Analytic solutions for nonlinear partial fractional differential equations

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Abstract

In this article, we improved the Fan algebraic direct method to construct the Jacobi elliptic solutions for nonlinear partial fractional partial differential equations based on the Jumarie's fractional derivatives. We use the improved direct proposed method to find the Jacobi elliptic solutions for some nonlinear fractional differential equation in mathematical physics namely the space–time fractional Hirota Satsuma KdV equations This method is powerful and effective for finding the Jacobi elliptic solutions to the nonlinear partial fractional differential equations. Jacobi elliptic solutions for nonlinear fractional differential equations degenerate the hyperbolic solutions and trigonometric solutions when the modulus $m \to 1$ and $m \to 0$ respectively. This method can be applied to many other nonlinear fractional partial differential equations in mathematical physics.

Keywords: Nonlinear fractional partial differential equations, Improved Fan's algebraic method, Exact solutions; Jumarie's fractional derivatives, Jacobi elliptic functions.

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1. Introduction

Nonlinear partial fractional equations are very effective for description of many physical phenomena such as theology, damping law , diffusion process and the nonlinear oscillation of earthquake can be modeled with fractional derivatives [1-2]. Also many applications of nonlinear partial fractional differential equations can be found in turbulence and fluid dynamics and nonlinear biological system [1-10]. There are many methods for finding the approximate solutions for nonlinear partial fractional differential equations such as Adomian decomposition method [3-5], variation iteration method [6], homotopy perturbation method [7,8,9] and homotopy analysis method [10] and so on . No analytical methods has been available before 1998 for nonlinear fractional differential equations. Li etal [11] have proposed the fractional complex transformation to convert the nonlinear partial fractional differential equations into ordinary differential equations so that all analytical methods devoted to advanced calculus can be applied to fractional calculus. Recently Zhang etal [12] have introduced a direct method called the sub-equation method to look for the exact solutions for nonlinear partial fractional differential equations. He [13] have extended the exp- function method to fractional partial differential equations in sense of modified Rieman Liouville derivative based on the fractional complex transform. Also Wang etal [14] have studied the symmetry properties of time fractional KdV equation in the sense of the Riemann-Liouville derivatives using the Lie group analysis method. There are many method for solving the nonlinear partial fractional differential equations such as [15,16]. Fan etal [17,18] , Zayed etal [19] and Hong etal [20,21] have proposed an algebraic method for nonlinear partial differential equations to obtain a series of exact wave solutions including the soliton, rational ,triangular periodic , Jacobi and Weierstrass doubly periodic solutions. In this paper, we will improve the extended proposed algebraic method to solve the nonlinear partial fractional differential equations. Also we use the improve extended proposed algebraic method to construct the Jacobi elliptic exact solutions for space–time fractional nonlinear Hirota Satsuma KdV equations in the following form[22] :

$$\begin{aligned}
D_t^{\alpha}u &= \frac{1}{2}D_x^{3\alpha}u - 3uD_x^{\alpha}u + 3vD_x^{\alpha}u + 3uD_x^{\alpha}w,\\
D_t^{\alpha}v &= -D_x^{3\alpha}v + 3uD_x^{\alpha}v;\\
D_t^{\alpha}w &= -D_x^{3\alpha}w + 3uD_x^{\alpha}w;
\end{aligned} \tag{1}$$

where $0 < \alpha \le 1$.

2. Preliminaries

There are many types of the fractional derivatives such as the Kolwankar- Gangal local fractional derivative [24], Chen's fractal derivative [25],Cresson's derivative [26], Jumarie's modified Riemann--Liouville derivative [27,28]. In this section, we give some basic definitions of fractional calculus theory which are be used in this work. Jumarie's derivative is defined as

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} (f(\xi) - f(0))\, d\xi, \quad 0 < \alpha < 1, \tag{2}$$

where $f : R \to R$, $x \mapsto f(x)$ denotes a continuous (but not necessarily first-order-differentiable) function. We can obtain the following properties:

Property 1. Let $f(x)$ satisfy the definition of the modified Riemann-Liouville derivative and $f(x)$ be a ($k\alpha$) th order differentiable function. The generalized Taylor series is given as [28,30]

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{(\alpha k)!} f^{(\alpha k)}(x), \quad 0 < \alpha < 1. \tag{3}$$

Property 2. Assume that $f(x)$ denotes a continuous $R \to R$ function. We use the following equality for the integral w.r.t. $(dx)^{\alpha}$ [29,30]:

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi) d\xi = \frac{1}{\Gamma(\alpha+1)} \int_0^x f(\xi)(dx)^{\alpha}, \quad 0 < \alpha \le 1. \tag{4}$$

Property 3. Some useful formula and important properties for the modified Riemann-Liouville derivative as follows [30-33]:

$$D_t^{\alpha} t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \ r > 0 \tag{5}$$

$$D_t^{\alpha} \left[f(t) g(t) \right] = f(t) D_t^{\alpha} g(t) + g(t) D_t^{\alpha} f(t) \tag{6}$$

$$D_t^{\alpha} \left[f(g(t)) \right] = f_g'(g(t)) D_t^{\alpha} g(t)$$

$$D_t^{\alpha} \left[f(g(t)) \right] = D_g^{\alpha} f(g(t)) [g'(t)]^{\alpha} \tag{7}$$

The function f(x) should be differentiable with respect to x(t) and x(t) is fractional differentiable in (7). The above results are employed in the following sections. The Liebnz rule is given (6) for modified Riemann- Liouville derivative which modified by Jumarie's in [30]. The modified Riemann-Liouville derivative has been successfully applied in probability calculus [31], fractional Laplace problems [32], the fractional variation approach with several variables [33], the fractional variational iteration method [34], the fractional variational approach with natural boundary conditions [35] and the fractional Lie group method [36].

