Exact Simulation of Generalised Vervaat Perpetuities

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Abstract

We consider a generalised Vervaat perpetuity of the form $X = Y_1W_1 + Y_2W_1W_2 + ...$, where $W_i \sim \mathcal{U}^{1/t}$ and $(Y_i)_{i\geq 0}$ is an i.i.d sequence of random variables independent from $(W_i)_{i\geq 0}$. Based on a distributional decomposition technique, we propose a novel method for exactly simulating the generalised Vervaat perpetuity. The general framework relies on the exact simulation of the truncated Gamma process, which we develop using a marked renewal representation for its paths. Furthermore, a special case arises when $Y_i = 1$, and X has the generalised Dickman distribution, for which we present an exact simulation algorithm using the marked renewal approach. In particular, this new algorithm is much faster than existing algorithms illustrated in Devroye and Fawzi (2010), Fill and Huber (2010), Chi (2012), and Cloud and Huber (2017) as well as being applicable to the general payments case. Examples and numerical analysis are provided to demonstrate the accuracy and effectiveness of our method.

Keywords: Vervaat perpetuities; Dickman distribution; Truncated Gamma process; Exact simulation; Marked Renewal Process.

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1 Introduction

A stochastic perpetuity is a random variable which takes the form

$$X = Y_1W_1 + Y_2W_1W_2 + Y_3W_1W_2W_3 + ... + Y_nW_1W_2...W_n + ..., \tag{1.1}$$

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where the W_i and Y_i are i.i.d random variables independent of each other. Perpetuities arise in a diverse range of fields such as economics (Embrechts et al., 1996), insurance mathematics (Nyrhinen, 2001), astrophysics (Chandrasekhar and Münch, 1950), analytic number theory (De Bruijn, 1951), and in the analysis of Hoare's selection algorithm (Mahmoud et al., 1995). We refer to Vervaat (1979) and Embrechts and Goldie (1994) for a huge variety of references and examples of applications, ranging from the brightness of the Milky Way to ARCH processes of financial modelling. In the context of economics, the quantity Y_i represents a random cashflow at time i and W_i represents the discount rate from time i to time $i-1$. The random variable X then represents the net present value of the stochastic cashflows.

When $W_i \stackrel{\mathcal{D}}{=} U^{1/t}$, for $U \sim Uniform(0,1)$, the random variable X is known as a generalised Vervaat perpetuity. It was shown in Vervaat (1979) that the series converges if and only if $\mathbb{E}\left(\log^+ |Y_i|\right) < \infty$, and it satisfies the distributional identity

$$X \stackrel{\mathcal{D}}{=} W(X+Y), \tag{1.2}$$

where W and Y represent random variables distributed like W_i and Y_i respectively. Vervaat perpetuity has been much studied. Vervaat (1979) showed that the random variable X is infinitely divisible, and obtained its Lévy Khintchine representation. This is the basis of our simulation framework, and we further show that under the mild assumption that the Laplace transform of Y exists, X turns out to be a Lévy process with Lévy measure proportional to $\frac{1}{y}\mathbb{P}(Y > y)$. There is also no loss of generality to consider only the case when $Y > 0$, as it can be shown that a two-sided perpetuity can be split into the difference of two one-sided perpetuities. Goldie and Grübel (1996) and Grübel and Rösler (1996) studied the tails of these perpetuities. Random variables of this type also appear as the limit of shot noise processes (Takács, 1954, 1955).

In this paper, we are interested in exactly simulating a generalised Vervaat perpetuity. Much attention has been given in the literature to the special case when $Y_i = 1$ almost surely, in which case X has the generalised Dickman distribution (Dickman, 1930; Arratia, 1998). Even so, explicit expressions for the distribution of X are difficult to obtain. The first simulation method is based on a density approximation given in Devroye (2001). Later, Fill and Huber (2010) and Devroye and Fawzi (2010) using a dominated coupling from the past procedure for Markov chains. Cloud and Huber (2017) proposed

an improvement by running both an upper and lower bound on the Markov chain. Chi (2012) suggested an alternative approach based on a rejection sampling procedure. However, these algorithms are efficient only for the Dickman distribution, where the parameter $t = 1$, and their speed slows down considerably when t gets larger than 1.

We develop a faster and more efficient simulation algorithm for the generalised Dickman distribution with parameter t. Since it is viewed here as a Lévy process over the range of parameters t, with Lévy measure $\nu(\mathrm{d}y) = \frac{1}{y}\mathbb{1}_{\{y<1\}}\mathrm{d}y$, we thus call it the Dickman process. The paths of the Dickman process can be characterised by a marked renewal process where the renewal intervals are the hitting times of level 1, and the jumps are the overshoots at this time. By simulating its path along each hitting time of level 1, we obtain a simulation algorithm for the Dickman process. Similar to our method, the approach in Chi (2012) also utilises the truncated density of the perpetuity. However, the advantage of our algorithm is that it is based on a marked renewal process, which not only provides a pathwise probabilistic interpretation of the corresponding Lévy process, it also means that the incremental random variables we need to simulate are i.i.d. pairs, allowing us to employ vectorisation to reduce total computation time. This is not the case for the algorithm suggested in Chi (2012), where the incremental variables depend on the previous ones. We provide comparisons of the simulation speed of our algorithm with the algorithms suggested by Devroye and Fawzi (2010), Fill and Huber (2010), Chi (2012), and Cloud and Huber (2017). We found that our method outperforms the other algorithms and even more substantially when t is large.

Blanchet and Sigman (2011) proposed a modification of the dominated coupling from the past algorithm to sample from generalised Vervaat perpetuities. The algorithm imposes similar regularity conditions as ours on the density of Y_i. However, it also requires Y_i to have a heavy tail. In this paper, we develop a framework for simulating from generalised Vervaat perpetuities. In particular, we prove a distributional decomposition result, which states that a generalised Vervaat perpetuity X can be represented as the sum of a Gamma or truncated Gamma process and independent compound Poisson processes. Thus, it can be sampled by generating each of these components separately and adding them up. The condition we need is

$$\lim_{y \downarrow 0} \frac{\mathbb{P}(Y < y)}{y} < \infty, \tag{1.3}$$

which holds for a wide range of distributions, such as the Gamma (shape parameter larger than 1), Pareto, Weibull (light tailed), Beta, and Normal distributions as well as all discrete distributions (to avoid technicalities 0 is not a possible value). Whenever a density exists, it should possess a finite derivative at 0.

Using the marked renewal representation, we develop an exact simulation algorithm for the truncated Gamma process, which we define to be the Lévy process with Lévy measure $\nu(\mathrm{d}y) = \frac{e^{-\mu y}}{y}\mathbb{1}_{\{y<b\}}\mathrm{d}y$ for some truncation level b. The truncated Gamma process can be obtained from the Gamma process by restricting the size of its jumps to be bounded from above by b. It is also a generalised Vervaat perpetuity with $Y_i \sim V \wedge 1$, where $V \sim Exp(\mu)$. This algorithm is important, as it forms the basis of other algorithms for more general distributions of Y_i. We provide specific examples to demonstrate the methodology, and numerical results and graphical illustrations are given for the cases when Y_i has a Gamma distribution, and when Y_i has a Normal distribution.

This paper will be organised as follows. Section 2 gives a representation of the generalised Vervaat perpetuities as Gamma related processes. We also prove that a two sided perpetuity is the difference of two positive perpetuities. In Section 3, we study the specific cases of a Dickman process and a truncated Gamma process. The marked renewal path representation is described and the exact simulation algorithms developed. Numerical analysis and comparison with existing algorithms are provided. In Section 4, we prove the distributional decomposition result, which enables us to exactly simulate a generalised Vervaat perpetuity. Numerical examples are also given to demonstrate the accuracy and efficiency of our algorithms.

2 Generalised Vervaat Perpetuities

We obtain the following distributional decomposition result for a positive perpetuity $X > 0$.

Theorem 2.1 *Consider the generalised Vervaat perpetuity X with $Y_i > 0$. We define the Laplace transforms $\hat{f}(\beta) = \mathbb{E}(\exp(-\beta X))$ and $\hat{g}(\beta) = \mathbb{E}(\exp(-\beta Y_i))$. Then, we have*

$$\hat{f}(\beta) = \exp\left(-t\int_0^\infty \frac{1-e^{-\beta y}}{y}\mathbb{P}(Y > y)\mathrm{d}y\right). \tag{2.1}$$

Hence, $X \stackrel{\mathcal{D}}{=} Z_t$, where Z_t is a Lévy process with Lévy measure

$$\nu(\mathrm{d}y) = \frac{1}{y}\mathbb{P}(Y > y)\mathrm{d}y. \tag{2.2}$$

Proof. For $W = U^{1/t}$, the density of W is tw^{t-1}. Then from (1.2), we have

$$\hat{f}(\beta) = \int_0^1 \hat{f}(\beta w)\hat{g}(\beta w)tw^{t-1}\mathrm{d}w, \tag{2.3}$$

and therefore,

$$\beta^t \hat{f}(\beta) = \int_0^\beta \hat{f}(w)\hat{g}(w)tw^{t-1}\mathrm{d}w.$$

Differentiating with respect to β on both sides, we have

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\hat{f}(\beta) = -t\frac{1-\hat{g}(\beta)}{\beta}\hat{f}(\beta).$$

Solving the first order differential equation with respect to β, we obtain

$$\begin{aligned}\hat{f}(\beta) &= \exp\left(-t\int_0^\beta \frac{1-\hat{g}(u)}{u}\mathrm{d}u\right) \\ &= \exp\left(-t\int_0^\infty \frac{1-e^{-\beta y}}{y}\mathbb{P}(Y > y)\mathrm{d}y\right),\end{aligned} \tag{2.4}$$

and the result (2.1) follows. Thus, $X \stackrel{\mathcal{D}}{=} Z_t$, where Z_t has Lévy measure defined in (2.2). ■

We can see that Z_t is a Gamma related Lévy process with Lévy measure density proportional to $\mathbb{P}(Y > y)y^{-1}$. For a two-sided perpetuity where Y takes values in the whole real line, the following result shows that it is the difference of two positive perpetuities and thus it suffices to study only positive perpetuities.

