Applied Mathematics and Mechanics
(English Edition, Vol.7, No.6, June. 1986)

Published by SUT,
Shanghai, China

# EXACT SOLUTION OF NAVIER-STOKES EQUATIONS—THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE UNDER DIRAC-PAULI REPRESENTATION AND ITS APPLICATION IN FLUID DYNAMICS (II)

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(Received Jan. 30, 1985 Communicated by Chien Wei-zang)

**Abstract**

*This work is the continuation of the discussion of ref. [1]. In ref. [1] we applied the theory of functions of a complex variable under Dirac-Pauli representation, introduced the Kaluza "Ghost" coordinate, and turned Navier-Stokes equations of viscofluid dynamics of homogeneous and incompressible fluid into nonlinear equation with only a pair of complex unknown functions. In this paper we again combine the complex independent variable except time, and cause it to decrease in a pair to the number of complex independent variables. Lastly, we turn Navier-Stokes equations into classical Burgers equation. The Cole-Hopf transformation join up with Burgers equation and the diffusion equation is Bäcklund transformation in fact, and the diffusion equation has the general solution as everyone knows. Thus, we obtain the exact solution of Navier-Stokes equations by Bäcklund transformation.*

## I. Introduction

In ref. [1] we cast aside the traditional quaternion theory and build up the theory of functions of a complex variable under Dirac-Pauli representation, then the multivariate Navier (1822)-Stokes (1845) equations become as nonlinear equation with only a pair of complex unknown functions. In fact, Sylvester[2] discovered the relation on the four elements of traditional quaternion with the Pauli matrix and $2\times2$ unit matrix long ago. Afterwards, A.S. Eddington[3](1946) again discovered that, these elements could be expressed by four $4\times4$ matrixes. Now we know that, these four $4\times4$ Eddington's matrixes relate to Dirac matrix[4-5].

Despite the discovery by Sylvester and Eddington, Cayley-Klein [6-7] and Branetz-Shmouglevsky[8] had only limited achievements in scientific research. Now, with the establishment of the theory of functions of a complex variable under Dirac-Pauli representation we conveniently solve the real mechanical and physical problems with it.

The solution of Navier-Stokes equations of viscofluid dynamics of homogeneous and incompressible fluid is the key problem of fluid dynamics[9]. At the same time this group of equations must be satisfied by the instant parameters of the turbance[10-13], and so is of great importance. It is a pity that we've not obtained te general solution of Navier-Stokes equations up to now. In normal textbooks only some special solutions (or simple exact solutions) for concrete flow problems are given.

In this paper we try the (general) exact solution of Navier-Stokes equations and cast a brick to attract jade. From ref. [1] we have a possibility for the achievement of this plan. From ref. [1] under introduction of kaluza "Ghost" coordinate we can write the Navier-Stokes equations in the theory of functions of complex variables under Dirac-Pauli representation as

$$\left.\begin{aligned} &\frac{\partial u_k}{\partial z_k}+\frac{\partial \bar{u}_k}{\partial \bar{z}_k}=0 \\ &\frac{\partial u_i}{\partial t}+\left(u_k\frac{\partial}{\partial z_k}+\bar{u}_k\frac{\partial}{\partial \bar{z}_k}\right)u_i=-\frac{1}{\rho}\frac{\partial p}{\partial \bar{z}_i}+2\nu\frac{\partial^2}{\partial z_k\partial \bar{z}_k}u_i \qquad (i,k=1,2) \end{aligned}\right\} \tag{1.1}$$

and its conjugate equations. Where $\nu$ is the viscid coefficient and is constant under condition of constant temperature; $z_k$ and $\bar{z}_k$ $(k=1.2)$ are two pairs of complex conjugate space coordinate: $u_k$ and $\bar{u}_k$ (K = 1.2) are two pairs of complex conjugate velocity components of fluid.

In ref. [1] we turn the above-mentioned Navier-Stokes equations into nonlinear equation with only a pair of complex conjugate unknown functions and two pairs of complex conjugate independent variables except time $t$. For this reason, our first problem is the great number of independent variables. In this paper we first combine the two pairs of complex conjugate independent variables into one pair, and from this turn the equation into nonlinear equation for only one real unknown functions. Second, we combine the pair of complex conjugate independent variables except time $t$ into one real independent variable, and change the equation into a simple form. This way of combining independent variables again is not an essential condition. So our solution is only the exact solution, and is not the general solution. But, from unapplication of initial-boundary conditions this exact solution has widespread applicability in this paper.

