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Exact solutions and approximations of MOND fields of disc galaxies

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ABSTRACT

We consider models of thin discs (with and without bulges) in the Bekenstein–Milgrom formulation of MOND as a modification of Newtonian gravity. Analytic solutions are found for the full gravitational fields of Kuzmin discs, and of disc-plus-bulge generalizations of them. For all these models a simple relation between the MOND potential field, ψ, and the Newtonian potential, φ_N, holds everywhere *outside the disc*: $\mu(|\nabla\psi|/a_o)\nabla\psi=\nabla\varphi_N$. We give exact expressions for the rotation curves for these models. We also find that this algebraic relation is a very good approximation for exponential discs. The algebraic relation outside the disc is then extended into the disc to derive an improved approximation for the MOND rotation curve of disc galaxies, requiring only knowledge of the Newtonian curve and the surface density.

Key words: gravitation – celestial mechanics, stellar dynamics – galaxies: structure – dark matter.

1 INTRODUCTION

There are two extreme interpretations of the modified Newtonian dynamics (MOND). One interpretation views MOND as a modification of inertia (Milgrom 1983, 1994a): gravitational fields of massive bodies remain Newtonian, but the equation of motion of a particle in the field is superseded by a MOND equation of motion. In this paper, however, we concentrate on the Bekenstein–Milgrom (BM) formulation of MOND (Bekenstein & Milgrom 1984, hereafter BM), which is an embodiment of MOND as a modification of gravity, leaving the Newtonian law of motion intact. The standard Poisson equation for the Newtonian gravitational potential, φ_N, $(\nabla\cdot\nabla\varphi_N=4\pi G\rho)$ induced by a mass density $\rho(R)$ is replaced by

$$\nabla\cdot[\mu(|\nabla\psi|/a_o)\nabla\psi]=4\pi G\rho, \quad (1)$$

with a_o the acceleration constant of MOND. This non-linear equation is hardly amenable to analytic solution beyond the simple cases of configurations with one-dimensional symmetry.

It would be very useful, for example, to have exact, or even approximate, analytic solutions for the gravitational field of model disc galaxies on which various ideas can be tested. Some problems whose study may benefit from the availability of such solutions are, for example, that of polar rings, and that of the motion and fate (disruption, capture etc.) of dwarf companions moving in the field of a mother galaxy.

Even more central is the problem of calculating the MOND rotation curves of disc galaxies. In formulations of MOND based on modification of inertia, the velocity on a circular orbit of radius r in the plane of disc galaxies is given exactly by

$$\mu(a/a_o)\,a=a_N, \quad (2)$$

where $a=v^2/r$, and a_N is the Newtonian acceleration at r (Milgrom 1994a). This has been the standard expression for calculating MOND rotation curves (e.g. Kent 1987; Milgrom 1988; Begeman, Broeils & Sanders 1991). It is not exact in the Bekenstein–Milgrom formulation, and had the status of being only an approximation before the work of Milgrom (1994a).

Here we describe a class of disc-galaxy models for which exact solutions of the MOND field equation are presented; this is done in Section 3. We also find (see Section 4) that an approximate analytic solution applies for a wider class of models, and we suggest a way to predict the adequacy of such an approximation, by studying only the Newtonian solution for the mass distribution (Section 2). In Section 5, we describe an approximation for the rotation curve in the BM formulation – based, like relation (2), only on knowledge of the Newtonian acceleration, but which is, generally, a better approximation. In Section 6 we mention further possible developments.

2 AN ALGEBRAIC RELATION BETWEEN THE NEWTONIAN AND MOND FIELDS

Subtracting the Poisson equation from the MOND equation (1) we get

$$\nabla\cdot[\mu(|\nabla\psi|/a_o)\nabla\psi-\nabla\varphi_N]=0, \quad (3)$$

by which the expression in parentheses is some curl field. For configurations with one-dimensional symmetry (spherical, cylindrical or plane) the curl field must vanish, and thus the MOND field is related to the Newtonian field by the algebraic relation

$$\mu(|\nabla\psi|/a_o)\nabla\psi=\nabla\varphi_N. \tag{4}$$

This affords a simple solution of the MOND problem, by solving first the Poisson equation for φ_N and then inverting equation (4) to get the MOND field. Relation (4) does not follow from the MOND equation (1), but the inverse is correct as the latter is just the divergence of the former.