Corollary 2.2 *Suppose now that $\mathbb{P}(Y < 0) > 0$. Then,*

$$\hat{f}(\beta) = \exp\left(-t\left(p\int_0^\infty \frac{1-e^{-\beta y}}{y}\mathbb{P}(Y > y|Y > 0)\mathrm{d}y + (1-p)\int_0^\infty \frac{1-e^{\beta y}}{y}\mathbb{P}(Y < -y|Y < 0)\mathrm{d}y\right)\right), \tag{2.5}$$

where $p = \mathbb{P}(Y > 0)$.

Proof. For two sided Y, the Laplace transform of the perpetuity X satisfies the following,

$$\begin{aligned}
& \exp\left(-t\int_0^\beta \frac{1-\hat{g}(u)}{u}\mathrm{d}u\right) \\
= \; & \exp\left(-t\left(\int_0^\beta\int_0^\infty \frac{1-e^{-uy}}{u}\mathbb{P}(Y\in \mathrm{d}y)\mathrm{d}u + \int_0^\beta\int_0^\infty \frac{1-e^{uy}}{u}\mathbb{P}(Y\in \mathrm{d}(-y))\mathrm{d}u\right)\right) \\
= \; & \exp\left(-t\left(\int_0^\beta\int_0^\infty e^{-uy}\mathbb{P}(Y>y)\mathrm{d}y\mathrm{d}u + \int_0^\beta\int_0^\infty e^{uy}\mathbb{P}(Y<-y)\mathrm{d}y\mathrm{d}u\right)\right) \\
= \; & \exp\left(-t\left(\int_0^\infty \frac{1-e^{-\beta y}}{y}\mathbb{P}(Y>y)\mathrm{d}y + \int_0^\infty \frac{1-e^{\beta y}}{y}\mathbb{P}(Y<-y)\mathrm{d}y\right)\right) \\
= \; & \exp\left(-t\left(p\int_0^\infty \frac{1-e^{-\beta y}}{y}\mathbb{P}(Y>y|Y>0)\mathrm{d}y + (1-p)\int_0^\infty \frac{1-e^{\beta y}}{y}\mathbb{P}(Y<-y|Y<0)\mathrm{d}y\right)\right),
\end{aligned}$$

which is easily seen as the difference of two positive perpetuities, one with $W_i \stackrel{\mathcal{D}}{=} \mathcal{U}^{1/pt}$ and $Y_i \stackrel{\mathcal{D}}{=} Y|Y>0$, and the other with $W_i \stackrel{\mathcal{D}}{=} \mathcal{U}^{1/(1-p)t}$ and $Y_i \stackrel{\mathcal{D}}{=} -Y|Y<0$. ■

Our aim is to simulate the generalised Vervaat perpetuity X for a given distribution of Y_i. In the simplest case when $Y_i = 1$ almost surely, X is known as the Vervaat perpetuity and it reduces to the Dickman process. When $Y_i \sim V$, where $V \sim Exp(\mu)$, we obtain the Gamma process with parameter μ, and when $Y_i \sim V \wedge 1$, we obtain the truncated Gamma process. Making use of a marked renewal representation for the paths, we develop fast and exact simulation algorithms for these perpetuities. These cases are important as they form the basis of the other algorithms. We will then give other examples to demonstrate how our methodology can be adapted to general random variables Y_i. Our method works for any Y_i satisfying (1.3) .

3 Generalised Dickman distribution and the Truncated Gamma process

When $Y \sim Exp(\mu)$, we have $\mathbb{P}(Y>y) = e^{-\mu y}$, and thus

$$\hat{f}(\beta) = \exp\left(-t\int_0^\infty (1-e^{-\beta y})\frac{1}{y}e^{-\mu y}\mathrm{d}y\right), \tag{3.1}$$

which is the Laplace transform of a Gamma process with parameter μ. This can be simulated easily by sampling from the Gamma distribution.

When $Y = 1$ almost surely, we have $\mathbb{P}(Y > y) = \mathbb{1}_{\{y<1\}}$, and thus

$$\hat{f}(\beta) = \exp\left(-t\int_0^1 \frac{1-e^{-\beta y}}{y}\mathrm{d}y\right), \tag{3.2}$$

which is the Laplace transform of a generalised Dickman distribution with parameter t. This is also a Lévy process Z_t with Lévy measure $\frac{1}{y}\mathbb{1}_{\{y<1\}}$, and hence we coin it the Dickman process.

When $Y = V \wedge 1$, where $V \sim Exp(\mu)$, we have $\mathbb{P}(Y > y) = e^{-\mu y}\mathbb{1}_{\{y<1\}}$, and thus

$$\hat{f}(\beta) = \exp\left(-t\int_0^1 (1-e^{-\beta y})\frac{e^{-\mu y}}{y}\mathrm{d}y\right), \tag{3.3}$$

which is the Laplace transform of a truncated Gamma process with parameter μ. The truncated Gamma process refers to the Gamma process with truncated Lévy measure $\frac{e^{-\mu y}}{y}\mathbb{1}_{\{y<1\}}$.

To simulate the Dickman process and the truncated Gamma process, we utilise a marked renewal process representation of its sample paths. We now describe the sample paths of truncated subordinators using a marked renewal process.

3.1 The marked renewal process

The paths of the truncated subordinator Z_t can be characterised by a marked renewal process. First, we define a sequence of hitting times $T_1, T_2, T_3, ...$, and denoting $S_i = \sum_{j=1}^{i} T_j$, let

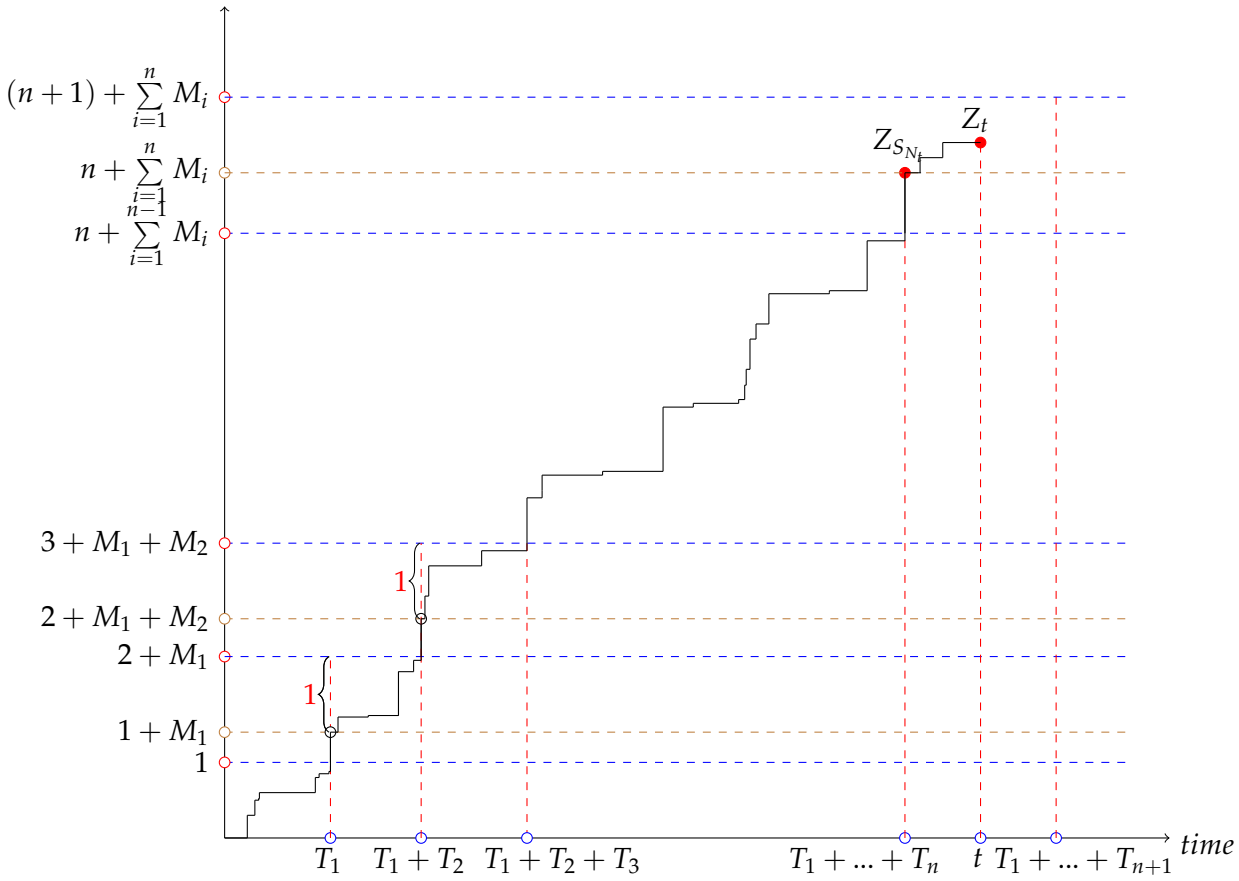
$$T_i := \inf\{t > S_{i-1} | Z_t > Z_{S_{i-1}} + 1\}, \qquad i = 2, 3, ..., \tag{3.4}$$

These $T_1, T_2, ...$ are treated as the holding times for the renewal process. We further define $M_1, M_2, ...$ to be the overshoots at time $S_1, S_2, ...$, i.e.

$$M_i := Z_{S_i} - Z_{S_{i-1}} - 1. \tag{3.5}$$

Hence, at time S_i the process will increase by $(1 + M_i)$ units for all i. Since the process Z_t has independent and stationary increments, each pair of (T_i, M_i) are independent and identically distributed with joint density given in Theorem 3.1. In addition, M_i will be bounded by 0 and 1 for all i as the jump sizes of the process are restricted by 1. The value of the process at time S_n will be $Z_{S_n} = \sum_{i=1}^{n}(1 + M_i)$.