3. Algebraic direct method for nonlinear partial fractional differential equations

Consider the following nonlinear partial fractional differential equation:

$$U(u, D_t^{\alpha}u, D_x^{\alpha}u, D_t^{2\alpha}u, D_x^{2\alpha}u, ...) = 0, \tag{8}$$

where u is an unknown function, U is a polynomial in u and its partial fractional derivatives in which the highest order fractional derivatives and the nonlinear terms are involved. We give the main steps of the algebraic direct method for nonlinear partial fractional differential equation.

Step 1. We use the travelling wave transformation

$$u(x,t) = u(\xi), \qquad \xi = x + ct, \tag{9}$$

where c is an arbitrary constant. The transformation (9) permits us to convert the partial fractional differential equations (8) to the fractional ODE in the following form

$$P(u, D_{\xi}^{\alpha}u, D_{\xi}^{2\alpha}u, ...) = 0, \tag{10}$$

where P is a polynomial in u and its total derivatives with respect to ξ.

Step 2. We suppose that Eq. (10) has the following solution

$$u(\xi) = \sum_{i=-N}^{N} \alpha_i \phi^i(\xi), \qquad \alpha_N \neq 0, \text{ or } \qquad \alpha_{-N} \neq 0, \tag{11}$$

where α_i are arbitrary constants to be determined later, while $\phi(\xi)$ satisfies the following nonlinear fractional first order differential equation:

$$[D_{\xi}^{\alpha}\phi(\xi)]^2 = e_0 + e_1\phi^2(\xi) + e_2\phi^4(\xi), \tag{12}$$

where e_0, e_1 and e_2 are arbitrary constants.

Step 3. We determine positive integer N of formal polynomial solution given in Eq. (11) by balancing nonlinear terms and highest order fractional derivatives in Eq.(10).

Step 4. Substituting Eqs. (11) and (12) into Eq. (10) and collecting the coefficients of $\phi(\xi)$, and setting the coefficients of $[\phi(\xi)]^i[D_{\xi}^{\alpha}\phi(\xi)]^j$ $(j = 0,1, \ i = 0,\pm1,\pm2,...)$ to be zero, we get an over-determined system of algebraic equations with respect to $a_i(i=0,\pm1,\pm2,...)$ and c.

Step 5. We solve the over-determined system of algebraic equations to determine $a_i(i=0,\pm1,\pm2,...)$ and c.

Step 6. In order to obtain the general solutions for Eq. (12), we suppose $\phi(\xi)=\psi(\eta)$ and a nonlinear fractional complex transformation $\eta=\frac{\xi^{\alpha}}{\Gamma(\alpha+1)}$. Then by Eq. (12) can be turned into the following nonlinear ordinary differential equation

$$[\psi'(\eta)]^2=e_0+e_1\psi(\eta)+e_2\psi^2(\eta) \tag{13}$$

The general solutions of (13) have been discussed in [37-39] as the following table

e_0	e_1	e_2	$\psi(\eta)$	$\phi(\xi)$
1	$-(1+m^2)$	m^2	$sn(\eta)$ or $cd(\eta)$	$sn(\frac{\xi^{\alpha}}{\Gamma(\alpha+1)})$ or $cd(\frac{\xi^{\alpha}}{\Gamma(\alpha+1)})$
$1-m^2$	$2m^2-1$	$-m^2$	$cn(\eta)$	$cn(\frac{\xi^{\alpha}}{\Gamma(\alpha+1)})$
m^2-1	$2-m^2$	-1	$dn(\eta)$	$dn(\frac{\xi^{\alpha}}{\Gamma(\alpha+1)})$
m^2	$-(1+m^2)$	1	$ns(\eta)$ or $dc(\eta)$	$ns(\frac{\xi^{\alpha}}{\Gamma(\alpha+1)})$ or $dc(\frac{\xi^{\alpha}}{\Gamma(\alpha+1)})$
$-m^2$	$2m^2-1$	$1-m^2$	$nc(\eta)$	$nc(\frac{\xi^{\alpha}}{\Gamma(\alpha+1)})$
-1	$2-m^2$	m^2-1	$nd(\eta)$	$nd(\frac{\xi^{\alpha}}{\Gamma(\alpha+1)})$
$1-m^2$	$2-m^2$	1	$cs(\eta)$	$cs(\frac{\xi^{\alpha}}{\Gamma(\alpha+1)})$
1	$2-m^2$	$1-m^2$	$sc(\eta)$	$sc(\frac{\xi^{\alpha}}{\Gamma(\alpha+1)})$
1	$2m^2-1$	$m^2(m^2-1)$	$sd(\eta)$	$sd(\frac{\xi^{\alpha}}{\Gamma(\alpha+1)})$
$m^2(m^2-1)$	$2m^2-1$	1	$ds(\eta)$	$ds(\frac{\xi^{\alpha}}{\Gamma(\alpha+1)})$
$\frac{1}{4}$	$\frac{1}{2}(1-2m^2)$	$\frac{1}{4}$	$ns(\eta)\pm cs(\eta)$	$ns(\frac{\xi^{\alpha}}{\Gamma(\alpha+1)})\pm cs(\frac{\xi^{\alpha}}{\Gamma(\alpha+1)})$

