R_-}$ then gives

$$\mathscr{C}_{R_+R_-}(\varphi)=\varphi_{\mathrm{inc}}+\varphi_{\mathrm{ref}},$$

where φ_{inc} and φ_{ref} are the analytic continuations of φ indicated on Fig. 13, whose Borel sums can be interpreted as an incoming wave $\phi_{\mathrm{inc}}\in\mathscr{W}_{+\infty}^{-}$ and a reflected wave $\phi_{\mathrm{ref}}\in\mathscr{W}_{+\infty}^{+}$ (where $\mathscr{W}_{+\infty}^{+}$ is the subspace of waves with positive classical momentum near $+\infty$).

Graphical conventions for connection paths: Here and in the sequel, we use full lines for those portions of path following the real q-axis with a positive real determination of p (wave travelling rightwards) or a positive imaginary determination of p (wave decaying leftwards).

B. Connection isomorphisms in the presence of bounded Stokes lines

Although they are ''non-generic,'' bounded Stokes lines often occur as a result of the symmetries of the potential function: for instance, if the potential is real, symmetry by complex

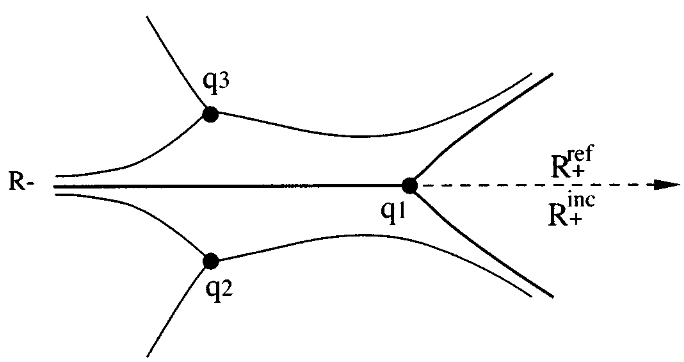


FIG. 12. The Stokes pattern for a cubic barrier.

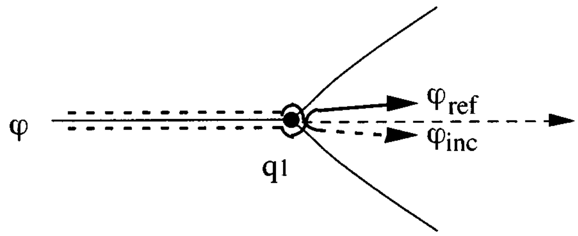


FIG. 13. Connection paths relating the symbols of the transmitted, incident and reflected waves.

conjugation implies that all classically forbidden segments of the real axis are Stokes lines, so that all ''tunnel segments'' are bounded Stokes lines. Of course this is true only because of our special way of choosing the ''resummation direction'' in the ''Borel plane'' (the plane of the complex variable ξ, dual to $1/\hbar$), taking it to be the positive real direction arg $\xi=0$. Replacing this direction by arg $\xi=\pm\epsilon$, with ϵ a small enough positive number, results in splitting the Stokes pattern into a simple one (with only unbounded Stokes lines). As shown in Ref. 12, one gets the connection isomorphism $\mathscr{C}^{+}_{R'R}$ (resp., $\mathscr{C}^{-}_{R'R}$) by applying the algorithm (Sec. II A) to this simple arg $\xi=+\epsilon$ (resp., arg $\xi=-\epsilon$) pattern.

The topology of this ''split'' pattern depends on the sign of arg ξ. One gets it by rotating each Stokes line anticlockwise (arg $\xi>0$) or clockwise (arg $\xi<0$) around its turning point, as illustrated by the examples below.

1. The parabolic barrier (subcritical case)

Let $V(q)=-q^2$ be our potential function, and let $E<0$ (Fig. 14).

Since the ''tunnel'' segment $[q_1;q_2]$ carries a bounded Stokes line, the corresponding Stokes pattern is singular as shown on Fig. 15.

Splitting this singular pattern gives the two simple patterns drawn on Fig. 16 and therefore two canonical sequences $(L_1^+;L_2^+)$ and $(L_1^-;L_2^-)$ of Stokes lines between R and R'.

The connection isomorphisms $\mathscr{C}^{+}_{R'R}$ and $\mathscr{C}^{-}_{R'R}$ can be computed explicitly as the products

$$\mathscr{C}^{-}_{R'R}=\mathscr{C}^{-1}_{L_2^-}\mathscr{C}_{L_1^-}, \quad \mathscr{C}^{+}_{R'R}=\mathscr{C}_{L_2^+}\mathscr{C}^{-1}_{L_1^+},$$

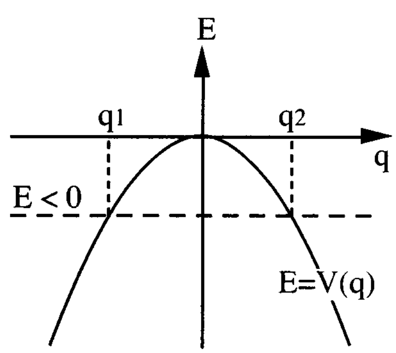


FIG. 14. The subcritical parabolic barrier.

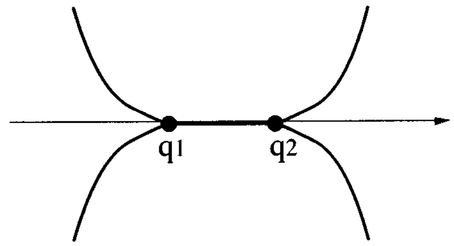


FIG. 15. The Stokes pattern of a subcritical parabolic barrier.

and pictographically represented by Fig. 17.

Being interested here in transmission and reflection of a wave ϕ coming from the right, we assume that along the left end of the real axis its canonical decomposition Sec. I B 1 has no positive p terms (no component ''coming from the left''). This amounts saying that it is exponentially decreasing along the Stokes line L_1^-, and the simplest way to ensure this is to take its symbol φ to be well normalized along L_1^-: therefore φ is Borel resummable on L_1^- and in the adjacent Stokes regions (including the left end of the real axis), where its Borel sum can be assumed to be exactly ϕ (this assumption amounts to suitably choosing the normalization of the transmitted wave).

From the pictogram of $\mathscr{C}^-_{R'R}$ (say) we thus infer that along the right end of the real axis the left symbol of ϕ is the sum of the two analytic continuations of φ shown on Fig. 18, which we have denoted by φ_{inc} and φ_{ref} because they can be interpreted, respectively, as the left symbol of the incoming, resp., reflected wave (working with right symbols would have given a more complicated decomposition, with four terms instead of two; we leave it to the reader to check which of the four belong to the incident wave, and which to the reflected wave).

2. The parabolic barrier (overcritical case)

Let again $V(q)=-q^2$, but assume now $E>0$ (Fig. 19). Here again the turning points q_1,q_2 (which are now complex conjugate) are tied by a bounded Stokes line as shown on Fig. 20.

Figures 21 and 22 show the corresponding split patterns, and the pictograms of the connection isomorphisms. Now the transmitted wave is exponentially decreasing along L_1^+, and the incoming and reflected waves are most easily described by their right symbols, which are shown on Fig. 23.

FIG. 16. Left splitting and right splitting of the singular pattern.

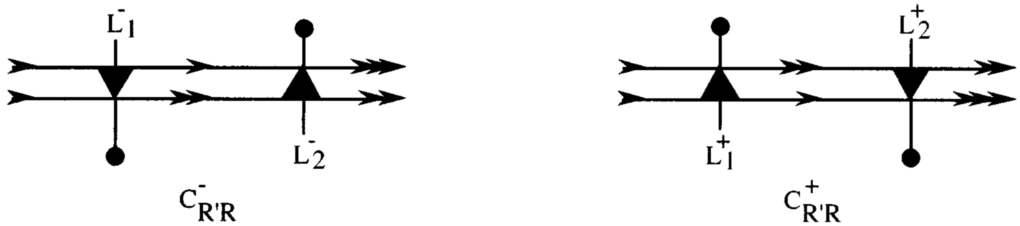


FIG. 17. Pictograms of left and right connection isomorphisms.

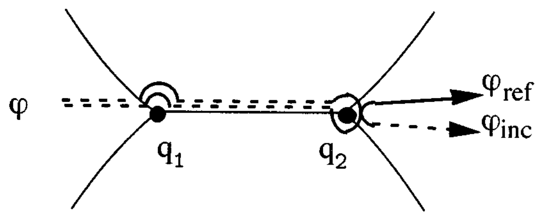


FIG. 18. Connection paths relating left symbols (the graphical conventions have been explained at the end of subsection II A).

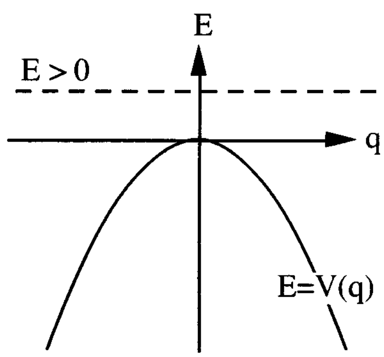


FIG. 19. The overcritical parabolic barrier.

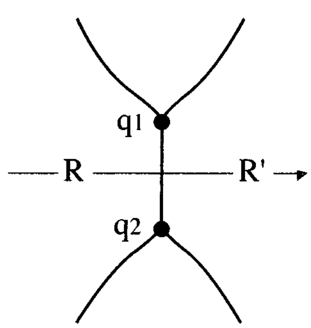


FIG. 20. The Stokes pattern of an overcritical parabolic barrier.

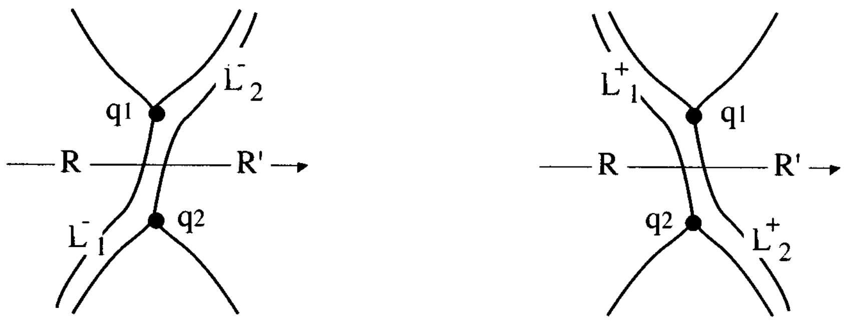


FIG. 21. Left splitting and right splitting of the singular pattern.

FIG. 22. Pictograms of left and right connection isomorphisms.

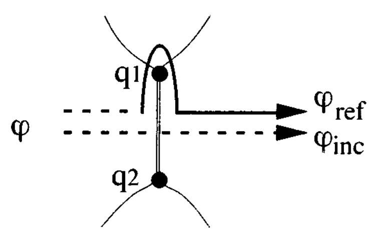


FIG. 23. Connection paths relating left symbols.

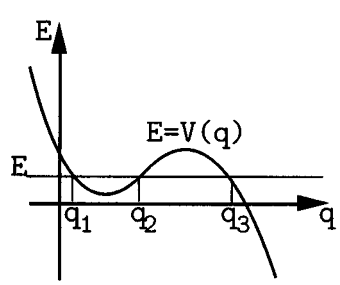


FIG. 24. The cubic barrier and the energy level E.

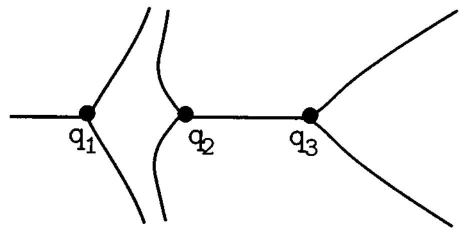


FIG. 25. The singular Stokes pattern for a cubic barrier.

3. The cubic barrier (resonance case)

Let us now consider a cubic barrier and a value E of the energy as on Fig. 24. The corresponding Stokes pattern (Fig. 25) is singular (the ''tunnel segment'' $[q_2;q_3]$ carries a bounded Stokes line). Splitting this singular pattern leads to the two simple patterns drawn on Fig. 26, and therefore to two canonical sequences $(L_1^+;L_2^+;L_3^+)$ and $(L_1^-;L_2^-;L_3^-)$ of Stokes lines between R and R'.

The connection isomorphisms $\mathscr{C}^+_{R'R}$ and $\mathscr{C}^-_{R'R}$ are explicitly given by the products

$$\mathscr{C}^-_{R'R}=\mathscr{C}^{-1}_{L_3^-}\mathscr{C}_{L_2^-}\mathscr{C}_{L_1^-},\quad \mathscr{C}^+_{R'R}=\mathscr{C}_{L_3^+}\mathscr{C}^{-1}_{L_2^+}\mathscr{C}_{L_1^+},$$

and pictographically represented by Fig. 27.

For the same reason as in Sec. II A we can assume that our wave function ϕ is, for large negative q, the Borel sum of a WKB expansion φ well normalized at $-\infty$ along the real axis.

The right, resp., left symbol of the reflected wave is deduced from φ by the right, resp., left Jost connection operator $\mathscr{T}^+$, resp., $\mathscr{T}^-$, which can be read directly on the pictograms of Fig. 27. One thus finds

$$\mathscr{T}^+=\lambda_1+gl_2+\mu,$$

$$\mathscr{T}^-=\lambda_1+\lambda_2,$$

where λ_1, λ_2, μ denote analytic continuation along the paths shown on Fig. 28.

Notice that any two of these paths differ only by a cycle on $\dot{\mathbf{C}}_2$, i.e. a path coming back where it started with the same determination of the momentum. For instance,

$$\lambda_2\lambda_1^{-1}=\gamma_{\text{osc}},$$

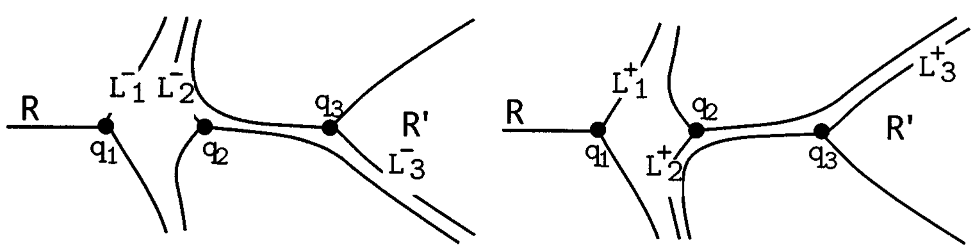


FIG. 26. Left and right splitting.

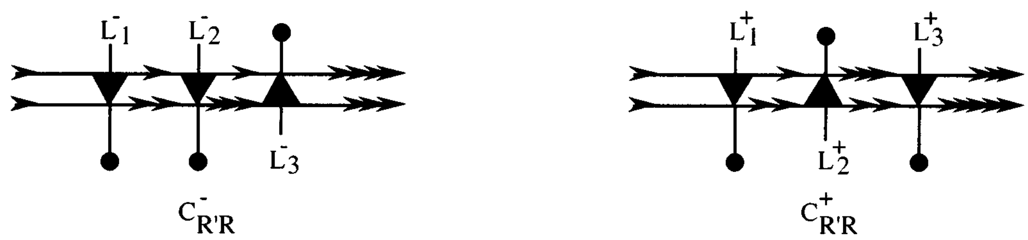


FIG. 27. Pictograms of left and right connection isomorphisms.

$$\mu\lambda_2^{-1}=\gamma_{\text{tun}},$$

where γ_{osc} and γ_{tun} are the "oscillator" and "tunnel" cycles represented on Fig. 29. We thus deduce from Sec. II A (noticing that these cycles have zero index, and therefore positive signature) the following rewriting of the Jost connection operators:

$$\mathscr{J}^+=(1+a^{\gamma_{\text{osc}}}+a^{\gamma_{\text{osc}}}a^{\gamma_{\text{tun}}})\lambda_1,$$

$$\mathscr{J}^-=(1+a^{\gamma_{\text{osc}}})\lambda_1.$$

Remembering that ϕ has been defined for large negative q as the Borel sum of φ, let us define ϕ^* for large positive q as the Borel sum of $\lambda_1\varphi$ (the latter symbol is indeed Borel resummable, because λ_1 crosses no bounded Stokes line, i.e. it does not cross the tunnel cycle). One thus gets a Jost basis (ϕ,ϕ^*) such that the right and left Jost symbols (i.e., the symbols of the Jost function) read as

$$J^+=1+a^{\gamma_{\text{osc}}}+a^{\gamma_{\text{osc}}}a^{\gamma_{\text{tun}}},$$

$$J^-=1+a^{\gamma_{\text{osc}}}.$$

Notice that the action period of the oscillator cycle is positive real, whereas the action period of the tunnel cycle is positive imaginary, so that $a^{\gamma_{\text{osc}}}$ has modulus 1 whereas $a^{\gamma_{\text{tun}}}$ is exponentially small.

Remark: Comparison of J^+ and J^- shows that

$$\mathfrak{G}a^{\gamma_{\text{osc}}}=(1+a^{\gamma_{\text{tun}}})a^{\gamma_{\text{osc}}},$$

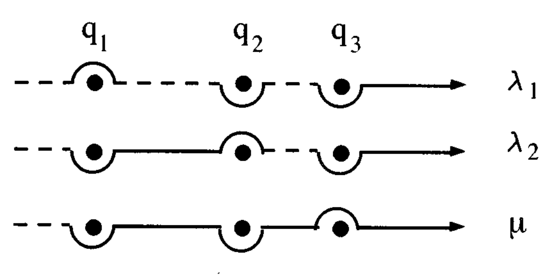


FIG. 28. Connection paths involved in the construction of the Jost connection operator.

