Exact small deviation asymptotics for some Brownian functionals

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Outline

L_2-small ball probabilities

Let $X(t)$, $0 \le t \le 1$, be a zero mean Gaussian process with covariance function $G(t,s) = \mathsf{E}X(t)X(s)$ and let ψ be a nonnegative function on $[0,1]$. Denote

$$\|X\|_\psi = \left(\int_0^1 X^2(t)\psi(t)dt \right)^{1/2}.$$

The problem is to describe the behavior of $\mathsf{P}\{\|X\|_\psi \le \varepsilon\}$ as $\varepsilon \to 0$.

L_2-small ball probabilities

The problem was solved by Sytaya (1974), but in an implicit way. Later Zolotarev; Dudley, Hoffmann-Jørgensen and Shepp; Ibragimov; W. Li; Dunker, Lifshits and Linde; Gao, Hannig, Lee and Torcaso; Nazarov and Nikitin, etc. improved and simplified the expression for $\mathrm{P}\{\|X\|_\psi \le \varepsilon\}$.

Karhunen-Loève expansion

Karhunen-Loève expansion implies

$$\|X\|_\psi^2 = \sum_{k=1}^{\infty} \lambda_k \xi_k^2,$$

where ξ_k are i.i.d. $N(0,1)$ r.v.'s,
$\lambda_k > 0$ are the eigenvalues of the integral equation

$$\lambda f(t) = \int_0^1 G(t,s)\sqrt{\psi(t)\psi(s)} f(s) ds, \quad t \in [0,1].$$

Comparison theorem

W. Li, 1992:

Comparison theorem. Let ξ_k be i.i.d. $N(0,1)$ r.v.'s, and let $a_k > 0$ and $b_k > 0$ be such that $\sum_{k=1}^{\infty} |1 - a_k/b_k| < \infty$. Then, as $\varepsilon \to 0$,

$$\mathrm{P}\left\{\sum_{k=1}^{\infty} a_k \xi_k^2 \le \varepsilon^2\right\} \sim \left(\prod_{k=1}^{\infty} \frac{b_k}{a_k}\right)^{1/2} \mathrm{P}\left\{\sum_{k=1}^{\infty} b_k \xi_k^2 \le \varepsilon^2\right\}.$$

F. Gao, J. Hannig, F. Torcaso, 2003:

$\sum_{k=1}^{\infty} |1 - a_k/b_k| < \infty$ was replaced by $0 < \prod_{k=1}^{\infty} (a_k/b_k) < \infty$.

Exact asymptotics

In typical cases the exact asymptotics has the form

$$\mathsf{P}\{\|X\|_\psi \le \varepsilon\} \sim C\varepsilon^\gamma \exp(-K\varepsilon^{-\alpha}).$$

If $\lambda_k = \big(\vartheta(k+\delta+O(k^{-1}))\big)^{-d}$, then

$$\alpha = \alpha(d), \quad K = K(d,\vartheta),$$

$$\gamma = \gamma(d,\delta), \quad C = C(\vartheta,d,\delta)\cdot C_{\text{dist}}(\lambda_k, k\in\mathbb{N}).$$

Exact asymptotics

Here

$$C_{\text{dist}} = \left(\prod_{k=1}^{\infty} \frac{(\vartheta(k+\delta))^{-d}}{\lambda_k} \right)^{1/2}.$$

When the eigenfunctions of the covariance can be expressed in terms of elementary or special functions, there exist explicit formulas for the distortion constants (A. Nazarov, 2003, A. Nazarov, R. Pusev, 2009).

Previous results for Wiener process

A. Nazarov, R. Pusev, 2009:

$$\mathsf{P}\left\{\int_0^1 \frac{W^2(t)}{(a^2+t^2)^2}\,dt \leqslant \varepsilon^2\right\} \sim \frac{4a^{1/2}(1+a^2)^{1/4}}{\pi^{1/2}\operatorname{arcctg} a}\varepsilon \exp\left(-\frac{(\operatorname{arcctg} a)^2}{8a^2\varepsilon^2}\right),$$

$$\mathsf{P}\left\{\int_0^1 \frac{W^2(t)}{(a^2-t^2)^2}\,dt \leqslant \varepsilon^2\right\} \sim \frac{8(a^2-1)^{1/4}a^{1/2}}{\pi^{1/2}\ln\frac{a+1}{a-1}}\varepsilon \exp\left(-\frac{\left(\ln\frac{a+1}{a-1}\right)^2}{32a^2\varepsilon^2}\right),$$

$$\mathsf{P}\left\{\int_0^1 (t+a)^{2\beta-2}W^2(t)\,dt \leqslant \varepsilon^2\right\} \sim$$

$$\sim \frac{(a(a+1)^{-1})^{(\beta-1)/4}\beta}{((a+1)^\beta - a^\beta)}\frac{4\varepsilon}{\sqrt{\pi}} \exp\left(-\frac{((a+1)^\beta - a^\beta)^2}{8\beta^2\varepsilon^2}\right).$$