$\frac{1}{4}(1-m^2)$	$\frac{1}{4}(1+m^2)$	$\frac{1}{4}(1-m^2)$	$nc(\eta)\pm sc(\eta)$	$nc(\frac{\xi^\alpha}{\Gamma(\alpha+1)})\pm sc(\frac{\xi^\alpha}{\Gamma(\alpha+1)})$
$\frac{m^2}{4}$	$\frac{1}{2}(m^2-2)$	$\frac{m^2}{4}$	$sn(\eta)\pm i\,cn(\eta)$	$sn(\frac{\xi^\alpha}{\Gamma(\alpha+1)})\pm i\,cn(\frac{\xi^\alpha}{\Gamma(\alpha+1)})$
$\frac{1}{4}$	$\frac{1}{2}(1-m^2)$	$\frac{1}{4}$	$\sqrt{1-m^2}\,sc(\eta)\pm dc(\eta)$	$\sqrt{1-m^2}\,sc(\frac{\xi^\alpha}{\Gamma(\alpha+1)})\pm dc(\frac{\xi^\alpha}{\Gamma(\alpha+1)})$
$\frac{1}{4}(m^2-1)$	$\frac{1}{2}(1+m^2)$	$\frac{1}{4}(m^2-1)$	$m\,sd(\eta)\pm nd(\eta)$	$m\ \ sd(\frac{\xi^\alpha}{\Gamma(\alpha+1)})\pm nd(\frac{\xi^\alpha}{\Gamma(\alpha+1)})$
$\frac{1}{4}$	$\frac{1}{2}(m^2-2)$	$\frac{m^2}{4}$	$\frac{sn(\eta)}{1\pm dn(\eta)}$	$sn(\frac{\xi^\alpha}{\Gamma(\alpha+1)})/(1\pm dn(\frac{\xi^\alpha}{\Gamma(\alpha+1)}))$

where $0<m<1$ is the modulus of the Jacobi elliptic functions and $i=\sqrt{-1}$.

Table 1

We put some of the general solutions of Eq. (13) have been discussed in table 1 and there are other cases which omitted here for convenience , (see [37]).

Step 6. Since the general solutions of (12) and (13) are discussed in the above table 1, then substituting $\alpha_i(i=0,\pm1,\ldots,\pm m), e_0$, e_1, e_2 and the general solutions of (12) and (13) into (11), we have obtained more new Jacobi elliptic exact solutions for nonlinear partial fractional derivatives equation (8).

4. Jacobi elliptic solutions for space-time fractional Hirota Satsuma KdV equations

In this section, we will construct the Jacobi elliptic wave solutions for the space – time fractional Hirota Satsuma KdV equations in the following form [22]:

$$
\begin{aligned}
&D_t^\alpha u=\frac{1}{2}D_x^{3\alpha}u-3uD_x^\alpha u+3vD_x^\alpha w+3wD_x^\alpha v,\\
&D_t^\alpha v=-D_x^{3\alpha}v+3uD_x^\alpha v;\\
&D_t^\alpha w=-D_x^{3\alpha}w+3uD_x^\alpha w;
\end{aligned}
\qquad (14)
$$

where $0<\alpha\le1$. Eq. (14) has been investigated in [22] using the fractional sub-equation method. Let us now solve Eq, (14) using the proposed method of Sec. 2. We use the traveling wave transformation

$$u=u(\xi),\quad v=v(\xi),\qquad w=w(\xi),\quad \xi=x+ct\,. \qquad (15)$$

where c is an arbitrary constant to be determined later. The transformation (15) permits us to convert the partial fractional Hirota Satsuma KdV equations (14) to the following nonlinear fractional ODE in the following form:

$$\begin{aligned} c^{\alpha} D_{\xi}^{\alpha} u &= \frac{1}{2} D_{\xi}^{\alpha}(D_{\xi}^{\alpha}(D_{\xi}^{\alpha} u)) - 3u D_{\xi}^{\alpha} u + 3v D_{\xi}^{\alpha} w + 3w D_{\xi}^{\alpha} v, \\ c^{\alpha} D_{\xi}^{\alpha} v &= -D_{\xi}^{\alpha}(D_{\xi}^{\alpha}(D_{\xi}^{\alpha} v)) + 3u D_{\xi}^{\alpha} v; \\ c^{\alpha} D_{\xi}^{\alpha} w &= -D_{\xi}^{\alpha}(D_{\xi}^{\alpha}(D_{\xi}^{\alpha} w)) + 3u D_{\xi}^{\alpha} w. \end{aligned} \tag{16}$$