Figure 1: Graphical illustration of a sample path of Z_t



The position of the process Z_t at time t can therefore be expressed as the sum of a marked renewal process with a random variable,

$$Z_t = \sum_{i=1}^{N_t}(1+M_i) + \left(Z_t - Z_{S_{N_t}}\right), \tag{3.6}$$

where $N_t = \sum_{i=1}^{\infty} \mathbb{1}_{\{S_i \leq t\}}$ is the renewal process determined via (3.4) such that $S_{N_t} \leq t < S_{N_t+1}$, and we have

$$Z_t - Z_{S_{N_t}} \overset{\mathcal{D}}{=} Z_{t-S_{N_t}} | S_{N_t+1} > t \overset{\mathcal{D}}{=} Z_t | T > t, \tag{3.7}$$

where the last equality is due to the strong Markov property of Z_t.

We illustrate the marked renewal framework graphically in Figure 1. The sum in the first part of (3.6) represents the position of the truncated process at time S_{N_t} just before it reaches t. The second term in (3.6) represents the final movement of the process within time $t - N_t$. Thus, Z_t can be simulated by generating pairs of hitting times and overshoots (T_i, M_i), stopping when the sum of T_i that have

been generated, say S_{N_t+1}, becomes larger than the input t. We then generate the final part $\{Z_t|T > t\}$ and return the entire sum (3.6).

3.2 Hitting times and overshoots

Now, we obtain analytically the first hitting times and overshoots of the truncated Gamma process and the Dickman process. This is given in the following Theorem 3.1.

Theorem 3.1 *Let T be the first hitting time of level 1 of a truncated Gamma process Z_t with $\mu \geq 0$, and M the overshoot at this time. Then the joint distribution of (T, M) is given by*

$$f_{T,M}(t,m) = \int_m^1 \frac{e^{\Gamma(0,\mu)t}\mu^t}{\Gamma(t)} y^{t-1} \frac{e^{-\mu(1+m)}}{1+m-y} \mathrm{d}y, \qquad \text{for } \mu > 0, \tag{3.8}$$

where $t \in (0,\infty)$, $m \in (0,1)$.

For the Dickman process, the joint density of (T, M) is the limit of (3.8) *as $\mu \to 0$,*

$$f_{T,M}(t,m) = \int_m^1 \frac{e^{-\gamma t}}{\Gamma(t)} \frac{y^{t-1}}{1+m-y} \mathrm{d}y, \tag{3.9}$$

where γ is the Euler–Mascheroni constant, $t \in (0,\infty)$, and $m \in (0,1)$.

Proof. Let $f(x;t)$ denote the density of Z_t. For $\mu > 0$, the density of Z_t restricted to $(0,1)$ can be derived though its Laplace transform. We have

$$\begin{aligned}
f(x;t) &= \mathcal{L}^{-1}\left\{\mathbb{E}\left[e^{-\beta Z_t}\right]\right\}\mathbb{1}_{\{0<x<1\}} \\
&= \mathcal{L}^{-1}\left\{\exp\left(-t\int_0^\infty \left(1-e^{-\beta s}\right)s^{-1}e^{-\mu s}\mathrm{d}s\right)\exp\left(t\int_1^\infty \left(1-e^{-\beta s}\right)s^{-1}e^{-\mu s}\mathrm{d}s\right)\right\}\mathbb{1}_{\{0<x<1\}} \\
&= \mathcal{L}^{-1}\left\{\left(1+\frac{\beta}{\mu}\right)^{-t}\exp\left(t\int_1^\infty s^{-1}e^{-\mu s}\mathrm{d}s\right)\sum_{k=0}^\infty \frac{(-t)^k}{k!}\left(\int_1^\infty e^{-\beta s}s^{-1}e^{-\mu s}\mathrm{d}s\right)\right\}\mathbb{1}_{\{0<x<1\}} \\
&= \mathcal{L}^{-1}\left\{e^{t\Gamma(0,\mu)}\left(1+\frac{\beta}{\mu}\right)^{-t} + e^{t\Gamma(0,\mu)}\left(1+\frac{\beta}{\mu}\right)^{-t}\sum_{k=1}^\infty \frac{(-t)^k}{k!}\left(\int_1^\infty e^{-\beta s}s^{-1}e^{-\mu s}\mathrm{d}s\right)\right\}\mathbb{1}_{\{0<x<1\}} \\
&= \mathcal{L}^{-1}\left\{e^{t\Gamma(0,\mu)}\left(1+\frac{\beta}{\mu}\right)^{-t}\right\}\mathbb{1}_{\{0<x<1\}} \\
&= \frac{e^{\Gamma(0,\mu)t}\mu^t}{\Gamma(t)}x^{t-1}e^{-\mu x}\mathbb{1}_{\{0<x<1\}}.
\end{aligned} \tag{3.10}$$

Using the strong Markov property of Lévy processes, we have

$$\begin{aligned} P\left(T \in dt, M > m\right) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} P\left(Z_{t-\epsilon} \leq 1, Z_t > 1 + m\right) \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^1 P(Z_{t-\epsilon} \in \mathrm{d}y) P(Z_\epsilon > 1 + m - y) \\ &= \int_0^1 \frac{e^{\Gamma(0,\mu)t} \mu^t}{\Gamma(t)} y^{t-1} e^{-\mu y} \int_{1+m-y}^{\infty} z^{-1} e^{-\mu z} \mathrm{d}z \mathrm{d}y, \end{aligned} \tag{3.11}$$

Differentiating (3.11) with respect to m, the joint density of (T, M) in (3.8) directly follows. For the Dickman process, we take the limit as $\mu \to 0$. The density of Z_t within $(0, 1)$ is obtain by taking limit of (3.10), i.e.

$$f(x,t) = \lim_{\mu \to 0} \frac{e^{[\Gamma(0,\mu) + \log(\mu)]t}}{\Gamma(t)} x^{t-1} e^{-\mu x} = \frac{e^{-\gamma t}}{\Gamma(t)} x^{t-1}, \qquad 0 < x < 1, \tag{3.12}$$

with γ being the Euler–Mascheroni constant. Hence, associated distribution of (T, M) directly follows (3.9). ■

Given that the first passage time of Z_t hitting level 1 is greater than t, the distribution of the truncated process Z_t is characterised via its density within $(0, 1)$. The details are in Theorem 3.2.

Theorem 3.2 *The density of $Z_t | T > t$ is given by*

$$f_{Z_t | T > t}(x; t) = \begin{cases} \frac{\mu^t}{\Gamma(t,\mu)} x^{t-1} e^{-\mu x}, & \mu > 0, \\ t x^{t-1}, & \mu = 0, \end{cases} \tag{3.13}$$

where $0 < x < 1$.

Proof. We know that

$$Z_t | T > t \stackrel{\mathcal{D}}{=} Z_t | Z_t < 1, \tag{3.14}$$

since $\{T > t\}$ is equivalent to $\{Z_t < 1\}$. The density therefore satisfies

$$f_{Z_t | T > t}(x; t) = f_{Z_t | Z_t < 1}(x; t) = \frac{f(x; t)}{\int_0^1 f(x; t) \mathrm{d}x}.$$

Thus, the density immediately follows (3.13) . ■

3.3 Exact simulation of the generalised Dickman distribution

Based on the marked renewal framework and choosing suitable acceptance-rejection (A/R) envelopes for $f_{T,M}(t,m)$ and $f_{Z_t|T>t}(x;t)$, we have the following simulation algorithm for the Dickman process, or equivalently, the generalised Dickman distribution.

