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Exact solution of a nonlinear fin problem of temperature-dependent thermal conductivity and heat transfer coefficient

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Abstract

This paper studies a class of nonlinear problems of convective longitudinal fins with temperature-dependent thermal conductivity and heat transfer coefficient. For thermal conductivity and heat transfer coefficient dominated by power-law nonlinearity, the exact temperature distribution is obtained analytically in an implicit form. In particular, the explicit expressions of the fin temperature distribution are derived explicitly for some special cases. An analytical expression for fin efficiency is given as a function of a thermogeometric parameter. The influences of the nonlinearity and the thermogeometric parameter on the temperature and thermal performance are analyzed. The temperature distribution and the fin efficiency exhibit completely different behavior for the power-law exponent of heat transfer coefficient being larger or less than negative unity.

Keywords: exact solution; temperature-dependent thermal conductivity; temperature-dependent heat transfer coefficient; nonlinear fin problem.

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Nomenclature

A Cross-sectional area
L Axial length
P Perimeter of cross-section
M Dimensionless thermogeometric parameter
m Power exponent of variable thermal conductivity
n Power exponent of variable heat transfer coefficient
T_a Temperature of a surrounding fluid
T_b Temperature at fin's base
$k(T)$ Temperature-dependent thermal conductivity
k_a Thermal conductivity at temperature of a surrounding fluid
$h(T)$ Temperature-dependent convection heat transfer coefficient
h_b Convection heat transfer coefficient at the base
X Axial distance measured from the fin's tip
x Dimensionless axial distance measured from the tip

Greek symbols
α A measure of thermal conductivity variation with temperature
δ Dimensionless constant of thermal conductivity
η Fin efficiency
θ Dimensionless temperature excess
θ_0 Dimensionless temperature excess at the fin tip

1 Introduction

Heat transfer is a typical phenomenon encountered frequently in both engineering application and daily life, and it has a long history of research [1]. Great progress in this area has made in recent decades. As an indispensable part in heat transfer problems, fins have been widely used in refrigeration and air conditioning and are capable of significantly enhancing heat transfer efficiency. Therefore, seeking the influence factors of heat transfer process and studying the method of enhancement of heat transfer efficiency are fundamental subjects in this field. To increase the heat transfer efficiency, many researches have been conducted in experiment and theory [2]. Heat transfer and pressure drop phenomena over a bank of micro pin fins were investigated [3]. Local heat transfer coefficient and fin efficiency of wavy fin-and-tube heat exchangers were studied via a numerical approach [4]. The effects of radiation and convection heat transfer in a rectangular profile fin were analyzed [5]. The thermal performance and efficiency of convective–radiative straight fins with various profiles were tackled [6].

For some fins, the thickness or cross-section is relatively small than the length, and the temperature change along the fin thickness-wise direction or cross-section is small and simply neglected.

Significant temperature change only occurs along the longitudinal direction. Such fins may be understood as one-dimensional (1D) fins, in which the temperature distribution is a function of one spatial variable, in addition to time variable unless for transient fin problems. On the other hand, due to large temperature change or temperature difference occurring for most fins, the heat transfer coefficient and the thermal conductivity cannot be assumed to be constant, but change with temperature. In other words, the thermal conductivity, heat transfer coefficient, and so on are dependent on temperature, and are nonlinear functions of unknown temperature. This leads to a difficulty in seeking an exact temperature distribution in the fins. Hitherto, many numerical methods have been presented to solve the temperature change of 1D fins. For instance, the fin efficiency of convective straight fins with temperature-dependent thermal conductivity was analyzed by the Adomian decomposition method [7–10], the homotopy analysis method [11, 12], the differential transformation method [6, 13–15], the variational iteration method [16, 17], the Taylor series method [18], the least squares method [19], the singular boundary method [20], the Variation of Parameters Method [21]. Besides, the effects of some physical parameters for the temperature distribution have been analyzed by using a hybrid approximate technique based on the differential transform method and finite difference method [15]. Owing to the difficulty of nonlinear fin problems, the above-mentioned researches mainly focus on numerical analysis. However, closed-form exact solutions are much desired since they are not only useful to design engineers but also to researchers as benchmark solutions for checking the convergence, validity, and accuracy of numerical methods. Therefore, to date some efforts have been made to obtain exact solutions of nonlinear fin problems with temperature-dependent thermal conductivity and/or heat transfer coefficient. Along this line, Moitsheki et al. employed the classical Lie symmetry technique to derived exact solutions of a nonlinear fin problem with power-law dominated temperature-dependent thermal conductivity and heat transfer coefficient [22]. For power-law temperature-dependent heat transfer coefficient, some exact solutions have been obtained and they are expressed in terms of the hypergeometric function [23], or in implicit form by solving a functional equation[24–26]. Furthermore, for power-law type temperature-dependent heat transfer coefficient and thermal conductivity, Abbasbandy and Shivanian also formulated several exact solutions of the fin temperature distribution for 1D straight fins [27].

In this paper, a more generalized and more realistic nonlinear problem for 1D straight fins is considered, where thermal conductivity and heat transfer coefficient are two power-law functions of temperature distribution. Various cases are discussed and analytical solutions are derived. In certain cases, explicit expressions for temperature are obtained. Finally, the fin efficiency is accurately evaluated. The influences of the power-law index in the thermal conductivity and the heat transfer coefficient on the temperature distribution and the fin efficiency are displayed graphically.

2 Statement of the problem

Consider a nonlinear heat transfer problem of a 1D convective longitudinal straight fin with cross-sectional area A, length L, and perimeter P, as shown in Fig. 1. The fin is attached to a base surface of temperature T_b, extends into a surrounding fluid of temperature T_a, and its tip is assumed to be insulated since heat transfer through the tip end is relatively small and is negligible. Namely, we have the following boundary conditions

$$\frac{dT}{dX} = 0, \text{ at } X = 0, \tag{1}$$

$$T = T_b, \text{ at } X = L. \tag{2}$$

When surface heat radiation is neglected and only surface heat convection through the surface is considered, the 1D steady-state heat balance equation reads

$$\frac{d}{dX}\left[k(T)\,A\frac{dT}{dX}\right] - Ph(T)(T-T_a) = 0, \; 0 < X < L \tag{3}$$

where the thermal conductivity $k(T)$ and the convection heat transfer coefficient $h(T)$ of the fin material are assumed to be temperature-dependent. In the present study, we assume that both of them are power-law functions of temperature, i.e.

$$k(T) = k_a\left[\delta + \alpha\left(\frac{T-T_a}{T_b-T_a}\right)^m\right], \tag{4}$$

$$h(T) = h_b\left(\frac{T-T_a}{T_b-T_a}\right)^n, \tag{5}$$

where k_a is the thermal conductivity when temperature takes T_a, δ is a dimensionless constant that takes 0 or 1 without loss of generality ($\delta = 0$ is often adopted in the previous studies for simplicity [27], and $\delta = 1$ is more realistic since the thermal conductivity takes k_a in the case of $T = T_a$, rather than zero), m and α are two parameters to describe the variation of the thermal conductivity depending on varying temperature, h_b is the convection heat transfer coefficient at the base surface, and the exponent n depends on the heat transfer mode and may vary in a wide range between −6.6 and 5. For example, $n = -4$ for R113, $n = -3$ for transition boiling, $n = 0$ for film boiling, $n = 1$ for convection, $n = 2$ for nucleate boiling, $n = 3$ for radiation into free space at zero absolute temperature [28–30]. For the power-law exponent of the thermal conductivity, a linear thermal conductivity is reduced if choosing $m = \delta = 1$ [25]. For other nonlinear thermal conductivity with $\delta = 1$, relative work is very limited, although some work in the case of $\delta = 0$ has been reported [22]. For the constant thermal conductivity with $\delta = 1$ and $\alpha = 0$, the nonlinear dependence only arises from the convection heat transfer coefficient. Obviously, if $m \neq 0, n \neq 0$, the above fin problem is related to multi-nonlinearity. Conversely, if $m = n = 0$, the problem reduces to a linear fin problem. The latter is quite simple. On the other hand, for practical problems, the thermal conductivity is positive, which implies that the parameters α and m must meet necessary conditions. Many studies focus on constant and linearly temperature-dependent thermal conductivity, in fact nonlinearly temperature-dependent thermal conductivity often takes place [31, 32]. Here more generalized nonlinearly temperature-dependent thermal conductivity and convection heat transfer coefficient will be treated.