Previous results for Wiener process

For

$$\psi(t) = (a^2 + t^2)^{-2}, \quad a > 0,$$

$$\psi(t) = (a^2 - t^2)^{-2}, \quad a > 1,$$

$$\psi(t) = (t + a)^{\beta}, \quad a > 0,$$

one has

$$\mathrm{P}\{\|W\|_\psi \leqslant \varepsilon\} \sim \frac{4\psi^{1/8}(0)}{\sqrt{\pi}\vartheta\psi^{1/8}(1)}\varepsilon \exp\left(-\frac{\vartheta^2}{8}\varepsilon^{-2}\right),$$

where

$$\vartheta = \int_0^1 \sqrt{\psi(t)}dt.$$

Question

Are these relations special or typical?

Regular-shaped weights

We show how to evaluate the distortion constant for some Gaussian processes when the eigenfunctions of the covariance are **unknown**.

Wiener process with regular-shaped weight

Theorem 1. *Let the weight* ψ*, defined on* $[0,1]$*, be positive and twice continuously differentiable. Put* $\vartheta = \int_0^1 \sqrt{\psi(t)}dt$*. Then as* $\varepsilon \to 0$

$$\mathsf{P}\{\|W\|_\psi \leqslant \varepsilon\} \sim \frac{4\psi^{1/8}(0)}{\sqrt{\pi}\vartheta\psi^{1/8}(1)}\varepsilon \exp\left(-\frac{\vartheta^2}{8}\varepsilon^{-2}\right).$$

We shall call the weight ψ satisfying the conditions of this Theorem *regular-shaped*.

Sketch of proof

The coefficients λ_k in the KL expansion satisfy $\lambda_k = \mu_k^{-1}$, where μ_k are the eigenvalues of the BVP

$$\begin{cases} -y'' = \mu\psi y \quad \text{on} \quad [0,1], \\ y(0) = y'(1) = 0. \end{cases}$$

We have

$$\mu_k = \left(\pi(k-1/2)/\vartheta + O(k^{-1})\right)^2,$$

and

$$\mathrm{P}\{\|W\|_\psi \le \varepsilon\} \sim C_{\mathrm{dist}}\,\mathrm{P}\left\{\sum_{k=1}^{\infty} \frac{\xi_k^2}{(\pi(k-1/2))^2} \le \left(\frac{\varepsilon}{\vartheta}\right)^2\right\}.$$

Sketch of proof

Denote by $\varphi_{1,2}(t,\zeta)$ the solutions of the equation $-y'' = \zeta^2\psi y$ satisfying the initial conditions

$$\varphi_1(0,\zeta) = 1, \qquad \varphi_1'(0,\zeta) = 0,$$
$$\varphi_2(0,\zeta) = 0, \qquad \varphi_2'(0,\zeta) = 1.$$

We have $\mu_k = x_k^2$, where $x_1 < x_2 < \dots$ are the positive roots of the function

$$F(\zeta) = \det \begin{bmatrix} \varphi_1(0,\zeta) & \varphi_2(0,\zeta) \\ \varphi_1'(1,\zeta) & \varphi_2'(1,\zeta) \end{bmatrix} = \varphi_2'(1,\zeta).$$

Sketch of proof

Using WKB approximation, we obtain

$$\varphi_2'(1,\zeta) = \frac{\psi^{1/4}(1)\cos(\vartheta\zeta)}{\psi^{1/4}(0)}(1 + O(\zeta^{-1})).$$

Applying Jensen's theorem to the functions $F(\zeta)$ and $\cos(\vartheta\zeta)$, we get

$$C^2_{\mathrm{dist}} = \frac{\psi^{1/4}(0)}{\psi^{1/4}(1)}.$$

Brownian bridge with regular-shaped weight

Let B be the Brownian bridge.