In this paper we simplify the Navier-Stokes equations into classical Burgers equation in final stage.It is well-known that M.J. Lighthill[14] approached the equations of ideal gas dynamics with one-dimension to the Burgers equation. His result is a perturbation soultion, and is not an exact solution. But, his result is as important as that obtained in this paper.

Owing to smooth conduct of the transfer of the Burgers equation into the diffusion equation by Cole-Hopf transformation, i.e. Bäcklund transformation, in fact, we can obtain the exact solution of the Navier-Stokes equations from the general solution of the diffusion equation by the Bäcklund transforation.

We do not set special limit to the Reynolds number in this paper.

The dummy index is the summation and depends on Einstein convention in this paper.

## II. Second Simplicity for Navier-Stokes Equations

In ref. [1] we applied the theory of functions of a complex variable under Dirac-Pauli representation, introduced the Kaluza "Ghost" coordinate, and turned the Navier-Stokes equations into nonlinear equation will a pair of complex unknown functions $\psi$ and $\bar{\psi}$, i.e.

$$\begin{aligned} \left(\frac{\partial}{\partial t}-2\nu\frac{\partial^2}{\partial z_k\partial \bar{z}_k}\right)\frac{\partial^2}{\partial z_i\partial \bar{z}_i}\psi=&\frac{\partial}{\partial \bar{z}_k}\left[\frac{\partial \psi}{\partial z_1}\left(\frac{\partial^2 \psi}{\partial z_2\partial z_k}\right)-\frac{\partial \psi}{\partial z_2}\left(\frac{\partial^2 \psi}{\partial z_1\partial z_k}\right)\right] \\ &+\frac{\partial}{\partial \bar{z}_k}\left[\frac{\partial \bar{\psi}}{\partial \bar{z}_1}\left(\frac{\partial^2 \psi}{\partial \bar{z}_2\partial z_k}\right)-\frac{\partial \bar{\psi}}{\partial \bar{z}_2}\left(\frac{\partial^2 \psi}{\partial \bar{z}_1\partial z_k}\right)\right] \end{aligned} \tag{2.1}$$

where $\psi$ is the "flow function", $u_1=\partial\psi/\partial z_2$, $u_2=-\partial\psi/\partial z_1$ ; and $\bar{\psi}$ is complex conjugate function of $\psi$ .

In Eq.(2.1), there is only one pair of complex unknown functions $\psi$ and $\bar{\psi}$, but two pairs of complex independent variables $z_k$ and $\bar{z}_k$ $(k=1,2)$ except time $t$. And we have the condition to combine the two pairs of complex independent variables into one pair, and lose no generality. The sole principle for the new combination of complex independent variables is to preserve basic form of Eq.(2.1). This method for new combination of complex independent variables corresponds to d'Alembert solution of wave equation. The multidimensional wave equation and diffusion equation both face a similar problem. This new combination of independent variables is appropriate, but not essential.

It is found that the new combination for preservation of basic form of Eq.(2.1) is

$$y = z_1 + i\bar{z}_2 \tag{2.2}$$

and

$$\bar{y} = \bar{z}_1 - iz_2 \tag{2.3}$$

From (2.17) of ref. [1] (2.2) and (2.3) lose no generality.

From the condition of complete differential we have

$$\frac{\partial y}{\partial z_1}=1, \quad \frac{\partial y}{\partial \bar{z}_2}=i, \quad \frac{\partial \bar{y}}{\partial \bar{z}_1}=1, \quad \frac{\partial \bar{y}}{\partial z_2}=-i \tag{2.4}$$

then

$$\frac{\partial^2}{\partial z_k \partial \bar{z}_k} = 2\frac{\partial^2}{\partial y \partial \bar{y}} \qquad (k=1,2) \tag{2.5}$$

We substitute (2.4) and (2.5) into (2.1), and have

$$\begin{aligned} 2\left(\frac{\partial}{\partial t} - 4\nu\frac{\partial^2}{\partial y\partial\bar{y}}\right)\frac{\partial^2}{\partial y\partial\bar{y}}\psi &= i\frac{\partial}{\partial\bar{y}}\left[\frac{\partial}{\partial\bar{y}}(\psi+\bar{\psi})\frac{\partial^2\psi}{\partial y^2}\right] - i\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(\psi+\bar{\psi})\frac{\partial^2\psi}{\partial\bar{y}^2}\right] \\ &+ i\frac{\partial}{\partial y}\left[\frac{\partial}{\partial\bar{y}}(\psi+\bar{\psi})\frac{\partial^2\psi}{\partial y\partial\bar{y}}\right] - i\frac{\partial}{\partial\bar{y}}\left[\frac{\partial}{\partial y}(\psi+\bar{\psi})\frac{\partial^2\psi}{\partial y\partial\bar{y}}\right] \end{aligned} \tag{2.6}$$