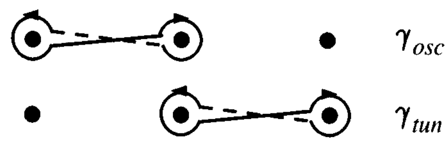


FIG. 29. The oscillator and tunnel cycles.

where $\mathfrak{S}$ is the Stokes automorphism (which transforms the left symbol of a function into the right symbol of the same function). This result is just a special case of theorem 3.1 in Ref. 11 (one immediately sees on Fig. 29 that the intersection number $\gamma_{\text{tun}} \cdot \gamma_{\text{osc}}$ is $+1$). By the same theorem, one has

$$\mathfrak{S}a^{\gamma_{\text{tun}}} = a^{\gamma_{\text{tun}}},$$

so that it is easy to compute the median symbol $J^{\text{med}} = \mathfrak{S}^{1/2} J^{-} = \mathfrak{S}^{-1/2} J^{+}$,

$$J^{\text{med}} = 1 + a^{\gamma_{\text{osc}}}(1 + a^{\gamma_{\text{tun}}})^{1/2} = 1 + a^{\gamma_{\text{osc}}} + \frac{1}{2} a^{\gamma_{\text{osc}}} a^{\gamma_{\text{tun}}} + \cdots .$$

4. The double well oscillator

Let us consider a double well potential V for a generic value of the energy (Fig. 30). The corresponding Stokes pattern is singular, because the tunnel segment $[q_2;q_3]$ carries a bounded Stokes line.

Being interested in the behavior of the solutions of the Schrödinger equation at infinity on the real axis, and since the half-lines $L =]-\infty;q_1]$ and $L' = [q_4;+\infty[$ are Stokes lines, we have to use either the right or the left resummation process. Splitting the singular picture above thus yields the two generic Stokes patterns drawn on Figs. 31 and 32.

On each of these pictures, the Stokes region R, resp., R' near $-\infty$, resp., $+\infty$ has been chosen to be that one which contains the real axis. Actually, this choice is of no consequence if one is only interested in computing the Jost connection operator, i.e. the operator connecting recessive symbols at $-\infty$ and dominant symbols at $+\infty$.

Reading the corresponding part of the pictogram of $\mathscr{C}^{+}_{R'R}$, resp., $\mathscr{C}^{-}_{R'R}$ we thus find that the right, resp., left Jost connection operator is given by

$$\mathscr{J}^{+} = \lambda_{-+} + \lambda_{+-} + \lambda_{--} + \lambda_{++} + \nu_{+},$$

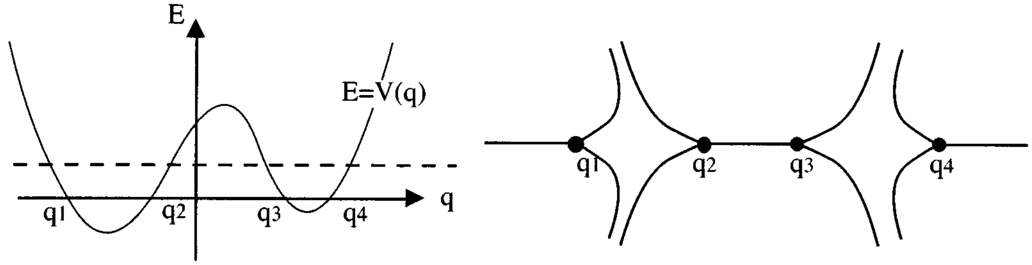


FIG. 30. A double well oscillator and its singular Stokes pattern.

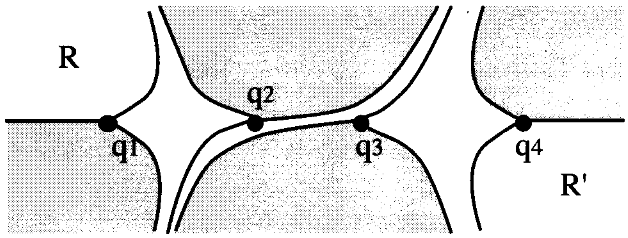


FIG. 31. Right splitting of the singular pattern.

resp.,

$$\mathscr{T}^{-}=\lambda_{-+}+\lambda_{+-}+\lambda_{--}+\lambda_{++}+\nu_{-},$$

where the paths of analytic continuations are those shown on Fig. 33.

Introducing the oscillator cycles γ_l, γ_r and the tunnel cycle γ of Figs. 34, 35, we have

$$\gamma_l=\lambda_{++}\lambda_{-+}^{-1},\quad \gamma_r=\lambda_{++}\lambda_{+-}^{-1},\quad \gamma=\nu_{+}\lambda_{++}^{-1},$$

and noticing that

$$\lambda_{--}\lambda_{++}^{-1}=-\gamma_r-\gamma_l$$

(the minus sign in front of γ_r and γ_ℓ indicates that the opposite orientation has been chosen. Whereas the composition of paths λ_i is denoted multiplicatively, as a composition of operators, the cycles make up a commutative group which we denote additively.), we get

$$\mathscr{T}^{+}=((1+a^{\gamma_l})(1+a^{-\gamma_r})+a^{\gamma_l}a^{\gamma})\lambda_{-+},$$

$$\mathscr{T}^{-}=((1+a^{\gamma_l})(1+a^{-\gamma_r})+a^{-\gamma_r}a^{\gamma})\lambda_{-+}.$$

Let us define a Jost basis (ϕ,ϕ^*) as follows

(1) ϕ is the Borel sum, for large negative q, of a WKB expansion φ well normalized at $-\infty$;
(2) ϕ^* is the right or left sum, for large positive q, of $\lambda_{-+}\varphi$.

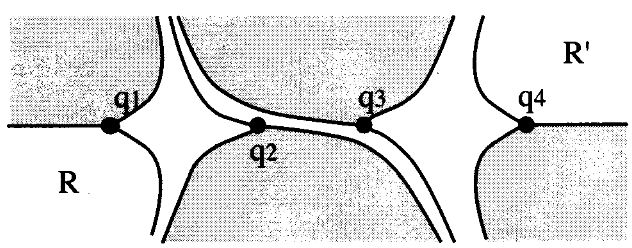


FIG. 32. Left splitting of the singular pattern.

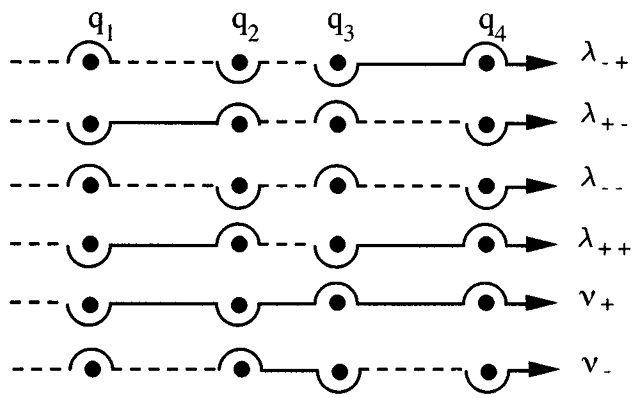


FIG. 33. Connection paths for the double well oscillator.

What makes such a choice convenient is the fact that λ_{-+} does not cross the tunnel segment (the only bounded Stokes line in this case), so that the right and left sums of $\lambda_{-+}\varphi$ coincide modulo $\mathscr{W}_{+\infty}$ (they do not exactly coincide because the positive real axis is a Stokes line).

With these conventions, the right, resp., left Jost symbol reads as

$$J^{+}=(1+a^{\gamma_l})(1+a^{-\gamma_r})+a^{\gamma_l}a^{\gamma},$$

$$J^{-}=(1+a^{\gamma_l})(1+a^{-\gamma_r})+a^{-\gamma_r}a^{\gamma}.$$

Notice that the action periods of the oscillator cycles γ_{ℓ},γ_r are positive real, whereas the action period of the tunnel cycle γ is positive imaginary, so that the ''tunnel'' contribution in the above expressions is exponentially small, as expected.

C. The principal part of the Jost symbol

The above results are easily generalized to any polynomial potential function. Assuming that the energy is so chosen that the real axis crosses at least one well, it can be readily seen on the corresponding pictogram that each of the oscillator cycles γ_i appearing in the (right, left or median) Jost symbol has index 0, so that this symbol reads as

$$J^{\pm}\quad\text{or}\quad J^{\text{med}}=\underbrace{(1+a^{\pm\gamma_1})(1+a^{\pm\gamma_2})\cdots(1+a^{\pm\gamma_k})}_{\text{principal part}}+\text{exponentially small correction},$$

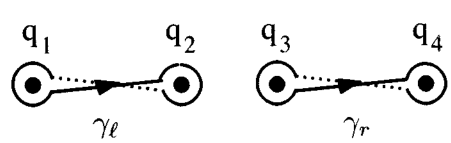


FIG. 34. The oscillator cycles (index ℓ, resp., r stands for left, resp., right).

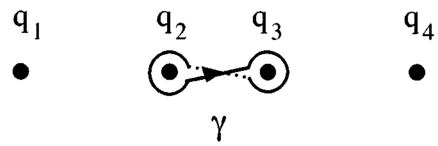


FIG. 35. The tunnel cycle.

where the "principal part" involves all the oscillator cycles $\gamma_1,\ldots,\gamma_k$, whereas the exponentially small correction is the contribution of cycles γ with $\mathscr{T}\omega_\gamma > 0$ (such as "tunnel cycles," or cycles tied to complex turning points).

III. SOLVING THE CONNECTION PROBLEM FOR CRITICAL ENERGIES

A. Rescaling E near a critical value

Studying the spectrum of the Schrödinger equation requires understanding how WKB expansions depend on E.

For non critical energies, i.e. as long as all turning points remain simple, this dependence is of course analytic. Even better, it is regular in the sense of Ref. 12, and this means that all usual operations on functions such as substitution, etc. can be performed without spoiling the resurgence properties. Therefore the results of Sec. I not only apply when E in equation (0) is a given constant: one can also substitute to E any resurgent expansion,

$$E = E_0 + E_1\hbar + E_2\hbar^2 + \ldots,$$

and make the corresponding substitution in the WKB expansions (the resulting Stokes lines are those of the $E=E_0$ case).

This nice behaviour may break down in the critical cases, i.e. when confluence of turning points occurs. More precisely, well normalized WKB expansions will be singular for those values of the energy for which the normalization path λ (used for defining the "good normalization": cf. Sec. I A) is "pinched" by the confluence of some turning points. Of course near a given critical energy E_{crit} it is always possible to choose a basis of WKB expansions such that their normalization paths are not pinched, so that their dependence on E is again regular in the above sense. But among all paths of analytic continuation in the q-plane which are involved in solving the connection problem, some will be pinched, so that regular dependence on E is not preserved by the connection isomorphisms.

The aim of the present section is to solve the connection problem in such critical cases. Allowing E in equation (0) to be a resurgent expansion in $\hbar$ as indicated above, we shall assume it to be "infinitely close" to a quadratic critical value of the potential, i.e. $E = E_{\mathrm{crit}} + 0(\hbar)$, where $E_{\mathrm{crit}} = V(q_{\mathrm{crit}})$, q_{crit} being a quadratic critical point of V (i.e., a double zero of $E_{\mathrm{crit}} - V$). It will turn out that no generality is lost by assuming the $0(\hbar)$ term to be linear, i.e.

$$E = E_{\mathrm{crit}} + E_r\hbar.$$

In fact, treating E_r as a free parameter (the "rescaled" energy), our solution of the connection problem will be built (via explicit special functions) from "simple" WKB expansions which will depend regularly on the rescaled energy E_r throughout the whole complex plane, allowing all further resurgent substitutions $E_r = E_r(\hbar)$.

The following terminology will be used throughout this section. By the rescaled Schrödinger equation we mean the Schrödinger equation with E replaced by $E_{\mathrm{crit}} + E_r\hbar$.

By a rescaled WKB expansion we mean any formal solution of the rescaled Schrödinger equation.

We shall start with so-called simple rescaled WKB expansions (''simple'' refers to the simple dependence on $\hbar$. Non simple expansions will be met in Sec. III C, Thm. III.3.1.), i.e. expansions of the form

$$\varphi(q)=(\varphi_0(q)+\varphi_1(q)\hbar+\varphi_2(q)\hbar^2+\dots)e^{(i/\hbar)S_{\text{crit}}(q)},$$

where $S_{\text{crit}}(q)$ is a primitive of

$$p_{\text{crit}}(q)=(E_{\text{crit}}-V(q))^{1/2}.$$

Such rescaled WKB expansions can be obtained from the usual ones by mere substitution $E=E_{\text{crit}}+E_r\hbar$, provided the normalization path of our ''usual'' WKB expansion is not pinched by the confluence of turning points.

Lemma III.1.1: The leading coefficient φ_0 of a simple rescaled WKB expansion reads as

$$\varphi_0=p_{\text{crit}}^{-1/2}e^{iE_rt},$$

where the ''time coordinate'' $t=t(q)$ satisfies

$$dt=\frac{dq}{2p_{\text{crit}}}$$

(the latter equation characterizes t up to an additive constant, ''the origin of time;'' rewriting it as $p_{\text{crit}}=1/2(dq/dt)$ shows why it deserves being called a ''time'').

Proof: Just solve the rescaled Schrödinger equation.

One may also look at the effect of the rescaling on a (usual) WKB expansion (assuming its normalization path is not pinched). Rescaling $S(q,E)=\int_{q_0}^q p(q',E)dq'$ gives

$$S(q,E_{\text{crit}}+E_r\hbar)=S(q,E_{\text{crit}})+\hbar E_r\,\frac{\partial S}{\partial E}\,(q,E_{\text{crit}})+0(\hbar^2),$$

with

$$\frac{\partial S}{\partial E}\,(q,E_{\text{crit}})=\int_{q_0}^q\frac{dq'}{2p_{\text{crit}}(q')}=t.$$

□

B. The monodromy exponent(s) of a double turning point

We analyze here the behaviour of WKB expansions near a double turning point.

For E close to E_{crit} (but different from it) the double turning point splits into two simple ones, and drawing a cut between them splits locally $\dot{\mathbf{C}}_2$ (the Riemann surface of p) into two copies of a cut disc, glued along the cut. Let us choose one of these copies, i.e. one determination of p, and let γ be a cycle in that cut disc, encircling the cut anticlockwise; we can draw it as close to the cut as we like, so that for $E=E_{\text{crit}}$ it becomes a circle of arbitrary small radius. We call γ the vanishing cycle associated to the chosen determination of p. Given any WKB expansion with that determination of p, analytic continuation along γ multiplies it by $e^{2i\pi s}$, where

$$s=s(E,\hbar)=\frac{1}{2\pi\hbar}\,\Omega_\gamma(E,\hbar^2)-\frac{1}{2},$$

$$\Omega_\gamma(E,\hbar^2)=\int_\gamma P(q,E,\hbar^2)dq=\omega_\gamma(E)+0(\hbar^2)$$

(cf. the notations of subsection I A).

We call s the monodromy exponent of the double turning point (for the chosen determination of p). Notice that choosing the opposite determination of p would result in changing the sign of Ω_γ (this amounts to reversing the orientation of the vanishing cycle), so that the two possible determinations of p define two monodromy exponents s_+, s_-, related by $s_+ + s_- = -1$.

Now comes the main point:

all the above (resurgent) expansions in $\hbar$ depend regularly on E near E_{crit} (this comes from the fact that the vanishing cycle is not pinched by the confluence of the turning points; in particular ω_γ is holomorphic at $E=E_{\mathrm{crit}}$).

This property allows us to perform the substitution $E=E_{\mathrm{crit}}+E_r\hbar$, thus obtaining the rescaled monodromy exponent (a resurgent series in $\hbar$, depending regularly on E_r in the whole complex plane). To the lowest order in $\hbar$, it reads as

$$s^{\mathrm{resc}}(E_r,\hbar)=-\frac{1}{2}+\frac{E_r T_\gamma}{2\pi}+0(\hbar),$$

where T_γ is a non zero constant, the ''time period'' of the vanishing cycle γ, defined by

$$T_\gamma=\frac{d\omega_\gamma}{dE}(E_{\mathrm{crit}})=\int_\gamma \frac{dq}{2p_{\mathrm{crit}}}$$

(an integral easily computable by the residue formula, since p_{crit} has a simple zero at q_{crit}).

N.B.- In the sequel the same notation s will be used for the monodromy exponent, whether rescaled or not.

Computation of rescaled monodromy exponents: Our aim here is to describe a simple algorithm for computing rescaled monodromy exponents to all orders in $\hbar$.