Theorem 2. *Let* ψ *be a regular-shaped weight on* $[0,1]$*. Then as* $\varepsilon \to 0$

$$\mathsf{P}\{\|B\|_\psi \le \varepsilon\} \sim \frac{2\sqrt{2}\psi^{1/8}(0)\psi^{1/8}(1)}{\sqrt{\pi\vartheta}} \exp\left(-\frac{\vartheta^2}{8}\varepsilon^{-2}\right).$$

Ornstein-Uhlenbeck process with regular-shaped weight

Denote by $U_{(\alpha)}(t)$ a usual stationary Ornstein-Uhlenbeck process, that is, a centered Gaussian process with covariance

$$G_{U_{(\alpha)}}(t,s) = e^{-\alpha|t-s|}/(2\alpha).$$

Theorem 3. *Let* ψ *be a regular-shaped weight on* $[0,1]$*. Then as* $\varepsilon \to 0$

$$\mathsf{P}\{\|U_{(\alpha)}\|_\psi \le \varepsilon\} \sim \frac{8\alpha^{1/2}e^{\alpha/2}}{\pi^{1/2}\vartheta^{3/2}\psi^{1/8}(0)\psi^{1/8}(1)}\varepsilon^2 \exp\left(-\frac{\vartheta^2}{8}\varepsilon^{-2}\right).$$

Ornstein-Uhlenbeck process with regular-shaped weight

Denote by $\mathring{U}_{(\alpha)}(t)$ the Ornstein-Uhlenbeck process starting at zero, that is, the centered Gaussian process with the covariance function

$$G_{\mathring{U}_{(\alpha)}}(t,s) = (e^{-\alpha|t-s|} - e^{-\alpha(t+s)})/(2\alpha).$$

Theorem 4. *Let* ψ *be a regular-shaped weight on* $[0,1]$*. Then as* $\varepsilon \to 0$

$$\mathsf{P}\{\|\mathring{U}_{(\alpha)}\|_\psi \le \varepsilon\} \sim \frac{4e^{\alpha/2}\psi^{1/8}(0)}{\sqrt{\pi}\vartheta\psi^{1/8}(1)}\varepsilon \exp\left(-\frac{\vartheta^2}{8}\varepsilon^{-2}\right).$$

Bogoliubov process with regular-shaped weight

Denote by $Y(t)$ the Bogoliubov process, that is, the centered Gaussian process with covariance

$$G_Y(t,s) = \frac{1}{2\omega \sinh(\omega/2)} \cosh\left(\omega|t-s| - \frac{\omega}{2}\right), \quad \omega > 0.$$

Theorem 5. *Let ψ be a regular-shaped weight on $[0,1]$. Then as $\varepsilon \to 0$*

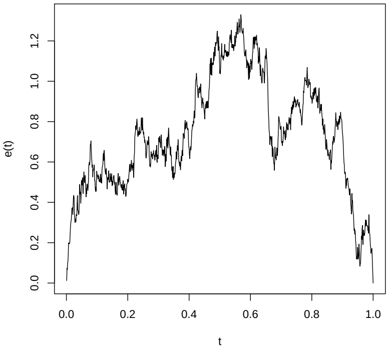
$$\mathsf{P}\{\|Y\|_\psi \le \varepsilon\} \sim \frac{8\sinh(\omega/2)\psi^{1/8}(0)\psi^{1/8}(1)}{\vartheta\pi^{1/2}\left(\psi^{1/2}(0)+\psi^{1/2}(1)\right)^{1/2}} \varepsilon \exp\left(-\frac{\vartheta^2}{8}\varepsilon^{-2}\right).$$

Non-Gaussian processes

For a non-Gaussian process $X(t)$ the use of random series of Karhunen-Loève-type becomes problematic. The problem simplifies if the process $X(t)$ can be expressed by means of simple Gaussian processes.

Brownian excursion

Denote by $\mathfrak{e}(t)$, $0 \le t \le 1$, a Brownian excursion.
We may imagine Brownian excursion on $[0, 1]$ as the Brownian motion starting at zero, conditioned to stay positive and to hit zero for the first time at time 1.



Brownian excursion

According to the Lévy-Williams identity,

$$\{\mathfrak{e}^2(t), 0 \le t \le 1\} \overset{law}{=} \{B_1^2(t) + B_2^2(t) + B_3^2(t), 0 \le t \le 1\},$$

where $B_1(t)$, $B_2(t)$, $B_3(t)$, $0 \le t \le 1$, are three independent Brownian bridges. Hence,

$$\|\mathfrak{e}\|_\psi^2 \overset{law}{=} \|B_1\|_\psi^2 + \|B_2\|_\psi^2 + \|B_3\|_\psi^2.$$

Brownian excursion

Theorem 6. *Let ψ be a regular-shaped weight on $[0,1]$. Then as $\varepsilon \longrightarrow 0$*

$$\mathsf{P}\{\|\mathfrak{e}\|_\psi \le \varepsilon\} \sim \frac{2\sqrt{6\vartheta}\psi^{3/8}(0)\psi^{3/8}(1)}{\sqrt{\pi}}\varepsilon^{-2}\exp\left(-\frac{9\vartheta^2}{8}\varepsilon^{-2}\right).$$

This result is new even for the unit weight $\psi \equiv 1$!