We begin by asking whether such a relation may hold for more general mass distributions, at least approximately. Because the function μ that appears in MOND is such that $I(x)\equiv x\mu(x)$ is monotonic, and varies between 0 and ∞ as x does so, $I(x)$ is invertible on the positive real axis. Equation (4) is thus equivalent to

$$\nabla\psi=\nu(|\nabla\varphi_N|/a_o)\nabla\varphi_N, \tag{5}$$

where $\nu(y)\equiv I^{-1}(y)/y$. A potential ψ that satisfies this equation exists if and only if the curl of the right-hand side vanishes, or, in other terms

$$\nabla|\nabla\varphi_N|\times\nabla\varphi_N=0 \tag{6}$$

(as $\nu'\neq 0$). This, in turn, is tantamount to $|\nabla\varphi_N|$ being some function of φ_N

$$|\nabla\varphi_N|=f(\varphi_N). \tag{7}$$

We find then a necessary and sufficient condition for equation (4) to hold for some ψ and some μ (with $\mu'\neq 0$); the condition is expressed solely in terms of the *Newtonian* field of the given mass distribution. By equation (4) the equipotentials for ψ and φ_N coincide, and ψ is thus a function of φ_N.

A potential ψ that satisfies equation (4) in some domain D is *the* MOND solution to the problem only if ψ also satisfies the correct boundary conditions. If the sphere at infinity is part of the boundary of D, then ψ automatically satisfies the correct boundary condition there. The same is true of the jump condition across a thin sheet of mass. If φ_N satisfies the condition then a ψ that obeys equation (4) (outside the mass sheet) satisfies the correct jump condition as well.

Concentrate now on mass distributions that model disc galaxies: an axisymmetric distribution, symmetric also about a mid-plane, made of a thin disc of surface density $\Sigma(r)$, and some bulge-like component. By the above arguments, if ψ satisfies equation (4) everywhere outside the disc it is *the* MOND solution of the problem: the boundary conditions are now satisfied automatically by a solution of equation (4). At infinity, $\nabla\psi\rightarrow(MGa_o)^{1/2}\boldsymbol{R}/R^2$, and just outside the surface of the disc

$$\mu(|\nabla\psi|/a_o)\partial_n\psi=\pm 2\pi\Sigma(r), \tag{8}$$

where ∂_n is the normal component of the gradient.

To assess the applicability of the algebraic relation for a given configuration we only have to find the Newtonian potential, and plot $|\nabla\varphi_N|$ versus φ_N for points outside the disc. If the points fall on a line, i.e. if $|\nabla\varphi_N|$ is a function of φ_N (a highly non-generic case), then $\nabla\psi$, as given by equation (4), is the exact MOND acceleration field *outside the disc*. If $|\nabla\varphi_N|$ and φ_N are correlated, with only a little scattering, equation (4) gives a good approximation to the MOND field (see Section 4 for examples).

3 EXACT SOLUTIONS FOR KUZMIN DISCS AND GENERALIZATIONS THEREOF

The two-parameter family of Kuzmin discs is described by a Newtonian gravitational potential

$$\varphi_K=-MG/[r^2+(|z|+h)^2]^{1/2} \tag{9}$$

(see e.g. Binney & Tremaine 1987), where we use cylindrical coordinates r, z. The potential above the disc ($z>0$) is that of a point mass M placed on the lower z-axis at $-\boldsymbol{h}\equiv(0,0,-h)$; the potential below the disc is produced by the same mass oppositely placed at $\boldsymbol{h}$. The surface density, $\Sigma_K(r)$, matches the jump in the z-gradient of the potential:

$$\Sigma_K(r)=(2\pi G)^{-1}\left.\frac{\partial\varphi_K}{\partial z}\right|_{z=0^+}=Mh/2\pi(r^2+h^2)^{3/2}. \tag{10}$$

Everywhere outside the disc the equipotential surfaces are concentric spheres centred at $\pm\boldsymbol{h}$. Equations (6) and (7) are thus satisfied (in this case $|\nabla\varphi_N|=\varphi_N^2/MG$), and, by the arguments of Section 2, the exact MOND solution for Kuzmin discs is given, outside the disc, by the algebraic relation equation (4). Thus, *outside the disc,*

$$\boldsymbol{g}=-\nabla\psi=a_oI^{-1}(g_N/a_o)\boldsymbol{g}_N/g_N, \tag{11}$$

where

$$\boldsymbol{g}_N=-MG(\boldsymbol{R}\pm\boldsymbol{h})/|\boldsymbol{R}\pm\boldsymbol{h}|^3 \tag{12}$$

is the Newtonian acceleration field above (+), and below (−) the disc. The MOND solution, above the disc, is simply that of a point mass located at $-\boldsymbol{h}$.