Algorithm 3.3 ***Simulation of Dickman process (Generalised Dickman distribution)***

1. *Set* $S = 0$, $\gamma = -digamma(1)$.
2. *Generate* (T, M) *via the following steps:*
   (a) *Generate* $U_1 \sim \mathcal{U}(0,1)$, *and set* $T = -\frac{1}{0.8}\log U_1$.
   (b) *Generate* $Y \sim Beta(T, 0.5)$, $U_2 \sim \mathcal{U}(0,1)$, *and set*
   $$M = Y - 1 + (1-Y)\exp\left(-U_2 \log(1-Y)\right).$$
   (c) *Generate* $V \sim \mathcal{U}(0,1)$, *and if*
   $$V \leq \frac{1}{2.35}\frac{\Gamma(0.5)e^{-(\gamma-0.8)T}}{0.8\Gamma(T+0.5)}\frac{(-\log(1-Y))}{(1-Y)^{-0.5}}, \tag{3.15}$$
   then accept (T, M). *Otherwise, reject this candidate and go back to Step 2(a).*
3. *If* $T > t$, *continue to Step 4. Otherwise, set* $S = S + 1 + M$, $t = t - T$, *and go back to Step 2.*
4. *Generate* $U \sim \mathcal{U}(0,1)$ *and set* $X = U^{\frac{1}{t}}$.
5. *Return* $S + X$.

Proof. For the joint density $f_{T,M}(t,m)$, we simulate (T, M, Y) jointly from the integral in (3.9),

$$f(t,m,y) = \frac{e^{-\gamma t}y^{t-1}}{\Gamma(t)}\frac{1}{1+m-y}, \tag{3.16}$$

for $t \in (0,\infty)$, $0 < m < y < 1$. The A/R envelope chosen is

$$g(t,m,y) = \sigma e^{-\sigma t}\frac{\Gamma(t+\eta)}{\Gamma(t)\Gamma(\eta)}y^{t-1}(1-y)^{\eta-1}\frac{\frac{1}{1+m-y}}{(-\log(1-y))}, \tag{3.17}$$

where $\sigma = 0.8$ and $\eta = 0.5$ are chosen numerically via two-dimensional optimisation. Furthermore, $f_{Z_t|T>t}(x;t)$ can be simulated directly via inverse transformation resulting in Step 4. ■

3.4 Exact simulation of the truncated Gamma process

For the truncated Gamma process, one could generate a Dickman process and apply a final A/R scheme. However, the algorithm is not very efficient, especially for large μ as the acceptance rate is exponentially decreasing in μ. Thereby, we propose the following algorithm based on the marked renewal framework to sample truncated Gamma process.

Algorithm 3.4 ***Simulation of truncated Gamma process (Marked renewal approach)***

1. *Set* $S = 0$.

2. *Generate* (T, M) *via the following steps:*

   (a) *Numerically minimise*

   $$C(\vartheta, \delta) = \frac{\Gamma(\delta)e^{-\mu}}{\vartheta(1-\delta)e} \frac{e^{\Gamma(0,\mu)\exp(\Gamma(0,\mu)+\vartheta+\log(\mu))}\mu^{\exp(\Gamma(0,\mu)+\vartheta+\log(\mu))}e^{\vartheta\exp(\Gamma(0,\mu)+\vartheta+\log(\mu))}}{\Gamma(\exp(\Gamma(0,\mu)+\vartheta+\log(\mu))+\delta)}, \quad (3.18)$$

   record the optimal value ϑ^* *and* δ^* *and set* $C = C(\vartheta^*, \delta^*)$.

   (b) *Generate* $U_1 \sim \mathcal{U}(0,1)$ *and set* $T = -\frac{1}{\vartheta^*}\log(U_1)$.

   (c) *Generate* $Y \sim Beta(T, \delta^*)$, $U_2 \sim \mathcal{U}(0,1)$ *and set*

   $$M = Y - 1 + (1-Y)\exp\left(-U_2\log(1-Y)\right).$$

   (d) *Generate* $V_1 \sim \mathcal{U}(0,1)$. *If*

   $$V_1 \leq \frac{1}{C}\frac{\Gamma(\delta^*)}{\vartheta^*}\frac{e^{\Gamma(0,\mu)T}\mu^T e^{\vartheta^* T}}{\Gamma(T+\delta^*)}\frac{-\log(1-Y)}{(1-Y)^{\delta^*-1}}e^{-\mu(1+M)}, \quad (3.19)$$

   then accept (T, M). *Otherwise, reject this candidate and go back to Step 2(b).*

3. *If* $T > t$, *continue to Step 4. Otherwise, set* $S = S + 1 + M$, $t = t - T$ *and go back to Step 2.*

4. *Generate* X *via the following steps:*

   (a) *Generate* $U_3 \sim \mathcal{U}(0,1)$ *and set* $X = U_3^{\frac{1}{t}}$.

   (b) *Generate* $V_2 \sim \mathcal{U}(0,1)$. *If*

   $$V_2 \leq \exp(-\mu X), \quad (3.20)$$

   then accept X. *Otherwise, reject this candidate and go back to Step 4(a).*

5. Return $S + X$.

Proof. For the joint density $f_{T,M}(t,m)$, we simulate (T,M,Y) jointly from the integral in (3.8) ,

$$f(t,m,y) = \frac{e^{\Gamma(0,\mu)t}\mu^t}{\Gamma(t)} y^{t-1} \frac{e^{-\mu(1+m)}}{1+m-y}, \tag{3.21}$$

where $t \in (0,\infty)$, $0 < m < y < 1$. We choose the envelope

$$g(t,m,y) = \vartheta e^{-\vartheta t} \frac{\Gamma(t+\delta)}{\Gamma(t)\Gamma(\delta)} y^{t-1}(1-y)^{\delta-1} \frac{\frac{1}{1+m-y}}{[-\log(1-y)]}. \tag{3.22}$$

The ratio of the densities is bounded by

$$\frac{f(t,m,y)}{g(t,m,y)} \leq \frac{\Gamma(\delta)e^{-\mu}}{\alpha(1-\delta)e} \frac{e^{\Gamma(0,\mu)t^*}\mu^{t^*}e^{\alpha t^*}}{\Gamma(t^*+\delta)} := C(\vartheta,\delta), \tag{3.23}$$

where we obtained by maximising the ratio with respect to t,

$$t^* = \exp\left(\Gamma(0,\mu) + \alpha + \log(\mu) + \mathcal{O}\left(\frac{1}{t^*}\right)\right) \approx \exp\left(\Gamma(0,\mu) + \alpha + \log(\mu)\right). \tag{3.24}$$

We then minimise $C(\vartheta,\delta)$ with respect to ϑ and δ via two dimensional numerical optimisation. Finally, $\{Z_t | T > t\}$ can be simulated directly via A/R scheme by choosing an envelop with density $g(x;t) = \frac{x^{t-1}}{t}\mathbb{1}_{\{0<x<1\}}$. ■

3.5 Numerical Results

In this section, we present some numerical results of the exact simulation methods developed in Algorithm 3.3, 3.4. Numerical validation and tests for our simulation algorithms are all based on the true mean. The associated errors are reported by the difference, the percentage errors, and *root mean square error* (RMSE)[1]. The detailed numerical results are reported in Table 1. We can see that each algorithm can achieve a very high level of accuracy, which is reflected by the difference in the theoretical mean and associated percentage errors. The simulation time is measured by the elapsed CPU time in seconds. We can see that both algorithms perform well. In particular, for Algorithm 3.4, the simulation time to generate the truncated Gamma process is not highly influenced by μ.

[1]RMSE is calculated by RMSE $= \sqrt{\text{SE}^2 + \text{Bias}^2}$

Table 1: Comparison between the true means and the associated simulation results for Algorithm 3.3, 3.4 based on the parameter setting $t = 0.5, 1, 2.5, 3$, $\mu = 0, 0.5, 1$, respectively.

Paths	True	Simulation	Difference	Error%	RMSE	Time	True	Simulation	Difference	Error%	RMSE	Time
			$t = 0.5$	$\mu = 0$					$t = 2.5$	$\mu = 0$		
1,000	0.5000	0.4879	-0.0121	-2.47%	0.0155	0.11	2.5000	2.4421	-0.0579	-2.37%	0.0343	0.08
4,000	0.5000	0.5057	0.0057	1.12%	0.0079	0.13	2.5000	2.5148	0.0148	0.59%	0.0178	0.11
16,000	0.5000	0.5011	0.0011	0.22%	0.0040	0.38	2.5000	2.5073	0.0073	0.29%	0.0089	0.36
64,000	0.5000	0.5008	0.0008	0.16%	0.0020	1.09	2.5000	2.4975	-0.0025	-0.10%	0.0044	1.17
256,000	0.5000	0.5002	0.0002	0.05%	0.0010	4.35	2.5000	2.5058	0.0058	0.23%	0.0022	4.43
1,024,000	0.5000	0.5009	0.0009	0.19%	0.0005	16.63	2.5000	2.5003	0.0003	0.01%	0.0011	17.19
			$t = 1$	$\mu = 0.5$					$t = 1$	$\mu = 1$		
1,000	0.7869	0.7890	0.0020	0.25%	0.0183	0.14	0.6321	0.6426	0.0105	1.63%	0.0165	0.19
4,000	0.7869	0.7909	0.0039	0.49%	0.0095	0.28	0.6321	0.6318	-0.0003	-0.04%	0.0081	0.28
16,000	0.7869	0.7809	-0.0060	-0.76%	0.0047	0.66	0.6321	0.6338	0.0017	0.26%	0.0041	0.83
64,000	0.7869	0.7891	0.0021	0.26%	0.0024	2.12	0.6321	0.6304	-0.0017	-0.27%	0.0020	2.18
256,000	0.7869	0.7855	-0.0014	-0.18%	0.0012	8.38	0.6321	0.6324	0.0002	0.03%	0.0010	8.55
1,024,000	0.7869	0.7865	-0.0004	-0.05%	0.0006	33.09	0.6321	0.6327	0.0005	0.08%	0.0005	31.48
			$t = 3$	$\mu = 0.5$					$t = 3$	$\mu = 1$		
1,000	2.3608	2.3699	0.0091	0.38%	0.0326	0.17	1.8964	1.9056	0.0092	0.48%	0.0289	0.16
4,000	2.3608	2.3431	-0.0177	-0.75%	0.0163	0.30	1.8964	1.8995	0.0031	0.16%	0.0139	0.17
16,000	2.3608	2.3731	0.0123	0.51%	0.0083	0.75	1.8964	1.8964	0.0000	0.00%	0.0070	0.50
64,000	2.3608	2.3629	0.0020	0.08%	0.0041	2.12	1.8964	1.8965	0.0002	0.01%	0.0035	1.94
256,000	2.3608	2.3608	0.0000	0.00%	0.0021	8.76	1.8964	1.8977	0.0013	0.06%	0.0018	6.98
1,024,000	2.3608	2.3579	-0.0029	-0.12%	0.0010	32.33	1.8964	1.8960	-0.0003	-0.01%	0.0009	31.60