Considering a simple rescaled WKB expansion φ in a neighborhood of a double turning point q_{crit}, let us denote by $(i/\hbar)Y$ its logarithmic derivative; then $Y(q,E_r,\hbar)$ is a formal solution of the following rescaled Riccati equation:

$$Y^2-i\hbar\frac{d}{dq}Y=p_{\mathrm{crit}}^2+\hbar E_r .$$

Setting

$$Y(q,E_r,\hbar)=Y_0(q)+\sum_{k\geqslant 1} Y_k(q,E_r)\hbar^k,$$

one gets the Y_k's by iteration;

$$Y_0=p_{\mathrm{crit}},$$

$$Y_1=\frac{1}{2Y_0}\left(E_r+i\frac{d}{dq}Y_0\right),$$

$$\text{for } n\geqslant 1,\quad Y_{(n+1)}=\frac{1}{2Y_0}\left(i\frac{d}{dq}Y_n-\sum_{1\leqslant k\leqslant n} Y_k Y_{(n-k)}\right).$$

We deduce the behaviour of the Y_k's in a neighborhood of q_{crit}: for $k\geqslant 1$:

$$Y_k \text{ is } \begin{cases} \text{polar of order at most } 2k-1 \text{ on } q_{\text{crit}}, \\ \text{polynomial with respect to } E_r \text{ of degree at most } k. \end{cases}$$

As a consequence we get that locally in a neighborhood of q_{crit},

$$\frac{d}{dq} \ln(\varphi) = \frac{i}{\hbar} Y = \frac{s(\hbar,E_r)}{q-q_{\text{crit}}} + \kappa(q,E_r,\hbar),$$

with κ uniform in a neighborhood of q_{crit}, without a simple pole, whereas the monodromy exponent s of φ reads as

$$s(\hbar,E_r) = s_0(E_r) + s_1(E_r)\hbar + s_2(E_r)\hbar^2 + \dots,$$

with $s_k(E_r)$ polynomial with respect to E_r of order at most $k+1$.

Observing now that the rescaled Schrödinger equation (or the corresponding rescaled Riccati equation) is invariant under the involution $(E_r,\hbar) \to (-E_r,-\hbar)$, and furthermore $s_0(-E_r) = -s_0(E_r) - 1$, we deduce that

$$s(-E_r,-\hbar) = -s(E_r,\hbar) - 1.$$

Putting pieces together this means that

(1) $s_0(E_r) + 1/2$ is an odd polynomial of order 1.
(2) for $k \geqslant 1$, $s_{2k-1}(E_r)$ is an even polynomial of order at most $2k$.
(3) for $k \geqslant 1$, $s_{2k}(E_r)$ is an odd polynomial of order at most $2k+1$.

C. The elementary connection operator

The pattern of Stokes lines in the critical case is obtained by writing $E = E_{\text{crit}}$: it does not depend on the rescaled energy E_r.

Whereas each simple turning point is tied to three Stokes lines, each double turning point is tied to four Stokes lines. With the same notations as in section II, the elementary connection isomorphism $\mathscr{C}_L := \mathscr{C}_{R'R}$ across any unbounded Stokes line L (coming from infinity) is given by its restrictions:

$$\mathscr{C}_L | \mathrm{WKB}^L(R) = L + \delta_L \quad \text{and} \quad \mathscr{C}_L | \mathrm{WKB}_L(R) = L,$$

where L means analytic continuation across L, and

$$\delta_L : \mathrm{WKB}^L(R) \to \mathrm{WKB}_L(R')$$

is the elementary connection operator across L.

In the case of a simple turning point, nothing has to be changed in the description of the elementary connection operator given in section II (Prop. II.1.1).

Consider now the case of a double turning point (Fig. 36).

Let φ^{resc} be a simple rescaled WKB expansion defined in a neighborhood of a simple Stokes line L fading into a double turning point q_{crit}. Assume φ^{resc} to be dominant on L. The following proposition describes how the elementary connection operator attached to L acts on φ^{resc}.

Theorem III.3.1: Denoting by $s = s(E_r,\hbar)$ the (rescaled) monodromy exponent of φ^{resc} at q_{crit} (cf. Sec. III B), one has

$$\delta_L \varphi^{\text{resc}} = \frac{\sqrt{2\pi}}{\Gamma(-s)} \hbar^{s+1/2} \delta_L^{\text{red}} \varphi^{\text{resc}},$$

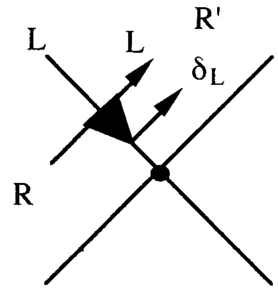


FIG. 36. Stokes lines and elementary connection operator across L.

where $\delta_L^{\text{red}}\varphi^{\text{resc}}$ is another simple rescaled WKB expansion (depending regularly on E_r) whose action exponent $S_{\text{crit}}^{\star}(q)$ is deduced from the action exponent $S_{\text{crit}}(q)$ of φ^{crit} by the symmetry of center $S_{\text{crit}}(q_{\text{crit}})$ (the value of the action at the turning point).

The important feature of this theorem (the proof of which is given in Ref. 12; see also Refs. 33, 34, 35, 36) is the fact that the connection operator δ_L spoils (or may spoil) two properties of our rescaled WKB expansions:

–the factor $\hbar^{s+1/2}$ spoils the simple character of WKB expansions.

–the denominator $\Gamma(-s)$ may spoil (when s_0 is a natural integer) the invertibility property of WKB expansions; this failure of invertibility will play a crucial role in section V, when we shall examine the quantization condition.

The numerical factor $\sqrt{2\pi}$ is there just for later convenience.

Definition III.3.1: The operator δ_L^{red} is called the ''reduced'' elementary connection operator.

How to compute $\delta_L^{\text{red}}\varphi^{\text{resc}}$: *the ''exact matching'' method:* The idea is to start from a non critical value E of the energy (close to E_{crit}), for which the connection operator can be computed as in section II; factoring out $\sqrt{2\pi}/\Gamma(-s)\hbar^{s+1/2}$ (expanded by means of Stirling's formula) yields a WKB expansion which turns out to depend regularly on E near E_{crit} (this fact, which is proved in Ref. 12, provides us with a rigorous interpretation of the formal computations below); from it, $\delta_L^{\text{red}}\varphi^{\text{resc}}$ is obtained by the rescaling $E=E_{\text{crit}}+E_r\hbar$.

Notice that generic values of E near E_{crit} are of two kinds, corresponding, respectively, to the two Stokes patterns of Fig. 9 (the harmonic oscillator can be viewed as the universal local model for the splitting of a double turning point): in one of these patterns our given ''critical'' Stokes line L splits in two, attached to either turning points; in the other pattern it does not split, remaining tied to one turning point only. This second way of choosing E is the most convenient for computing the connection operator, because we only have one Stokes line L to cross, so that Proposition II.1.1 directly applies, yielding δ_L as the analytic continuation operator along a loop l_q (with base point q) around the relevant simple turning point.

Let us illustrate this strategy by exhibiting the leading term of $\delta_L^{\text{red}}\varphi^{\text{resc}}$ (examples of computations to higher order in $\hbar$ will be given in section IV). Let

$$\varphi(q,E)=p(q,E)^{-1/2}e^{(i/\hbar)S(q,E)}(1+0(\hbar))$$

be the WKB expansion (depending regularly on E) from which φ^{resc} is obtained by rescaling. Analytic continuation along the loop l_q multiplies $p^{-1/2}$ by $-i$, and acts on S as the symmetry of centre ΔS where

$$\Delta S(E)=\int_{l_{q_0}} p(q',E)\,dq'.$$

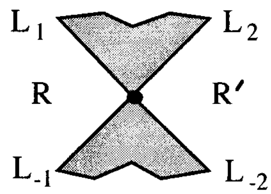


FIG. 37. Crossing a double turning point.

When E tends to E_{crit} this term is singular, because the loop l_{q_0} is pinched by the confluence of turning points: more precisely, denoting by γ the corresponding vanishing cycle, oriented in such a way that $\Re\omega_\gamma > 0$ (our generic choice of E ensures that $\Re\omega_\gamma \neq 0$, otherwise the two turning points would be tied by a bounded Stokes line), one has

$$\Delta S(E) = -\omega_\gamma(E)\,\frac{\ln\,\omega_\gamma(E)}{2\pi i} + \text{hol}(E),$$

where the function hol(E) is holomorphic for E near E_{crit}. Now it turns out that factoring out $\sqrt{2\pi}/\Gamma(-s)\hbar^{s+1/2}$ just cancels out the singular part: more precisely, noticing that with the above choices $-s \approx \omega_\gamma/2\pi\hbar - \frac{1}{2}$ has a positive real part which goes to infinity as $\hbar \to 0$ (for fixed E), Stirling's formula gives

$$\frac{\sqrt{2\pi}}{\Gamma(-s)}\,\hbar^{s+1/2} = (-\hbar s)^{s+1/2} e^{-s}(1+0(\hbar)) = \left(\frac{\omega_\gamma}{2\pi}\right)^{-\omega_\gamma/2\pi\hbar} e^{\omega_\gamma/2\pi\hbar}(1+0(\hbar)).$$

Factoring out this expression in $\delta_L\varphi = -ip^{-1/2}e^{i/\hbar(-S+\Delta S)}(1+0(\hbar))$ thus amounts to replacing the singular term $\Delta S(E)$ by the ''renormalized'' expression

$$\Delta^{\text{ren}}S(E) = \Delta S(E) - \frac{\hbar}{i}\ln\left(\frac{\sqrt{2\pi}}{\Gamma(-s)}\,\hbar^{s+1/2}\right) \approx \Delta S(E) + \frac{\omega_\gamma(E)}{2\pi i}\left[\ln\frac{\omega_{\gamma(E)}}{2\pi} - 1\right]$$

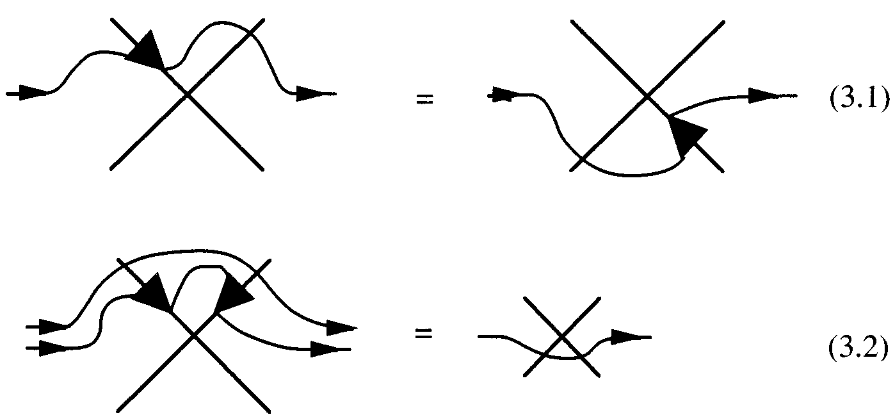


FIG. 38. Basic relations between the ''connection paths'' across a double turning point.

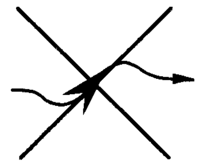


FIG. 39. Pictorial representation of the connection operator (3.1).

(the quantities inside the logarithms have positive real parts, and ln must be understood as the principal determination of the logarithm, which is real on the positive real axis), which is holomorphic indeed for $E \approx E_{\text{crit}}$. Substituting $E_{\text{crit}} + E_r \hbar$ for E in the renormalized expression thus gives the leading term of $\delta_L^{\text{red}} \varphi^{\text{resc}}$, as stated in the following lemma.

Lemma III.3.1: With the above notations one has

$$\delta_L^{\text{red}} \varphi^{\text{resc}} = -ip_{\text{crit}}^{-1/2} e^{iE_r t^*} e^{(i/\hbar) S^*_{\text{crit}}(q)} (1 + 0(\hbar)),$$

where the time coordinate t^* of $\delta_L^{\text{red}} \varphi^{\text{resc}}$ is deduced from the time coordinate t of φ^{resc} (cf. Lemma III.1.1) by

$$t^* + t = \lim_{E \to E_{\text{crit}}} \left[\int_{l_{q_0}} \frac{dq'}{2p(q',E)} + \frac{T_\gamma}{2\pi i} \ln \frac{\omega_\gamma(E)}{2\pi} \right],$$

where T_γ stands for the time period of the vanishing cycle (cf. Sec. III B). □

Remark: The higher order terms in $\hbar$ can be computed by the same strategy, using the expanded Stirling formula.

D. Local relations between connection operators

Relations between elementary connection operators: Among the four Stokes regions incident to a double turning point q_{crit}, let R and R' be two ''opposite'' ones, as on Fig. 37.

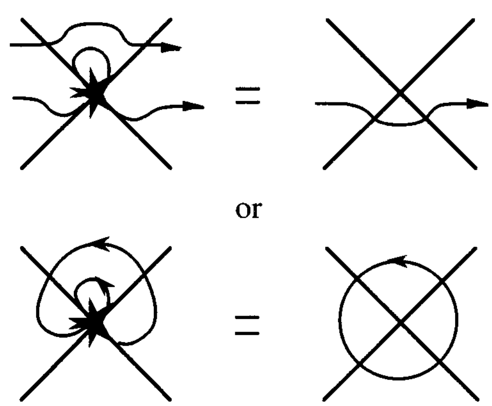


FIG. 40. Redrawing of relation (3.2).

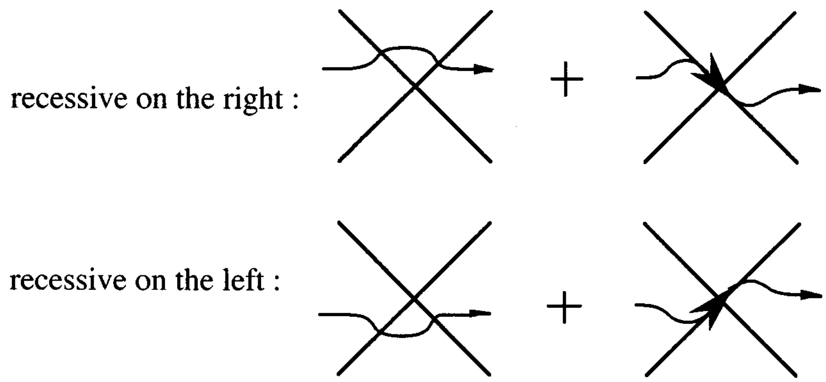


FIG. 41. The connection operator across a double turning point.

Assuming that all four Stokes lines incident to q_{crit} are unbounded, let φ^{resc} be a simple rescaled WKB expansion, dominant on L_1 (say). Then, using the same notations as in Sec. II, one has

$$\mathscr{C}_{R'R}(\varphi^{\text{resc}})=\delta_{L_2}\delta_{L_1}(\varphi^{\text{resc}})+L_2L_1(\varphi^{\text{resc}})+L_2\delta_{L_1}(\varphi^{\text{resc}})$$

(cross L_1 then L_2), but also

$$\mathscr{C}_{R'R}(\varphi^{\text{resc}})=L_{-2}L_{-1}(\varphi^{\text{resc}})+\overline{\delta_{L_{-2}}}L_{-1}(\varphi^{\text{resc}})$$

(cross L_{-1} then L_{-2}). Comparing both formulas yields

$$L_2\delta_{L_1}(\varphi^{\text{resc}})=\overline{\delta_{L_{-2}}}L_{-1}(\varphi^{\text{resc}}), \tag{3.1}$$

$$\delta_{L_2}\delta_{L_1}(\varphi^{\text{resc}})+L_2L_1(\varphi^{\text{resc}})=L_{-2}L_{-1}(\varphi^{\text{resc}}). \tag{3.2}$$

Representing elementary connection operators δ_L, $\overline{\delta_L}$ by the same pictograms as in Sec. II, Fig. 3, we can transcribe these equations pictorially as on Fig. 38 by considering connection paths (see Ref. 12, Sec. 2.3 for a precise definition).

It will be convenient to have a common pictorial representation for both sides of (3.1), as shown on Fig. 39. On this picture, the arrow ''threaded through the (double) turning point'' can be understood as representing the connection operator from $\text{WKB}_{L_{\text{in}}}$, the space of WKB symbols

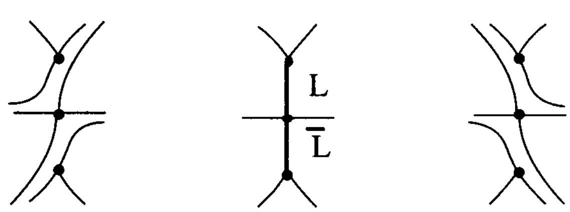


FIG. 42. Singular and split Stokes patterns of $V(q)=q^2+q^4$ for the critical energy $E=0$.