For very-low-acceleration Kuzmin discs (with $MG/h^2\ll a_o$), we have $\mu(x)\approx x$, so $I^{-1}(x)\approx x^{1/2}$. Then, the MOND potential is

$$\psi_K\approx(MGa_o)^{1/2}\ln[r^2+(|z|+h)^2]^{1/2}, \tag{13}$$

which can be obtained by direct integration of equation (11). The MOND rotation curve of a Kuzmin disc is, by equation (11),

$$v^2(r)=a_oI^{-1}[g_N(r,0^+)/a_o]\,r^2/(r^2+h^2)^{1/2}, \tag{14}$$

where $g_N(r,0^+)=MG/(r^2+h^2)$. It is clear then that we can write

$$v^2(r)=v_\infty^2\eta(\zeta,u), \tag{15}$$

where $v_\infty\equiv(MGa_o)^{1/4}$ is the asymptotic rotational speed, $\zeta\equiv MG/h^2a_o$ is a measure of how deep in the MOND regime we are, and $u\equiv r/h$. If we take, for instance, $\mu(x)=x/(1+x^2)^{1/2}$, then $I^{-1}(y)=[y^2/2+(y^2+y^4/4)^{1/2}]^{1/2}$, and

$$v^2(r)=v_\infty^2\frac{u^2}{1+u^2}\left\{\left[1+\frac{\zeta^2}{4(1+u^2)^2}\right]^{1/2}+\frac{\zeta}{2(1+u^2)}\right\}^{1/2}. \tag{16}$$

In the limit of very-low-acceleration discs, $\zeta\rightarrow 0$, one has, *independently of the exact form of* $\mu(x)$,

$$v^2(r)=v_\infty^2u^2/(1+u^2), \tag{17}$$

as in this limit $I^{-1}(y)=y^{1/2}$.

Milgrom 1994b has proved a virial-like relation for self-gravitating, low-acceleration systems, in the BM formulation. For thin discs this relation reads (Milgrom 1994b)

$$\frac{2}{3}M^{3/2}(Ga_o)^{1/2}=\int_0^\infty 2\pi r\Sigma(r)\,v^2(r)\,\mathrm{d}r, \tag{18}$$

where $v(r)$ is the circular rotation speed; it can be readily verified to hold for the pair of $\Sigma(r)$ and $v(r)$, given by equations (10) and (17), respectively.

Kuzmin discs may be generalized into a family of disc-plus-bulge models that are exactly solvable in MOND. These may be generated in a number of equivalent ways.

For example, beginning with the Newtonian potential of a Kuzmin disc $\varphi_K(\boldsymbol{R})$, we define a new mass distribution whose Newtonian potential is

$$\varphi=U(\varphi_K). \tag{19}$$

We choose U such that $U(x)\rightarrow x$ for $x\rightarrow 0$; thus, at spatial infinity φ has the same behavior as φ_K, and satisfies the correct boundary behaviour for a potential of a mass M. The potential φ is produced, outside the disc, by a mass distribution

$$\rho(\boldsymbol{R})=(4\pi G)^{-1}\nabla^2\varphi=(4\pi G)^{-1}U''(\varphi_K)(\nabla\varphi_K)^2, \tag{20}$$

where we have made use of the fact that $\nabla^2\varphi_K=0$. From equation (20), the equidensity surfaces coincide with the equipotential surfaces (common to φ and φ_K), because $(\nabla\varphi_K)^2$ is a function of φ_K.

In addition, a disc is needed at $z=0$, with surface density

$$\Sigma(r)=(2\pi G)^{-1}\left.\frac{\partial\varphi}{\partial z}\right|_{z=0^+}=U'[\varphi_K(r,0)]\,\Sigma_K(r), \tag{21}$$

with $\varphi_K(r,0)=-MG/(r^2+h^2)^{1/2}$. For ρ to be non-negative we must have $U''\geq 0$; thus, U' is an increasing function. Since the maximum value of φ_K is 0, and there $U'=1$, we have $U'\leq 1$ everywhere, or $\Sigma(r)\leq\Sigma_K(r)$. The total mass (bulge plus disc) contained within an equipotential surface φ_K is

$$M(\varphi_K)=U'(\varphi_K)\,M_K(\varphi_K), \tag{22}$$

where M_K is the mass within φ_K for the generating Kuzmin disc. This can be seen by applying the Gauss's theorem to the equipotential surface.

All the potentials φ defined by equation (19) satisfy equations (6) and (7) because φ_K does. Thus the algebraic relation (4) gives the MOND solutions for all these model galaxies in terms of the Newtonian field $\nabla\varphi=U'(\varphi_K)\nabla\varphi_K$.