For the generalised Dickman distribution, we implement a comparison between Algorithm 3.3 against the algorithms suggested by Devroye and Fawzi (2010), Fill and Huber (2010), Chi (2012), and Cloud and Huber (2017). The detailed numerical results based on different parameter settings against t are reported in Table 2 and the associated log-log plots for the RMSE against the CPU time are presented in Figure 2. In addition, we also compared the computation time to generate $100,000$ samples using these five algorithms for different values of t, and details are presented in Table 3. We can see that sampling based on Algorithm 3.3 is much faster than algorithms in Devroye and Fawzi (2010), Fill and Huber (2010), Chi (2012), and Cloud and Huber (2017). For instance, when $t = 3$, Algorithm 3.3 is 4 times faster than Devroye and Fawzi (2010), 50 times faster than Fill and Huber (2010), 25 times faster than Chi (2012), and 7 times faster than Cloud and Huber (2017). Compared with the other algorithms, the main advantage of Algorithm 3.3 is that the incremental random variables $(T_i, M_i)_{i\geq 0}$ are i.i.d pairs, and this means that we can employ vectorisation to reduce total computation time, especially for large values of t. Although the expected simulation time is unbounded for Algorithm 3.3, the simulation idea based on the marked renewal approach still outperforms the dominated coupling approach and rejection sampling approach, and even more so when t is large.

Table 2: Comparison for Algorithm 3.3 and Algorithms suggested in Devroye and Fawzi (2010), Fill and Huber (2010), Chi (2012), and Cloud and Huber (2017) based on the parameter setting $t = 1, 3$ respectively.

Paths	True	Simulation	Difference	Error%	RMSE	Time	True	Simulation	Difference	Error%	RMSE	Time
Algorithm 3.3			$t = 1$						$t = 3$			
1,000	1.0000	0.9813	-0.0187	-1.91%	0.0218	0.07	3.0000	3.0031	0.0031	0.10%	0.0379	0.09
4,000	1.0000	1.0057	0.0057	0.57%	0.0114	0.17	3.0000	3.0030	0.0030	0.10%	0.0192	0.14
16,000	1.0000	1.0043	0.0043	0.43%	0.0056	0.28	3.0000	2.9979	-0.0021	-0.07%	0.0096	0.41
64,000	1.0000	0.9997	-0.0003	-0.03%	0.0028	0.97	3.0000	2.9967	-0.0033	-0.11%	0.0048	1.19
256,000	1.0000	0.9985	-0.0015	-0.15%	0.0014	4.30	3.0000	3.0041	0.0041	0.14%	0.0024	4.51
1,024,000	1.0000	0.9994	-0.0006	-0.06%	0.0007	16.69	3.0000	2.9978	-0.0022	-0.07%	0.0012	18.84
Devroye and Fawzi (2010)			$t = 1$						$t = 3$			
1,000	1.0000	0.9923	-0.0077	-0.77%	0.0221	0.03	3.0000	2.9731	-0.0269	-0.90%	0.0038	0.14
4,000	1.0000	0.9977	-0.0023	-0.22%	0.0113	0.13	3.0000	3.0006	0.0006	0.02%	0.0192	0.61
16,000	1.0000	1.0011	0.0011	0.11%	0.0056	0.57	3.0000	2.9935	-0.0065	-0.21%	0.0097	2.00
64,000	1.0000	1.0057	0.0057	0.56%	0.0028	1.95	3.0000	2.9951	-0.0049	-0.16%	0.0048	5.54
256,000	1.0000	1.0034	0.0034	0.33%	0.0014	7.99	3.0000	3.0014	0.0014	0.04%	0.0024	23.93
1,024,000	1.0000	0.9993	-0.0007	-0.07%	0.0007	37.51	3.0000	3.0014	0.0014	0.04%	0.0012	101.61
Fill and Huber (2010)			$t = 1$						$t = 3$			
1,000	1.0000	0.9851	-0.0149	-1.50%	0.0027	0.10	3.0000	2.9516	-0.0484	-1.64%	0.0398	1.31
4,000	1.0000	0.9977	-0.0023	-0.23%	0.0113	0.28	3.0000	2.9444	-0.0556	-1.88%	0.0195	4.56
16,000	1.0000	1.0004	0.0004	0.04%	0.0056	1.09	3.0000	3.0175	0.0175	0.58%	0.0099	18.69
64,000	1.0000	0.9968	-0.0032	-0.32%	0.0028	2.67	3.0000	2.9995	-0.0005	-0.02%	0.0049	75.23
256,000	1.0000	0.9977	-0.0023	-0.23%	0.0013	11.14	3.0000	2.9969	-0.0031	-0.10%	0.0025	325.37
1,024,000	1.0000	0.9994	-0.0006	-0.06%	0.0006	57.37	3.0000	3.0131	0.0131	0.43%	0.0012	1358.24
Chi (2012)			$t = 1$						$t = 3$			
1,000	1.0000	1.0070	0.0070	0.70%	0.0233	0.33	3.0000	3.0109	0.0109	0.36%	0.0392	0.62
4,000	1.0000	1.0100	0.0100	0.99%	0.0115	0.78	3.0000	3.0208	0.0208	0.69%	0.0197	2.19
16,000	1.0000	0.9955	-0.0045	-0.45%	0.0057	2.83	3.0000	3.0107	0.0107	0.35%	0.0099	8.44
64,000	1.0000	1.0026	0.0026	0.26%	0.0029	12.24	3.0000	3.0027	0.0027	0.09%	0.0050	31.16
256,000	1.0000	1.0018	0.0018	0.18%	0.0014	46.27	3.0000	3.0016	0.0016	0.05%	0.0025	132.28
1,024,000	1.0000	1.0036	0.0036	0.36%	0.0007	193.00	3.0000	2.9978	-0.0002	-0.01%	0.0012	514.12
Cloud and Huber (2017)			$t = 1$						$t = 3$			
1,000	1.0000	0.9832	-0.0168	-1.71%	0.0220	0.09	3.0000	2.9376	-0.0624	-2.12%	0.0379	0.28
4,000	1.0000	1.0153	0.0153	1.50%	0.0113	0.33	3.0000	2.9986	-0.0014	-0.05%	0.0195	0.92
16,000	1.0000	1.0059	0.0059	0.58%	0.0056	0.84	3.0000	3.0108	0.0108	0.36%	0.0096	2.31
64,000	1.0000	1.0076	0.0076	0.75%	0.0028	3.07	3.0000	3.0014	0.0014	0.05%	0.0048	8.61
256,000	1.0000	1.0013	0.0013	0.13%	0.0014	12.51	3.0000	3.0021	0.0021	0.07%	0.0024	36.47
1,024,000	1.0000	1.0017	0.0017	0.17%	0.0007	58.19	3.0000	3.0038	0.0038	0.13%	0.0012	155.93

4 More general Vervaat perpetuities

In this section, we present a general method for simulating Vervaat perpetuities, for more general random variables Y. This method is applicable to a wide range of distributions, and we give several examples to demonstrate this. We have the following result which decomposes the Vervaat perpetuity X into the sum of a Gamma process (or truncated Gamma process), and compound Poisson processes. For simplicity, we will let Y have a continuous density g.