We add the Eq.(2.6) to its complex conjugate equation, and let

$$\varphi = \psi + \bar{\psi} \tag{2.7}$$

and obtain the nonlinear equation with only one real unknown function $\varphi$ :

$$\left(\frac{\partial}{\partial t} - 4\nu\frac{\partial^2}{\partial y\partial\bar{y}}\right)\frac{\partial^2\varphi}{\partial y\partial\bar{y}} = i\left(\frac{\partial\varphi}{\partial\bar{y}}\frac{\partial}{\partial y} - \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\bar{y}}\right)\frac{\partial^2\varphi}{\partial y\partial\bar{y}} \tag{2.8}$$

We notice that $\varphi$ is a real unknown function from (2.7). Eq.(2.8) degenerates into the flow function equation for incompressible planar flows or axisymmetric flows.

If we get the solution of Eq.(2.8), then from Eq(2.6), i.e. from

$$\begin{aligned} 2\left(\frac{\partial}{\partial t} - 4\nu\frac{\partial^2}{\partial y\partial\bar{y}}\right)\frac{\partial^2\psi}{\partial y\partial\bar{y}} &= 2i\left(\frac{\partial\varphi}{\partial\bar{y}}\frac{\partial}{\partial y} - \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\bar{y}}\right)\frac{\partial^2\psi}{\partial y\partial\bar{y}} \\ &+ i\left(\frac{\partial^2\varphi}{\partial\bar{y}^2}\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2\psi}{\partial\bar{y}^2}\right) \end{aligned} \tag{2.9}$$

we can obtain the complex unknown function $\psi$ . We notice that Eq.(2.9) is a linear equation under premise for known function $\varphi$ .

Thus we have

**Theorem 1** The exact solution of Navier-Stokes equations can be obtained by linear Eq.(2.9). Where $\varphi$ satisfies nonlinear Eq (2.8), and the complex independent variable is given by (2.2) and (2.3).

Now we build the whole problem into a solution for nonlinear Eq.(2.8) with only one real unknown function $\varphi$ .

## III. Exact Solution of Navier-Stokes Equation

We investigate Eq.(2.8) and find that, it is the sum of the two complex conjugate equation. One of the equation is

$$\left(\frac{\partial}{\partial t}-4\nu\frac{\partial^2}{\partial y\partial\bar{y}}\right)\frac{\partial^2\varphi}{\partial y\partial\bar{y}}=2i\left[\frac{\partial\varphi}{\partial\bar{y}}\frac{\partial}{\partial y}\left(\frac{\partial^2\varphi}{\partial y\partial\bar{y}}\right)+\left(\frac{\partial^2\varphi}{\partial y\partial\bar{y}}\right)^2\right]$$

i.e.

$$\left(\frac{\partial}{\partial t}-4\nu\frac{\partial^2}{\partial y\partial\bar{y}}\right)\frac{\partial^2\varphi}{\partial y\partial\bar{y}}=2i\frac{\partial}{\partial y}\left[\frac{\partial\varphi}{\partial\bar{y}}\left(\frac{\partial^2\varphi}{\partial y\partial\bar{y}}\right)\right] \qquad (3.1)$$

The other equation is complex conjugate equation of (3.1). At separation of Eq.(2.8) into (3.1) and its conjugate equation, we notice (2.7), i.e. $\varphi$ is a real function. Eq.(3.1) can be written (in full condition) as

$$\left(\frac{\partial}{\partial t}-4\nu\frac{\partial^2}{\partial y\partial\bar{y}}\right)\frac{\partial\varphi}{\partial\bar{y}}=2i\frac{\partial\varphi}{\partial\bar{y}}\left(\frac{\partial^2\varphi}{\partial y\partial\bar{y}}\right) \qquad (3.2)$$

Let

$$\phi=\frac{\partial\varphi}{\partial y}\qquad\left(\text{in this time}\qquad\bar{\phi}=\frac{\partial\varphi}{\partial\bar{y}}\right) \qquad (3.3)$$

then Eq.(3.2) be comes as

$$\left(\frac{\partial}{\partial t}-4\nu\frac{\partial^2}{\partial y\partial\bar{y}}\right)\phi=-2i\phi\frac{\partial\phi}{\partial\bar{y}} \qquad (3.4)$$

In Eq.(3.4) the complex unknown function $\phi$ is only one, but the number of the complex independent variables $y$ and $\bar{y}$ except time $t$ is two. And we have the condition for combination of two complex independent variables into one real independent variable, and lose no generality. The sole principle for new combination of complex independent variables preserve the same basic form of Eq.(3.4). This new combination of independent variables is also appropriate, but not essential.