A different approach, which generates the same family of solvable models, starts with some spherical density distribution that is centred at $-\boldsymbol{h}$: $\rho(\boldsymbol{R})=\hat{\rho}(q)$, $q\equiv[r^2+(|z|+h)^2]^{1/2}$. Take the MOND potential in the $z>0$ region to coincide with that of $\rho(\boldsymbol{R})$. In the $z<0$ region the potential is defined symmetrically. For spherical systems the MOND field is related to the Newtonian field by the algebraic relation (4). Thus, this is also the case for the model under construction. The 'bulge' density that produces the potential is just the part of the spherical density distribution $\rho(\boldsymbol{R})$ that is above the mid-plane; we can dictate it at will. A disc with surface density $\Sigma(r)$ must supplement the bulge to match the jump in the z-gradient. If $M(q)\equiv\int_0^q 4\pi\lambda^2\hat{\rho}(\lambda)\,\mathrm{d}\lambda$ is the spherical mass within distance $q=(r^2+h^2)^{1/2}$ from the centre of $\rho(\boldsymbol{R})$, then

$$\Sigma(r)=\frac{M(q)\,h}{2\pi q^3}. \tag{23}$$

$\Sigma(r)$ is just the surface density of a Kuzmin disc with the same h and a mass equal to the total spherical mass within the sphere going through the point at r on the disc. Comparing with equation (21) we find the corresponding $U'(\varphi_K)=M(q)/M(\infty)$, with $q=-MG/\varphi_K$.

A third approach, which we shall not detail here, is to start with the MOND potential for the Kuzmin disc, ψ_K, and construct new potentials $\psi=S(\psi_K)$.

We reiterate that in all the above models, the bulge equidensity surfaces coincide with equipotential surfaces of the model. This means that we can readily construct, for the bulge, distribution functions with isotropic velocity distributions. These are of the form $f(E)$, with $E=v^2/2+\psi(r)$, for which $\rho(\boldsymbol{r})=\int\mathrm{d}^3v\,f(E)=F[\psi(\boldsymbol{r})]$.

4 SOME OTHER DISC-GALAXY MODELS

How good an approximation is the algebraic relation in general? There clearly are disc models for which it fails rankly. Consider, for example, a disc whose surface density vanishes at the centre. Then, $|\nabla\varphi_N|$ vanishes both near the centre and at infinity, while the potential, which vanishes at infinity, is non-zero at the centre. Thus $|\nabla\varphi_N|$ is anything but a function of φ_N, and the algebraic approximation must break appreciably.

We have found that for the very pertinent case of a disc with an exponential surface-density law, $\Sigma(r)=\Sigma_0\exp(-r/h)$, the algebraic approximation holds very well. A disc for which it holds less well is the so-called Kalnajs disc (characterized by a constant angular velocity on circular orbits inside the material disc), whose surface density is $\Sigma(r)=\Sigma_0[1-(r/h)^2]^{1/2}$. We now discuss these two examples in more detail.

As explained in Section 2, to be able to foretell the quality of the algebraic approximation for a given disc, it is enough to look at the tightness of the relation $|\nabla\varphi_N|$ versus φ_N. In Fig. 1 we show this relation for both of the above surface-density distributions, as obtained from numerical calculations using a multigrid scheme. The code is capable of solving the non-linear MOND equation, and is described in detail in Brada 1995 (in preparation). For reference we also show in Fig. 1(a) the numerical results for the Kuzmin disc, which show that the numerical scattering about the expected exact relation, $|\nabla\varphi_N|=\varphi_N^2/MG$ (marked by crosses), is quite negligible (the slight departure from the exact relation is numerical, and stems from the cut-off in the disc at the end of the mesh). The correlation for the exponential disc (Fig. 1b) is also rather tight (but does not follow the asymptotic relation). We thus expect the algebraic approximation to be rather good for the MOND field, *for all values of the mean acceleration*. We plot in Fig. 2 the relative departure, δ, from the algebraic relation:

$$\boldsymbol{\delta}\equiv\frac{\mu(|\nabla\psi|/a_o)\nabla\psi-\nabla\varphi_N}{|\nabla\varphi_N|}. \tag{24}$$

For a very-low-acceleration Kuzmin disc we see that $\delta\approx 0$ everywhere, as expected. For an exponential disc in the same