By balancing the highest order fractional derivatives with the nonlinear terms in Eqs. (16) we have the formal solutions of Eq.(16) as following:

$$\begin{aligned} u(\xi) &= a_0 + a_1\phi(\xi) + a_2\phi^2(\xi) + \frac{a_3}{\phi(\xi)} + \frac{a_4}{\phi^2(\xi)}, \\ v(\xi) &= b_0 + b_1\phi(\xi) + b_2\phi^2(\xi) + \frac{b_3}{\phi(\xi)} + \frac{b_4}{\phi^2(\xi)}, \\ w(\xi) &= L_0 + L_1\phi(\xi) + L_2\phi^2(\xi) + \frac{L_3}{\phi(\xi)} + \frac{L_4}{\phi^2(\xi)}, \end{aligned} \tag{17}$$

where $a_i, b_i, L_i, i = 0,1,...,4$ are constants to be determined later, such that $a_2 \neq 0$ or $a_4 \neq 0$, $b_2 \neq 0$ or $b_4 \neq 0$ and $L_2 \neq 0$ or $L_4 \neq 0$. Substituting (17) along with Eq. (12) into (16), collecting all the terms of the same orders $\phi^i(\xi), i = 0,\pm1,\pm2,...$ and setting each coefficient to be zero, we have obtained a set of algebraic equations which can be solved by using Maple or Mathematica to obtain the following cases:

Case 1.

$$\begin{aligned} &a_0 = \frac{c^{\alpha}}{3} + \frac{4e_1}{3}, \qquad a_2 = 4e_2, \qquad a_4 = 4e_0, \\ &b_0 = -\frac{4e_0}{3L_4^2}(-2L_4e_1 - 2c^{\alpha}L_4 + 3L_0e_0), \qquad b_2 = \frac{4e_2e_0}{L_4}, \qquad b_4 = \frac{4e_0^2}{L_4}, \\ &L_2 = \frac{e_2L_4}{e_0}, \\ &a_1 = b_1 = a_3 = b_3 = L_1 = L_3 = 0, \end{aligned} \tag{18}$$

where e_0, e_1, e_2 are arbitrary constants.

Let us now write down the following exact solutions of the space-time fractional Hirota Satsuma equations (14) for case 1.

$$u(\xi)=\frac{c^{\alpha}}{3}+\frac{4e_1}{3}+4e_2\phi^2(\xi)+\frac{4e_0}{\phi^2(\xi)}\quad ,$$
$$v(\xi)=-\frac{4e_0}{3L_4^2}(-2L_4e_1-2c^{\alpha}L_4+3L_0e_0)+\frac{4e_2e_0}{L_4}\phi^2(\xi)+\frac{4e_0^2}{L_4\phi^2(\xi)}\quad , \tag{19}$$
$$w(\xi)=L_0+\frac{e_2L_4}{e_0}\phi^2(\xi)+\frac{L_4}{\phi^2(\xi)}.$$

The general solutions of Eq. (13) dependent on the values of e_0,e_1,e_2, consequently we get the following families of exact solutions :

Family 1. $e_0=1,\ e_1=-(1+m^2),$ and $e_2=m^2$ the Jacobi elliptic exact solutions for Eq.(14) take the following form:

$$u_1(\xi)=\frac{c^{\alpha}}{3}-\frac{4(1+m^2)}{3}+4m^2sn^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+4\,ns^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),$$
$$v_1(\xi)=-\frac{4}{3L_4^2}[2L_4(1+m^2)-2c^{\alpha}L_4+3L_0]+\frac{4m^2}{L_4}sn^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+\frac{4}{L_4}ns^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})\quad ,$$
$$w_1(\xi)=L_0+m^2L_4sn^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+L_4ns^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}).$$

(20)

Or

$$u_1(\xi)=\frac{c^{\alpha}}{3}-\frac{4(1+m^2)}{3}+4m^2cd^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+4\,dc^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),$$
$$v_1(\xi)=-\frac{4}{3L_4^2}[2L_4(1+m^2)-2c^{\alpha}L_4+3L_0]+\frac{4m^2}{L_4}cd^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+\frac{4}{L_4}dc^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),$$
$$w_1(\xi)=L_0+m^2L_4cd^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+L_4dc^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}).$$

(21)

Family 2. $e_0=1-m^2,\ e_1=2m^2-1,$ and $e_2=-m^2$ the Jacobi elliptic exact solutions for Eq.(14) take the following form:

$$u_2(\xi)=\frac{c^{\alpha}}{3}+\frac{4(2m^2-1)}{3}-4m^2cn^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+4(1-m^2)\ nc^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),$$

$$v_2(\xi)=-\frac{4(1-m^2)}{3L_4^2}[-2L_4(2m^2-1)-2c^{\alpha}L_4+3L_0(1-m^2)]$$
$$-\frac{4m^2(1-m^2)}{L_4}cn^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+\frac{4(1-m^2)^2}{L_4}nc^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}), \quad (22)$$

$$w_2(\xi)=L_0-\frac{m^2L_4}{1-m^2}cn^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+L_4\ nc^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}).$$

Family 3. $e_0=m^2-1,\ e_1=2-m^2,$ and $e_2=-1$ the Jacobi elliptic exact solutions for Eq.(14) take the following form:

$$u_3(\xi)=\frac{c^{\alpha}}{3}+\frac{4(2-m^2)}{3}-4dn^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+4(m^2-1)\ nd^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),$$

$$v_3(\xi)=-\frac{4(m^2-1)}{3L_4^2}[-2L_4(2-m^2)-2c^{\alpha}L_4+3L_0(m^2-1)]$$
$$-\frac{4(m^2-1)}{L_4}dn^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+\frac{4(m^2-1)^2}{L_4}nd^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}), \quad (23)$$

$$w_3(\xi)=L_0-\frac{L_4}{(m^2-1)}dn^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+L_4\ nd^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}).$$