For convenience of later analysis, let us introduce the following dimensionless parameters

$$\theta = \frac{T - T_a}{T_b - T_a}, \quad x = \frac{X}{L}, \quad k(\theta) = \frac{k(T)}{k_a}, \quad M = L\sqrt{\frac{h_b P}{k_a A}}, \tag{6}$$

where the dimensionless parameter M is a thermogeometric parameter, and it plays a significant role in the design and application of fins. In fact, its physical significance is seen from the following relationship

$$M = \sqrt{\frac{L/(k_a A)}{1/(h_b P L)}} = \sqrt{\frac{\text{internal thermal resistance along the fin length } L \text{ direction}}{\text{gross external thermal resistance}}}. \tag{7}$$

Using the above-introduced these dimensionless parameters, the 1D nonlinear heat transfer balance equation (3) may be rewritten as

$$\frac{d}{dx}\left[k(\theta)\frac{d\theta}{dx}\right] - M^2\theta^{n+1} = 0, \quad 0 < x < 1 \tag{8}$$

or

$$(\delta + \alpha\theta^m)\theta'' - M^2\theta^{n+1} + m\alpha\theta^{m-1}(\theta')^2 = 0, \tag{9}$$

where the prime denotes differentiation with respect to the argument, i.e. $\theta'(x) = d\theta/dx$. The boundary conditions (1) and (2) can be stated in the dimensionless form:

$$\theta'(0) = 0, \quad \theta(1) = 1. \tag{10}$$

3 Exact solutions

In order to solve the above-stated problem more conveniently, we introduce a new unknown function through $u = d\theta/dx$. Then we find

$$\frac{d^2\theta}{dx^2} = \frac{du}{dx} = \frac{du}{d\theta}\frac{d\theta}{dx} = u\frac{du}{d\theta}. \tag{11}$$

With the above result, after some manipulation the ordinary differential equation (9) can be converted to the following form:

$$(\delta + \alpha\theta^m)udu + (m\alpha\theta^{m-1}u^2 - M^2\theta^{n+1})d\theta = 0. \tag{12}$$

Next, after multiplying both sides of the above differential equation (12) by $\delta + \alpha\theta^m$, from (12) we readily get

$$(\delta + \alpha\theta^m)^2udu + m\alpha\theta^{m-1}u^2(\delta + \alpha\theta^m)d\theta - M^2(\delta\theta^{n+1} + \alpha\theta^{m+n+1})d\theta = 0. \tag{13}$$

In the following, we seek exact solutions for some combined cases of the power-law exponents m and n.

3.1 The case of $n \neq -2$

3.1.1 $m + n \neq -2$

Firstly, we consider the case of $n \neq -2$ and $m + n \neq -2$. For this case, after integrating both sides of equation (13), one has

$$\frac{1}{2}\left(\delta + \alpha\theta^m\right)^2 u^2 - \frac{M^2\delta}{n+2}\theta^{n+2} - \frac{M^2\alpha}{m+n+2}\theta^{m+n+2} = C, \tag{14}$$

where C is an unknown integration constant. Taking into account $u = d\theta/dx$, one may rewrite the above-resulting equation (14) as follows

$$\frac{1}{2}(\delta + \alpha\theta^m)^2\left(\frac{d\theta}{dx}\right)^2 - \frac{M^2\delta}{n+2}\theta^{n+2} - \frac{M^2\alpha}{m+n+2}\theta^{m+n+2} = C. \tag{15}$$

Now, using the first boundary condition in (10), one acquires

$$-\frac{M^2\delta}{n+2}\theta_0^{n+2} - \frac{M^2\alpha}{m+n+2}\theta_0^{m+n+2} = C, \tag{16}$$

where θ_0 is specified by $\theta\left(0\right)$. Eliminating the constant C through subtracting (16) from (15) leads to

$$\frac{1}{2}(\delta + \alpha\theta^m)^2\left(\frac{d\theta}{dx}\right)^2 = \frac{M^2\delta}{n+2}(\theta^{n+2} - \theta_0^{n+2}) + \frac{M^2\alpha}{m+n+2}(\theta^{m+n+2} - \theta_0^{m+n+2}), \tag{17}$$

which is further rewritten as

$$\frac{(\delta+\alpha\theta^m)d\theta}{\sqrt{\frac{2M^2\delta}{n+2}(\theta^{n+2}-\theta_0^{n+2})+\frac{2M^2\alpha}{m+n+2}(\theta^{m+n+2}-\theta_0^{m+n+2})}}=dx, \tag{18}$$

where another equation with a negative sign before dx is removed since it has no physical meaning in practical application.

Integrating both sides of equation (18) and imposing the boundary condition we get

$$\sqrt{2}Mx=\int_{\theta_0}^{\theta}\frac{(\delta+\alpha z^m)dz}{\sqrt{\frac{\delta}{n+2}(z^{n+2}-\theta_0^{n+2})+\frac{\alpha}{m+n+2}(z^{m+n+2}-\theta_0^{m+n+2})}}. \tag{19}$$

Notice that in the above equation, θ_0 is still unknown. To determine it, in view of the boundary condition $\theta(1)=1$, one finds that θ_0 must satisfy the following equation

$$\sqrt{2}M=\int_{\theta_0}^{1}\frac{(\delta+\alpha z^m)dz}{\sqrt{\frac{\delta}{n+2}(z^{n+2}-\theta_0^{n+2})+\frac{\alpha}{m+n+2}(z^{m+n+2}-\theta_0^{m+n+2})}}. \tag{20}$$

Equation (20) provides us with a relation between θ_0 and M in terms of integration, and its solution is easily determined through commercial software such as Mathematica and Matlab, provided that the parameters α,δ,m,n are prescribed. Once the θ_0 value is determined via (20), we put its value into (19) to obtain the dependence of the temperature change on the certain physical quantities of interest.

In the following let us further consider a special case of $\delta=0$ and $m=n$. we denote

$$M_*=\frac{M}{\sqrt{\alpha}}. \tag{21}$$

Under such circumstances, the parameter α is required to be positive, and equation (19) then reduces to

$$\frac{M_*x}{\sqrt{n+1}}=\int_{\theta_0}^{\theta}\frac{z^n dz}{\sqrt{z^{2n+2}-\theta_0^{2n+2}}}, \tag{22}$$

for $n>-1$, and

$$\frac{M_*x}{\sqrt{|n+1|}}=\int_{\theta_0}^{\theta}\frac{z^n dz}{\sqrt{\theta_0^{2n+2}-z^{2n+2}}}, \tag{23}$$

for $n<-1$. It is noted that owing to our assumption in the present case under scrutiny, the case of $m=n=-1$ does not occur and will be considered in the subsequent analysis. A straight integration allows us to arrive at

$$\sqrt{n+1}M_*x=\ln\frac{\theta^{n+1}+\sqrt{\theta^{2n+2}-\theta_0^{2n+2}}}{\theta_0^{n+1}},\quad \text{if } n>-1, \tag{24}$$

$$\sqrt{|n+1|}M_*x=\cos^{-1}\left(\frac{\theta^{n+1}}{\theta_0^{n+1}}\right),\qquad \text{if } n<-1. \tag{25}$$

If expressing the temperature change in terms of x, from the above results we easily obtain

$$\theta = \begin{cases} \left[\dfrac{\cosh\left(\sqrt{n+1}M_* x\right)}{\cosh\left(\sqrt{n+1}M_*\right)}\right]^{1/(n+1)}, \text{ if } n > -1, \\ e^{\frac{M_*^2(x^2-1)}{2}}, \text{ if } n = -1, \\ \left[\dfrac{\cos\left(\sqrt{|n+1|}M_* x\right)}{\cos\left(\sqrt{|n+1|}M_*\right)}\right]^{1/(n+1)}, \quad \text{if } n < -1, \end{cases} \tag{26}$$

where the solution for $n = -1$ will be given in the following, i.e. (36). In the above solution, besides the parameter n, the temperature distribution is expressed explicitly in terms of the parameters M and α. It is interesting to note that when $n < -1$, the parameter M must satisfy $M \leq \pi\sqrt{\alpha}/2\sqrt{|n+1|}$ to avoid multivalue appearance.