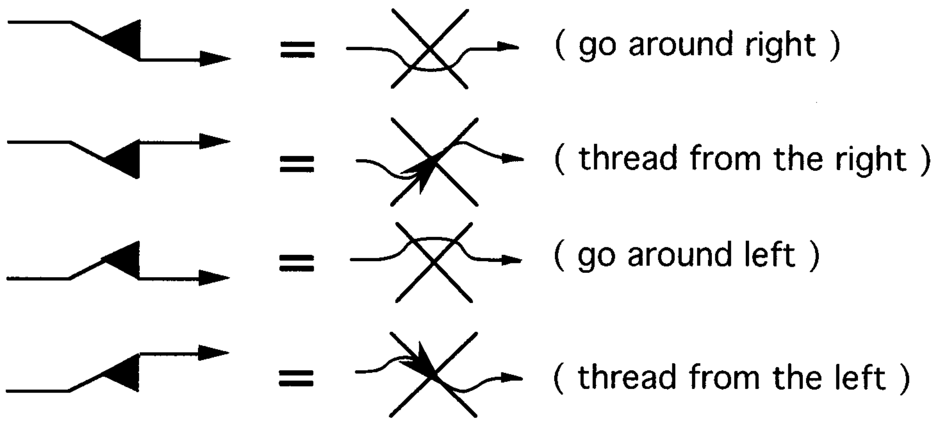


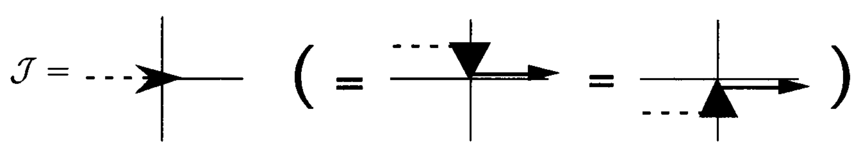
FIG. 43. Pictograms for connection operators through a double turning point.

which are recessive on the ''ingoing'' Stokes line L_{in} (here L_{-1}), to $\mathrm{WKB}^{L_{\mathrm{out}}}$ the space of WKB symbols which are dominant on the ''outgoing'' Stokes line L_{out} (here L_2).

Using that convention, relation (3.2) can be drawn as shown on Fig. 40. The connection operator $\mathscr{C}_{R'R}$ can be drawn as shown on Fig. 41.

Application: The Jost connection operator through the bottom of a simple well: Let our critical energy correspond globally to a strict minimum of the real valued potential function V, say at $q=0$ $[V(q)>V(0)$ for $q\neq 0$; of course we assume $V''(0)\neq 0]$.

Proposition III.4.1: In this situation the right and left Jost connection operators are equal, and given by



Proof: This easily follows from the above considerations, taking for L_{in} the negative real axis and for L_{out} the positive real axis. □

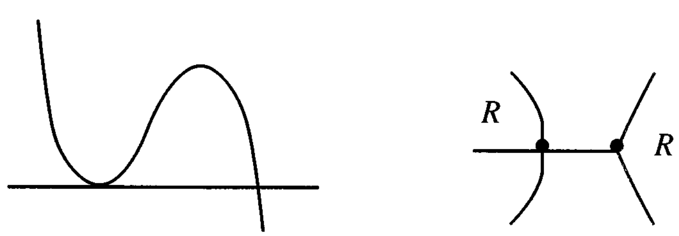


FIG. 44. The bottom of a cubic well and its Stokes pattern.

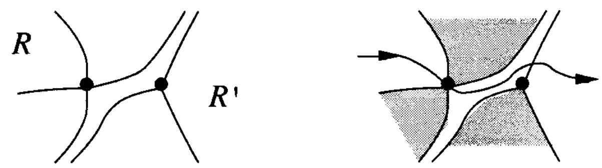


FIG. 45. Split pattern and canonical sequence.

Notice that since 0 is a strict minimum of V both L_{in} and L_{out} are unbounded Stokes lines. But it is not necessary to assume that the two other (complex conjugate) Stokes lines tied to 0 are also unbounded. Consider for instance the case when V is even, and has a pair of complex conjugate zeros on the imaginary axis, e.g. $V(q)=q^2+q^4$. The Stokes pattern has two complex conjugate bounded Stokes lines $L,\ \bar{L}$ (cf. Fig. 42, middle). Replacing this singular pattern by its right and left split patterns (Fig. 42, right and left), one immediately sees that as far as the Jost connection operator is concerned one can completely forget about the Stokes lines tied to the other turning points.

E. Composing connection operators

Let R and R' be two Stokes regions. We shall assume here that the Stokes pattern is simple (no bounded Stokes lines), so that

$$\mathscr{C}_{R'R}=\mathscr{C}^{+}_{R'R}=\mathscr{C}^{-}_{R'R}$$

(the case when the Stokes pattern is singular can be reduced to this one by the ''splitting algorithm'' sketched in Sec. II B, yielding two different right and left connection isomorphisms $\mathscr{C}^{+}_{R'R}$ and $\mathscr{C}^{-}_{R'R}$).

Here again (for the same reasons as in Sec. II) there exists a ''canonical way'' of computing the connection isomorphism between R and R', as described by the following algorithm.

• Consider one simple (resp., double) turning point and the three (resp., four) unbounded Stokes lines linked to this turning point. These Stokes lines split the complex plane into three (resp., four) connected regions. Shade those regions which contain neither R nor R'.

• Do the same for all turning points. We thus get a finite ordered sequence of (distinct) turning points $q_1,q_2,\dots,q_n$ and a finite ordered sequence of (distinct) Stokes regions $R=R_1,R_2,\dots,R_{n+1}=R'$ such that for any couple of successive Stokes regions R_i,R_{i+1},

–either $R_i,\ R_{i+1}$ are separated by a unique Stokes line L_i linked to q_i.

–or $R_i,\ R_{i+1}$ lie opposite with respect to a double turning point q_i.

• Then draw a path λ running successively through $R_1,\ R_2,\dots,R_{n+1}$; if $R_i,\ R_{i+1}$ are mutually

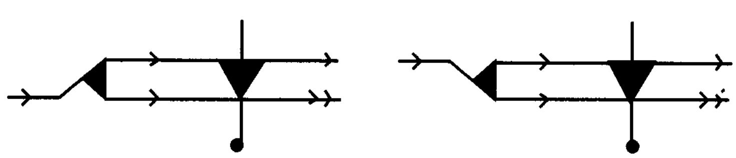


FIG. 46. Pictogram of the right connection operator.

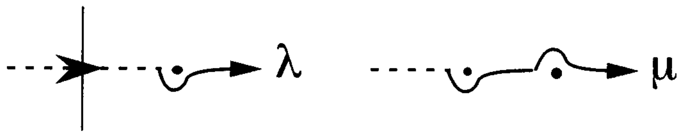


FIG. 47. Connection paths at the bottom of a cubic well.

''opposite'' with respect to a double turning point q_i, we shall draw λ through that turning point q_i, and interpret the crossing of q_i as the connection isomorphism described in Sec. III D (Fig. 41).

• The global connection isomorphism $\mathscr{C}_{R'R}$ is described by a pictogram analogous to those of Sec. II, simply obtained by pasting together the elementary pictograms described hereafter: when two Stokes regions R_i, R_{i+1} are separated by a (unbounded) Stokes line, the pictographic representation for $\mathscr{C}_{R_{i+1},R_i}$ can be taken to be the same as in Sec. II A 1 (fig. 3), irrespective of the simple or double character of the concerned turning point; when R_i, R_{i+1} lie ''opposite'' with respect to a double turning point we shall use the conventions shown on Fig. 43.

F. Metastable equilibrium: The bottom of a cubic well

The splitting algorithm for $\mathscr{C}^{+}_{R'R}$, with R and R' as on Fig. 44, yields the split pattern drawn on Fig. 45 with the canonical sequence obtained by an algorithm (Sec. III E). This leads immediately to the pictogram of the connection operator $\mathscr{C}^{+}_{R'R}$ (Fig. 46).

The pictogram drawn on fig. 46 (right) shows that the right Jost connection operator $\mathscr{J}^{+}$ reads as

$$\mathscr{J}^{+}=\lambda+\mu,$$

where λ and μ are the ''connection paths'' represented on Fig. 47.

Similarly, fig. 46 (left) shows that the left Jost connection operator $\mathscr{J}^{-}$ reads as

$$\mathscr{J}^{-}=\lambda,$$

where λ is the same connection path as before.

G. Two-state equilibrium: The bottom of a double well

Consider the case of Fig. 48. The splitting algorithm yields the generic Stokes pattern drawn on Fig. 49 and the corresponding canonical sequences related to a choice of two Stokes regions R and R'.

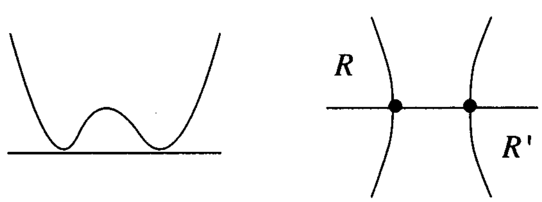


FIG. 48. The bottom of a double well and its Stokes pattern.

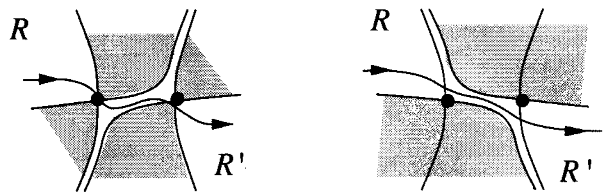


FIG. 49. Right splitting and left splitting.

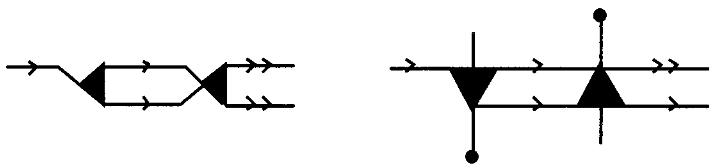


FIG. 50. Pictogram for $\mathscr{C}^{+}_{R'R}$. Pictogram for $\mathscr{C}^{-}_{R'R}$.

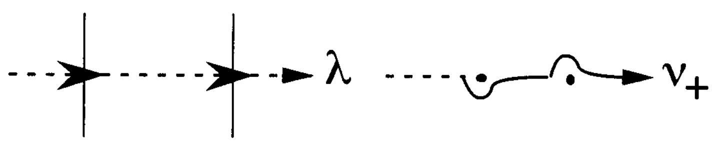


FIG. 51. Connection paths at the bottom of a double well.

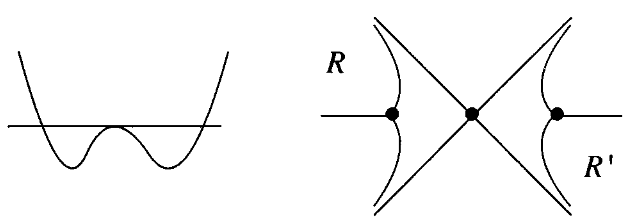


FIG. 52. The top of a double well and its Stokes pattern.

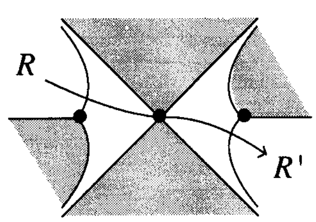


FIG. 53. Canonical sequence.

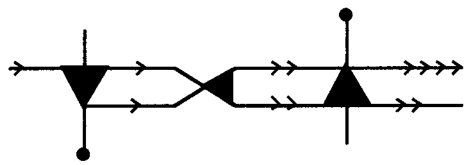


FIG. 54. Pictogram for $\mathscr{C}_{R'R}$.

Let us describe for example the isomorphisms $\mathscr{C}^{\pm}_{R'R}$ restricted to the subspace $\mathrm{WKB}_{-\infty}$ of those WKB expansions which are recessive on the Stokes line coming from $-\infty$. Translating the previous canonical sequences leads immediately to the pictograms drawn in Fig. 50.

Using these pictograms we see that the right, resp., left Jost connection operator is given by

$$\mathscr{J}^{\pm}=\lambda+\nu_{\pm}\,.$$

Here λ and ν_+ are the ''connection paths'' represented on Fig. 51, and ν_- is the path deduced from ν_+ by complex conjugation.

As a consequence, notice that

$$\mathfrak{G}(\lambda+\nu_-)=\lambda+\nu_+ \quad mod. \quad \mathrm{WKB}_{+\infty}\,,$$

where the equality holds only modulo the space of recessive symbols at $+\infty$, because the paths λ and $\nu_{\pm}$ end along the real axis which is a Stokes line.

H. The top of a double well

Consider the top of a double well. The corresponding Stokes pattern is non singular (Fig. 52). Figure 53 shows the canonical sequence related to the choice of the two Stokes regions R and R'.

Here again we shall focus on the isomorphism $\mathscr{C}_{R'R}$ restricted to the subspace $\mathrm{WKB}_{-\infty}$ of those WKB expansions which are recessive on the Stokes line coming from $-\infty$. Translating the previous canonical sequence leads immediately to the pictograms drawn in Fig. 54.

It follows from the pictogram of $\mathscr{C}_{R'R}$ that the Jost connection operator is given by a sum of four connection paths:

$$\mathscr{J}=\lambda_{-+}+\lambda_{+-}+\mu_{+}+\mu_{-}\,,$$

drawn on Fig. 55.

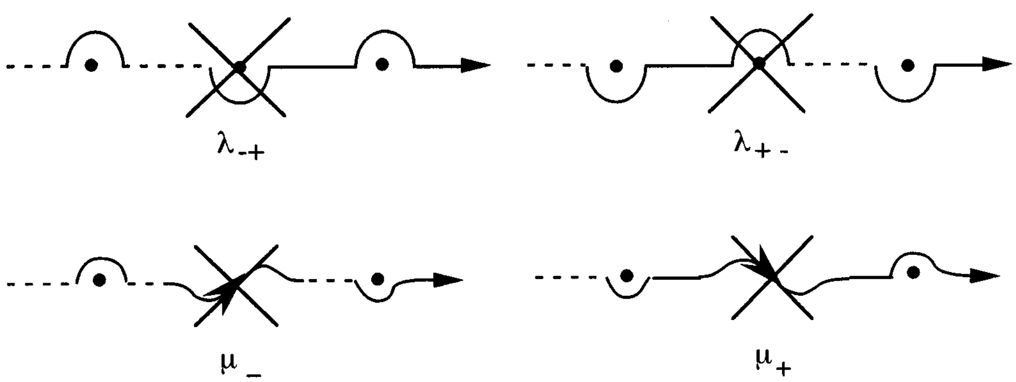


FIG. 55. Connection paths at the top of a double well.

Notice that this collection of connection paths is globally stable under complex conjugation; this property is a consequence of the fact that the symbols corresponding to these four connection paths are Borel resummable.

IV. COMPUTING JOST SYMBOLS FOR CRITICAL ENERGIES

The aim of this section is to give formulae for Jost symbols in critical cases, exhibiting their singular behaviour in a way which will be useful in the next section.

Recall that the definition of the Jost function depends on the choice of a ''Jost basis'' (ϕ,ϕ^*), where $\phi \in \mathscr{W}_{-\infty}$ (cf. Sec. I B 3). The first element ϕ of that basis can be taken for large negative q to be the Borel sum of a well normalized WKB expansion φ (well normalized at $-\infty$ along the left end of the real axis, when the latter is classically forbidden). The second element $\phi^\star$ can be chosen to be the (right, left or median) sum, for large positive q, of some WKB symbol deduced from φ in a natural way, e.g. analytic continuation along some path of $\dot{\mathbf{C}}_2$, when possible (in critical cases $\dot{\mathbf{C}}_2$ is not always connected!). With such choices of $\phi^\star$, the only difference with the generic case is the more complicated form of the Voros multipliers a^γ of the ''connection cycles'' γ going through double turning points (what we mean by a ''connection cycle'' is any connection path ending at the same place where it started, with the same determination of the momentum). Such Voros multipliers read as

$$a^\gamma = \left(\prod \frac{\sqrt{2\pi}}{\Gamma(-s)} \hbar^{s+1/2} \right) a^\gamma_{\text{reg}},$$

where the product runs over all the double turning points involved (s being their monodromy exponents); a^γ_{reg} is a simple resurgent expansion,

$$a^\gamma_{\text{reg}} = (a_0 + a_1\hbar + a_2\hbar^2 + \dots) e^{i/h\omega_\gamma} (a_i \in \mathbf{C}),$$

which can be exactly computed by the ''exact matching method'' explained in subsection III C: starting from a non critical value E of the energy (close to E_{crit}), the problem can be reduced to computing the Voros multiplier of a cycle of the type considered in section II, and factoring out the singular factors, via the (expanded) Stirling formula.

Let us illustrate the above ideas on the examples of the previous section.

A. The bottom of a cubic well

The basic ingredients for constructing Jost symbols will be the monodromy exponent s of the double turning point, and the Voros multiplier $a^{[L]}$ of the connection cycle $[L]$ drawn on Fig. 56, right (denoting by L the bounded Stokes line between the two turning points, $[L]$ is the ''connection cycle associated with L,'' in the terminology of Ref. 12).

Whenever (as is the case here) a double turning point q_{crit} corresponds to the bottom of a well, we shall use for defining its monodromy exponent that determination of p such that $\lim_{q\to q_{\text{crit}}}(p/(q-q_{\text{crit}}))$ is positive imaginary. The monodromy exponent is a resurgent power

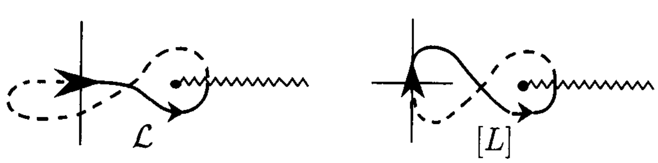


FIG. 56. Connection cycles at the bottom of a cubic well.

expansion in $\hbar$, with real coefficients (this can be easily deduced from the algorithm described in Sec. III B), depending regularly on the rescaled energy E_r, and starting like this:

$$s=-\frac{1}{2}+\frac{T_0}{2\pi}E_r+0(\hbar),$$

where T_0 (a positive real number) is the time period of the corresponding ''vanishing oscillator cycle'' γ_0.