Family 4. $e_0=1-m^2,\ e_1=2-m^2,$ and $e_2=1$ the Jacobi elliptic exact solutions for Eq.(14) take the following form:

$$u_4(\xi)=\frac{c^{\alpha}}{3}+\frac{4(2-m^2)}{3}+4cs^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+4(1-m^2)\ \ sc^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),$$

$$v_4(\xi)=-\frac{4(1-m^2)}{3L_4^2}[-2L_4(2-m^2)-2c^{\alpha}L_4+3L_0(1-m^2)]$$
$$+\frac{4(1-m^2)}{L_4}cs^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+\frac{4(1-m^2)^2}{L_4}sc^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}), \quad (24)$$

$$w_4(\xi)=L_0+\frac{L_4}{1-m^2}cs^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+L_4\ sc^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}).$$

Family 5. $e_0=1,\ e_1=2m^2-1,$ and $e_2=m^2(m^2-1)$ the Jacobi elliptic exact solutions for Eq.(14) take the following form:

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$$u_5(\xi)=\frac{c^{\alpha}}{3}+\frac{4(2m^2-1)}{3}+4m^2(m^2-1)sd^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+4\,ds^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),$$

$$v_5(\xi)=-\frac{4}{3L_4^2}(-2L_4(2m^2-1)-2c^{\alpha}L_4+3L_0)+\frac{4m^2(m^2-1)}{L_4}sd^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+\frac{4}{L_4}ds^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),$$

$$w_5(\xi)=L_0+m^2(m^2-1)L_4sd^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+L_4\,ds^2(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}).$$

(25)

Family 6. $e_0=\frac{1}{4}$, $e_1=\frac{1}{2}(1-2m^2)$, and $e_2=\frac{1}{4}$ the Jacobi elliptic exact solutions for Eq.(14) take the following form:

$$u_6(\xi)=\frac{c^{\alpha}}{3}+\frac{2(1-2m^2)}{3}+[ns(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+cs(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^2+[ns(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+cs(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-2},$$

$$v_6(\xi)=-\frac{1}{3L_4^2}(-L_4(1-2m^2)-2c^{\alpha}L_4+\frac{3}{4}L_0)+\frac{1}{4L_4}[ns(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+cs(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^2$$
$$+\frac{1}{4L_4}[ns(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+cs(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-2},$$

$$w_6(\xi)=L_0+L_4[ns(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+cs(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^2+L_4[ns(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+cs(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-2}.$$

(26)

Family 7. $e_0=\frac{1}{4}(1-m^2)$, $e_1=\frac{1}{4}(1+m^2)$, and $e_2=\frac{1}{4}(1-m^2)$ the Jacobi elliptic exact solutions for Eq.(14) take the following form:

$$u_7(\xi)=\frac{c^{\alpha}}{3}+\frac{(1+m^2)}{3}+(1-m^2)[nc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^2$$
$$+(1-m^2)[nc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-2},$$

$$v_7(\xi)=-\frac{(1-m^2)}{3L_4^2}[-\frac{1}{2}L_4(1+m^2)-2c^{\alpha}L_4+\frac{3}{4}L_0(1-m^2)]+\frac{(1-m^2)^2}{4L_4}[nc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^2$$
$$+\frac{(1-m^2)^2}{4L_4}[nc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-2},$$

$$w_7(\xi)=L_0+L_4[nc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^2+L_4[nc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-2}.$$

(27)

Family 8. $e_0=\frac{m^2}{4},\ e_1=\frac{1}{4}(m^2-2),$ and $e_2=\frac{m^2}{4}$ the Jacobi elliptic exact solutions for Eq.(13) take the following form:

$$u(\xi)=\frac{c^{\alpha}}{3}+\frac{(m^2-2)}{3}+m^2[sn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+i\,cn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^2$$
$$+m^2\,[sn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+i\,cn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-2}\quad ,$$
$$v(\xi)=-\frac{m^2}{3L_4^2}[-\frac{1}{2}L_4(m^2-2)-2c^{\alpha}L_4+\frac{3}{4}L_0m^2]+\frac{m^4}{4L_4}[sn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+i\,cn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^2$$
$$+\frac{m^4}{4L_4}[sn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+i\,cn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-2},$$
$$w(\xi)=L_0+L_4[sn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+i\,cn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^2+L_4\,[sn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+i\,cn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-2}. \tag{28}$$

Family 9. $e_0=\frac{1}{4},\ e_1=\frac{1}{2}(1-m^2),$ and $e_2=\frac{1}{4}$ the Jacobi elliptic exact solution for Eq.(13) takes the following form:

$$u(\xi)=\frac{c^{\alpha}}{3}+\frac{2(1-m^2)}{3}+[\sqrt{1-m^2}\,sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})\pm dc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^2$$
$$+[\sqrt{1-m^2}\,sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})\pm dc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-2},$$
$$v(\xi)=-\frac{1}{3L_4^2}[-L_4(1-m^2)-2c^{\alpha}L_4+\frac{3}{4}L_0]+\frac{1}{4L_4}[\sqrt{1-m^2}\,sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})\pm dc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^2$$
$$+\frac{1}{4L_4}\,[\sqrt{1-m^2}\,sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})\pm dc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-2}\ ,$$
$$w(\xi)=L_0+L_4[\sqrt{1-m^2}\,sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})\pm dc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^2+L_4\,[\sqrt{1-m^2}\,sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})\pm dc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-2}. \tag{29}$$

Also , we can construct more families of the exact Jacobi elliptic solutions for the case 1, we are omitted here for convenience to the reader.