Lemma 4.1 *Consider a perpetuity X defined in* (1.1)*, with $Y \in \mathbb{R}^+$. If the density of Y satisfies* (1.3)*, then there exists $k \in \mathbb{R}^+$ and $b \in \mathbb{R}^+$ such that $\mathbb{P}(Y > y) \geq e^{-ky}$ for all $y < b$. We then have the following two cases:*

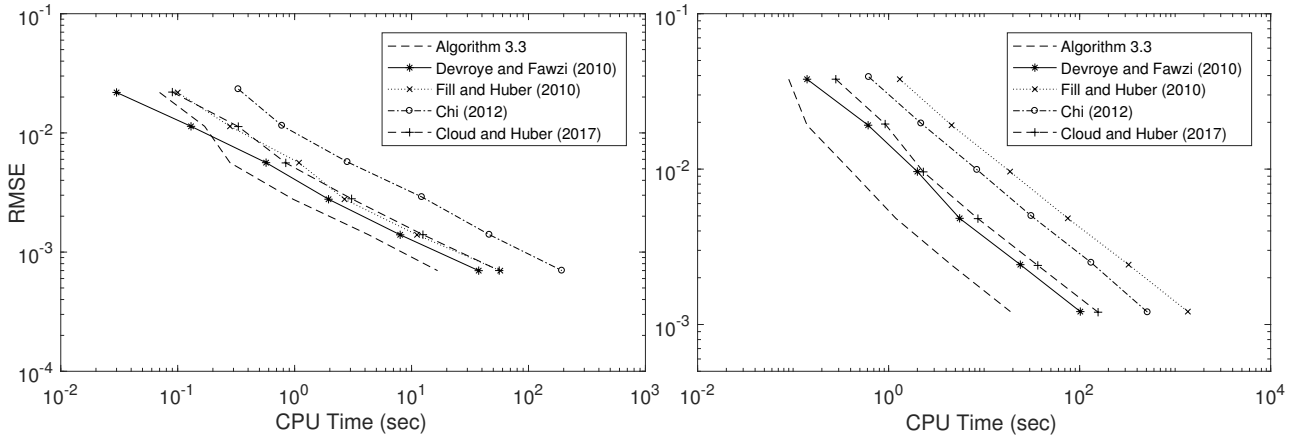


Figure 2: Convergence analysis of Algorithm 3.3 and Algorithms suggested in Devroye and Fawzi (2010), Fill and Huber (2010), Chi (2012), and Cloud and Huber (2017) based on parameter setting $t = 1, 3$, respectively, with associated detailed numerical results reported in Table 2.

Table 3: Comparison CPU time (sec) for Algorithms suggested in Devroye and Fawzi (2010), Fill and Huber (2010), Chi (2012) and Cloud and Huber (2017) with Algorithm 3.3 for 100,000 replications.

Input t	0.40	0.60	0.80	1.00	1.20	1.40	1.60	1.80	2.00	2.20	2.40	2.60	2.80	3.00	3.20	3.40
Algorithm 3.3	1.85	1.81	1.68	1.60	1.71	1.79	1.82	1.83	1.78	2.03	1.72	1.77	1.83	2.02	1.83	1.95
Devroye and Fawzi (2010)	3.02	2.23	3.19	3.94	5.66	5.47	5.23	5.31	5.93	8.48	7.22	7.51	7.97	8.47	10.88	11.50
Fill and Huber (2010)	1.87	2.53	3.14	4.12	5.68	7.34	10.33	13.64	18.28	20.76	36.78	51.22	70.23	100.75	140.91	200.04
Chi (2012)	3.92	6.28	9.62	18.52	21.25	23.53	27.45	31.80	34.78	36.92	38.41	41.57	55.31	50.54	55.83	59.23
Cloud and Huber (2017)	2.45	3.09	3.93	4.71	5.66	6.71	7.57	9.70	11.16	11.82	13.45	13.72	14.71	15.54	16.99	17.25

Case 1: *There exists $k \in \mathbb{R}^+$ such that $\mathbb{P}(Y > y) \geq e^{-ky}$ for all $y \in \mathbb{R}^+$. Then we have*

$$B = \int_0^\infty \frac{\mathbb{P}(Y > y) - e^{-ky}}{y} \mathrm{d}y < \infty, \tag{4.1}$$

and thus

$$X \stackrel{\mathcal{D}}{=} \Gamma_t + \sum_{i=1}^{N_t} J_i, \tag{4.2}$$

where

- *Γ_t is a Gamma process such that $\Gamma_t \sim \Gamma(t, k)$;*
- *$\sum_{i=1}^{N_t} J_i$ is a compound Poisson process such that*
  - *N_t is a poisson process with rate tB,*
  - *$\{J_i\}_{i=1,2,\ldots}$ are i.i.d jumps with density*

$$g_{J_i}(y) = \frac{\mathbb{P}(Y \geq y) - e^{-ky}}{By}, \qquad y \in \mathbb{R}^+.$$

Case 2: *Otherwise, for the case that there only exists $k \in \mathbb{R}^+$ such that $\mathbb{P}(Y > y) \geq e^{-ky}$ for $y < b$. We have*

$$D = \int_0^b \frac{\mathbb{P}(Y > y) - e^{-ky}}{y} \mathrm{d}y < \infty, \tag{4.3}$$

and thus

$$X \stackrel{\mathcal{D}}{=} Z_t + \sum_{i=1}^{N_t^1} J_i^{(1)} + \sum_{k=1}^{N_t^2} J_k^{(2)}, \tag{4.4}$$

where

- *Z_t is a truncated Gamma process with Lévy measure*

$$\nu(\mathrm{d}z) = z^{-1} e^{-kz} \mathbb{1}_{\{0<z<b\}} \mathrm{d}z;$$

- *$\sum_{i=1}^{N_t^1} J_i^{(1)}$ is a compound Poisson process such that*
  - *N_t^1 is a poisson process with rate tD,*
  - *$\{J_i^{(1)}\}_{i=1,2,\ldots}$ are i.i.d jumps with density*

$$g_{J_i^{(1)}}(y) = \frac{\mathbb{P}(Y \geq y) - e^{-ky}}{Dy}, \qquad y \in (0, b).$$

- *$\sum_{k=1}^{N_t^2} J_k^{(2)}$ is a compound Poisson process such that*
  - *N_t^2 is a poisson process with rate tE, where*

$$E = \int_b^\infty \frac{\mathbb{P}(Y \geq y)}{y} \mathrm{d}y \tag{4.5}$$

  - *$\{J_k^{(2)}\}_{i=1,2,\ldots}$ are i.i.d jumps with density*

$$g_{J_k^{(2)}}(y) = \frac{\mathbb{P}(Y \geq y)}{Ey}, \qquad y \in (b, \infty).$$

Proof. If the density of Y satisfies (1.3), then $g(0) < \infty$. Hence, the survival function of Y, $\mathbb{P}(Y > y)$, decays at an exponential rate or slower at 0. This implies that there exists a $k \in \mathbb{R}^+$ such that $\mathbb{P}(Y > y) \geq e^{-ky}$ for all $y < b$, for some $b \in \mathbb{R}^+$.

Case 1: If, furthermore, there exists such $k \in \mathbb{R}^+$ so that $\mathbb{P}(Y > y) \geq e^{-ky}$ holds for all $y \in \mathbb{R}^+$, then

for B defined as in (4.1), we have that the integral B is finite since

$$\int_0^\infty \frac{\mathbb{P}(Y>y)-e^{-ky}}{y}\mathrm{d}y = \int_0^1 \frac{\mathbb{P}(Y>y)-e^{-ky}}{y}\mathrm{d}y + \int_1^\infty \frac{\mathbb{P}(Y>y)-e^{-ky}}{y}\mathrm{d}y.$$

The second term is finite since we have assumed that the mean of Y exists, and we also have that

$$\begin{aligned}\int_0^1 \frac{\mathbb{P}(Y>y)-e^{-ky}}{y}\mathrm{d}y &= \left[\log y\left(\mathbb{P}(Y>y)-e^{-ky}\right)\right]_0^1 + \int_0^1 \log y\left(g(y)-\frac{e^{-ky}}{k}\right)\mathrm{d}y \\ &= \int_0^1 \log y\left(g(y)-\frac{e^{-ky}}{k}\right)\mathrm{d}y < \infty\end{aligned}$$

since $g(0)<\infty$. Hence, the Laplace transform of X can be expressed as

$$\begin{aligned}\mathbb{E}\left[e^{-\beta X}\right] &= \exp\left(-t\int_0^\infty (1-e^{-\beta y})\frac{e^{-ky}}{y}\mathrm{d}y\right)\exp\left(-tB\int_0^\infty (1-e^{-\beta y})\frac{\mathbb{P}(Y>y)-e^{-ky}}{By}\mathrm{d}y\right) \\ &= \mathbb{E}\left[e^{-\beta\Gamma_t}\right]\mathbb{E}\left[e^{-\beta\sum_{i=1}^{N_t}J_i}\right], \qquad (4.6)\end{aligned}$$

and we have the distributional decomposition result given in (4.2).

Case 2: In the case where we only have $\mathbb{P}(Y>y)\geq e^{-ky}$ for $y<b$ for some $b\in\mathbb{R}^+$, we define the integral D as in (4.3) and E as in (4.5). A similar argument as above would show that they are finite, and we thus have

$$\begin{aligned}\mathbb{E}\left[e^{-\beta X}\right] &= \exp\left(-t\int_0^b (1-e^{-\beta y})\frac{\mathbb{P}(Y>y)}{y}\mathrm{d}y\right)\exp\left(-t\int_b^\infty (1-e^{-\beta y})\frac{\mathbb{P}(Y>y)}{y}\mathrm{d}y\right) \\ &= \exp\left(-t\int_0^b (1-e^{-\beta y})\frac{e^{-ky}}{y}\mathrm{d}y\right)\exp\left(-tD\int_0^b (1-e^{-\beta y})\frac{\mathbb{P}(Y>y)-e^{-ky}}{Dy}\mathrm{d}y\right) \\ &\quad\times\exp\left(-tE\int_b^\infty (1-e^{-\beta y})\frac{\mathbb{P}(Y>y)}{Ey}\mathrm{d}y\right) \\ &= \mathbb{E}\left[e^{-\beta Z_t}\right]\mathbb{E}\left[e^{-\beta\sum_{i=1}^{N_t^1}J_i^{(1)}}\right]\mathbb{E}\left[e^{-\beta\sum_{k=1}^{N_t^2}J_k^{(2)}}\right]. \qquad (4.7)\end{aligned}$$

The distributional decomposition result (4.4) thus follows. ■

We now have a general method for exactly simulating generalised Vervaat perpetuities for a wide range of distributions for Y. In particular, Case 2 is useful when the distribution of Y has finite support, or when Y has a tail which decays faster than exponential, in which case there does not exist $k\in\mathbb{R}^+$ such that $\mathbb{P}(Y>y)\geq e^{-ky}$ for all y. In the rest of this section, we give some examples to demonstrate

the methodology.