The Voros multiplier $a^{[L]}$ is also real, and reads as

$$a^{[L]}=\frac{\sqrt{2\pi}}{\Gamma(s+1)}\left(\frac{c}{\hbar}\right)^{s+1/2}e^{-u_L/\hbar}\underbrace{a(\hbar,E_r)}_{1+0(\hbar)},\tag{4.1}$$

where u_L (a positive real number) is $1/i$ times the action integral along the tunnel cycle, whereas $a(\hbar,E_r)$ is a simple resurgent expansion depending regularly on E_r; c is a positive constant with the dimension of an action, which we call the critical action multiplier; using the ''exact matching method'' explained in subsection III C (looking at leading terms only), we find the following formula for c:

$$c=\lim_{E\overset{>}{\to}E_{\text{crit}}}\left(\frac{\omega_{\gamma_0(E)}}{2\pi}e^{2\pi i[T(E)/T_0]}\right),\tag{4.2}$$

where $\omega_{\gamma_0(E)}=\int_{\gamma_0(E)}p\,dq$ is the action integral along the ''vanishing oscillator cycle'' $\gamma_0(E)$, whereas $T(E)=\int_{\gamma(E)}dq/2p$ is the (positive imaginary) time period of the tunnel cycle $\gamma(E)$; notice that when $E\to E_{\text{crit}}$ one has $\omega_{\gamma_0(E)}\sim T_0(E-E_{\text{crit}})$, whereas $T(E)$ tends to infinity like $(T_0/2\pi i)\ln\omega_{\gamma_0(E)}$, so that the above limit is a finite number.

Example: We consider the general cubic oscillator,

$$V(q)=-q^3+\alpha q,$$

where α is assumed to be real positive. Rescaling the energy near the bottom of this cubic well $(E=(2/3\sqrt{3})\alpha^{3/2}+E_r\hbar)$, one gets

$$a^{[L]}=\frac{\sqrt{2\pi}}{\Gamma(s+1)}\left(\frac{c}{\hbar}\right)^{s+1/2}e^{-u_L\hbar}a(\alpha,E_r,\hbar),$$

where

$$u_L(\alpha)=\frac{8}{5}3^{1/4}\alpha^{5/4},$$

whereas

$$c(\alpha)=60u_L(\alpha)$$

is the ''critical action multiplier.'' The ''quasi-homogeneity'' property of the (rescaled) Schrödinger equation allows us to write

$$s(\alpha,E_r,\hbar)=s\left(1,\frac{E_r}{\alpha^{1/4}},\frac{\hbar}{\alpha^{5/4}}\right)\quad\text{and}\quad a(\alpha,E_r,\hbar)=a\left(1,\frac{E_r}{\alpha^{1/4}},\frac{\hbar}{\alpha^{5/4}}\right).$$

Now the ''exact matching method'' leads to the following results (see Ref. 12, Sec. 5):

$$s(1,E_r,\hbar)+\frac{1}{2}=\frac{3^{3/4}}{6}E_r+\left(\frac{7}{576}3^{3/4}+\frac{5}{192}3^{1/4}E_r^2\right)\hbar+\left(\frac{455}{18432}3^{1/4}E_r+\frac{385}{55296}3^{3/4}E_r^3\right)\hbar^2$$
$$+\left(\frac{119119}{10616832}3^{1/4}+\frac{95095}{5308416}3^{3/4}E_r^2+\frac{85085}{10616832}3^{1/4}E_r^4\right)\hbar^3+O(\hbar^4),$$

while $a(1,E_r,\hbar):=\exp(-D(1,E_r,\hbar))$ with

$$D(1,E_r,\hbar)=\left(\frac{77}{1152}3^{3/4}+\frac{47}{384}3^{1/4}E_r^2\right)\hbar+\left(\frac{15911}{110592}3^{1/4}E_r+\frac{11947}{331776}3^{3/4}E_r^3\right)\hbar^2+O(\hbar^3).$$

Computation of Jost symbols: Let us now come back to the connection paths λ and μ of Fig. 46, from which the right and left connection operators were built (Sec. III E 1). They are related by $\lambda=a^{\mathscr{L}}\mu$, where $\mathscr{L}$ is the connection cycle represented on Fig. 56 (left). By relation (3.2) of Sec. III D (fig. 40), $a^{\mathscr{L}}$ is related to $a^{[L]}$ by

$$a^{\mathscr{L}+[L]}=a^{\mathscr{L}}a^{[L]}=1-e^{-2i\pi s}.$$

This allows us to rewrite the right and left connection operators as follows:

$$\mathscr{T}^+=\lambda+\mu=\left(\frac{1-e^{-2i\pi s}}{a^{[L]}}+1\right)\mu,$$

$$\mathscr{T}^-=\lambda=\frac{1-e^{-2i\pi s}}{a^{[L]}}\mu.$$

To define the Jost function J we shall choose the following Jost basis $(\phi,\phi^\star)$:

(1) let ϕ be the Borel sum, for large negative q, of a WKB expansion φ well normalized at $-\infty$;

(2) let $\phi^\star$ be the right sum, for large positive q, of $\mu\varphi$.

Proposition IV.1.1: With the above conventions the right, left and median symbols of the Jost function, respectively, read as

$$J^+=\frac{1-e^{-2i\pi s}}{a^{[L]}}+1, \tag{4.3)+}$$

$$J^-=(1+a^{[L]})\,\frac{(1-e^{-2i\pi s})}{a^{[L]}}, \tag{4.3)−}$$

$$J^{\text{med}}=(1+a^{[L]})^{1/2}\left(\frac{(1-e^{-2i\pi s})}{a^{[L]}}+\frac{(1+a^{[L]})^{1/2}-1}{a^{[L]}}\right). \tag{4.3)med}$$

Proof: The formula for J^+ is an immediate consequence of the formula for $\mathscr{T}^+$. Computation of J^- requires more care, because the path μ intersects the tunnel cycle $[L]$ with a non zero index, so that the left symbol of $\phi^\star$ is not equal to its right symbol $\mu\varphi$. By Theorem 2.5.1 of Ref. 12 (using the fact that this intersection index is equal to $+1$), we find that the Stokes automorphism acts in the following way:

$$\mathfrak{S}\mu\varphi=(1+a^{[L]})\mu\varphi\mod \text{WKB}_{+\infty},$$

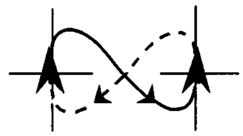


FIG. 57. The ''connection cycle'' $[L]$ for the bottom of a double well.

whereas

$$\mathfrak{G} a^{[L]} = a^{[L]},$$

so that modulo $\mathrm{WKB}_{+\infty}$ the left symbol of $\phi^{\star}$ is equal to $(1+a^{[L]})^{-1}\mu\varphi$, and formula $(4.3)^{-}$ immediately follows.

Another way of proving it consists in deducing J^{-} from J^{+} by $J^{-}=\mathfrak{G}^{-1}J^{+}$, noticing that

$$\mathfrak{G} e^{2i\pi s} = (1+a^{[L]})e^{2i\pi s} \tag{4.4}$$

(this again follows from Theorem 2.5.1 of Ref. 12, remembering that $-e^{2i\pi s}$ is a^{γ_0}, the Voros multiplier of the vanishing cycle γ_0).

Similarly, formula $(4.3)^{\mathrm{med}}$ is easily proved by computing $J^{\mathrm{med}}=\mathfrak{G}^{1/2}J^{-}$ (or $\mathfrak{G}^{-1/2}J^{+}$). □

B. The bottom of a double well

The basic ingredients will now be the monodromy exponents s_1, s_2 of the two well bottoms, and the Voros multiplier $a^{[L]}$, where $[L]$ is the connection cycle associated to the bounded Stokes line L shown on Fig. 57.

As was the case in subsection IV A, the monodromy exponents are resurgent expansions with real coefficients, depending regularly on the rescaled energy E_r,

$$s_1 = -\frac{1}{2} + \frac{T_1}{2\pi} E_r + 0(\hbar), \quad s_2 = -\frac{1}{2} + \frac{T_2}{2\pi} E_r + 0(\hbar),$$

where T_1 (resp., T_2) is a positive number, the time period of the bottom of the first (resp., second) well.

By the same reasoning as in subsection IV A, one finds that the Voros multiplier $a^{[L]}$ has the following form:

$$a^{[L]} = \frac{2\pi}{\Gamma(s_1+1)\Gamma(s_2+1)} \left(\frac{c}{\hbar}\right)^{(s_1+s_2+1)} e^{-u_L/\hbar} a(\hbar, E_r), \tag{4.5}$$

where u_L (a positive real number) is $1/i$ times the action integral along the tunnel cycle, whereas $a(\hbar, E_r)$ is a simple resurgent expansion depending regularly on E_r; c is a positive constant with the dimension of an action, the critical action multiplier, defined by

$$c = \lim_{E \overset{>}{\to} E_{\mathrm{crit}}} \left[\left(\frac{\omega_{\gamma_1(E)}}{2\pi}\right)^{T_1/(T_1+T_2)} \left(\frac{\omega_{\gamma_2(E)}}{2\pi}\right)^{T_2/(T_1+T_2)} e^{2\pi i[T(E)/(T_1+T_2)]} \right], \tag{4.6}$$

where $\omega_{\gamma_i(E)}$ $(i=1,2)$ is the action integral along the ''vanishing oscillator cycle'' $\gamma_i(E)$, whereas $T(E)=\int_{\gamma(E)}dq/2p$ is the (pure imaginary) time period of the tunnel cycle $\gamma(E)$ [here again the factor in front of the exponential vanishes as $E\to E_{\text{crit}}$, in such a way as to cancel the divergence of $T(E)$].

Example: Considering the symmetrical quartic oscillator,

$$V(q)=q^4-\alpha q^2,$$

for a real positive α and rescaling the energy near the bottom of this double well ($E=-\alpha^2/4+E_r\hbar$), one gets $s_1=s_2:=s$ and

$$a^{[L]}=\frac{2\pi}{\Gamma^2(s+1)}\left(\frac{c}{\hbar}\right)^{2s+1}e^{-u_L/\hbar}a(\alpha,E_r,\hbar),$$

where

$$u_L(\alpha)=\frac{2\sqrt{2}}{3}\alpha^{3/2},$$

whereas

$$c(\alpha)=6u_L(\alpha)$$

is the ''critical action multiplier.'' Now the (rescaled) Schrödinger equation presents a ''quasi-homogeneity'' property which allows to write

$$s(\alpha,E_r,\hbar)=s\left(1,\frac{E_r}{\alpha^{1/2}},\frac{\hbar}{\alpha^{3/2}}\right)\quad\text{and}\quad a(\alpha,E_r,\hbar)=a\left(1,\frac{E_r}{\alpha^{1/2}},\frac{\hbar}{\alpha^{3/2}}\right).$$

Applying the ''exact matching method'' we get (see Ref. 12, Sec. 5):

$$s(1,E_r,\hbar)+\frac{1}{2}=\frac{1}{2\sqrt{2}}\left(E_r+\left(\frac{1}{4}+\frac{3E_r^2}{8}\right)\hbar+\left(\frac{25E_r}{32}+\frac{35E_r^3}{64}\right)\right.$$
$$\left.\times\hbar^2+\left(\frac{175}{256}+\frac{735E_r^2}{256}+\frac{1155E_r^4}{1024}\right)\hbar^3+O(\hbar^4)\right),$$

and $a(1,E_r,\hbar):=\exp(-D(1,E_r,\hbar))$ with

$$D(1,E_r,\hbar)=\frac{1}{8\sqrt{2}}\left(\left(\frac{19}{3}+\frac{17E_r^2}{2}\right)\hbar+\left(\frac{187E_r}{4}+\frac{227E_r^3}{16}\right)\hbar^2+O(\hbar^3)\right).$$

Computation of Jost symbols: Let us now come back to the Jost connection operators $\mathscr{T}^{\pm}$: in Sec. III E 2 they were expressed in terms of the connection paths λ, ν_+, ν_-, the last two of which are invertible operators; remembering that these two are complex conjugate to each other, and noticing that they are related by

$$\nu_+=e^{2i\pi(s_1+s_2)}\nu_-,$$

it is convenient to factor out in $\mathscr{T}^{\pm}$ the real connection path,

$$\nu_0=e^{-i\pi(s_1+s_2)}\nu_+=e^{i\pi(s_1+s_2)}\nu_-.$$

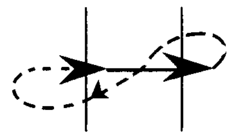


FIG. 58. The connection cycle $\mathscr{L}$ for the bottom of a double well.

Proposition IV.2.1: One has

$$\mathscr{J}^{\pm}=\left(-\frac{4\sin(\pi s_1)\sin(\pi s_2)}{a^{[L]}}+e^{\pm i\pi(s_1+s_2)}\right)\nu_0 .$$

Proof: Denoting by $\mathscr{L}$ the connection cycle represented on Fig. 58, we obviously have $\lambda = a^{\mathscr{L}}\nu_+$, so that

$$\mathscr{J}^{+}=(1=a^{\mathscr{L}})\nu_+ .$$

But the basic relations of Sec. III D (fig. 40) easily yield

$$a^{\mathscr{L}+[L]}=a^{\mathscr{L}}a^{[L]}=(1-e^{-2i\pi s_1})(1-e^{-2i\pi s_2}),$$

from which the proposition immediately follows. □

Besides being real, the invertible connection path ν_0 enjoys the nice property of being ''invariant under the Stokes automorphism $\mathfrak{G}$, modulo exponentially decreasing functions at $+\infty$.''

Lemma IV.2.1:

$$\mathfrak{G}\nu_0=\nu_0 \text{ mod } \mathrm{WKB}_{+\infty} .$$

Proof: By Theorem 2.5.1 in Ref. 12 one has

$$\mathfrak{G}\nu_+=(1+a^{[L]})\nu_+ ,\ \text{mod } \mathrm{WKB}_{+\infty} ,$$

$$\mathfrak{G}\nu_-=(1+a^{[L]})^{-1}\nu_- ,\ \text{mod } \mathrm{WKB}_{+\infty} ,$$

whereas

$$\mathfrak{G}a^{[L]}=a^{[L]},$$

$$\mathfrak{G}e^{2i\pi s_1}=(1+a^{[L]})e^{2i\pi s_1},$$

$$\mathfrak{G}e^{2i\pi s_2}=(1+a^{[L]})e^{2i\pi s_2}.$$

□

To define the Jost function J we shall choose the following Jost basis $(\phi,\phi^{\star})$.

(1) Let ϕ be the Borel sum, for large negative q, of a WKB expansion φ well normalized at $-\infty$;
(2) let $\phi^{\star}$ be, for large positive q, the right or left sum of $\nu_0\varphi$.

By the above lemma, choosing for $\phi^{\star}$ the right or left sum of $\nu_0\varphi$ makes no difference in the definition of the Jost function J, and Proposition IV.2.1 can be re-expressed by saying that the right and left symbols of this function read as

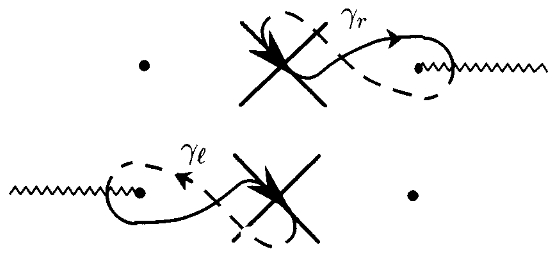


FIG. 59. The right and left oscillator cycles at the top of a double well.

$$J^{\pm} = -\frac{4\sin(\pi s_1)\sin(\pi s_2)}{a^{[L]}} + e^{\pm i\pi(s_1+s_2)}. \tag{4.7$^{\pm}$}$$

Addenum: The median symbol of the Jost function: With the above conventions the Jost function is real, so that its right and left symbols are complex conjugate to each other. If one likes to keep track of reality properties it is convenient to replace right and left symbols by the median symbol, defined by

$$J^{\mathrm{med}} = \mathfrak{G}^{1/2} J^{-} = \mathfrak{G}^{-1/2} J^{+}.$$

Lemma IV.2.2: One has

$$J^{\mathrm{med}} = -\frac{4\sin(\pi s_1)\sin(\pi s_2)}{a^{[L]}} + \frac{2\cos(\pi(s_1+s_2))}{1+(1+a^{[L]})^{1/2}}. \tag{4.7med}$$

Proof: Given that the automorphism $\mathfrak{G}$ leaves $a^{[L]}$ invariant, and multiplies $e^{i\pi s_1}$ and $e^{i\pi s_2}$ by $(1+a^{[L]})$, the lemma follows by a straightforward computation. □

C. The top of a double well

Consider now the situation of Sec. III F 3. All symbols in that case are Borel resummable, and the Jost connection operator is the sum of the four connection paths shown on Fig. 55. Two of these four are invertible, namely λ_{-+} and λ_{+-}, and this allows us to define the following connection cycles:
the right oscillator cycle $\gamma_r = \mu_+ \lambda_{+-}^{-1}$;
the left oscillator cycle $\gamma_l = \mu_+ \lambda_{-+}^{-1}$;
(cf. Fig. 59), which we call that way because they follow, respectively, the right and left real component of the (critical) classical trajectory in the (p,q)-plane; notice that these cycles are so oriented that the corresponding ‘‘critical action integrals,’’

$$\omega_r = \int_{\gamma_r} p_{\mathrm{crit}}(q)\,dq,$$

$$\omega_l = \int_{\gamma_l} p_{\mathrm{crit}}(q)\,dq \quad (p_{\mathrm{crit}} = (E_{\mathrm{crit}} - V(q))^{1/2}),$$

are positive real numbers.