Case 2.

$$a_0 = \frac{1}{3}(c^\alpha + e_1 + 6\sqrt{e_2 e_0}), \qquad a_2 = 2e_2, \qquad a_4 = 2e_0,$$

$$b_0 = -\frac{2L_0 e_0}{3L_3^2}(2c^\alpha - e_1 + 6\sqrt{e_2 e_0}), \qquad b_1 = -\frac{2\sqrt{e_2 e_0}}{3L_3}(2c^\alpha - e_1 + 6\sqrt{e_2 e_0}),$$

$$b_3 = \frac{2e_0}{3L_3}(2c^\alpha - e_1 + 6\sqrt{e_2 e_0}), \qquad L_1 = \frac{L_3\sqrt{e_2 e_0}}{e_0}, \tag{30}$$

$$a_1 = a_3 = b_2 = b_4 = L_2 = L_4 = 0,$$

where L_0, L_3, e_0, e_1 and e_2 are arbitrary constants. Let us now write down the following exact solutions of the space-time fractional Hirota Satsuma equations (14) for case 2:

$$u(\xi) = \frac{1}{3}(c^\alpha + e_1 + 6\sqrt{e_2 e_0}) + 2e_2\phi^2(\xi) + \frac{2e_0}{\phi^2(\xi)},$$

$$v(\xi) = -\frac{2L_0 e_0}{3L_3^2}(2c^\alpha - e_1 + 6\sqrt{e_2 e_0}) - \frac{2\sqrt{e_2 e_0}}{3L_3}(2c^\alpha - e_1 + 6\sqrt{e_2 e_0})\phi(\xi) + \frac{2e_0(2c^\alpha - e_1 + 6\sqrt{e_2 e_0})}{3L_3\phi(\xi)}, \tag{31}$$

$$w(\xi) = L_0 + \frac{L_3\sqrt{e_2 e_0}}{e_0}\phi(\xi) + \frac{L_3}{\phi(\xi)}.$$

The general solutions of Eq. (13) dependent on the values of e_0, e_1, e_2, consequently we get the following families of exact solutions :

Family 1. $e_0 = 1,\ e_1 = -(1+m^2)$, and $e_2 = m^2$ the Jacobi elliptic exact solutions for Eq.(14) take the following form:

$$u_{10}(\xi) = \frac{1}{3}(c^\alpha - (1+m^2) \pm 6m) + 2m^2 sn^2(\frac{(x+ct)^\alpha}{\Gamma(\alpha+1)}) + 2\,ns^2(\frac{(x+ct)^\alpha}{\Gamma(\alpha+1)}),$$

$$v_{10}(\xi) = -\frac{2L_0}{3L_3^2}(2c^\alpha + (1+m^2) \pm 6m) \mp \frac{2m}{3L_3}(2c^\alpha + (1+m^2) \pm 6m)sn(\frac{(x+ct)^\alpha}{\Gamma(\alpha+1)}) + \frac{2(2c^\alpha + (1+m^2) \pm 6m)}{3L_3}ns(\frac{(x+ct)^\alpha}{\Gamma(\alpha+1)}), \tag{32}$$

$$w_{10}(\xi) = L_0 \pm 6mL_3\, sn(\frac{(x+ct)^\alpha}{\Gamma(\alpha+1)}) + L_3 ns(\frac{(x+ct)^\alpha}{\Gamma(\alpha+1)}).$$

Or

$$u_{10}(\xi)=\frac{1}{3}(c^{\alpha}-(1+m^{2})\pm 6m)+2m^{2}cd^{2}(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+2\,dc^{2}(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),$$

$$v_{10}(\xi)=-\frac{2L_{0}}{3L_{3}^{2}}(2c^{\alpha}+(1+m^{2})\pm 6m)\mp\frac{2m}{3L_{3}}(2c^{\alpha}+(1+m^{2})\pm 6m)\ cd(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})$$

$$+\frac{2(2c^{\alpha}+(1+m^{2})\pm 6m)}{3L_{3}}dc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}), \quad (33)$$

$$w_{10}(\xi)=L_{0}\pm 6mL_{3}\ cd(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+L_{3}dc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}).$$