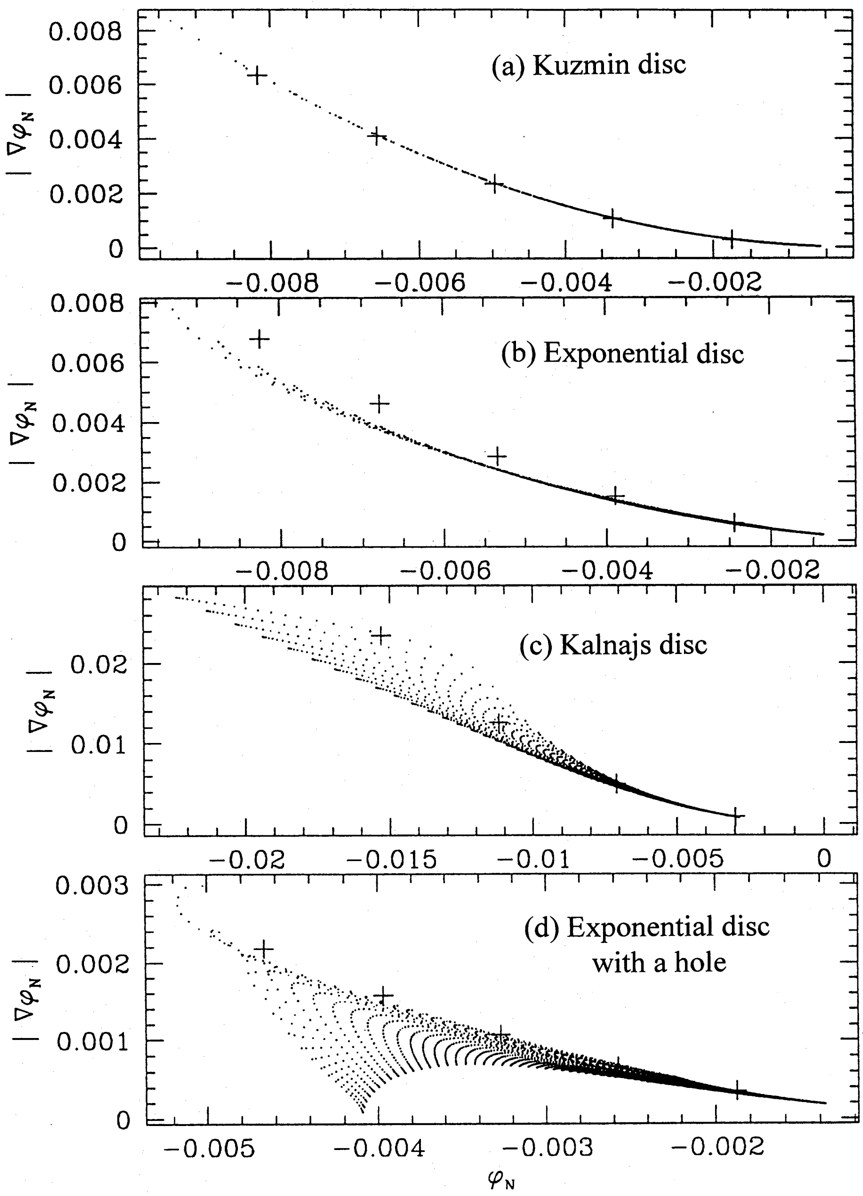

Figure 1. Plots of $|\nabla\varphi_N|$ versus φ_N for Kuzmin (a), exponential (b), and Kalnajs (c) discs, and for an exponential disc cut-off below one-and-a-half scalelengths. The crosses mark the relation $|\nabla\varphi_N| = \varphi_N^2/MG$.

limit ($\Sigma_0 \ll a_o/G$), we see that $|\boldsymbol{\delta}| \ll 1$ everywhere outside the disc, in keeping with the tight $|\nabla\varphi_N|$ versus φ_N relation. For the Kalnajs disc, we see in Fig. 1(c) that the $|\nabla\varphi_N|$ versus φ_N relation has rather more scattering, and indeed the plot of $\boldsymbol{\delta}$, shown in Fig. 2(c) (again for $\Sigma_0 \ll a_o/G$), evinces a more substantial departure from the algebraic approximation. An exponential disc with a hole within one-and-a-half scalelengths is an even more extreme case, as shown in Figs 1(d) and 2(d).