Brownian excursion

Theorem 7. *As* $\varepsilon \to 0$

$$\mathsf{P}\left\{\int_0^1 \frac{\mathfrak{e}^2(t)}{t(1-t)}\,dt \le \varepsilon^2\right\} \sim \frac{9\pi^3}{\varepsilon^5}\exp\left(-\frac{9\pi^2}{8\varepsilon^2}\right).$$

Let $\beta > -2$. Then as $\varepsilon \to 0$

$$\mathsf{P}\left\{\int_0^1 t^\beta \mathfrak{e}^2(t)\,dt \le \varepsilon^2\right\} \sim \frac{4\pi^{1/4}}{3^{(\beta-4)/(4(\beta+2))}\Gamma^{3/2}\left(\frac{\beta+3}{\beta+2}\right)} \times$$

$$\times\,((\beta+2)\varepsilon)^{-\frac{\beta+8}{2(\beta+2)}}\exp\left(-\frac{9}{2}\,((\beta+2)\varepsilon)^{-2}\right).$$

Brownian excursion

Denote by $Wat(\mathfrak{e})$ the Watson-type functional of the Brownian excursion:

$$Wat(\mathfrak{e}) = \int_0^1 \left(\mathfrak{e}(t) - \int_0^1 \mathfrak{e}(x)dx \right)^2 dt.$$

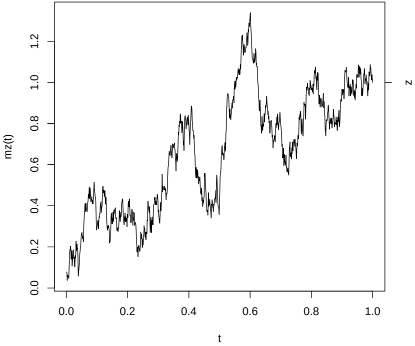
Theorem 8. *As* $\varepsilon \to 0$

$$\mathrm{P}\left\{ Wat(\mathfrak{e}) \leq \varepsilon^2 \right\} \sim \sqrt{\frac{2}{\pi}} \varepsilon^{-1} \exp\left(-\frac{1}{8\varepsilon^2} \right).$$

Brownian meander

Denote by $\mathfrak{m}^z(t)$, $0 \leq t \leq 1$, a Brownian meander taking the value $z \geq 0$ at the point 1.

We may think of $\mathfrak{m}^z(t)$ as the Brownian motion starting at zero, conditioned to stay positive and to take the value z at time 1.



Brownian meander

For the Brownian meander taking the value $z \geq 0$ at the point 1, one has (J. Bertoin, J. Pitman, J. Ruiz de Chavez, 1999)

$$(\mathfrak{m}^z(t))^2 \overset{law}{=} B_1^2(t) + B_2^2(t) + (B_3(t) + zt)^2,\ 0 \leq t \leq 1,$$

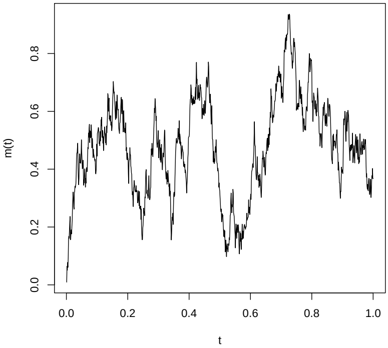
where B_1, B_2, and B_3 are three independent Brownian bridges.