Family 2. $e_0=1-m^2$, $e_1=2m^2-1$, and $e_2=-m^2$ the Jacobi elliptic exact solutions for Eq.(14) take the following form:

$$u_{11}(\xi)=\frac{1}{3}(c^{\alpha}+2m^{2}-1+6\sqrt{m^{2}(m^{2}-1)})-2m^{2}cn^{2}(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+2(1-m^{2})\,nc^{2}(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),,$$

$$v_{11}(\xi)=-\frac{2L_{0}(1-m^{2})}{3L_{3}^{2}}[2c^{\alpha}-(2m^{2}-1)\pm 6\sqrt{m^{2}(m^{2}-1)}]-\frac{2\sqrt{m^{2}(m^{2}-1)}}{3L_{3}}[2c^{\alpha}-(2m^{2}-1)$$

$$\pm 6\sqrt{m^{2}(m^{2}-1)}]\,cn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+\frac{2(1-m^{2})}{3L_{3}}[2c^{\alpha}-(2m^{2}-1)\pm 6\sqrt{m^{2}(m^{2}-1)}]\ nc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),$$

$$w_{11}(\xi)=L_{0}+\frac{L_{3}\sqrt{m^{2}(m^{2}-1)}}{1-m^{2}}cn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+L_{3}\ nc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}). \quad (34)$$

Family 3. $e_0=m^2-1$, $e_1=2-m^2$, and $e_2=-1$ the Jacobi elliptic exact solutions for Eq.(14) take the following form:

$$u_{12}(\xi)=\frac{1}{3}(c^{\alpha}+(2-m^{2})+6\sqrt{1-m^{2}})-2dn^{2}(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+2(m^{2}-1)\,nd^{2}(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),$$

$$v_{12}(\xi)=-\frac{2L_{0}(m^{2}-1)}{3L_{3}^{2}}[2c^{\alpha}-(2-m^{2})+6\sqrt{1-m^{2}}]-\frac{2\sqrt{1-m^{2}}}{3L_{3}}[2c^{\alpha}-(2-m^{2})+6\sqrt{1-m^{2}}]\,dn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})$$

$$+\frac{2(m^{2}-1)}{3L_{3}}[2c^{\alpha}-(2-m^{2})+6\sqrt{1-m^{2}}]\ nd(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),$$

$$w_{12}(\xi)=L_{0}+\frac{L_{3}\sqrt{1-m^{2}}}{m^{2}-1}dn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+L_{3}\ nd(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}). \quad (35)$$

Family 4. $e_0=1-m^2$, $e_1=2-m^2$, and $e_2=1$ the Jacobi elliptic exact solutions for Eq.(14) take the following form:

$$u_{13}(\xi)=\frac{1}{3}[c^{\alpha}+2-m^{2}+6\sqrt{1-m^{2}}]+2\,cs^{2}(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+2(1-m^{2})\,sc^{2}(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),$$

$$v_{13}(\xi)=-\frac{2L_{0}(1-m^{2})}{3L_{3}^{2}}[2c^{\alpha}-(2-m^{2})+6\sqrt{1-m^{2}}]-\frac{2\sqrt{1-m^{2}}}{3L_{3}}[2c^{\alpha}-(2-m^{2})+6\sqrt{1-m^{2}}]\,cs(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})$$

$$+\frac{2(1-m^{2})}{3L_{3}}[2c^{\alpha}-(2-m^{2})+6\sqrt{1-m^{2}}]\,sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),$$

$$w_{13}(\xi)=L_{0}+\frac{L_{3}\sqrt{1-m^{2}}}{1-m^{2}}cs(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+L_{3}sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}).$$

(36)

Family 5. $e_0=1$, $e_1=2m^2-1$, and $e_2=m^2(m^2-1)$ the Jacobi elliptic exact solutions for Eq.(14) take the following form:

$$u_{14}(\xi)=\frac{1}{3}[c^{\alpha}+(2m^{2}-1)+6\sqrt{m^{2}(m^{2}-1)}]+2m^{2}(m^{2}-1)\,sd^{2}(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+2\,ds^{2}(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),$$

$$v_{14}(\xi)=-\frac{2L_{0}}{3L_{3}^{2}}[2c^{\alpha}-(2m^{2}-1)+6\sqrt{m^{2}(m^{2}-1)}]-\frac{2\sqrt{m^{2}(m^{2}-1)}}{3L_{3}}[2c^{\alpha}-(2m^{2}-1)$$

$$+6\sqrt{m^{2}(m^{2}-1)}]\,sd(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+\frac{2}{3L_{3}}[2c^{\alpha}-(2m^{2}-1)+6\sqrt{m^{2}(m^{2}-1)}]\,ds(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}),$$ (37)

$$w_{14}(\xi)=L_{0}+L_{3}\sqrt{m^{2}(m^{2}-1)}\;sd(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+L_{3}\,ds(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}).$$

Family 6. $e_0=\frac{1}{4}$, $e_1=\frac{1}{2}(1-2m^2)$, and $e_2=\frac{1}{4}$ the Jacobi elliptic exact solutions for Eq.(14) take the following form:

$$u_{15}(\xi)=\frac{1}{3}(c^{\alpha}+\frac{1}{2}(1-2m^{2})+\frac{6}{4})+\frac{1}{2}[ns(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+cs(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{2}+\frac{1}{2}[ns(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+cs(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-2},$$

$$v_{15}(\xi)=-\frac{L_{0}}{6L_{3}^{2}}[2c^{\alpha}-\frac{1}{2}(1-2m^{2})+\frac{6}{4}]-\frac{1}{6L_{3}}[2c^{\alpha}-\frac{1}{2}(1-2m^{2})+\frac{6}{4}][ns(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+cs(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]$$

$$+\frac{1}{6L_{3}}(2c^{\alpha}-\frac{1}{2}(1-2m^{2})+\frac{6}{4})[ns(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+cs(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-1},$$

$$w_{15}(\xi)=L_{0}+L_{3}[ns(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+cs(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]+L_{3}[ns(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+cs(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-1}.$$