Example 1: $Y \sim Pareto(\alpha, \sigma)$. From Theorem 2.1, the Laplace transform of X is

$$E\left[e^{-\beta X}\right] = \exp\left(-t\int_0^{\sigma}\frac{1-e^{-\beta y}}{y}\mathrm{d}y\right)\exp\left(-t\int_{\sigma}^{\infty}\frac{1-e^{-\beta y}}{y}\left(\frac{\sigma}{y}\right)^{\alpha}\mathrm{d}y\right). \tag{4.8}$$

Hence, $X \overset{\mathcal{D}}{=} Z_t + CP$, where $Z_t \sim$ Dickman process, and CP is a Compound Poisson process, independent of each other. The Dickman process can be generated using Algorithm 3.3, and the Compound Poisson process can be generated easily via an A/R scheme.

Example 2: $Y \sim Gamma(\alpha, \beta)$. Since $g(0) < \infty$ for all $\alpha \geq 1$, we can simulate all Gamma perpetuities with $\alpha \geq 1$ and all values of β. Furthermore, for all $\alpha \geq 1$, there exists a k such that $\mathbb{P}(Y > y) \geq e^{-ky}$ for all y, thus it falls under Case 1 of Lemma 4.1. Thus the Gamma perpetuity X can be split into $X \overset{\mathcal{D}}{=} Z_t + CP$, where Z_t is a Gamma process, and CP is a Compound Poisson process independent of Z_t by Lemma 4.1. Figure 3 shows the histograms of the Vervaat perpetuity X for $Y \sim \Gamma(\alpha, \beta)$ with different shape parameters α.

Example 3: $Y \sim Weibull(\kappa, \lambda)$. Since $g(0) < \infty$ for all $\kappa \geq 1$, we can simulate all Weibull perpetuities with $\kappa \geq 1$ and all values of λ. Furthermore, there only exists a k such that $\mathbb{P}(Y > y) \geq e^{-ky}$ for $y \in (0, b)$ for some $b \in \mathbb{R}^+$. Hence, the perpetuity falls into Case 2 of Lemma 4.1, and can be split into $X \overset{\mathcal{D}}{=} Z_t + CP_1 + CP_2$, where Z_t is a truncated Gamma process, CP_1 and CP_2 are Compound Poisson processes, independent of each other and Z_t. The truncated Gamma process can be generated using Algorithm 3.4 and the two compound Poisson processes can be easily generated via A/R schemes.

Example 4: $Y \sim Beta(\alpha, \beta)$. Since $g(0) < \infty$ for all $\alpha \geq 1$, we can simulate all Beta perpetuities with $\alpha \geq 1$ and all values of β. Furthermore, since the distribution has a finite support, there only exists a k such that $\mathbb{P}(Y > y) \geq e^{-ky}$ for $y \in (0, b)$ for some $b \in \mathbb{R}^+$. Hence, the perpetuity falls into Case 2 of Lemma 4.1, and the Lévy measure needs to be truncated. The Beta perpetuity can be split into $X \overset{\mathcal{D}}{=} Z_t + CP_1 + CP_2$ according to Lemma 4.1, where Z_t is a truncated Gamma process, and CP_1 and CP_2 are two compound Poisson processes, independent of each other and Z_t.

Example 5: $Y \sim Normal(\mu, \sigma^2)$. The Normal perpetuity X is two-sided. According to Corollary 2.2, one could sample independent X_1 and X_2, each of which are half-normal perpetuities, and the

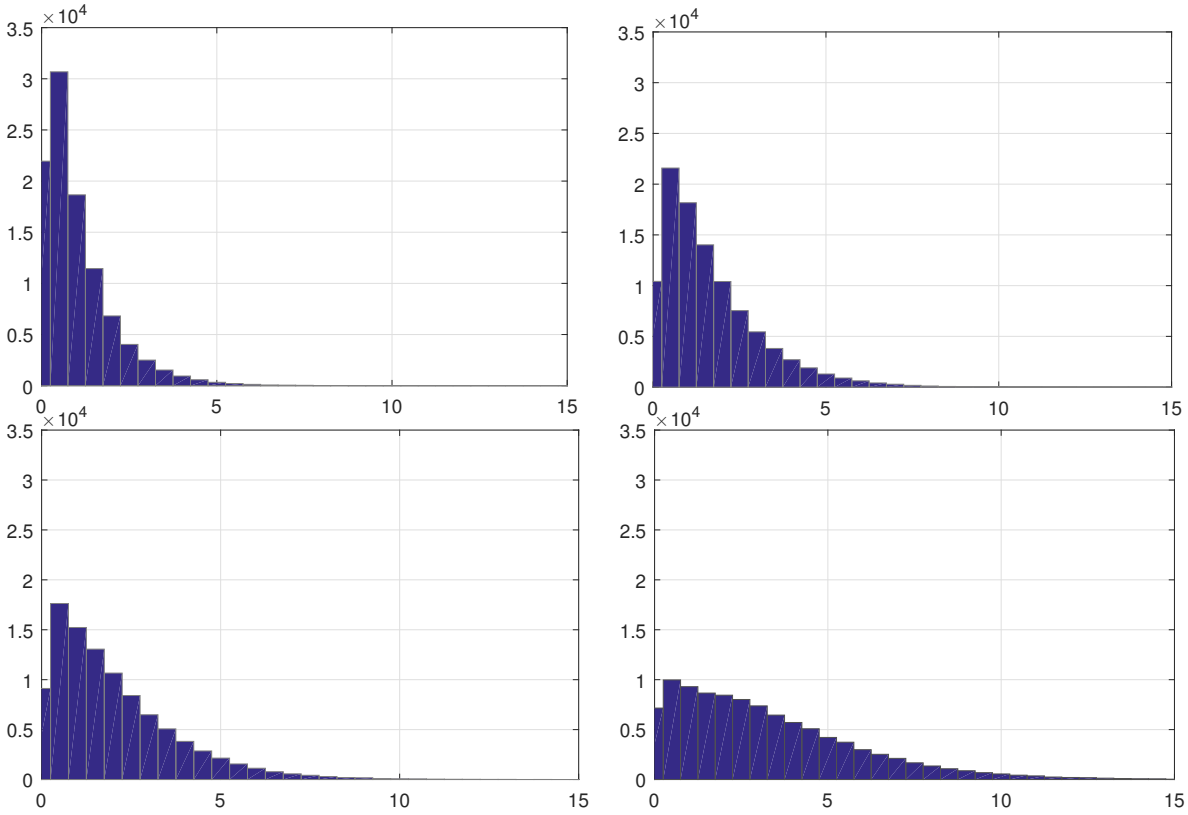


Figure 3: Histograms of perpetuity X with $Y \sim \Gamma(\alpha, \beta)$ under the parameter setting $\alpha = 1, 1.5, 2, 2.5$ and $\beta = 1$.

difference $X_1 - X_2$ is the realisation of the two-sided perpetuity X.

For a half-normal perpetuity X_i, where $Y_i \sim \textit{Half-Normal}(\mu, \sigma)$, there only exists a k such that $\mathbb{P}(Y > y) \geq e^{-ky}$ for $y \in (0, b)$ for some $b \in \mathbb{R}^+$. Hence, the perpetuity X_i falls into Case 2 and can be split into a truncated Gamma process and two compound Poisson processes according to Lemma 4.1. The truncated Gamma process can be generated via Algorithm 3.4 and the two compound Poisson processes can be easily generated via A/R schemes. Figure 4 illustrates a sample density plot for the perpetuity X_i with $Y_i \sim$ Half-Normal$(0, 1)$. Figure 5 compares the distribution of X_i under different parameter settings.

Now, we can simulate the Normal perpetuity X by generating two independent half-normal perpetuities X_1 and X_2 and taking the difference $X \overset{\mathcal{D}}{=} X_1 - X_2$. Figure 6 and 7 demonstrate the distribution behaviour of the two sided perpetuity X via its histogram and density plot. We can also obtain other variations of the Normal perpetuities by introducing a new variable $B \sim \textit{Bernoulli}(p)$ and set-

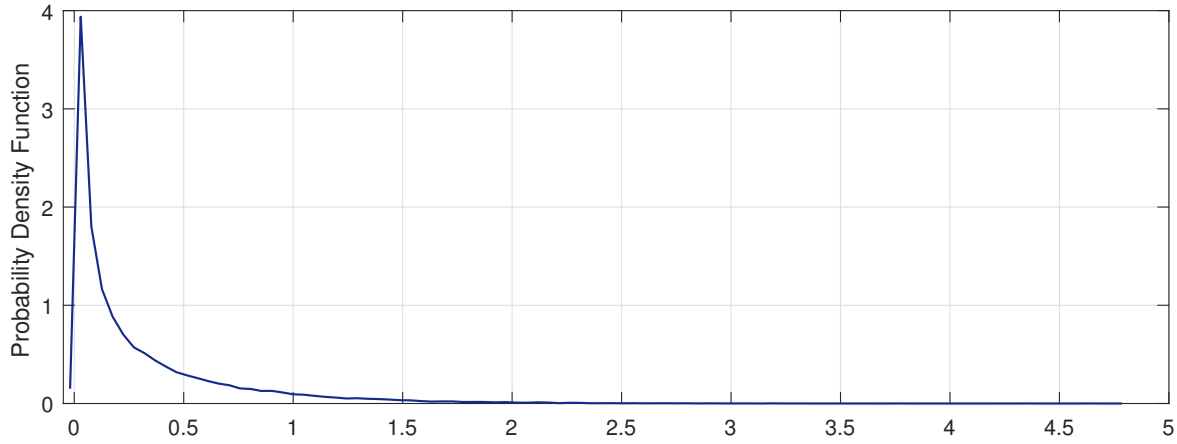


Figure 4: Probability density function of Perpetuity X_i with $Y_i \sim$ Half-Normal$(0,1)$.