Notice also that the corresponding (critical) Voros multipliers are related to each other by

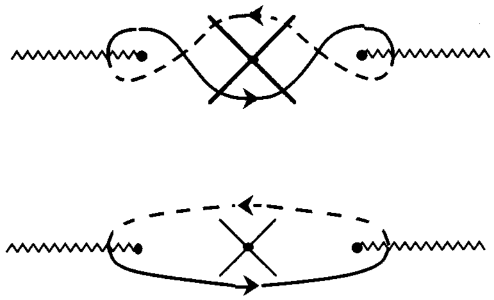


FIG. 60. The cycle $\gamma = \lambda_{-+}\lambda_{+-}^{-1}$ (top), and a homologous one (bottom).

$$a^{\gamma_r} = a^{\gamma_l} e^{(i/\hbar)(\omega_r - \omega_l)}.$$

To prove it, notice that $\gamma_r = \gamma_l + \lambda_{-+}\lambda_{+-}^{-1}$, check that the cycle $\gamma = \lambda_{-+}\lambda_{+-}^{-1}$ is homologous to a large circle (cf. Fig. 60), and apply the residue theorem at infinity.

Besides these two oscillator cycles it is also natural to introduce their complex conjugates, defined by

$$\overline{\gamma_r} = \mu_- \lambda_{-+}^{-1}, \quad \overline{\gamma_l} = \mu_- \lambda_{+-}^{-1}.$$

By the basic relations of Sec. III D (fig. 40), they are related to the previous ones by

$$a^{\gamma_r} a^{\overline{\gamma_r}} = a^{\gamma_l} a^{\overline{\gamma_l}} = 1 + e^{U},$$

where U is a real resurgent expansion in $\hbar$, depending regularly on E_r, defined by

$$U = \frac{1}{i\hbar} \, \Omega_{\gamma_0}(E_{\text{crit}} + E_r \hbar) \left(= E_r \, \frac{T_0}{i} + 0(\hbar) \right),$$

where γ_0 is the vanishing cycle around the double turning point, associated to that determination of p for which $(d\omega_{\gamma 0}/dE)(E_{\text{crit}}) = T_0$ is positive imaginary (when using the basic relations of Sec. III D one should remember that the monodromy exponent of the double turning point is $s = iU/2\pi - 1/2$).

Proposition IV.3.1: One has

$$a^{\gamma_r} = \frac{\sqrt{2\pi} e^{U/4}}{\Gamma\left(\frac{1}{2} + i \, \frac{U}{2\pi} \right)} \left(\frac{c}{\hbar} \right)^{i(U/2\pi)} e^{i\omega_r/\hbar} a(\hbar, E_r) \tag{4.8}$$

(and a similar equation for a^{γ_l}, with ω_r replaced by ω_l), where $a(\hbar, E_r) = 1 + 0(\hbar)$ is a simple resurgent function depending regularly on E_r; the ''critical action multiplier'' c (a positive real number with the dimension of an action) is defined by

$$c = \lim_{\substack{> \\ E \to E_{\text{crit}}}} \left(\frac{\omega_{\gamma_0(E)}}{2\pi i} \, e^{2\pi i [T(E)/T_0]} \right), \tag{4.9}$$

where $T(E)$ is the (positive real) time period of the real trajectory for E real $>E_{\mathrm{crit}}$; recall that T_0 is positive imaginary, and that $\omega_{\gamma_0(E)} \sim T_0 \cdot (E-E_{\mathrm{crit}}.)$

Proof: This again follows from Proposition III.3.1. The ''exact matching method'' of Sec. III C gives the announced expression for c by matching leading terms only, and the full expansion of $a(\hbar,E_r)$ by matching higher order terms.

Example: Considering the symmetrical quartic oscillator

$$V(q)=q^4-\alpha q^2,$$

where α is real positive, we get

$$a^{\gamma_r}=a^{\gamma_l}=\frac{\sqrt{2\pi}e^{U/4}}{\Gamma\left[\frac{1}{2}+i(U/2\pi)\right]}\left(\frac{c}{\hbar}\right)^{i(U/2\pi)} e^{i\omega/\hbar}a(\alpha,E_r,\hbar).$$

The periods $\omega_r=\omega_l:=\omega(\alpha)$ are proportional to $\alpha^{3/2}$,

$$\omega(\alpha)=\frac{2}{3}\alpha^{3/2},$$

while the ''critical action multiplier'' c is given by

$$c(\alpha)=12\omega(\alpha).$$

Now the ''quasi-homogeneity'' property of the (rescaled) Schrödinger equation induces the equalities

$$U(\alpha,E_r,\hbar)=U\left(1,\frac{E_r}{\alpha^{1/2}},\frac{\hbar}{\alpha^{3/2}}\right) \quad \text{and} \quad a(\alpha,E_r,\hbar)=a\left(1,\frac{E_r}{\alpha^{1/2}},\frac{\hbar}{\alpha^{3/2}}\right).$$

The ''exact matching method'' leads to the following results, as proved in Ref. 12, Sec. 5:

$$U(1,E_r,\hbar)=\pi\left(E_r+\left(\frac{3}{8}-\frac{3E_r^2}{8}\right)\hbar+\left(-\frac{85E_r}{64}+\frac{35E_r^3}{64}\right)\right.$$
$$\left.\times\hbar^2+\left(-\frac{1995}{1024}+\frac{2625E_r^2}{512}-\frac{1155E_r^4}{1024}\right)\hbar^3+O(\hbar^4)\right),$$

while $a(1,E_r,\hbar):=\exp(iD(1,E_r,\hbar)/2\pi)$ with

$$D(1,E_r,\hbar)=\frac{\pi}{16}\left(\left(-\frac{67}{3}+17E_r^2\right)\hbar+\left(\frac{671E_r}{8}-\frac{227E_r^3}{8}\right)\hbar^2+O(\hbar^3)\right).$$

Computation of Jost symbols: Let us now come back to the Jost connection operator,

$$\mathscr{J}=\lambda_{-+}+\lambda_{+-}+\mu_{+}+\mu_{-}.$$

Factoring out λ_{-+} yields

$$\mathscr{J}=(1+e^{i/\hbar(\omega_l-\omega_r)}+a^{\gamma_l}+a^{\overline{\gamma_r}})\lambda_{-+}.$$

Factoring out λ_{+-} yields

$$\mathscr{J}=(e^{(i/\hbar)(\omega_r-\omega_l)}+1+a^{\gamma_r}+a^{\overline{\gamma_l}})\lambda_{+-}.$$

If one likes to keep track of the reality property of $\mathscr{J}$, it is more convenient to factorize the real connection cycle λ_0 defined by

$$\lambda_0=e^{(i/\hbar)[(\omega_l-\omega_r)/2]}\lambda_{-+}=e^{(i/\hbar)[(\omega_r-\omega_l)/2]}\lambda_{+-}.$$

Taking Proposition IV.3.1 into account, one thus gets the following result.

Proposition IV.3.2: One has $\mathscr{J}=J\lambda_0$, where J is the real valued, Borel resummable symbol,

$$J=2\cos\frac{\omega_r-\omega_l}{2\hbar}+a(\hbar,E_r)e^{U/4}\left[\frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}+i\frac{U}{2\pi}\right)}\left(\frac{c}{\hbar}\right)^{i(U/2\pi)}e^{(i/\hbar)[\omega_r+\omega_l/2]}+\text{complex conj.}\right].$$

□

Taking the Borel sum of $\lambda_0\varphi$ for our generator of $\mathscr{W}$ mod. $\mathscr{W}_{+\infty}$, we obtain the corresponding Jost function as the Borel sum of the above symbol.

V. BOUND STATE SPECTRUM AND RESONANCES

From the symbol of the Jost function we shall now derive information on the zeros of that function, which are the bound state or resonance energies (cf. Sec. I B 3). Our main tool will be the implicit function theorem, and more precisely its resurgent version presented in Refs. 37 and 22.

A. Quantization of simple oscillators

1. The generic case

Consider now a range of generic energies such that the real axis crosses only one well, as in Fig. 61 (‘‘stable case’’), or in Fig. 24 (‘‘metastable case,’’ where tunnelling occurs).

We shall denote by γ_0 the corresponding ‘‘oscillator cycle’’ and by $\omega_{\gamma_0}=\int_{\gamma_0}pdq$ its period (a positive real number, increasing with E).

Proposition V.1.1: In any such energy range there is a uniquely defined analytic change of variable (this idea of ‘‘changing the energy variable’’ was suggested to us by Colin de Verdière),

$$E\mapsto\Omega(E)=\omega_{\gamma_0}(E)+O(\hbar),$$

such that in this new variable Ω the zeros of the Jost function are given by the ‘‘exact Bohr–Sommerfeld quantization rule,’’

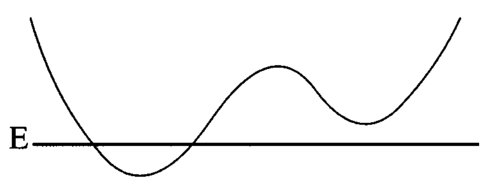


FIG. 61. A simple oscillator (generic, stable case).

$$\Omega=(n+\tfrac{1}{2})2\pi\hbar,$$

where n runs over all natural integers such that the corresponding values of E stay in the energy range considered.

In the stable case this change of variable is real analytic, and increasing. In the metastable case it is almost so, up to exponentially small imaginary corrections.

Proof: The idea is to rewrite the Jost function under the form

$$J=1+e^{(i/\hbar)\Omega},$$

where $\Omega=\Omega(E,\hbar)$ is an analytic function with the above properties, depending analytically on the small parameter $\hbar$, such that

$$\frac{d\Omega}{dE}=\frac{d\omega_{\gamma_0}}{dE}+O(\hbar).$$

We shall explain the construction in detail in the special case of the cubic barrier, from which the idea of the general construction will be clear (this case is more instructive than the stable case, because of tunnelling effects). The construction will be made on symbols, which we could choose to be the right or left symbols. Here we shall work with the median symbols, which will provide us with more readable information on the imaginary part of the energy levels (resonance widths). As we have seen in Sec. II B 3, the median symbol of the Jost function reads as

$$J^{\text{med}}=1+a^{\gamma_0}(1+a^{\gamma})^{1/2},$$

where $\gamma_0=\gamma_{\text{osc}}$, $\gamma=\gamma_{\text{tun}}$. Setting $u=1/i\int_\gamma p\,dq$(a positive real number, decreasing with E), one has

$$a^{\gamma_0}=a_{\gamma_0}e^{(i/\hbar)\omega_{\gamma_0}},\quad a^{\gamma}=a_{\gamma}e^{-u/\hbar},$$

where a_{γ_0} and a_γ are resurgent series with regular dependence on E, of the form $1+0(\hbar)$. Setting

$$\Omega^{\text{med}}=\omega_\gamma+\frac{\hbar}{i}\left(\ln a_{\gamma_0}+\frac{1}{2}\ln(1+a^{\gamma})\right),$$

one gets a resurgent symbol with regular dependence on E, such that

$$J^{\text{med}}=1+e^{(i/\hbar)\Omega^{\text{med}}}.$$

Defining Ω as the median resummation of this symbol, we therefore get the announced form for the Jost function, and there only remains to prove that the function $\Omega(E)$ has a non zero derivative, a fact which is easily checked on its symbol. Let us do it in detail here, so as to precise also the reality properties of the functions involved. One has

$$\frac{d\Omega^{\text{med}}}{dE}=\frac{d\omega_{\gamma_0}}{dE}+\frac{\hbar}{i}\left(\frac{d}{dE}\ln a_{\gamma_0}-\varepsilon\right),$$

where

$$\epsilon = \frac{1}{2}\frac{d}{dE}\ln(1+a^{\gamma}) = \frac{1}{2}(1+a^{\gamma})^{-1}\frac{d}{dE}a^{\gamma} \sim -\frac{1}{2}\frac{du}{dE}e^{-u\hbar},$$

is a strictly positive, exponentially small quantity. Since $d\omega_{\gamma_0}/dE$ is a positive real, appreciably large quantity (the time period $T_{\gamma 0}$ of the oscillator cycle), this ends the proof of the proposition.□

Furthermore it easily follows from formula (1.3) in subsection I A that for any well cycle γ_0 the term $\ln a^{\gamma_0}$ is pure imaginary, so that the imaginary part of $d\Omega^{\text{med}}/dE$ is just $-\hbar\epsilon$. Since median resummation commutes with complex conjugation, we thus see that

$$\mathfrak{I}\frac{d\Omega}{dE} = -\epsilon,$$

so that $d\Omega/dE$ is a complex number close to T_{γ_0}, slightly below the positive real axis. It follows that the resonance energies are slightly above the real axis, as could be expected from physical considerations (recall that with our conventions the time dependence of the wave functions is $e(i/\hbar)Et$).

2. Stable equilibrium

Let now the energy be close to an absolute (quadratic) minimum of the potential function. Rescaling the energy as explained in Section III, let

$$s = -\frac{1}{2} + \frac{T_0}{2\pi}E_r + 0(\hbar)$$

be the monodromy exponent of the well bottom.

For every natural integer n the equation

$$s(E_r,\hbar) = n \tag{5.1}$$

obviously has a unique formal power series solution,

$$E_n(\hbar) = \frac{2\pi}{T_0}\left(n+\frac{1}{2}\right) + E_{n,1}\hbar + E_{n,2}\hbar^2 + \cdots \tag{5.2}$$

(the Rayleigh–Schrödinger series).

Theorem V.1.1: The Rayleigh–Schrödinger series is Borel resummable, and its Borel sum (defined when n is not too large compared with $1/\hbar$) gives the n-th energy level.

Proof: Let $(\phi,\phi^{\star})$ be some Jost basis such that

(1) ϕ is the Borel sum, for large negative q, of a simple WKB expansion well normalized at $-\infty$;

(2) $\phi^{\star}$ is the right or left (or median) sum, for large positive q, of another simple WKB expansion, well normalized along some path which crosses no bounded Stokes line.

Then it immediately follows from Proposition III.4.1 that the corresponding Jost symbol is Borel resummable. By Theorem III.3.1 it reads as

$$J = \frac{\sqrt{2\pi}}{\Gamma(-s)}\hbar^{(s+1/2)}c(E_r,\hbar), \tag{5.3}$$

where $c(E_r,\hbar)$ is an invertible resurgent power series in $\hbar$, depending regularly on E_r.

Since the monodromy exponent $s=s(E_r,\hbar)$ is also a resurgent power series in $\hbar$ depending regularly on E_r, the formal operation of substituting any resurgent power series to E_r in the right hand side of (5.3) can be interpreted as an operation on true functions: all one has to do is replace all formal objects by their right sum (or by their left sum, if one prefers). Since $c(E_r,\hbar)$ is invertible, the only way to get the zero function $J=0$ is to choose $E_r=E_r^n(\hbar)$ in such a way that $s(E_r,\hbar)=n$, a natural integer. Formally speaking this determines the series (5.2) unambiguously, and by the implicit resurgent function theorem this series is indeed resurgent.

We can thus conclude that the right (say) sum of the Rayleigh–Schrödinger series (5.2) is the n-th zero of the Jost function. But since the same reasoning holds for the left sum, it therefore follows that the Rayleigh–Schrödinger series is Borel resummable. □

Remark: From the fact that the symbol (5.3) is Borel resummable one should not infer that each individual factor in the right-hand side of (5.3) is Borel resummable. This holds only in the simple pattern case (Ref. 12, Sec. 2.5.1), because in that case s is Borel resummable. In the singular pattern case it is easily checked that s is not Borel resummable: more precisely, by Theorem 2.5.1 of Ref. 12 one has

$$\mathfrak{S}e^{2i\pi s}=(1+a^{[L]})^{-2}e^{2i\pi s},$$

where $[L]$ is the connection cycle associated to the bounded Stokes line L of Fig. 42 (or its complex conjugate $\overline{L}$, yielding the same Voros multiplier $a^{[\overline{L}]}=a^{[L]}$). Taking logarithms of both sides, one gets

$$\mathfrak{S}s=s-\frac{1}{i\pi}\ln(1+a^{[L]}), \tag{5.4}$$

showing that s is not Borel resummable.

But since the cycle $[L]$ ''goes through'' the double turning point, similar arguments as those of section IV show that $a^{[L]}$ contains a $[\sqrt{2\pi}/\Gamma(-s)]\hbar^{(s+1/2)}$ factor, which vanishes when the Rayleigh–Schrödinger series is substituted to E_r. This explains the apparent ''paradox'' that the solution of equation (5.1) is Borel resummable, although the equation itself is not.

Computation of the Rayleigh–Schrödinger series: For every given $n\in N$, the implicit equation $s(\hbar,E_r)=n$ can be formally solved by the following algorithm (in the same spirit see also Ref. 38):

We construct the formal series

$$E_n(\hbar)=\sum_{k\geqslant 0}E_{n,k}\hbar^k,$$

jointly with the formal series $Y(q,E_r,\hbar)$ introduced in Sec. III B (computation of the monodromy exponent), by demanding that the residue of iY at the double turning point $q_{\rm crit}$ should equal n.