It is pointed out that our solution with $\alpha = 1$ for $n \geq -1$ are in exact agreement with those derived in [22]. However, the solution for $n < -1$ seems not to be reported before, to the best of the authors' knowledge. In an alternative way, we also give the temperature distribution θ as a function of θ_0 at the fin tip, and the parameters M and α do not appear, namely

$$\theta = \begin{cases} \theta_0^{1-x}\left[\dfrac{\left(1+\sqrt{1-\theta_0^{2n+2}}\right)^x + \left(1-\sqrt{1-\theta_0^{2n+2}}\right)^x}{2}\right]^{1/(n+1)}, \text{ if } n > -1, \\ \theta_0^{1-x^2}, \text{ if } n = -1, \\ \theta_0 \cos^{1/(n+1)}\left(x\cos^{-1}\left(\theta_0^{-n-1}\right)\right), \quad \text{if } n < -1, \end{cases} \tag{27}$$

if θ_0 is prescribed. Of course, θ_0 is dependent on the parameters M and α, which can be seen by setting $x = 0$ in (26) or (24) and (25). That is, the dependence of θ_0 on M and α is given by

$$M = \begin{cases} \sqrt{\frac{\alpha}{n+1}} \ln\left[\theta_0^{-n-1} + \sqrt{\theta_0^{-2(n+1)} - 1}\right], \text{ if } n > -1, \\ \sqrt{-2\alpha\ln\theta_0}, \text{ if } n = -1, \\ \sqrt{\frac{\alpha}{|n+1|}} \cos^{-1}\left(\theta_0^{-n-1}\right), \quad \text{if } n < -1, \end{cases} \tag{28}$$

where a negative M value has been removed since it has no physical meaning.

3.1.2 $m + n = -2$

Both sides of the ordinary differential equation (13) are integrated, yielding

$$\frac{1}{2}(\delta + \alpha\theta^m)^2 u^2 + \frac{M^2\delta}{m\theta^m} - M^2\alpha\ln\theta = C. \tag{29}$$

Imposing the boundary condition at $x = 0$ can get

$$\frac{M^2\delta}{m\theta_0^m} - M^2\alpha\ln\theta_0 = C. \tag{30}$$

We plug equation (30) into equation (29) to eliminate the constant C,

$$\frac{1}{2}(\delta+\alpha\theta^m)^2u^2+\frac{M^2\delta}{m}(\frac{1}{\theta^m}-\frac{1}{\theta_0^m})-M^2\alpha\ln{(\theta/\theta_0)}=0. \tag{31}$$

In view of $u=d\theta/dx$ in equation (31), we immediately obtain

$$\frac{(\delta+\alpha\theta^m)d\theta}{\sqrt{\alpha\ln\frac{\theta}{\theta_0}-\frac{\delta}{m}\left(\frac{1}{\theta^m}-\frac{1}{\theta_0^m}\right)}}=\sqrt{2}M dx. \tag{32}$$

Integrating both sides of equation (32) results in

$$\sqrt{2}Mx=\int_{\theta_0}^{\theta}\frac{(\delta+\alpha z^m)dz}{\sqrt{\alpha\ln\frac{z}{\theta_0}-\frac{\delta}{m}\left(\frac{1}{z^m}-\frac{1}{\theta_0^m}\right)}}. \tag{33}$$

Taking the boundary condition $\theta\,(1)=1$ in the above resulting equation, one has

$$\sqrt{2}M=\int_{\theta_0}^{1}\frac{(\delta+\alpha z^m)dz}{\sqrt{\alpha\ln\frac{z}{\theta_0}-\frac{\delta}{m}\left(\frac{1}{z^m}-\frac{1}{\theta_0^m}\right)}}. \tag{34}$$

Once the θ_0 value is determined via (34), we put its value into (33) to obtain the dependence of the temperature change on certain physical quantities of interest.

In particular, consider a special case of $\delta=0$ and $m=-1$, meaning $m=n=-1$. Under such circumstances, the above exact solution (33) further becomes

$$x=\frac{\sqrt{2\ln{(\theta/\theta_0)}}}{M_*}. \tag{35}$$

Expressing the temperature change in terms of x, equation (35) can be transformed into the following form

$$\theta=e^{\frac{M_*^2(x^2-1)}{2}}. \tag{36}$$

3.2 The case of $n=-2$

3.2.1 $m\neq 0$

For this case, we integrate both sides of the ordinary differential equation (12), yielding

$$\frac{1}{2}(\delta+\alpha\theta^m)^2u^2-M^2\delta\ln\theta-\frac{M^2\alpha}{m}\theta^m=C. \tag{37}$$

The first boundary condition in (10) allows us to obtain

$$-M^2\delta\ln\theta_0-\frac{M^2\alpha}{m}\theta_0^m=C. \tag{38}$$

Eliminating the constant C from (37) and (38) leads to

$$\frac{1}{2}(\delta+\alpha\theta^m)^2u^2 = M^2\delta\ln\frac{\theta}{\theta_0} + \frac{M^2\alpha}{m}(\theta^m - \theta_0^m). \tag{39}$$

Bearing $u = d\theta/dx$ in mind, the above ordinary differential equation (39) can be rewritten as follows:

$$\frac{1}{2}(\delta+\alpha\theta^m)^2\left(\frac{d\theta}{dx}\right)^2 = M^2\delta\ln\frac{\theta}{\theta_0} + \frac{M^2\alpha}{m}(\theta^m - \theta_0^m), \tag{40}$$

or

$$\sqrt{2}M\,dx = \frac{(\delta+\alpha\theta^m)d\theta}{\sqrt{\delta\ln\frac{\theta}{\theta_0} + \frac{\alpha}{m}(\theta^m - \theta_0^m)}}. \tag{41}$$

Integrating both sides of equation (41) and imposing the boundary condition $\theta = \theta_0$ as $x = 0$, one finally obtains

$$\sqrt{2}Mx = \int_{\theta_0}^{\theta}\frac{(\delta+\alpha z^m)dz}{\sqrt{\delta\ln\frac{z}{\theta_0} + \frac{\alpha}{m}(z^m - \theta_0^m)}}. \tag{42}$$

Inserting the boundary condition $\theta\,(1) = 1$ into equation (42), the undetermined parameter θ_0 should meet the following equation

$$\sqrt{2}M = \int_{\theta_0}^{1}\frac{(\delta+\alpha z^m)dz}{\sqrt{\delta\ln\frac{z}{\theta_0} + \frac{\alpha}{m}(z^m - \theta_0^m)}}. \tag{43}$$

Once the θ_0 value is determined via (43), we put its value into (42) to obtain the dependence of the temperature change on certain physical quantities of interest.

In the following, we consider a special case $\delta = 0$. In this case, equation (42) simplifies to

$$\sqrt{2}M_*x = \int_{\theta_0}^{\theta}\sqrt{\frac{m}{z^m - \theta_0^m}}z^m dz. \tag{44}$$

Next, we also give explicit expressions for the exact solution of several values of m. For $m = -2$, equation (44) can be integrated in a closed form:

$$x = \frac{1}{M_*}\cos^{-1}\frac{\theta_0}{\theta}, \tag{45}$$

which further gives an explicit expression below

$$\theta = \frac{\cos(M_*)}{\cos(M_*x)}. \tag{46}$$

In the case of $m = -1$, omitting the detail, one finally obtains the exact solution as follows:

$$x = \frac{\sqrt{2\theta_0}}{M_*}\ln\left(\sqrt{\frac{\theta}{\theta_0}} + \sqrt{\frac{\theta}{\theta_0} - 1}\right), \tag{47}$$

which can be further rewritten as the function of θ

$$\theta = \left[\frac{\cosh\left(\frac{M_* x}{\sqrt{2\theta_0}}\right)}{\cosh\left(\frac{M_*}{\sqrt{2\theta_0}}\right)}\right]^2, \tag{48}$$

where θ_0 satisfies the following equation

$$\frac{M_*}{\sqrt{2\theta_0}} = \ln\left(\sqrt{\frac{1}{\theta_0}} + \sqrt{\frac{1}{\theta_0} - 1}\right), \tag{49}$$

which is further rewritten as

$$\theta_0 = \frac{2}{1 + \cosh\left(\sqrt{\frac{2}{\theta_0}} M_*\right)}. \tag{50}$$