Theorem 9. *For any $z \geq 0$, we have as $\varepsilon \to 0$*

$$\mathrm{P}\{\|\mathfrak{m}^z\| \leq \varepsilon\} \sim \frac{2\sqrt{2(z^2+3)}}{\sqrt{\pi}}\ \varepsilon^{-2}\ \exp\left(-\frac{(z^2+3)^2}{8}\varepsilon^{-2} + \frac{z^2}{2}\right).$$

Brownian meander

Denote by $\mathfrak{m}(t)$, $0 \le t \le 1$, a Brownian meander.
The Brownian meander can be thought of as a Brownian motion conditioned to stay positive up to time 1,



Brownian meander

J. Pitman, M. Yor, 1996: for the Brownian meander $\mathfrak{m}$, one has

$$\{\mathfrak{m}^2(t), 0 \le t \le 1\} \stackrel{law}{=} \{B^2(t) + W_1^2(t) + W_2^2(t), 0 \le t \le 1\},$$

where the Brownian bridge B and two Brownian motions W_1 and W_2 are independent.

Theorem 10. *For a regular-shaped weight function ψ on $[0,1]$, we have*

$$\mathsf{P}\{\|\mathfrak{m}\|_\psi \le \varepsilon\} \sim 4\sqrt{\frac{2}{3\pi}}\frac{\psi^{3/8}(0)}{\vartheta^{1/2}\psi^{1/8}(1)} \exp\left(-\frac{9\vartheta^2}{8}\varepsilon^{-2}\right).$$

Brownian meander

Theorem 11. *Let* $\beta > -2$. *As* $\varepsilon \to 0$

$$P\left\{\int_0^1 t^\beta \mathfrak{m}^2(t)dt \le \varepsilon^2\right\} \sim \frac{2^{2+\frac{\beta}{2(\beta+2)}}\pi^{\frac{1}{2}}}{3^{\frac{1}{2}+\frac{3\beta}{4(\beta+2)}}\Gamma\left(\frac{1}{\beta+2}\right)\Gamma^{1/2}\left(\frac{\beta+3}{\beta+2}\right)} \times$$

$$\times\left((\beta+2)\varepsilon\right)^{\frac{3\beta}{2(\beta+2)}} \exp\left(-\frac{9}{2}((\beta+2)\varepsilon)^{-2}\right).$$

Brownian local time

Let $L_t^x(B)$ be the jointly continuous local time of a Brownian bridge B at the point $x \in \mathbb{R}$ up to time $t \in [0,1]$.

M. Csörgő, Z. Shi, M. Yor, 1999: for any $m \in \mathbb{N}$,

$$\int_{-\infty}^{\infty} (L_1^x(B))^m dx \overset{law}{=} 2^{m-1} \int_0^1 (\mathfrak{e}(t))^{m-1} dt.$$

They also found the logarithmic asymptotics for

$$\mathrm{P}\left\{\int_{-\infty}^{\infty} (L_1^x(B))^2 dx \le \varepsilon\right\} \text{ and } \mathrm{P}\left\{\int_{-\infty}^{\infty} (L_1^x(B))^3 dx \le \varepsilon\right\}.$$

Brownian local time

Theorem 12. *As* $\varepsilon \to 0$

$$\mathrm{P}\left\{\int_{-\infty}^{\infty}(L_1^x(B))^3 dx \le \varepsilon\right\} \sim \frac{8\sqrt{6}}{\sqrt{\pi}}\varepsilon^{-1}\exp\left(-\frac{9}{2}\varepsilon^{-1}\right),$$

$$\mathrm{P}\left\{\int_{-\infty}^{\infty}(L_1^x(B))^2\ dx \le \varepsilon\right\} \sim \frac{8}{3}a_1^{3/2}\varepsilon^{-2}\exp\left(-\frac{8a_1^3}{27\varepsilon^2}\right),$$

where $a_1 \approx 2.3381$ is the absolute value of the first zero of the standard Airy function.

Brownian local time

Let $T_1 = \inf\{t : W(t) = 1\}$ be the first hitting time of 1 by the Brownian motion. For $x \in [0,1]$ consider the local time process in x up to the moment T_1:

$$Z(x) = L^x_{T_1}(W), \quad 0 \le x \le 1.$$

Theorem 13. *Let ψ be a regular-shaped weight on $[0,1]$. Then as $\varepsilon \to 0$*

$$\mathrm{P}\left\{\int_0^1 L^x_{T_1}(W)\psi(x)\,dx \le \varepsilon^2\right\} \sim \frac{2\sqrt{2}\psi^{1/4}(0)}{\sqrt{\pi}\vartheta\psi^{1/4}(1)}\varepsilon \exp\left(-\frac{\vartheta^2}{2}\varepsilon^{-2}\right).$$

More details and more results can be found in the paper

Nikitin, Ya. Yu. and Pusev, R. S. (2011). Exact L_2-small deviation asymptotics for some Brownian functionals. Preprint, 26 pp.

Available at `http://arxiv.org/abs/1104.2891`