The first couple $Y_1(q,E_r)$, $E_{n,0}$ is given by the equation

$$Y_1=\frac{1}{2Y_0}\left(E_{n,0}+i\frac{d}{dq}Y_0\right),$$

where iY_1 is required to have residue n at $q_{\rm crit}$. Then for every $l\geqslant 1$ the couple $Y_{l+1}(q,E_r)$, $E_{n,l}$ is given by the equation

$$Y_{(l+1)}=\frac{1}{2Y_0}\left(E_{n,l}+i\frac{d}{dq}Y_n-\sum_{1\leqslant k\leqslant l}Y_kY_{(l-k)}\right),$$

where iY_{l+1} is required to have residue 0 at $q_{\rm crit}$.

Of course it is enough to work with the Laurent expansions of the Y_l's. For every $l \geqslant 1$, since Y_l has a pole of order (at most) $(2l-1)$ in q_{crit}, computing $E_{n,l}$ requires computing the $(2l+1)$ first terms of the Laurent expansions of each of the Y_k's, $1 \leqslant k \leqslant l$.

It follows from this analysis that each $E_{n,l} = E_l(n)$ is a polynomial in n, of degree (at most) $(l+1)$, with real coefficients (and even rational coefficients if the Taylor expansion of p_{crit} at q_{crit} has rational coefficients).

Besides, relation $s(-E_r, -\hbar) + 1/2 = -s(E_r, \hbar) - 1/2$ implies that the expansion $E_n(\hbar) = E(n, \hbar)$ satisfies the functional relation

$$E(\mu - \tfrac{1}{2}, \hbar) = -E(-\mu - \tfrac{1}{2}, -\hbar),$$

so that for every $l \geqslant 0$,

$$E_l(\mu - \tfrac{1}{2}) = P_l(\mu),$$

where P_l is an odd (resp., even) polynomial of degree (at most) $(l+1)$ if l is even (resp., odd).

Example: The simple anharmonic oscillator: We consider after Bender and Wu[2] the following simple anharmonic oscillator $V(q) = 1/4(q^2 + q^4)$. The preceding algorithm implemented under Maple yields the following result where we have set $\mu = n + 1/2$:

$$E_n = \mu + \left(\frac{3}{2}\mu^2 + \frac{3}{8}\right)\hbar - \left(\frac{17}{4}\mu^3 + \frac{67}{16}\mu\right)\hbar^2 + \left(\frac{375}{16}\mu^4 + \frac{1707}{32}\mu^2 + \frac{1539}{256}\right)\hbar^3$$
$$-\left(\frac{10689}{64}\mu^5 + \frac{89165}{128}\mu^3 + \frac{305141}{1024}\mu\right)\hbar^4 + O(\hbar^5).$$

Coming back to the n variable, this gives

$$E_n = \left(n + \frac{1}{2}\right) + \left(\frac{3}{2}n^2 + \frac{3}{2}n + \frac{3}{4}\right)\hbar - \left(\frac{17}{4}n^3 + \frac{51}{8}n^2 + \frac{59}{8}n + \frac{21}{8}\right)\hbar^2$$
$$+\left(\frac{375}{16}n^4 + \frac{375}{8}n^3 + \frac{177}{2}n^2 + \frac{1041}{16}n + \frac{333}{16}\right)\hbar^3$$
$$-\left(\frac{10689}{64}n^5 + \frac{53445}{128}n^4 + \frac{71305}{64}n^3 + \frac{80235}{64}n^2 + \frac{111697}{128}n + \frac{30885}{128}\right)\hbar^4 + O(\hbar^5).$$

The case $n=0$ corresponds to the case of the ground state, already computed by Bender and Wu.[2] In the same article Bender and Wu have estimated numerically the asymptotic growth of the sequence of $E_{0,k}$ for large k,

$$E_{0,k} \sim (-1)^{k+1}\left(\frac{6}{\pi^3}\right)^{1/2}\Gamma\left(k + \frac{1}{2}\right)3^k,$$

a result proven afterwards by Harrel and Simon[39] (see also Refs. 40, 13).

Let us show how resurgence theory yields another rigorous proof of this result, and more generally of the following large k asymptotic formula for the n-th energy level,

$$E_{n,k} \sim (-1)^{k+1}\left(\frac{6}{\pi^3}\right)^{1/2} 12^n \frac{\Gamma(n+k+1/2)}{\Gamma(n+1)} 3^k.$$

The idea is to analyze the singularities of the Borel transform of the monodromy exponent s, and then use Ecalle's ''alien calculus'' to deduce from it the singularities of the Borel transform of

E_n. Using again Theorem 2.5.1 of Ref. 12 it is easily checked that the Borel transform of s has singularities in two directions only: the positive real direction, where they are described by formula (5.4), and the negative real direction, where they are given by a similar formula,

$$\mathfrak{S}_{(\pi)} s = s + \frac{1}{i\pi} \ln(1 + a^{\mathscr{L}}), \tag{5.4$'$}$$

where the connection cycle $\mathscr{L}$ bears the same relationship with $[L]$ of fig. 42 as the cycle of fig. 58 does with that of fig. 57: one thus has

$$a^{\mathscr{L}} a^{[L]} = e^{2i\pi s} - 1,$$

with

$$a^{[L]} = \frac{\sqrt{2\pi}}{\Gamma(-s)} \left(\frac{4}{\hbar}\right)^{-(s+1/2)} e^{-u_L/\hbar}(1 + O(\hbar))$$

(here $u_L = 1/3$). Translating equation (5.4$'$) in terms of alien derivatives (cf. for instance Ref. 12) yields

$$\dot{\Delta}_{-lu_L} s = \frac{(-1)^{l+1}}{i\pi l} (a^{\mathscr{L}})^l (l = 1,2,\ldots).$$

Taking alien derivatives of equation $s(E_n,\hbar) = n$,

$$\dot{\Delta}_{-lu_L} s(E_r,\hbar)|_{E_r = E_n} + (\dot{\Delta}_{-lu_L} E_n) \frac{\partial}{\partial E_r} s(E_r,\hbar)|_{E_r = E_n} = 0,$$

we thus get in particular (for $l = 1$),

$$\dot{\Delta}_{-u_L} E_n = -\frac{1}{i\pi} \frac{a^{\mathscr{L}}}{\frac{\partial}{\partial E_r} s(E_r,\hbar)} |_{E_r = E_n} = -\frac{1}{i\pi} \frac{\sqrt{2\pi}}{\Gamma(n+1)} \left(\frac{4e^{i\pi}}{\hbar}\right)^{(n+1/2)} e^{u_L/\hbar}(1 + O(\hbar)). \tag{5.5}$$

In other words the Borel transform of E_n has its closest singularity at $-u_L$ (recall that it has no singularity along the positive real axis, by the above remark), and the nature of this singularity can be read on formula (5.5); from this ''resurgence formula'' in the sense of Ecalle we immediately get what Berry and Howls[24,25] call a ''resurgence formula in the sense of Dingle:''

$$E_{n,k} \sim -\frac{1}{2i\pi} \frac{(-1)^{k+1}\Gamma\left(n+k+\frac{1}{2}\right)}{u_L^{n+k+1/2}} \frac{1}{i\pi} \frac{\sqrt{2\pi}}{\Gamma(n+1)} 4^{(n+1/2)},$$

a formula equivalent to the announced one. Remark here that this strategy could be exploited for double wells, thus giving an answer to a question of Simon.[40]

The knowledge on the asymptotic growth of the $E_{n,k}$'s yields a precise numerical computation by ''resummation to the least term.'' Actually many formal and numerical procedure for resummation are available, see for instance Ref. 42. In this way it would be interesting to compare the numerical computations based on Padé approximants,[41,40] with the powerful hyperasymptotics methods of Refs. 25 and 26.

3. Resonance energies near a metastable equilibrium

Looking now near the bottom of a ''cubic well'' (subsections III F and IV A), we expect the zeros of the Jost function to be complex, and interpretable as resonance energies.

Working with left symbols will make the formulas look very similar to those of the previous subsection: using formula (4.1) for the Voros multiplier $a^{[L]}$, it is easily seen on formula $(4.3)^-$ (using Euler's reflection formula) that the left Jost symbol J^- equals $1/\Gamma(-s)$ times an invertible factor. The quantization condition for left symbols therefore reads as

$$s(\hbar,E_r)=n \quad \text{(a natural integer)}, \tag{5.6}$$

exactly as in Sec. V A 2, and for the same reason it admits a unique formal solution,

$$E_n^-(\hbar)=\frac{2\pi}{T_0}\left(n+\frac{1}{2}\right)+E_{n,1}\hbar+E_{n,2}\hbar^2+\dots \tag{5.7}$$

(the Rayleigh–Schrödinger series) which is resurgent, and whose left-sum is a zero of the Jost function. Since this zero reads $(2\pi/T_0)(n+1/2)+0(\hbar)$, it can be interpreted as the rescaled energy of the n-th resonance level.

Now the big difference with Sec. V A 2 is the fact that this (real valued) resurgent series is not Borel resummable (if it were, its Borel sum would be real, contradicting physical expectations). To understand the mathematical reason for that difference, notice that by equation (4.4) one has

$$\mathfrak{G}s=s+\frac{1}{2i\pi}\ln(1+a^{[L]}),$$

a formula looking like formula (5.4), with the important difference that $a^{[L]}$ no longer vanishes for $s=n$, because formula (4.1) for $a^{[L]}$ now contains a $1/\Gamma(s+1)$ factor instead of a $1/\Gamma(-s)$ factor (the connection cycle $[L]$ of fig. 56 crosses the double turning point on the opposite sheet, compared to that of fig. 42).

Another way of understanding this is to compare the result of the above computation with what we would get by solving equation $J^+=0$. Using equation $(4.3)^+$, the vanishing of the right Jost symbol J^+ is easily seen to be equivalent to the equation

$$-\frac{\sqrt{2\pi}}{\Gamma(-s)}\,e^{-i\pi(s+1/2)}=\epsilon(\hbar,E_r), \tag{5.8}$$

where

$$\epsilon(\hbar,E_r)=\left(\frac{c}{\hbar}\right)^{s+1/2}e^{-u_L/\hbar}\underbrace{a(\hbar,E_r)}_{1+0(\hbar)}. \tag{5.9}$$

Since ϵ is exponentially small this condition can be satisfied only for $s\simeq n$, a natural integer. Noticing that the left-hand side of (5.8), considered as a function of s, has a simple zero at $s=n$, with coefficient $-i\sqrt{2\pi}n!$, we have

$$\frac{\partial}{\partial E_r}[\text{left-hand side of (5.8)}]\Big|_{\hbar=0}^{E_r=E_n^-(\hbar)}=-i\sqrt{2\pi}n!\,\frac{T_0}{2\pi}\neq 0,$$

thus warranting the existence of a unique formal solution of equation (5.8), of the form

$$E_n^+(\hbar)=E_n^-(\hbar)+\sum_{\geqslant 1} E_n^{(k)}(\hbar)\epsilon_n^k, \tag{5.10}$$

with

$$\epsilon_n=\left(\frac{c}{\hbar}\right)^{n+1/2} e^{-u_L/\hbar},$$

whereas $E_n^{(k)}(\hbar)$ is a polynomial of degree $(k-1)$ in $\ln\hbar$ [this polynomial dependence on $\ln\hbar$ stems from the fact that the power series expansion of $(c/\hbar)^{s+1/2}$ with respect to E_r has a $\ln\hbar$ factor in every term of positive degree], with integral power series of $\hbar$ as its coefficients; in particular, $E_n^{(1)}(0)=(i/\sqrt{2\pi}n)(2\pi/T_0)$.

Remembering that both sides of equation (5.8) depend regularly on E_r, the implicit resurgent function theorem (in its extended form shown in appendix 2) allows us to conclude that the formal expansion (5.9) is a ''regular'' resurgent symbol, and that the right-sum of this symbol is a zero of the Jost function J, which is nothing of course but the rescaled energy of the n-th resonance level.

In other words, E_n^- [eq. (5.7)] and E_n^+ [eq. (5.10)] are, respectively, the left and right symbol of the (rescaled) n-th resonance level. More readable information on the real and imaginary part of the energy can be seen on the median symbol, which can be computed in completely analogous fashion, solving the equation $J^{\text{med}}=0$: using formula $(4.3)^{\text{med}}$ one easily checks that the ''median'' quantization condition can be written in the same form as eq. (5.8), with ϵ replaced by

$$\epsilon^{\text{med}}(\hbar,E_r)=\frac{e^{i\pi s}}{2\cos(\pi s)}\left(\frac{c}{\hbar}\right)^{s+1/2} e^{-u_L/\hbar}\underbrace{a(\hbar,E_r)}_{1+0(\hbar)},$$

yielding for E_n^{med} an expression similar to (5.10),

$$E_n^{\text{med}}(\hbar)=E_n^-(\hbar)+\sum_{k\geqslant 1} E_n^{\text{med}(k)}(\hbar)\epsilon_n^k, \tag{5.11}$$

with $E_n^{\text{med}(k)}(\hbar)$ a polynomial of degree $(k-1)$ in $\ln\hbar$ with an integral power series of $\hbar$ as its coefficients; in particular, $E_n^{(1)}(0)=(i/2\sqrt{2\pi}n!)(2\pi/T_0)$, yielding the principal part of the resonance width.

Example: We consider the cubic oscillator $V(q)=-q^3+q$. According to the results of Sec. IV A, Example, we get

$$E_0^-(\hbar)=3^{1/4}-\frac{11}{48}\hbar-\frac{155}{2304}3^{3/4}\hbar^2-\frac{39709}{331776}\sqrt{3}\hbar^3+O(\hbar^4),$$

$$\epsilon_0=\left(96\frac{3^{1/4}}{\hbar}\right)^{1/2} e^{-(8/5)(3^{1/4}/\hbar)},$$

$$E_0^{(1)}(\hbar)=\frac{i\sqrt{2}}{\sqrt{\pi}}\left(3^{1/4}-\frac{169}{192}\hbar+O(\hbar^2)\right)$$

$$E_0^{(2)}(\hbar)=-\frac{3^{1/4}}{\pi}\left((i\pi+\gamma)(1+O(\hbar))+\ln\left(96\frac{3^{1/4}}{\hbar}\right)(1+O(\hbar))\right),$$

where γ is the Euler's constant, whereas

$$E_0^{\mathrm{med}(1)}(\hbar)=\frac{i\sqrt{2}}{2\sqrt{\pi}}\left(3^{1/4}-\frac{169}{192}\hbar+O(\hbar^2)\right),$$

$$E_0^{\mathrm{med}(2)}(\hbar)=-\frac{3^{1/4}}{4\pi}\left((2i\pi+\gamma)(1+O(\hbar))+\ln\left(96\frac{3^{1/4}}{\hbar}\right)(1+O(\hbar))\right).$$

B. Quantization of double oscillators (proof of the Zinn-Justin conjecture)

Let us go back to the results of Sec. IV B which described the right, left and median Jost symbols for a double absolute minimum (two quadratic wells at the same level, separated by a ''tunnel''): finding bound states amounts to equating to zero these Jost symbols; using Euler's reflection formula and formula (4.5) for the Voros multiplier $a^{[L]}$, one easily see that the quantization condition for right and left symbols are complex conjugated and read as

$$\frac{2\pi}{\Gamma(-s_1)\Gamma(-s_2)}\frac{1}{e^{\pm i\pi(s_1+s_2)}}=\epsilon(\hbar,E_r), \tag{5.12$^\pm$}$$

where $\pm=+$ for the right symbols and $\pm=-$ for the left symbols, whereas

$$\epsilon(\hbar,E_r)=\left(\frac{c}{\hbar}\right)^{(s_1+s_2+1)}e^{-u_L/\hbar}\underbrace{a(\hbar,E_r)}_{1+0(\hbar)}. \tag{5.13}$$

One shall recover the reality properties of the energy by working with median symbols: the quantization condition near a two-fold absolute minimum is therefore equivalent to the equation

$$\frac{1}{\cos\,\pi(s_1+s_2)}\frac{2\pi}{\Gamma(-s_1)\Gamma(-s_2)}=\epsilon^{med}(\hbar,E_r), \tag{5.14}$$

with
$$\varepsilon^{\mathrm{med}}(\hbar,E_r)=\frac{2}{1+(1+a^{[L]})^{1/2}}\left(\frac{c}{\hbar}\right)^{(s_1+s_2+1)}e^{-u_L/\hbar}\underbrace{a(\hbar,E_r)}_{1+0(\hbar)}, \tag{5.15}$$

i.e.

$$\frac{\varepsilon^{\mathrm{med}}(\hbar,E_r)}{\left(\frac{c}{\hbar}\right)^{(s_1+s_2+1)}e^{-u_L/\hbar}a}=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(n+2)}\left(\frac{2\pi\left(\frac{c}{\hbar}\right)^{(s_1+s_2+1)}e^{-u_L/\hbar}a}{\Gamma(s_1+1)\Gamma(s_2+1)}\right)^n.$$

Generically the monodromy exponents s_1 and s_2 differ: since ε (or $\varepsilon^{\mathrm{med}}$) is exponentially small, each bottom of well will contribute, in the principal part of the spectral symbols, a factor analogous to the simple absolute minimum case, corresponding to $s_1\sim n$ or $s_2\sim n$, where n is a natural integer. Choosing one of them, say

$$s_1=-\frac{1}{2}+\frac{T_0}{2\pi}E_r+0(\hbar)\sim n,$$

we thus define a sequence of simple formal real resurgent functions $(E_n^{(0)}(\hbar))_{n\in N}$,

$$E_n^{(0)}(\hbar)=\frac{2\pi}{T_0}\left(n+\frac{1}{2}\right)+E_{n,1}^{(0)}\hbar+E_{n,2}^{(0)}\hbar^2+\ldots,$$

which we call the principal values of energy levels in the corresponding well (the Rayleigh–Schrödinger series).