In the case of $m = 1$, using the same procedure one gets the exact solution below

$$x = \frac{(\theta + 2\theta_0)\sqrt{2(\theta - \theta_0)}}{3M_*}. \tag{51}$$

Furthermore, θ may be rewritten as a cubic equation

$$y^3 + 3\theta_0 y - t = 0, \tag{52}$$

with

$$y = \sqrt{\theta - \theta_0}, \quad t = \frac{3M_*}{\sqrt{2}} x. \tag{53}$$

With the aid of the roots of a cubic equation, a real root is found to be

$$\sqrt{\theta - \theta_0} = \left(\frac{t + \sqrt{4\theta_0^3 + t^2}}{2}\right)^{1/3} - \theta_0 \left(\frac{2}{t + \sqrt{4\theta_0^3 + t^2}}\right)^{1/3}, \tag{54}$$

or

$$\theta = \left(\frac{t + \sqrt{4\theta_0^3 + t^2}}{2}\right)^{2/3} + \theta_0^2 \left(\frac{2}{t + \sqrt{4\theta_0^3 + t^2}}\right)^{2/3} - \theta_0. \tag{55}$$

In particular, if letting $x = 1$, from $\theta(1) = 1$ one gets

$$\theta_0 = \frac{1}{2}\left(B + \frac{1}{B}\right), \tag{56}$$

where

$$B = \left(1 - \frac{9M_*^2}{2} + \sqrt{\left(\frac{9}{2}M_*^2\right)^2 - (3M_*)^2}\right)^{1/3}. \tag{57}$$

Once substitution of the expression (57) for B into (55), an explicit expression for temperature θ depending on M is derived, which is omitted here for saving space. Alternatively, by eliminating the parameter M we express θ in terms of θ_0 below

$$\theta = E^{2/3} + \theta_0^2 E^{-2/3} - \theta_0, \tag{58}$$

with

$$E = \frac{\sqrt{1+3\theta_0-4\theta_0^3}x+\sqrt{(1+3\theta_0)\,x^2+4\theta_0^3\,(1-x^2)}}{2} \tag{59}$$

Finally, let us consider the case of $m = 2$. After integration of (44) we obtain an exact solution

$$x = \frac{1}{2M_*}\left(\theta\sqrt{\theta^2-\theta_0^2}+\theta_0^2\ln\frac{\theta+\sqrt{\theta^2-\theta_0^2}}{\theta_0}\right). \tag{60}$$

For the above result, it is inconvenient to express θ as an explicit function of x. However, we still may plot the dependence of θ on x only if θ_0 is determined by taking $x = 1$ in (60). In other words, the θ_0 value is related to M through the following relationship

$$2M_* = \sqrt{1-\theta_0^2}+\theta_0^2\ln\frac{1+\sqrt{1-\theta_0^2}}{\theta_0}. \tag{61}$$

Thus, from (60) one can give the dependence relationship of θ as a function of x. An alternative dependence of θ on θ_0 is easily determined from (60) and (61), viz.

$$x = \frac{\theta\sqrt{\theta^2-\theta_0^2}+\theta_0^2\ln\frac{\theta+\sqrt{\theta^2-\theta_0^2}}{\theta_0}}{\sqrt{1-\theta_0^2}+\theta_0^2\ln\frac{1+\sqrt{1-\theta_0^2}}{\theta_0}}. \tag{62}$$

3.2.2 $m = 0$

The remaining case is $n = -2$ and $m = 0$, which in fact corresponds to a constant thermal conductivity. Equation (12) in this case reduces to

$$(\delta+\alpha)udu - M^2\theta^{-1}d\theta = 0. \tag{63}$$

A straight integration leads to

$$(\delta+\alpha)\left(\frac{d\theta}{dx}\right)^2 - 2M^2\ln\theta = C. \tag{64}$$

Obviously, setting $x = 0$ in the above equation, due to $\theta'(0) = 0$, we have $C = -2M^2\ln\theta_0$. Subsequently, the above first-order ordinary differential equation (64) can be further rewritten

$$\frac{d\theta}{\sqrt{\ln(\theta/\theta_0)}} = M\sqrt{\frac{2}{\delta+\alpha}}dx. \tag{65}$$

Introducing $\theta = \theta_0 e^{s^2}$, from (65) one sees

$$e^{s^2}ds = \sqrt{\frac{1}{2(\delta+\alpha)}}\frac{M}{\theta_0}dx, \tag{66}$$

or

$$x = \frac{\theta\sqrt{2(\delta+\alpha)}D\left(\sqrt{\ln\left(\theta/\theta_0\right)}\right)}{M}, \tag{67}$$

where $D(x)$ is the Dawson function, defined by

$$D(x) = e^{-x^2}\int_0^x e^{s^2}ds. \tag{68}$$

Finally, in terms of θ_0, we easily give the relationship of θ as follows:

$$x\int_0^{\sqrt{\ln(1/\theta_0)}} e^{s^2}ds = \int_0^{\sqrt{\ln(\theta/\theta_0)}} e^{s^2}ds. \tag{69}$$

4 Fin efficiency

The fin efficiency is the ratio of the actual heat transferred from the fin surface to the surrounding liquid to the amount of heat to be transferred if the entire fin area is at the base temperature. In fact, the actual heat transferred from the fin surface to the surrounding liquid is equal to the heat conducted through the base at the position $X = L$ or $x = 1$ through the fin, the fin efficiency can be written as follows

$$\eta = \frac{k\left(T_b\right)A\left.\frac{dT}{dX}\right|_{X=L}}{PLh\left(T_b\right)\left(T_b-T_a\right)} = \frac{\delta+\alpha}{M^2}\theta'(1). \tag{70}$$

Therefore, $\theta'(1)$ can be directly obtained from the previous results. That is, with the help of (18), (41), (32), and (65) one has

$$\frac{d\theta}{dx} = \begin{cases} \dfrac{M\sqrt{2(m+n+2)\delta(\theta^{n+2}-\theta_0^{n+2})+2\alpha(n+2)(\theta^{m+n+2}-\theta_0^{m+n+2})}}{\sqrt{(m+n+2)(n+2)}(\delta+\alpha\theta^m)}, & n\neq-2, n\neq-m-2 \\ \dfrac{M\sqrt{2m\alpha(\ln\theta-\ln\theta_0)-2\delta\left(\dfrac{1}{\theta^m}-\dfrac{1}{\theta_0^m}\right)}}{\sqrt{m}(\delta+\alpha\theta^m)}, & n\neq-2, n=-m-2 \\ \dfrac{M\sqrt{2m\delta(\ln\theta-\ln\theta_0)+2\alpha(\theta^m-\theta_0^m)}}{\sqrt{m}(\delta+\alpha\theta^m)}, & n=-2, m\neq0 \\ M\sqrt{\dfrac{2\ln\left(\theta/\theta_0\right)}{\delta+\alpha}}, & n=-2, m=0 \end{cases} \tag{71}$$

Putting the above results (71) into (70) leads to the fin efficiency as follows:

$$\eta = \begin{cases} \frac{1}{M}\sqrt{\dfrac{2\delta}{n+2}(1-\theta_0^{n+2})+\dfrac{2\alpha}{m+n+2}(1-\theta_0^{m+n+2})}, & n\neq-2, n\neq-m-2 \\ \frac{1}{M}\sqrt{\dfrac{2\delta\left(\theta_0^{-m}-1\right)}{m}-2\alpha\ln\theta_0}, & n\neq-2, n=-m-2 \\ \frac{1}{M}\sqrt{\dfrac{2\alpha(1-\theta_0^m)}{m}-2\delta\ln\theta_0}, & n=-2,\ m\neq0 \\ \frac{1}{M}\sqrt{-2\ln\theta_0(\delta+\alpha)}, & n=-2, m=0. \end{cases} \tag{72}$$