5 ROTATION CURVES BASED ON THE ALGEBRAIC APPROXIMATION

If the algebraic relation (4) holds outside the disc it cannot be correct in the mid-plane of the thin disc; so, we cannot use equation (2) $[\mu(a/a_o)a = a_N]$ to obtain the rotation curve of the model galaxy. Rather, we have to follow the following procedure: we need the radial acceleration, a_r, in the mid-plane of the disc. As the acceleration component parallel to the disc is continuous across the thin disc, a_r is the same as a_r^+, the radial acceleration just outside the disc. This can be obtained from the algebraic relation in terms of the total Newtonian acceleration just outside the disc a_N^+, and its radial component. The latter can again be equated to its value in the mid-plane of the disc (as it too is continuous), and so we obtain

$$v^2(r)/r = a_r = a_r^+ = \frac{a_{rN}}{\mu(a^+/a_o)} = \frac{a_{rV}}{\mu[I^{-1}(a_N^+/a_o)]}. \qquad (25)$$

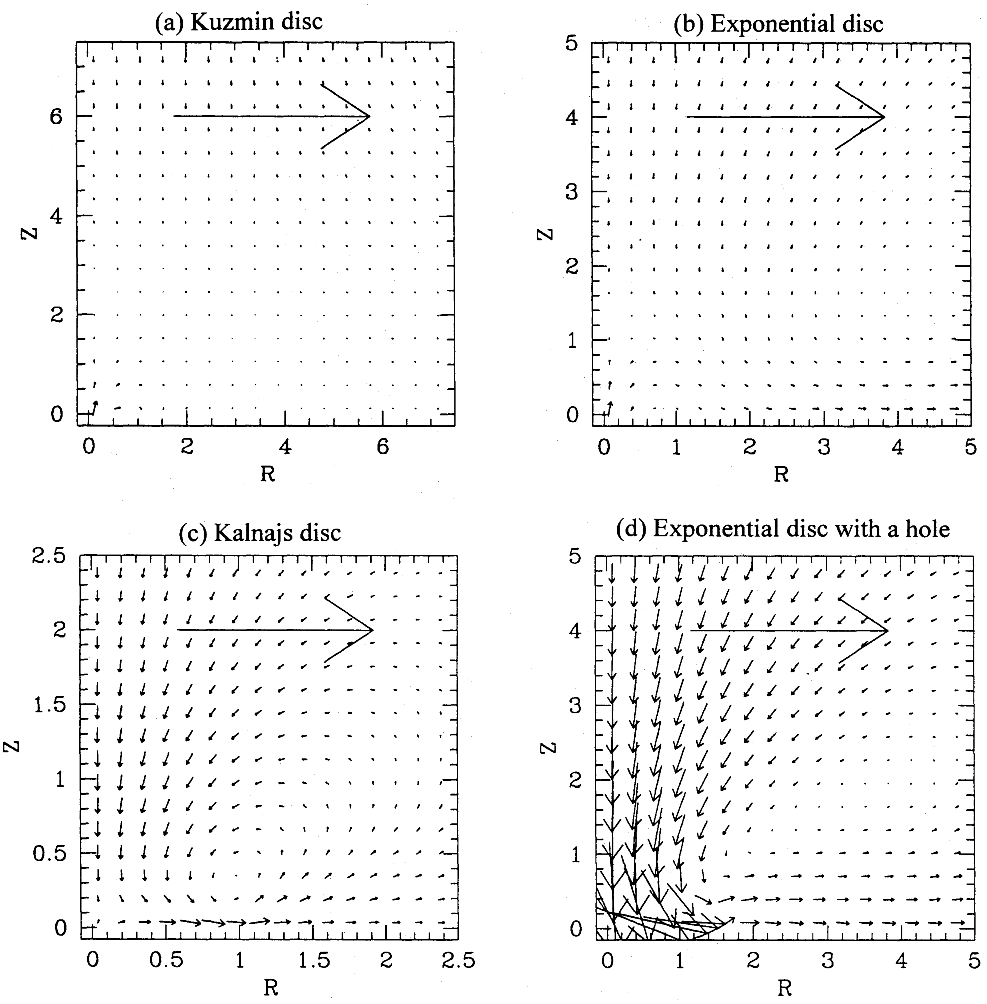

Figure 2. A plot of δ – a measure of the departure from the algebraic relation – for the four discs as in Fig. 1. The large arrow is of unit length.

To complete the expression we express a_N^+ in terms of Newtonian radial acceleration in the mid-plane, directly related to the Newtonian rotation curve $a_N^+ = [a_{rN}^2 + (2\pi G\Sigma)^2]^{1/2}$. The MOND rotation curve is thus given by a simple function of the corresponding Newtonian quantity. The correction to equation (2) involves the addition of the $2\pi G\Sigma$ term in the argument of I^{-1} in equation (25).

Relation (2) was found numerically (Milgrom 1986) to constitute a good approximation for a large class of bulge-plus-disc galaxy models, but, as we said, it is not exact in the BM formulation, even for configurations for which it is correct outside the disc. For example, for the low-surface-density Kuzmin disc, relation (2) gives for the rotation speed

$$v^2(r) = (MGa_o)^{1/2} r^{3/2}/(r^2 + h^2)^{3/4}, \quad (26)$$

to be compared with the somewhat different exact expression (17) [where the r dependence is $r^2/(r^2 + h^2)$]. This latter expression is obtained from equation (25).