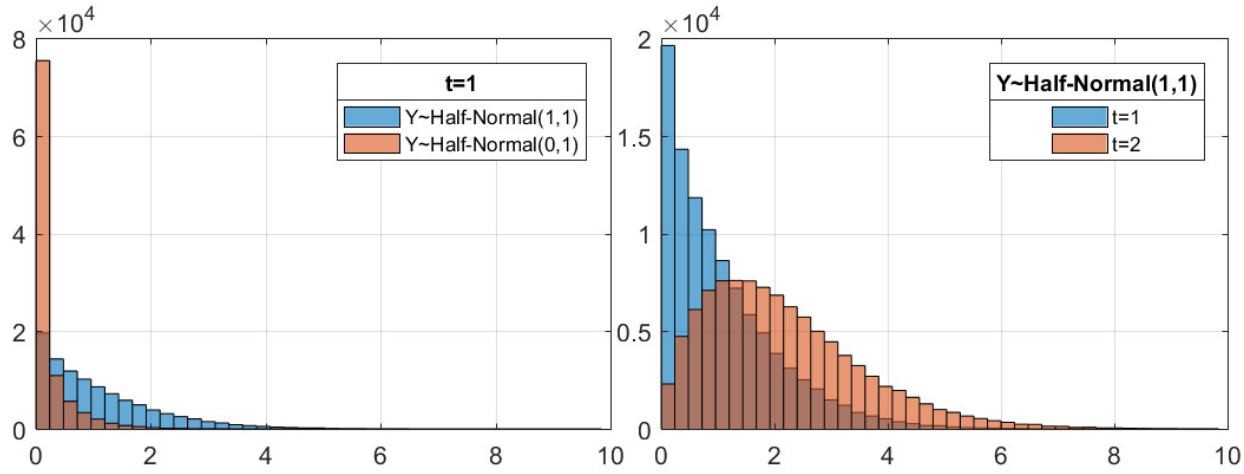


Figure 5: Histograms of Perpetuity X_i with $Y_i \sim$ Half-Normal$(0,1)$ and $Y_i \sim$ Half-Normal$(1,1)$ with $t = 1, 2$, respectively.

ting $\hat{X} \overset{\mathcal{D}}{=} pX_1 - (1-p)X_2$. When $p = \frac{1}{2}$ and $X_i \sim Normal(0,1)$, this is identical to the $Normal(0,1)$ perpetuity. Figure 8 illustrates the differences of the perpetuity $\hat{X}$ and X in terms of their histograms.

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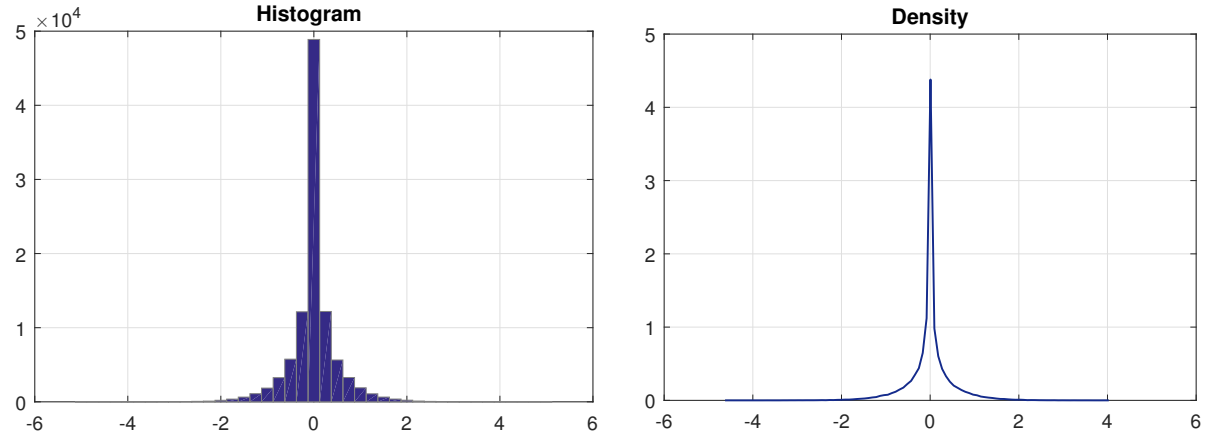

Figure 6: Histogram and Density Plot of Perpetuity X with $Y \sim \text{Normal}(0, 1)$.

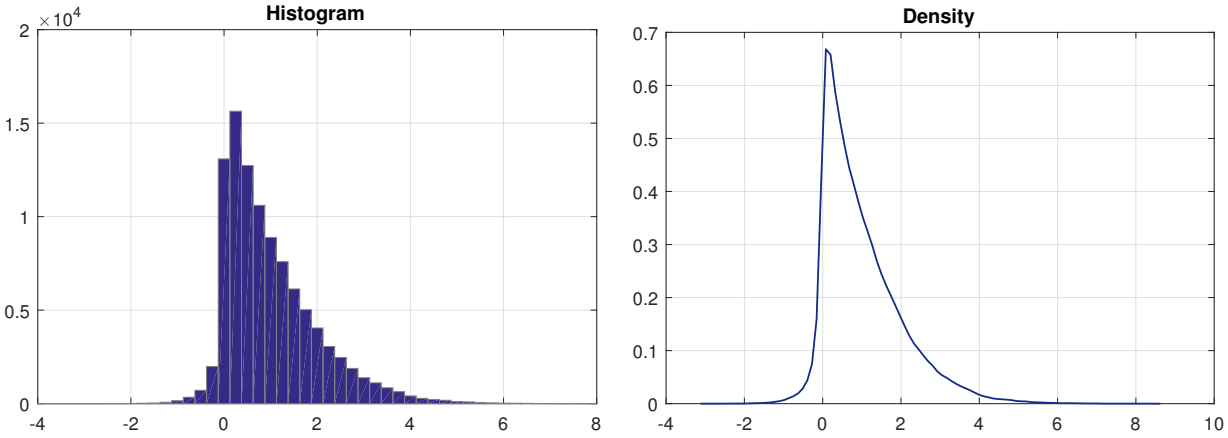

Figure 7: Histogram and Density Plot of Perpetuity X with $Y \sim \text{Normal}(1, 1)$.

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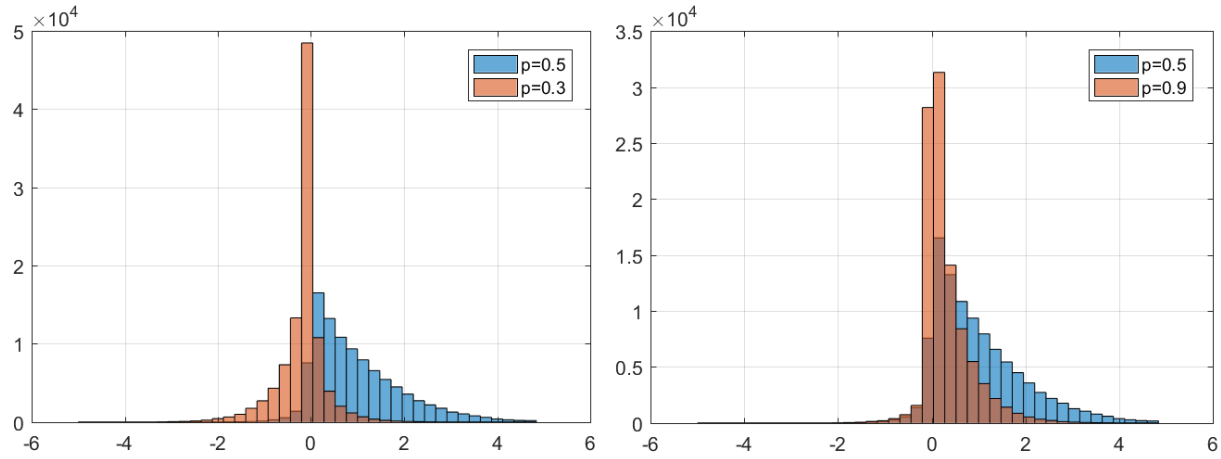

Figure 8: Histogram of Perpetuity $\hat{X} = BX_1 - (1-B)X_2$ with $B \sim Bernoulli(p)$ under parameter setting $p = 0.3, 0.5, 0.9$.

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