In what follows, we turn our attention to a special case of $\delta = 0$. For this special case, some explicit expressions for the fin efficiency can be given. For instance, for $m = n$, we take the derivative of both sides of equation (26), and get the following results

$$\frac{d\theta}{dx} = \begin{cases} \dfrac{M_* \sinh\left(\sqrt{n+1}M_* x\right)}{\sqrt{n+1}\cosh\left(\sqrt{n+1}M_*\right)}\left[\dfrac{\cosh\left(\sqrt{n+1}M_* x\right)}{\cosh\left(\sqrt{n+1}M_*\right)}\right]^{-n/(n+1)}, & \text{if } n > -1, \\ M_*^2 x e^{\frac{M_*^2(x^2-1)}{2}}, \text{ if } n = -1, \\ \dfrac{M_* \sin\left(\sqrt{|n+1|}M_* x\right)}{\sqrt{|n+1|}\cos\left(\sqrt{|n+1|}M_*\right)}\left[\dfrac{\cos\left(\sqrt{|n+1|}M_* x\right)}{\cos\left(\sqrt{|n+1|}M_*\right)}\right]^{-n/(n+1)}, & \text{if } n < -1. \end{cases} \tag{73}$$

By substituting the above equation (73) into equation (70), we can obtain the following exact expressions for the fin efficiency

$$\eta = \begin{cases} \dfrac{\tanh\left(\sqrt{n+1}M_*\right)}{\sqrt{n+1}M_*}, \text{ if } n > -1, \\ 1, \text{ if } n = -1, \\ \dfrac{\tan\left(\sqrt{|n+1|}M_*\right)}{\sqrt{|n+1|}M_*}, \text{ if } n < -1. \end{cases} \tag{74}$$

From the above results, one readily finds that if $n < -1$, the fin efficiency is larger than unity, whereas $n > -1$, it is less than unity. For frequently encountered situations, we have $n > -1$ [29, 30], and the fin efficiency is always lower than the unity. In an alternative manner, we have

$$\eta = \begin{cases} \frac{\sqrt{1-\theta_0^{2(n+1)}}}{\ln\left[\theta_0^{-n-1}+\sqrt{\theta_0^{-2(n+1)}-1}\right]}, \text{ if } n > -1, \\ 1, \text{ if } n = -1, \\ \frac{\sqrt{\theta_0^{2(n+1)}-1}}{\cos^{-1}\left(\theta_0^{-n-1}\right)}, \text{ if } n < -1. \end{cases} \tag{75}$$

Furthermore, we omit the detail and give the fin efficiency in the case of $\delta = 0$ and $n = -2$,

$$\eta = \begin{cases} \frac{\sqrt{\theta_0^{-2}-1}}{\cos^{-1}\theta_0}, \text{ if } m = -2, \\ \frac{\sqrt{\theta_0^{-1}-1}}{\sqrt{\theta_0}\ln\left(\sqrt{\theta_0^{-1}}+\sqrt{\theta_0^{-1}-1}\right)}, \text{ if } m = -1, \\ \frac{\sqrt{-\ln\theta_0}}{D\left(\sqrt{\ln(1/\theta_0)}\right)}, \text{ if } m = 0, \\ \frac{3}{1+2\theta_0}, \text{ if } m = 1, \\ \frac{2\sqrt{1-\theta_0^2}}{\sqrt{1-\theta_0^2}+\theta_0^2\ln\left(\theta_0^{-1}+\sqrt{\theta_0^{-1}-1}\right)}, \text{ if } m = 2. \end{cases} \tag{76}$$

5 Results and discussion

Based on the results derived above, the exact analytical solutions were obtained in implicit form for most cases. For certain special cases, explicit expressions for the fin temperature distribution and the fin efficiency are possible. For the case of $m = n$, Fig. 2(a,b) shows the temperature distribution for a given thermogeometric parameter $M/\sqrt{\alpha} = 1.1$ or tip temperature $\theta_0 = 0.6$. The temperature at each position on the fin increases with n rising. From a physical viewpoint, it is because the increase of parameters m and n implies that the thermal conductivity and heat transfer coefficient become lower, then the heat emitted to the external environment decreases or less heat is dissipated. This conclusion is in agreement with that obtained for linear thermal conductivity in [6]. In particular, if choosing $M/\sqrt{\alpha} = 1.1$, the temperature excess θ may arrive at a value lower than 0.2 for $n = -3$, which is much less than those for other n values. On the other hand, if limiting the dimensionless tip temperature excess $\theta_0 = 0.6$, from Fig. 2b one also finds that a less n value causes a lower temperature excess. For the case of unequal m and n, Fig. 3 shows the temperature distribution along the fin length if taking $n = -2, \delta = 0$. The increase of m also causes the fin temperature to rise, and the reason is the same as the above. That is, the increase of m indicates a reduction in the property of thermal conductivity.

The fin efficiency is also discussed for the above cases. Fig. 4 shows the fin efficiency η as a function of M for different n values when $\alpha = 0.4, \delta = 0, m = n$. From Fig. 4, it is seen that the fin efficiency strongly depends on the index n. With M rising, the fin efficiency significantly increases if $n < -1$ such as transition boiling, decreases if $n > -1$ such as film and nucleate boiling, and remains unchanged if $n = -1$. Such a trend is similar to that observed for a linear thermal conductivity in [25]. In other words, the contribution mainly arises from the convection heat transfer. For a special case of $n = -2$, Fig. 5 displays the variation of the fin efficiency η against θ_0 for different m values with $\alpha = 0.4$. With the increase of temperature θ_0 at the fin tip, there is a declination in the fin efficiency, irrespective of the thermal conductivity index parameter m. Moreover, the smaller the m value, the larger the η value. It turns out that a fin with lower thermal conductivity index m corresponding to strong thermal conductivity has larger fin efficiency.

Next, we examine the influence of the power-law index in the thermal conductivity and heat transfer coefficient on the temperature distribution and the fin efficiency. Here, we only consider several typical cases of the thermal conductivity: $m = 0, 1, 2$, where $m = 0$ corresponds to constant thermal conductivity, $m = 1$ to a linear dependence on varying temperature, and $m = 2$ to a parabolic or nonlinear dependence on varying temperature.

In order to examine the temperature distribution and the fin efficiency, there is a need to get the θ_0 value for prescribed parameters M and α. For temperature variation in a fin of 500 K, the value of α takes approximately 0.4 for AISI 302 stainless steel, and about -0.4 for aluminum [33]. For this reason, in what follows, we take $\alpha = \pm 0.4$, unless otherwise stated. Fig. 6(a-c) displays the variation of the dimensionless temperature θ_0 at the fin tip as a function of the parameter M if taking $\alpha = 0.4$. Obviously, by comparing solid lines (for $\delta = 1$) with dashed lines (for $\delta = 0$) in Fig. 6(a-c), for each given θ_0 value, the M value as a function of θ_0 for $\delta = 0$ is much underestimated as compared to that for $\delta = 1$ for a large range of θ_0. This reveals that the neglect of the parameter δ in the thermal conductivity may give rise to large errors.

In the case of $\delta = 1$, Fig. 7(a-c) gives a comparison of the temperature excess θ_0 at the fin tip against the parameter M when taking $\alpha = \pm 0.4$. By inspection, one finds that for each θ_0 value, the M values corresponding to $\alpha = -0.4$ are always lower than those to $\alpha = 0.4$. Moreover, from Figs. 6 and 7, the fin tip temperature θ_0 is sensitive to the magnitude of M. That is, for different values of n, the role of M affecting θ_0 is completely different. For $n < -1$, there are two corresponding temperature values θ_0 for each curve if M is lower than a critical value, denoted as M_{cr}, while there is only a temperature value θ_0 for each M value and the above-mentioned critical value M_{cr} disappears. For convenience, we list several pairs of such points, denoted as (M_{cr}, θ_0^*) in Tables 1 and 2. Such a phenomenon only occurs in the case of the convection heat transfer coefficient behaving like θ^n with $n < -1$. For $n \geq -1$, a unique temperature value θ_0 corresponds to each M value. For the same n value, the change in thermal conductivity index m slightly affects the tip temperature. Nonetheless, for the same m value, the change in convection heat transfer coefficient index n strongly alters the tip temperature. This also implies a larger influence of convection heat transfer coefficient than thermal conductivity in changing the tip temperature.

Only if the dimensionless temperature θ_0 at $x = 0$ is obtained, the temperature distribution along the length direction of the fin can be determined. Fig. 8(a,b) shows the dimensionless temperature distribution for a fin with $m = 2$, $M = 0.4$, $\delta = 1$,and $\alpha = \pm 0.4$. As pointed out above, for certain cases, e.g. $n < -1$, there are two corresponding temperature excess values if $M < M_{cr}$. Fig. 8a presents the distribution of a unique temperature and the larger temperature value if two temperature excess values occur if $M < M_{cr}$, and Fig. 8b only displays the lower temperature excess value if two temperature excess values occur if $M < M_{cr}$. From Fig. 8a, the fin temperature increases with n rising, which is due to less heat loss on the fin surface.