It is found that the new combination for preservation of basic form of Eq.(3.4) is

$$\xi=y+\bar{y}$$

From (2.17) of ref. [1], (3.5) loses no generality. And we must notice that $\xi$ is a real independent variable.

From conditions of complete differential we have

$$\frac{\partial \xi}{\partial y}=1, \qquad \frac{\partial \xi}{\partial \bar{y}}=1 \tag{3.6}$$

then

$$\frac{\partial^2}{\partial y \partial \bar{y}}=\frac{\partial^2}{\partial \xi^2} \tag{3.7}$$

We substitute (3.6) and (3.7) into Eq.(3.4), and have the classical Burgers equation

$$\frac{\partial \phi}{\partial t}+2 i \phi \frac{\partial \phi}{\partial \xi}-4\nu \frac{\partial^2 \phi}{\partial \xi^2}=0 \tag{3.8}$$

By the Cole-Hopf transformation

$$\phi=4 i \nu \frac{1}{w} \frac{\partial w}{\partial \xi} \tag{3.9}$$

the Burgers equation is related to the diffusion equation

$$\frac{\partial w}{\partial t}=4\nu \frac{\partial^2 w}{\partial \xi^2} \tag{3.10}$$

and the Burgers Eq.(3.8) can be solved. From the result of refs. [1], [17] and this paper, we call diffusion equation (3.10) the course equation of incompressible viscofluid dynamics.

In fact we can write the Cole-Hopf transformation as[15]

$$\frac{\partial w}{\partial \xi}=-\frac{i}{4\nu} \phi w, \quad \frac{\partial w}{\partial t}=\left(-i \frac{\partial \phi}{\partial \xi}-\frac{1}{4\nu} \phi^2\right) w \tag{3.11}$$

Eq.(3.11) is the Bäcklund transformation. Thus we have

**Theorem 2** The exact solution of Navier-Stokes equations of incompressible viscofluid dynamics can be obtained by the linear equation

$$\begin{aligned} 2\left(\frac{\partial}{\partial t}-4\nu \frac{\partial^2}{\partial y \partial \bar{y}}\right) \frac{\partial^2 \psi}{\partial y \partial \bar{y}}= & 2 i\left(\bar{\phi} \frac{\partial}{\partial y}-\phi \frac{\partial}{\partial \bar{y}}\right) \frac{\partial^2 \psi}{\partial y \partial \bar{y}} \\ & +i\left(\frac{\partial \bar{\phi}}{\partial \bar{y}} \frac{\partial^2 \psi}{\partial y^2}-\frac{\partial \phi}{\partial y} \frac{\partial^2 \psi}{\partial \bar{y}^2}\right) \end{aligned} \tag{3.12}$$

and the Burgers equation

$$\frac{\partial \phi}{\partial t}+2 i \phi \frac{\partial \phi}{\partial \xi}-4\nu \frac{\partial^2 \phi}{\partial \xi^2}=0 \tag{3.13}$$

and its conjugate equation. Where

$$y=z_1+i\bar{z}_2, \quad \bar{y}=\bar{z}_1-i z_2, \quad \xi=y+\bar{y} \tag{3.14}$$

The exact solution of the Burgers equation is related to the general solution of the diffusion equation by Cole-Hopf transformation. And the exact solution of the Navier-Stokes equation can be obtained by the general solution of the diffusion equation

$$\frac{\partial w}{\partial t}=4\nu \frac{\partial^2 w}{\partial \xi^2} \tag{3.15}$$

and the Bäcklund transformation

$$\frac{\partial w}{\partial \xi} = -\frac{i}{4\nu}\phi w, \quad \frac{\partial w}{\partial t} = \left(-i\frac{\partial \phi}{\partial \xi} - \frac{1}{4\nu}\phi^2\right)w \tag{3.16}$$

From the Bäcklund transformation in Theorem 2 we can solve the Navier-Stokes equation by inverse scattering transformation. Thus, the problem for exact solution of Navier-Stokes equations is related to the guantum eigenvalues problem.

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