(38)

Family 7. $e_0=\frac{1}{4}(1-m^2)$, $e_1=\frac{1}{4}(1+m^2)$, and $e_2=\frac{1}{4}(1-m^2)$ the Jacobi elliptic exact solutions for Eq.(14) take the following form:

$$u_{16}(\xi)=\frac{1}{3}[c^{\alpha}+\frac{1}{4}(1+m^{2})+\frac{6}{4}(1-m^{2})]+\frac{1}{2}(1-m^{2})[nc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{2}$$
$$+\frac{1}{2}(1-m^{2})[nc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-2},$$
$$v_{16}(\xi)=-\frac{L_0(1-m^{2})}{6L_3^{2}}[2c^{\alpha}-\frac{1}{4}(1+m^{2})+\frac{6}{4}(1-m^{2})]-\frac{(1-m^{2})}{6L_3}[2c^{\alpha}-\frac{1}{4}(1+m^{2}) \qquad (39)$$
$$+\frac{6}{4}(1-m^{2})][nc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]+\frac{(1-m^{2})}{6L_3}[2c^{\alpha}-\frac{1}{4}(1+m^{2})$$
$$+\frac{6}{4}(1-m^{2})][nc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-1},$$
$$w_{16}(\xi)=L_0+L_3[nc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]+L_3[nc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-1}.$$

Family 8. $e_0=\frac{m^2}{4}$, $e_1=\frac{1}{4}(m^2-2)$, and $e_2=\frac{m^2}{4}$ the Jacobi elliptic exact solutions for Eq.(13) take the following form:

$$u_{17}(\xi)=\frac{1}{3}[c^{\alpha}+\frac{1}{4}(m^{2}-2)+\frac{6}{4}m^{2}]+\frac{1}{2}m^{2}[sn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+i\,cn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{2}$$
$$+\frac{1}{2}m^{2}[sn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+i\,cn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-2},$$
$$v_{17}(\xi)=-\frac{L_0m^{2}}{6L_3^{2}}[2c^{\alpha}-\frac{1}{4}(m^{2}-2)+\frac{6}{4}m^{2}]-\frac{m^{2}}{6L_3}[2c^{\alpha}-\frac{1}{4}(m^{2}-2)+\frac{6}{4}m^{2}][sn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)}) \qquad (40)$$
$$+i\,cn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]+\frac{m^{2}}{6L_3}[2c^{\alpha}-\frac{1}{4}(m^{2}-2)+\frac{6}{4}m^{2}][sn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+i\,cn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-1},$$
$$w_{17}(\xi)=L_0+L_3[sn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+i\,cn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]+L_3[sn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})+i\,cn(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-1}.$$

Family 9. $e_0=\frac{1}{4}$, $e_1=\frac{1}{2}(1-m^2)$, and $e_2=\frac{1}{4}$ the Jacobi elliptic exact solution for Eq.(13) takes the following form:

$$u_{18}(\xi)=\frac{1}{3}(c^{\alpha}+\frac{1}{2}(1-m^2)+\frac{6}{4})+\frac{1}{2}[\sqrt{1-m^2}\,sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})\pm dc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^2$$
$$+\frac{1}{2}[\sqrt{1-m^2}\,sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})\pm dc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-2},$$
$$v_{18}(\xi)=-\frac{L_0}{6L_3^2}[2c^{\alpha}-\frac{1}{2}(1-m^2)+\frac{6}{4}]-\frac{1}{6L_3}[2c^{\alpha}-\frac{1}{2}(1-m^2)+\frac{6}{4}][\sqrt{1-m^2}\,sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})\pm dc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]$$
$$+\frac{1}{6L_3}[2c^{\alpha}-\frac{1}{2}(1-m^2)+\frac{6}{4}][\sqrt{1-m^2}\,sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})\pm dc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-1},$$
$$w_{18}(\xi)=L_0+L_3[\sqrt{1-m^2}\,sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})\pm dc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]+L_3[\sqrt{1-m^2}\,sc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})\pm dc(\frac{(x+ct)^{\alpha}}{\Gamma(\alpha+1)})]^{-1},$$
(41)

Also , we can construct more of the exact Jacobi elliptic solutions for the case 2, we are omitted here for convenience to the reader.

4. Some conclusions and discussions

In this article, an algebraic direct method are used to find the exact solutions for nonlinear partial fractional differential equations. Successfully we have been obtained the analytical Jacobi elliptic solutions for some nonlinear partial fractional differential equations in mathematics physics. The reliability of this method and reduction in computations give this method a wider applicability**.** Algebraic direct method is powerful method for constructing many new type of Jacobi elliptic solutions for many nonlinear partial fractional differential equations in mathematical physics. Jacobi elliptic solutions are generalized the hyperbolic exact solutions and trigonometric exact solutions when the modulus m take some special values . This method is clearly a very efficient and powerful technique for finding the exact solutions for nonlinear partial fractional differential equations in mathematical physics. Maple and Mathematica have been used for computations in this paper.

5. **Conflict of interests**

The authors declare that there is no conflict of interests regarding the publication of this article.

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