Fig. 9 examines the effect of the parameter M on the dimensionless temperature distribution for a fin with nonlinearly temperature-dependent thermal conductivity ($m = 2$) and nonlinearly temperature-dependent convection heat transfer coefficient ($n = 2$) with $\delta = 1$, $\alpha = \pm 0.4$. Since $m = 0$ corresponds to constant thermal conductivity, and $m = 1$ to linearly temperature-dependent thermal conductivity, some analyses for both cases have been reported in [12, 27]. Here for both nonlinear thermal properties: $m = n = 2$, Fig. 9 presents an illustration of the effect of the parameter M on the dimensionless temperature distribution. Clearly, the larger the parameter M is, the less the temperature θ at any position of the fin. This provides us with choosing appropriate M values to achieve an optimal state.

Figs. 10 and 11 depict the influence of the parameter M on the fin efficiency for various values m and n and given α and δ, respectively. For comparison, $\alpha = 0.5$ is chosen, and the fin efficiency is computed for $\delta = 0, 1$. Clearly, our results with $m = 1$ in Fig. 10b are in agreement with those in [25]. For the case of $\delta = 1$, the fin efficiency is displayed in Fig. 11. The fin efficiency is significantly larger than unity for $n < -1$, irrespective of constant, linear, or nonlinear thermal conductivity: $m = 0, 1, 2$. If $n = -1$, the fin efficiency is always equal to unity for any m, α and δ, and in this case, the values of m, α and δ do not affect the fin efficiency. Nevertheless, for most practical cases, $n > -1$, and the fin efficiency is lower than unity [1].

6 Conclusions

In this paper, the temperature distribution and the fin efficiency for a nonlinear fin problem were analyzed. The nonlinearity contains temperature-dependent thermal conductivity and temperature-dependent convection heat transfer coefficient, both of which exhibit a generalized power-law form. Exact analytical solutions in the implicit form were given. For certain special cases, the fin temperature distribution and the fin efficiency were obtained explicitly. The influence of the parameters on the temperature distribution and the fin efficiency were presented graphically and discussed. The temperature distribution and the fin efficiency strongly depend on the power-law index of the convection heat transfer coefficient. The distinct differences of the temperature and the fin efficiency under the assumptions of constant, linearly, and nonlinearly temperature-dependent thermal conductivity and convection heat transfer coefficient were displayed.

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Declaration of interest statement

The authors declare that they have no conflict of interest.

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List of Figure Captions

Table 1. Critical parameter M_{cr} and corresponding temperature θ_0^* pair: (M_{cr}, θ_0^*) with $\alpha = 0.4$

δ	n	$m = 0$	$m = 1$	$m = 2$
0	-4	$(0.3162, 0.707)$	$(0.2959, 0.732)$	$(0.2794, 0.752)$
	-3	$(0.3742, 0.612)$	$(0.3420, 0.666)$	$(0.3177, 0.683)$
	-2	$(0.4839, 0.426)$	$(0.4216, 0.500)$	$(0.3794, 0.552)$
1	-4	$(0.5916, 0.707)$	$(0.5806, 0.714)$	$(0.5716, 0.719)$
	-3	$(0.7000, 0.612)$	$(0.6827, 0.622)$	$(0.6694, 0.628)$
	-2	$(0.9053, 0.426)$	$(0.8718, 0.443)$	$(0.8495, 0.450)$

Table 2. Critical parameter M_{cr} and corresponding temperature θ_0^* pair: (M_{cr}, θ_0^*) with $\delta = 1$

α	n	$m = 0$	$m = 1$	$m = 2$
-0.4	-4	$(0.3873, 0.707)$	$(0.4042, 0.692)$	$(0.4179, 0.685)$
	-3	$(0.4583, 0.612)$	$(0.4848, 0.590)$	$(0.5043, 0.581)$
	-2	$(0.5927, 0.426)$	$(0.6432, 0.394)$	$(0.6737, 0.391)$
0.4	-4	$(0.5916, 0.707)$	$(0.5806, 0.714)$	$(0.5716, 0.719)$
	-3	$(0.7000, 0.612)$	$(0.6827, 0.622)$	$(0.6694, 0.628)$
	-2	$(0.9053, 0.426)$	$(0.8718, 0.443)$	$(0.8495, 0.450)$

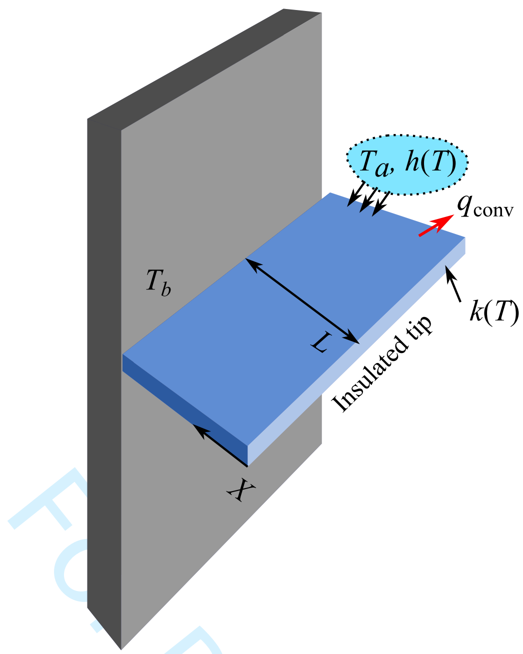


Fig.1

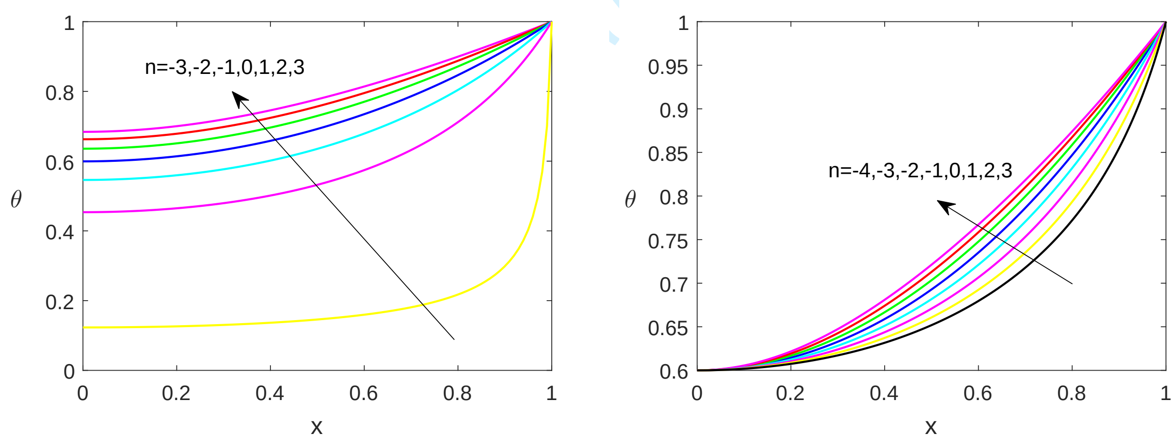
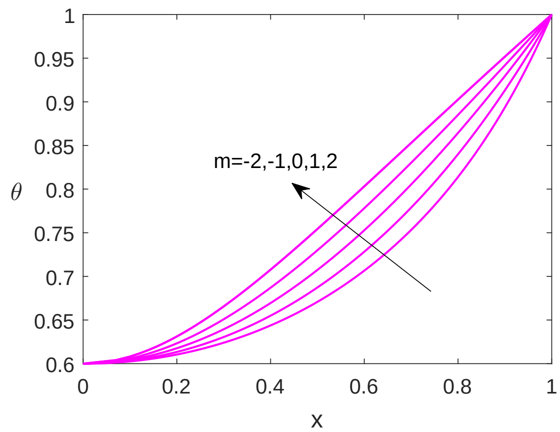