We suggest that equation (25) is, generically, a better approximation for the rotation curve of disc galaxies in the BM formulation than equation (2) is, even when the algebraic approximation is not so good outside the disc (see some examples below); it is as easy to apply as the latter formulation. [We remind the reader that in the formulation of MOND as a modification of inertia (Milgrom 1994a) relation (2) gives the rotation curve exactly.]

In Fig. 3, we give three rotation curves for each of a few galaxy models. The galaxy models presented are the bare Kuzmin disc, a bare exponential disc, and a Kalnajs disc, all in the deep MOND limit. We give the exact rotation curve calculated numerically, the curve calculated from the approximation expression (2), and that calculated from what we propose as an improved approximation (25). We expect the performance of equation (2) to be worst for pure discs in the deep MOND limit: adding a spherical component, and/or going nearer to the Newtonian regime can only improve performance of equation (2) [but not that of approximation (25)].

Interestingly, the MOND rotation curve for a Kalnajs disc seems to be given exactly by $v \propto r$ – as in the Newtonian limit – no matter how deep in the MOND regime we are. We do not yet understand the origin of this behaviour. Once this is accepted as fact, the proportionality factor, i.e., the constant angular velocity, Ω, may be calculated for very-low-accelera-

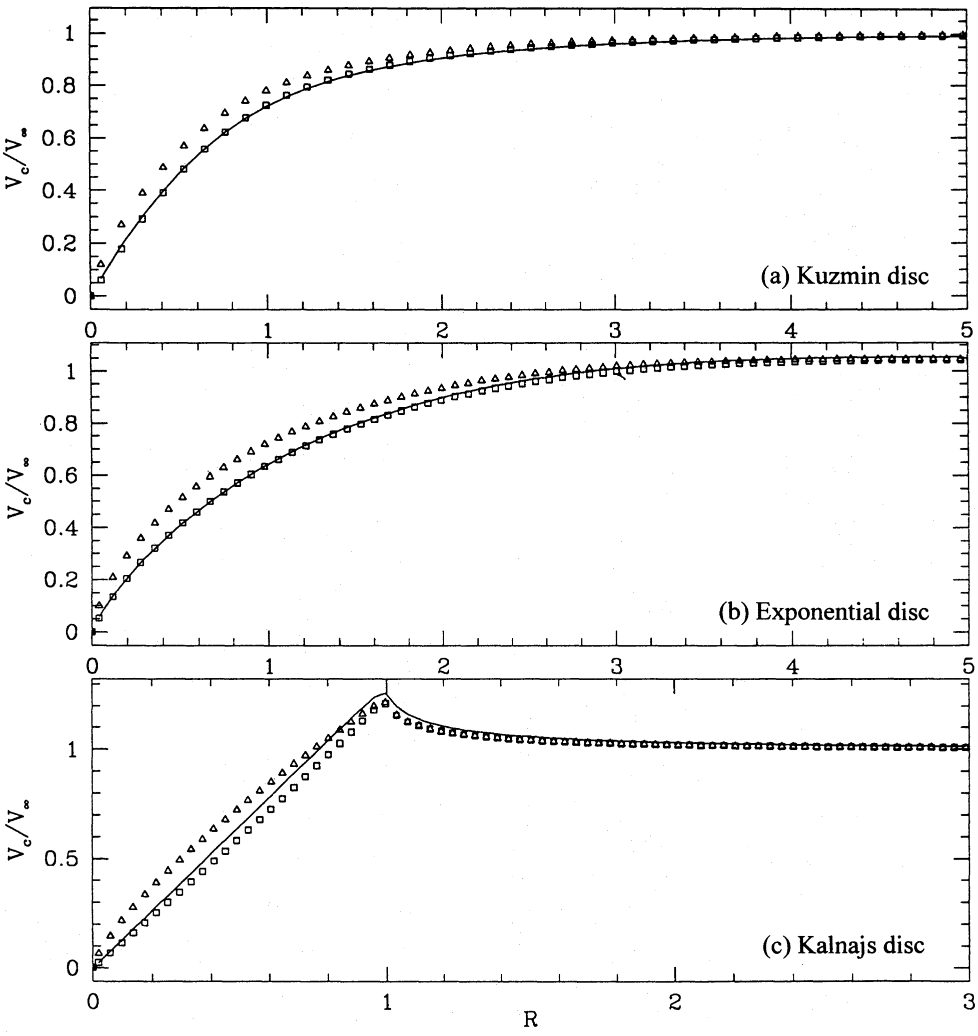

Figure 3. The rotation curves for the first three disc models of Fig. 1: the line is the exact curve; triangles and squares mark the curves calculated, respectively, by the algebraic relation and by the improved approximation.

tion discs from the virial relation (18) for discs to get $\Omega^2 = 5\times 3^{-3/2}(2\pi\Sigma_0 G a_o)^{1/2}/h$, compared with the Newtonian angular velocity which is $\Omega_N^2 = \pi^2 G\Sigma_0/2h$.