For the spectral symbols to vanish exactly we must add to those principal values suitable exponentially small corrections, yielding the ''multi-instanton expansions.'' Special attention has to be taken near those values of the potential where both s_1 and s_2 are similar to natural integers for some values of the energy, leading to the so-called ''avoided crossing phenomena'' (cf. Ref. 18). For the sake of simplicity we shall focus only on the symmetrical double well.

Symmetrical double well: In that case the monodromy exponents are equal, $s_1=s_2=s$. Equation ($5.12^{\pm}$) then factorizes to give the Zinn-Justin formula,[43–45,13,14]

$$\frac{\sqrt{2\pi}}{\Gamma(-s)}\frac{1}{e^{\pm i\pi s}}=\epsilon(\hbar,E_r), \tag{$5.16^{\pm}$}$$

where

$$\epsilon(\hbar,E_r)=\mathrm{sgn}\sqrt{\varepsilon}(\hbar,E_r)=\mathrm{sgn}\left(\frac{c}{\hbar}\right)^{(s+1/2)}e^{-u_L/2\hbar}\underbrace{a^{1/2}(\hbar,E_r)}_{1+0(\hbar)}, \tag{5.17}$$

whereas sgn$=\pm$. The same factorization occurs with equation (5.14) as well, leading to the following equality for median symbols:

$$\frac{1}{\sqrt{\cos 2\pi s}}\frac{\sqrt{2\pi}}{\Gamma(-s)}=\epsilon^{\mathrm{med}}(\hbar,E_r), \tag{5.18}$$

with

$$\epsilon^{\mathrm{med}}(\hbar,E_r)=\mathrm{sgn}\sqrt{\varepsilon^{\mathrm{med}}}(\hbar,E_r),$$

that is

$$\epsilon(\hbar,E_r)=\mathrm{sgn}\sqrt{\varepsilon}(\hbar,E_r)=\mathrm{sgn}\left(\frac{c}{\hbar}\right)^{(s+1/2)}e^{-u_L/2\hbar}\underbrace{a^{1/2}(\hbar,E_r)}_{1+0(\hbar)}, \tag{5.19}$$

i.e.

$$\frac{\epsilon^{\mathrm{med}}(\hbar,E_r)}{\mathrm{sgn}\left(\frac{c}{\hbar}\right)^{(s+1/2)} e^{-u_L/2\hbar}a^{1/2}}=1-\frac{1}{8}\,\frac{2\pi(c/\hbar)^{(2s+1)}e^{-u_L/\hbar}a}{\Gamma^2(s+1)}+\frac{7}{128}\left(\frac{2\pi(c/\hbar)^{(2s+1)}e^{-u_L/\hbar}a}{\Gamma^2(s+1)}\right)^2+\dots\,.$$

Equations ($5.16^{\pm}$) and (5.18) are quite similar to equation (5.8) and can be solved in the same way: noticing that

$$\frac{\partial}{\partial E_r}\,[\text{left-hand side of } (5.16^{\pm})]\big|_{E_r=E_n^{(0)}(\hbar)}=\frac{\partial}{\partial E_r}\,[\text{left-hand side of } (5.18)]\big|_{E_r=E_n^{(0)}(\hbar)}$$

and

$$\frac{\partial}{\partial E_r}\,[\text{left-hand side of } (5.16) \text{ or } (5.18)]\big|_{\hbar=0}^{E_r=E_n^{(0)}(\hbar)}=-\sqrt{2\pi}n!\,\frac{T_0}{2\pi}\neq 0,$$

we thus deduce the existence of a unique formal solution of equation (5.16), of the form (see also Ref. 13)

$$E_n^{\pm}(\hbar)=E_n^{(0)}(\hbar)+\sum_{k\geqslant 1}E_n^{(\pm k)}(\hbar)\epsilon_n^k, \tag{$5.20^{\pm}$}$$

where E_n^{+} (resp., E_n^{-}) stands for right (resp., left) symbols, with

$$\epsilon_n=\mathrm{sgn}\left(\frac{c}{\hbar}\right)^{n+1/2} e^{-u_L/2\hbar}, \tag{5.21}$$

and similarly the existence of a unique formal solution of equation (5.18), of the form

$$E_n^{\mathrm{med}}(\hbar)=E_n^{(0)}(\hbar)+\sum_{k\geqslant 1}E_n^{(\mathrm{med}\ k)}(\hbar)\epsilon_n^k. \tag{5.22}$$

The $E_n^{(\pm k)}(\hbar)$ and $E_n^{(\mathrm{med}\ k)}(\hbar)$ are polynomials of degree $(k-1)$ in $\ln\hbar$ with integral power series of $\hbar$ as its coefficients. As a remark notice the reality of the series $E_n^{(\pm 1)}(\hbar)$; furthermore the following equality holds:

$$E_n^{(+1)}(\hbar)=E_n^{(-1)}(\hbar)=E_n^{(\mathrm{med}\ 1)}(\hbar)$$

(this result, which follows from the above computation, can be also directly deduced by an analysis of the resurgence structure), and in particular,

$$E_n^{(\pm 1)}(0)=E_n^{(\mathrm{med}\ 1)}(0)=-\frac{1}{\sqrt{2\pi}n!}\,\frac{2\pi}{T_0}.$$

Now the regularity on E_r of both equations (5.16) and (5.18) allows us to conclude that the formal expansion ($5.20^{\pm}$) and (5.22) are regular resurgent symbols, as a consequence of the (extended) implicit resurgent function theorem (cf. appendix 2). The right-sum-resp., left-sum, resp., median-sum- of the formal expansion (5.20^{+})-resp., (5.20^{-}), resp., (5.22)-is the rescaled energy of the n-th bound state level. Remark here that the choice of the sign sgn in (5.21) determines the parity of the corresponding eigenfunction. We have thus proved a conjecture of Zinn-Justin.[43–45,13,14]

Numerical example: Let us consider the following quartic oscillator $V(q)$:

$$V(q)=q^4-q^2.$$

Following the results of Sec. IV B, example, we get

$$E_n^{(0)}(\hbar)=(2n+1)\sqrt{2}-(3n^2+3n+1)\hbar-\frac{\sqrt{2}}{8}(34n^3+51n^2+35n+9)\hbar^2$$

$$-\left(\frac{375}{16}n^4+\frac{375}{8}n^3+\frac{99}{2}n^2+\frac{417}{16}n+\frac{89}{16}\right)\hbar^3+O(\hbar^4),$$

and

$$\epsilon_n=\mathrm{sgn}\left(\frac{4\sqrt{2}}{\hbar}\right)^{n+1/2}e^{-\sqrt{2}/3\hbar},$$

while for instance,

$$E_0^{(\pm 1)}(\hbar)=-\frac{2}{\sqrt{\pi}}\left(1-\frac{71\sqrt{2}}{48}\hbar\right)+O(\hbar^2)$$

and

$$E_0^{(+2)}(\hbar)=\frac{\sqrt{2}}{\pi}\left(i\pi+\gamma+\ln\left(\frac{4\sqrt{2}}{\hbar}\right)\right)+O(\hbar).$$

APPENDIX: RESURGENT FUNCTIONS DEFINED BY IMPLICIT EQUATIONS

1. Implicit equations in rings of formal power series

Given a (commutative) ring R, one denotes by $R[[X_1,\ldots,X_n]]$ the ring of formal power series in n indeterminates $X_1,\ldots,X_n$, with coefficients in R. This ring is naturally ''filtered'' by the ''order'' of its elements, the order of a formal power series being defined as the smallest degree of its constituent monomials; in particular the elements of order 0 are those with non vanishing constant term; those of strictly positive order will be called the small elements.

A sequence $(g_n)_{n=0,1,2,\ldots}$ of elements of $R[[X_1,\ldots,X_n]]$ is said to converge formally to an element $g\in R[[X_1,\ldots,X_n]]$ if $\lim_{n\to\infty}\mathrm{ord}(g-g_n)=\infty$. In that case g is of course unique. The following obvious statement is a formal version of the Cauchy convergence criterion:

for a sequence $(g_n)_{n=0,1,2,\ldots}$ to converge formally in $R[[X_1\ldots,X_n]]$, it is necessary and sufficient that $\lim_{n\to\infty}\mathrm{ord}(g_{n+1}-g_n)=\infty$.

The invertible elements of $R[[X_1,\ldots,X_n]]$ are those elements of order 0 whose constant term is invertible in R. In particular when R is a field (e.g., $R=\mathbb{R}$ or $\mathbb{C}$) the invertible elements are all elements of order 0 (except 0, which by convention has arbitrary order but is of course not invertible). In the applications we have in mind the ring R of coefficients will either be a field, or a ring of formal power series over a field: e.g., we can write $\mathbb{C}[[X,Y]]=R[[Y]]$ with $R=\mathbb{C}[[X]]$; in such a case one should carefully distinguish the filtration in $R[[Y]]$ (where X is considered as a ''constant'') from the so-called total filtration (where ord $X=$ord $Y=1$).

An important example of formal convergence is provided by the operation of substituting a small element to the indeterminate Y of a ring of formal power series: let R be a ring of formal power series (say, in n indeterminates $X_1,\ldots,X_n$), and let $f\in R[[Y]]$, where Y is another indeterminate; then any small element $u\in R$ can be substituted to Y in f, yielding an element of R which will be denoted by $f(u)\in R$.

Example: For any small $u \in R$, the inverse of $1-u$ in R can be obtained by substituting u to Y in the formal power series $f=1+Y+Y^2+\ldots\in R[[Y]]$.

With the same hypothesis on R, we have the following (see Refs. 37, 22).

Implicit function theorem: Let $f=f_0+f_1Y+f_2Y^2+\ldots\in R[[Y]]$ be such that f_0 is small, and f_1 is invertible. Then there exists a unique small element $u\in R$ such that $f(u)=0$.

Proof: The equation $f(Y)=0$ can be rewritten as a ''fixed point problem,''

$$Y=-\frac{f_0}{f_1}-\frac{f_2}{f_1}Y^2+\ldots,$$

which can be solved formally by iteration (formal convergence in R being insured by the hypothesis that f_0 is small). □

Example A.1.1: Consider the implicit equation for E_r,

$$s(\hbar,E_r)=n,$$

whose formal solution is the Rayleigh–Schrödinger series (cf. Sec. V A 2). Setting $R=\mathbb{C}[[\hbar]]$ (or $\mathbb{R}[[\hbar]]$), and changing the unknown E_r to $Y=E_r-(n+1/2)(T_0/2\pi)$, we are exactly under the hypotheses of the implicit function theorem.

Example A.1.2: More generally, consider the quantization condition (5.8) of Sec. V A 3. Setting $\epsilon=(c/\hbar)^{n+1/2}/e^{-u/\hbar}$ (here we denote for short by ϵ what was denoted in Sec. V A 3 by ϵ_n), define the new unknown Y by

$$E_r=E^{RS}+\epsilon Y,$$

where $E^{RS}\in R=\mathbb{C}[[\hbar]]$ is the Rayleigh–Schrödinger series. Expanding $s(\hbar,E_r)$ in powers of ϵY,

$$s(\hbar,E_r)=n+\tfrac{1}{2}+a_1(\hbar)\epsilon Y+a_2(\hbar)\epsilon^2Y^2+\ldots,$$

one gets an expansion,

$$\begin{aligned}\left(\frac{c}{\hbar}\right)^{s+1/2}e^{-u/\hbar}&=\epsilon e^{(a_1\epsilon Y+a_2\epsilon^2Y^2+\ldots)\ln(c/\hbar)}\\&=\epsilon\left(1+a_1\epsilon\ln\left(\frac{c}{\hbar}\right)Y+\left(a_2\epsilon^2\ln\left(\frac{c}{\hbar}\right)+\frac{a_1^2}{2!}\epsilon^2\ln^2\left(\frac{c}{\hbar}\right)\right)Y^2+\ldots\right)\\&=\epsilon(1+\alpha),\alpha\in\tilde{\epsilon}YR[[\epsilon,\tilde{\epsilon},Y]],\end{aligned}$$

where we have used the notation

$$\tilde{\epsilon}=\epsilon\ln\frac{c}{\hbar}.$$

Treating $\hbar$, ϵ, $\tilde{\epsilon}$ as independent variables (which they are, from a formal algebraic point of view), one easily checks that the quantization condition satisfies the hypotheses of the implicit function theorem, yielding a unique formal solution,

$$Y \in \epsilon R[[\epsilon, \tilde{\epsilon}]] = \epsilon \mathbb{C}[[\hbar, \epsilon, \tilde{\epsilon}]].$$

Taking the relation between ϵ and $\tilde{\epsilon}$ into account, one can expand this solution as a formal power series in ϵ,

$$Y = \epsilon \sum_{k=0}^{\infty} Y_k(\hbar, \ln \hbar) \epsilon^k, \tag{$\star$}$$

where Y_k is a formal power series in $(\hbar, \ln \hbar)$, polynomial of degree $\leqslant k$ in $\ln \hbar$.

2. Resurgent version of the above results

Resurgent power series in $\hbar$ make up a subring $\mathscr{R} \subset \mathbb{C}[[\hbar]]$. Similarly, we shall denote by $\mathscr{R}_{(u)}$ the subring of $\mathbb{C}[[\hbar, u_1, \ldots, u_n]]$ consisting of the resurgent power series in $\hbar$ which depend regularly on the parameter $u = (u_1, \ldots, u_n)$ (in a neighbourhood of $0 \in \mathbb{C}^n$).

The property of being resurgent (resp., resurgent with regular dependence on u) is stable by all formal operations considered in the Appendix, Sec. 1, i.e.

(1) *Invertibility:* The inverse of a formally invertible resurgent series is again resurgent (and regular dependence on parameters is preserved).

(2) *Substitution:* Let $f = f(\hbar, u, y) \in \mathscr{R}_{(u,y)}$; then the operation of substituting to y a small element of $\mathscr{R}_{(u)}$ yields an element $f(\hbar, u, y(\hbar, u)) \in \mathscr{R}_{(u)}$.

(3) *Implicit function theorem:* Let $f = f(\hbar, u, y) \in \mathscr{R}_{(u,y)}$ be such that $f|_{u=y=0}$ is small, and $\partial f / \partial y|_{u=0}$ is invertible. Then the equation $f(\hbar, u, y) = 0$ has a unique small solution $y = y(\hbar, u) \in \mathscr{R}_{(u)}$.

Furthermore, all the above operations are compatible with (right, left, median) resummation, yielding the corresponding operations on true functions of $\hbar, u, y$ (for small enough $\hbar, u, y$).

Example A.2.1: The Rayleigh–Schrödinger series (cf. Example A.1.1) is resurgent.

Extensions of the above results: Everything which has just been said about resurgent power series still holds true for the more general notion of ''formal resurgent function'' (cf. Ref. 12 Sec. 0.2), allowing us to build resurgent objects of a more general nature. For instance, starting from a resurgent power series in $\hbar$ depending regularly on a parameter λ, and performing the substitution $\lambda = \ln \hbar$, we get a formal resurgent function which is a formal power series in $\hbar$ and $\ln \hbar$; the set of all power series in $\hbar$ and $\ln \hbar$ obtained in this way is a subring of the ring of all formal resurgent functions, which we shall hereafter denote by $\tilde{\mathscr{R}}$.

Still more general than ''formal resurgent functions'' are resurgent symbols (cf. Ref. 12, Sec. 0.4), which are essentially formal combinations of exponentials with coefficients in some ring of resurgent series.

Example A.2.2: Consider again Example A.1.2.

• When the parameters ϵ, $\tilde{\epsilon}$ are treated as independent variables, the quantization condition is easily seen to depend regularly on these parameters, so that its solution Y belongs to $\mathscr{R}_{(\epsilon, \tilde{\epsilon})}$.

• Now regular dependence on the parameters allows us to make the suitable substitutions $\epsilon = \epsilon(\hbar)$, $\tilde{\epsilon} = \tilde{\epsilon}(\hbar)$ (tending to zero when $\hbar \to 0$).

First performing the substitution $\tilde{\epsilon} = \epsilon \ln(c/\hbar)$, we can thus consider Y as an element of $\tilde{\mathscr{R}}_{(\epsilon)}$ [this means not only that each coefficient Y_k in the expansion ($\star$) belongs to $\tilde{\mathscr{R}}$ but that the dependence on the parameter ϵ is regular].

Finally remembering what ϵ actually stands for, we thus see that $Y = Y(\hbar)$ is a resurgent symbol. But it is a resurgent symbol of a very peculiar kind, built from an element of $\tilde{\mathscr{R}}_{(\epsilon)}$ by substituting a small exponential to the parameter ϵ: the important feature to remember is the initial regular dependence on ϵ, which we shall summarize by saying that $Y(\hbar)$ is a regularly built resurgent symbol. This implies that for every small enough $\hbar$ not only are all the Y_k's in ($\star$) (right, left, and median) resummable, but that replacing all of them by their (right, left or median)-sums yields a convergent series, whose sum is an exact solution of the (right, left or median) resummed implicit equation.

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