Fig.2(a,b)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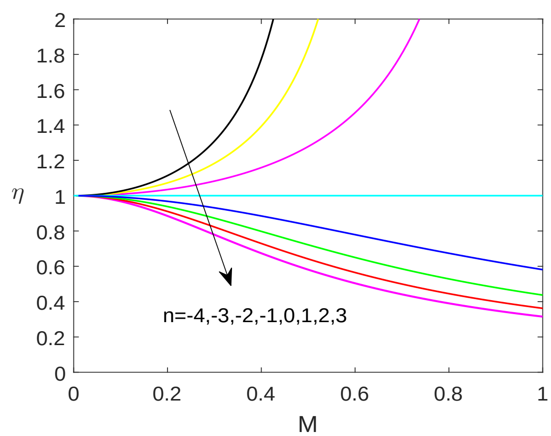
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
θ
m=-2,-1,0,1,2
0
0.2
0.4
0.6
0.8
1
x

Fig.3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1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
η
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0
0.2
0.4
0.6
0.8
1
M

Fig.4

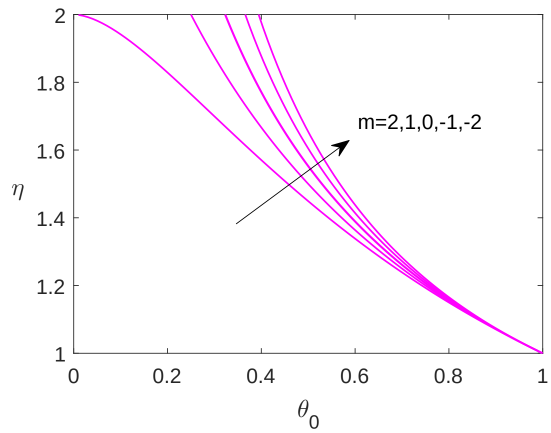

Fig.5

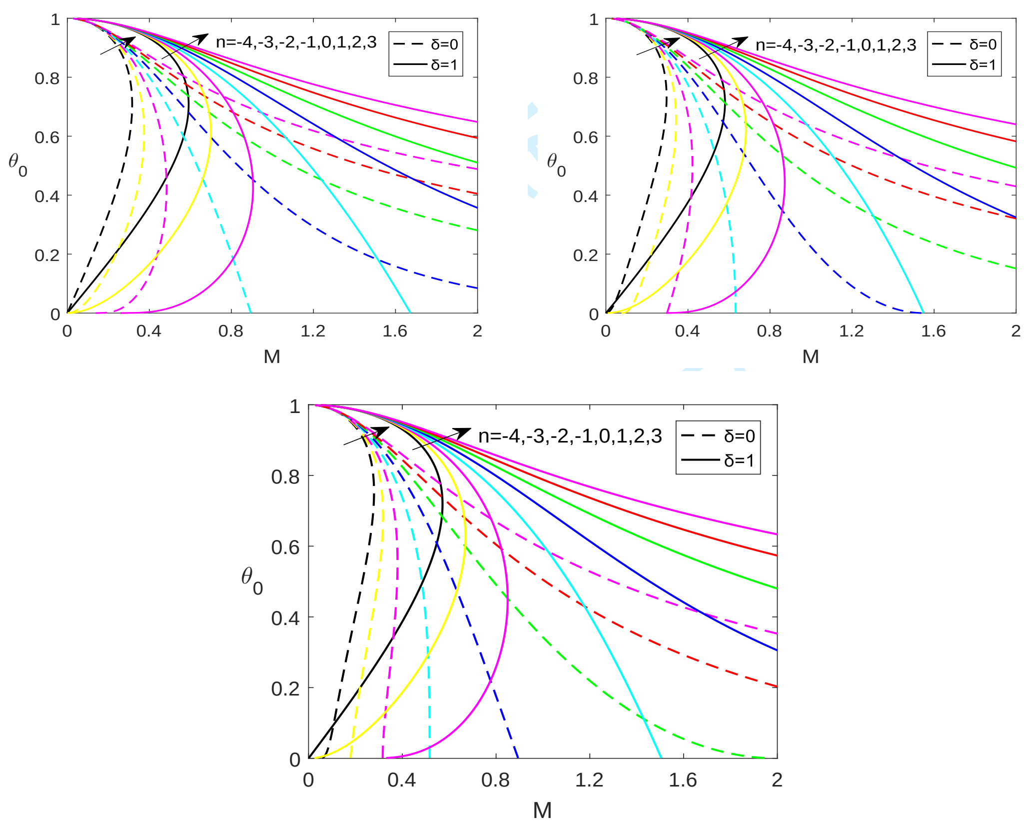

Fig.6(a,b,c)

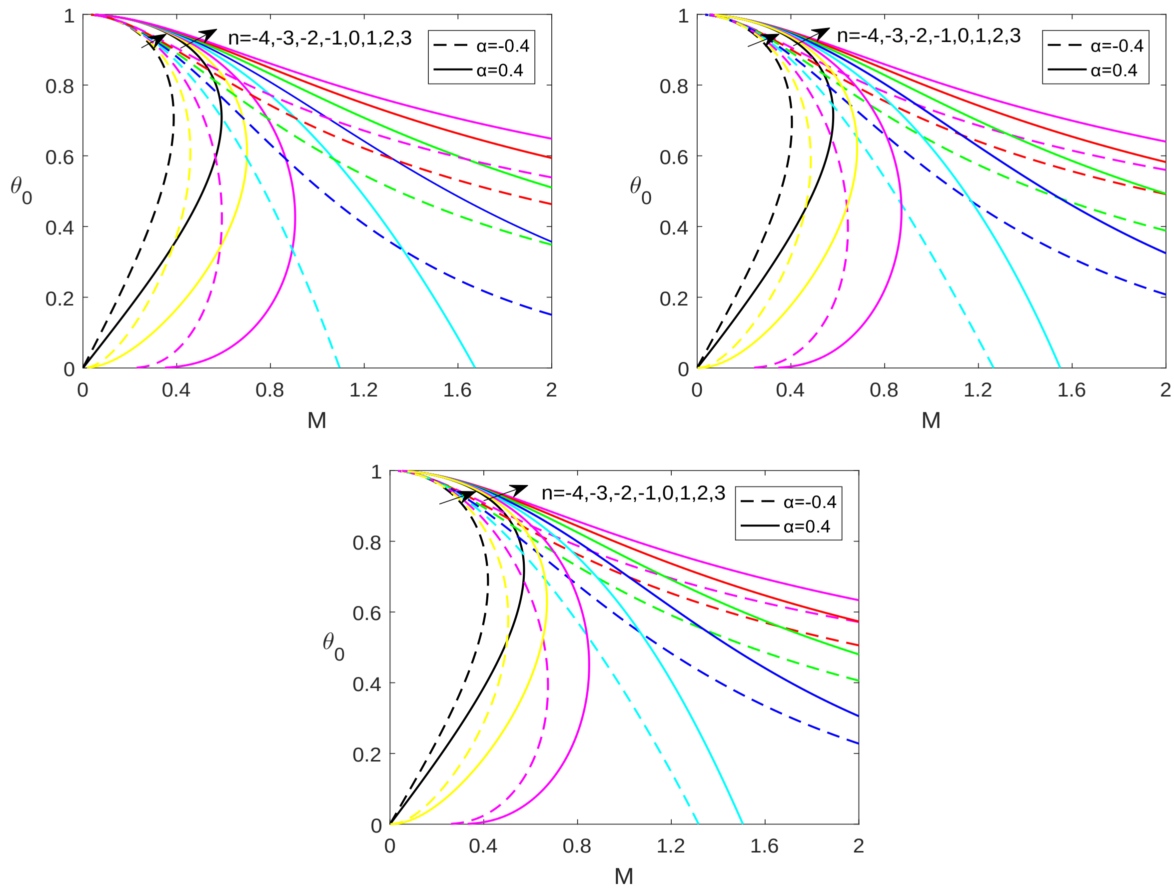

Fig.7(a,b,c)