6 DISCUSSION

We have described models of disc galaxies for which exact solutions of the Bekenstein–Milgrom field equation can be obtained in the form of a simple algebraic relation between the MOND solution and the Newtonian field of the same mass distribution. This relation holds approximately for a wider class of configurations, which include exponential discs. We have given a simple criterion to assess the validity, or near validity of this relation; the use of this criterion assumes knowledge of the Newtonian field φ_N only: it requires that $|\nabla\varphi_N|$ be tightly correlated with φ_N outside the disc. We have also suggested an improved approximation – inspired by the above approximation – for calculating rotation curves in the BM formulation.

When accuracy beyond the algebraic approximation is needed it may serve as a first approximation around which we can linearize the MOND equation in small increments. We may, for instance, proceed as follows: suppose that the $|\nabla\varphi_N|$ versus φ_N has some scattering, but that we can reasonably define a mean relation $|\nabla\varphi_N| \approx f(\varphi_N)$. The acceleration field that is derived from the algebraic relation, which is to serve as our zeroth order approximation, is not derivable from a potential, in general. So, it is more convenient to work with accelerations, not with potentials. Define then

$$\boldsymbol{q} \equiv \mu(|\nabla\psi|/a_o)\nabla\psi, \tag{27}$$

which is inverted, as in equation (5), to give

$$\nabla\psi = \nu(q/a_o)\boldsymbol{q}, \tag{28}$$

where $q = |\boldsymbol{q}|$. The algebraic relation would equate $\boldsymbol{q}$ to $\nabla\varphi_N$, but we now write

$$\boldsymbol{q} = \nabla\varphi_N + \boldsymbol{\eta}, \tag{29}$$

with $\boldsymbol{\eta} = |\nabla\varphi_N|\boldsymbol{\delta}$ being a curl field which is assumed to be small compared with $\nabla\varphi_N$, and which we shall treat to first-order. By the MOND equation we now have

$$\nabla \cdot \boldsymbol{\eta} = 0, \tag{30}$$

and from equation (28)

$$\nabla \times [\nu(q/a_o)\boldsymbol{q}] = 0. \tag{31}$$

Equations (30) and (31) are equivalent to the original MOND equation for the potential ψ (see Milgrom 1986). We now substitute equation (29) into equation (31), and take the first-order in $\boldsymbol{\eta}$ (noting that $\nabla|\nabla\varphi_N| \times \nabla\varphi_N$, which measures the departure from the algebraic relation, is also first order) to get

$$\nabla \times \boldsymbol{\eta} + \hat{\nu}\boldsymbol{e} \times [\nabla(\boldsymbol{e}\cdot\boldsymbol{\eta}) - f'(\varphi_N)\boldsymbol{\eta}] = \hat{\nu}\nabla|\nabla\varphi_N| \times \boldsymbol{e}, \tag{32}$$

where $\boldsymbol{e}(\boldsymbol{r}) \equiv -\nabla\varphi_N/|\nabla\varphi_N|$ is a unit vector in the direction of the local Newtonian acceleration, and $\hat{\nu}(\boldsymbol{r})$ is the logarithmic derivative of ν calculated at $|\nabla\varphi_N|/a_o$ ($\hat{\nu}=1$ in the deep MOND limit). The linear equations (30) and (32) determine $\boldsymbol{\eta}$.

Our construction of the solvable disc models began with a known MOND solution which does not involve a disc, such as a point mass, or, in general, a spherical mass distribution. We then place that mass distribution anywhere relative to the $z=0$ plane; then we take the MOND potential, above the plane only, to be that of the mass in question, defining the potential below the plane as the minor image of the one above. The disc that matches the jump of the z-gradient of the potential is then found; hence a family of solvable disc models is born. Clearly, we may start with any axisymmetric mass distribution for which the MOND solution is known analytically or numerically – not just a spherical one – and get a new family of disc models. Such initial non-disc MOND solutions can be found by starting from a potential, then calculating the density distribution from equation (1), making sure that the resulting ρ is positive everywhere, and otherwise has a reasonable value.

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