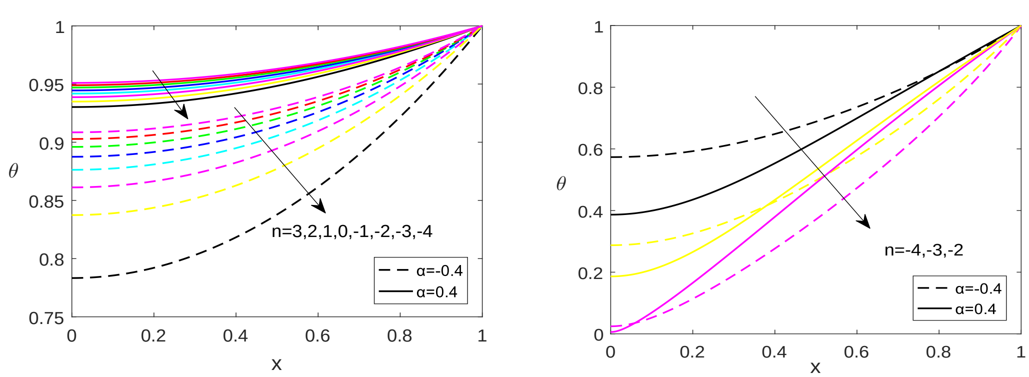

Fig.8(a,b)

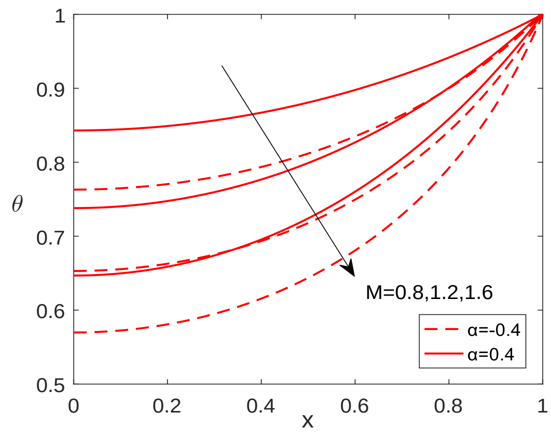

Fig.9

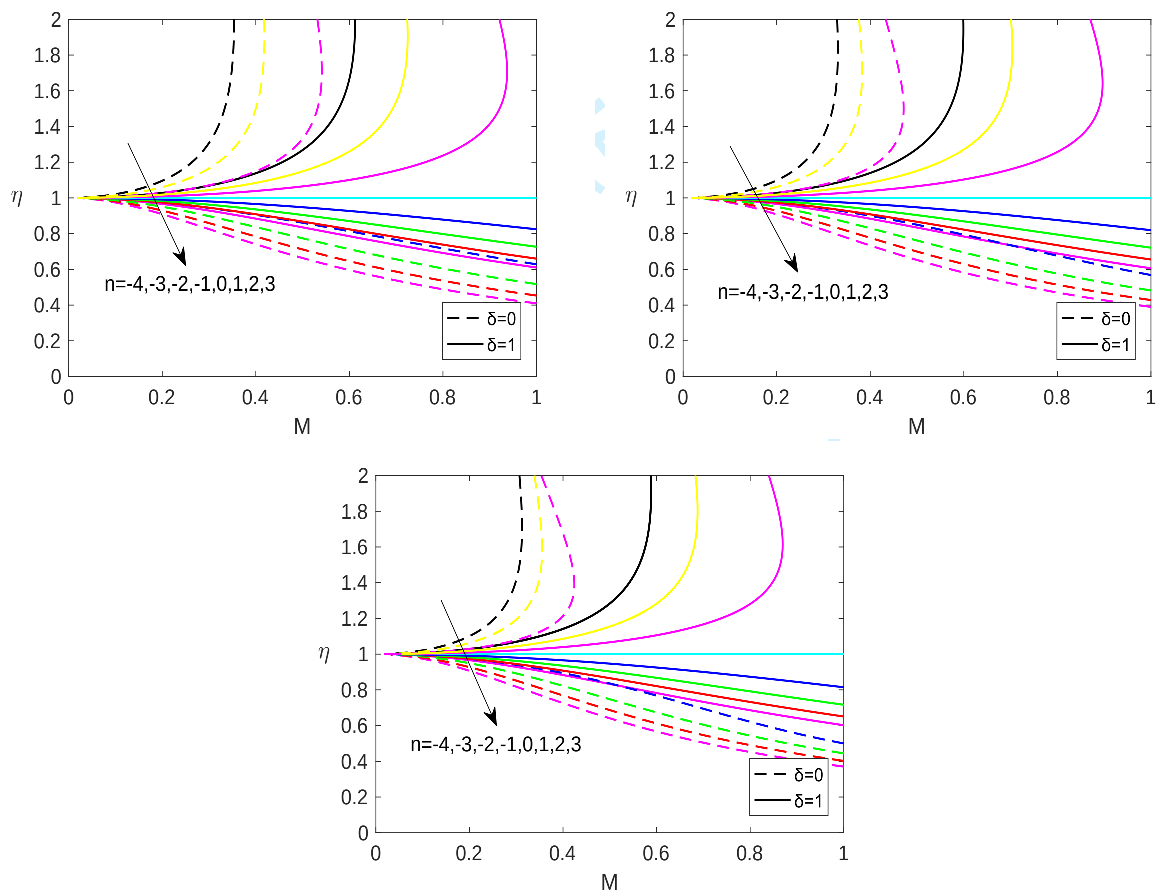

Fig.10(a,b,c)

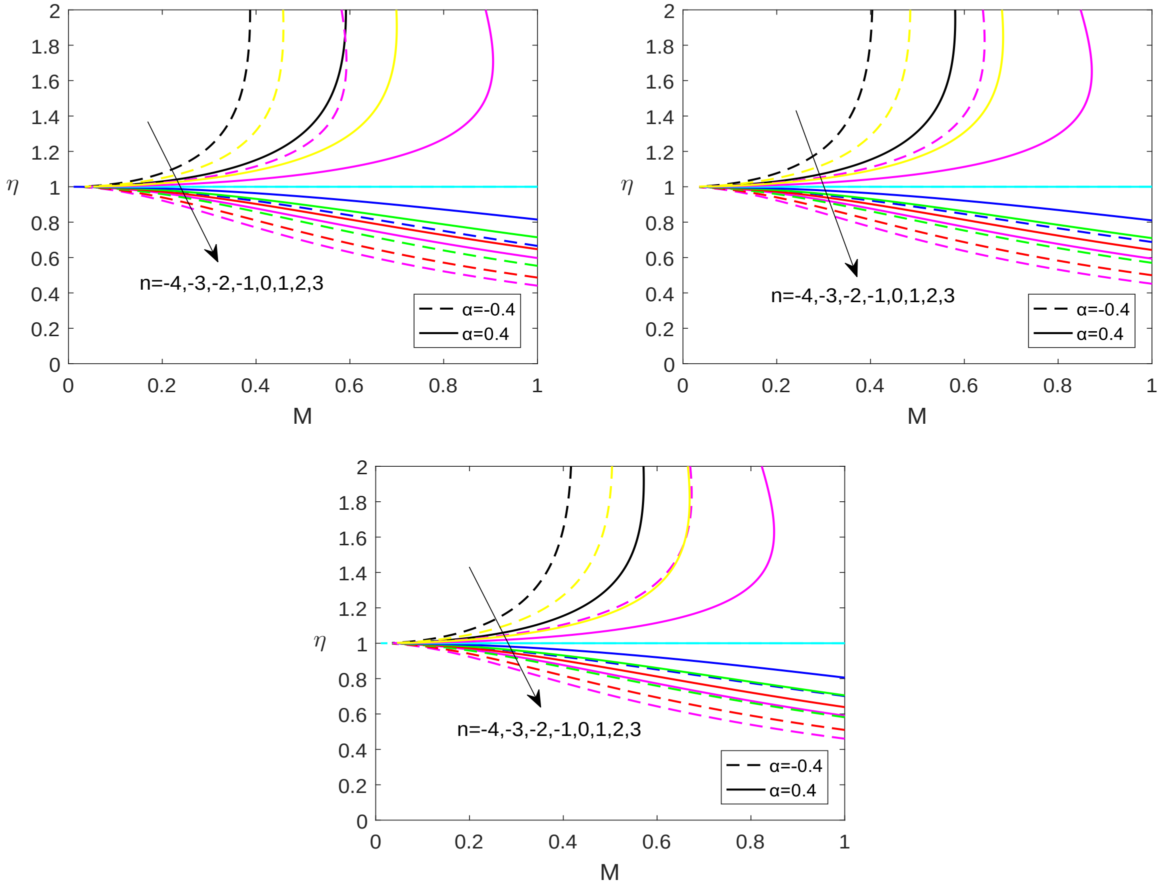

Fig.11(a,b,c)