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EXACT SOLUTIONS FOR OSCILLATING SPHERES IN GENERAL RELATIVITY

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Summary

A class of exact interior solutions is given for adiabatic spherically symmetric motion of a perfect fluid of uniform density but non-uniform pressure. This is matched to the Schwarzschild exterior solution at a moving boundary. The solutions include cases of oscillating motion in which both the pressure and the density are always positive, and the metric non-singular. Such cases are possible models for quasar oscillations, provided it is permissible to ignore radiation flux; the period is calculated in an example.

The Schwarzschild static interior solution is shown to be stable to perturbations in which the density remains uniform, provided the ratio mass/radius is not too great.

1. *Introduction.* A great many authors have written on spherically symmetric motions in relativistic hydrodynamics. Until quite recently, most of the work had been on pressure-free perfect fluids, though note should be taken of the pioneering paper of Wyman (1946) on motions with a pressure gradient. Within the last few years a number of papers have appeared taking account of the fluid pressure. These include papers on general theory (Misner & Sharp 1964; Hernandez & Misner 1966) and others on solutions, mostly with special reference to gravitational collapse (May & White 1966; McVittie 1964, 1967). We have derived particular benefit from the latter paper of McVittie, and the metric we present is a member of a class given in that paper.

Our attitude in this paper is rather different from that of most other recent authors. Our main interest is not in gravitational collapse, but *in the existence of non-static, physically plausible solutions which are non-singular for all time*. These might represent an oscillating star or quasar. If general relativity is a good physical theory, such solutions must exist. However, Hoyle & Narlikar (1964) have thought it necessary to invoke the *C*-field theory to obtain such oscillatory solutions.

We give a class of solutions for the radial adiabatic motion of perfect fluid spheres of uniform density but non-uniform pressure. In the class are included solutions of collapse, and also oscillatory solutions. It is easy enough to get oscillatory fields if one allows non-physical matter—for instance one can get oscillatory Robertson-Walker cosmological models by introducing a negative pressure. In our oscillatory solutions therefore we require that the density ρ and the pressure p satisfy

$$\rho > 0, \qquad (1.1)$$

$$p \geq 0; \qquad (1.2)$$

we shall also sometimes impose the more stringent condition

$$0 \leq p \leq \tfrac{1}{3}\rho, \qquad (1.3)$$

(in relativistic units). We also show that our interior solution joins correctly on to the Schwarzschild exterior solution of the surrounding empty space. In this task we were much assisted by the paper of Cocke (1966), referred to below.

We give the solutions in Section 2, and consider the physical interpretation in Section 3. In Section 4 we prove the existence of exact oscillatory solutions satisfying (1.1)–(1.3), and calculate the period in a special case. Section 5 is devoted to the theory of small oscillations of static Schwarzschild spheres, and Section 6 considers briefly a case of collapse. There is a summarizing section, Section 7, and an Appendix on the boundary conditions.

2. *The solutions*. We are concerned with the field equations

$$R_{ik}-\tfrac{1}{2}g_{ik}R = -8\pi T_{ik}, \tag{2.1}$$

$$T_{ik} = (p+\rho)v_i v_k - g_{ik}p, \tag{2.2}$$

g_{ik} and R_{ik} being the metric and Ricci tensors, T_{ik} the material energy tensor, and v_i the four-velocity of matter.

Let space-time be divided into an interior and an exterior region, denoted by $\mathscr{R}_i$ and $\mathscr{R}_e$ respectively, separated by a time-like hypersurface S. Let the coordinates in $\mathscr{R}_i$ be $x_{(i)}{}^m = r, \theta, \phi, t$ and those in $\mathscr{R}_e$ be $x_{(e)}{}^m = \bar{r}, \bar{\theta}, \bar{\phi}, \bar{t}$. The solutions are as follows.

Interior region $\mathscr{R}_i$. The line element is

$$ds^2 = -\alpha^2 F^{2\alpha+2} Y^{-2}(dr^2+r^2 d\theta^2+r^2\sin^2\theta\, d\phi^2)+Z^2 Y^{-2}\,dt^2 \tag{2.3}$$

$$(0\le r\le r_0,\ 0\le\theta\le\pi,\ 0\le\phi\le 2\pi,\ -\infty\le t\le\infty),$$

and the physical properties of the matter present are given by

$$8\pi\rho = \frac{6mY_0{}^3}{r_0{}^3\alpha^3F^{3\alpha+3}}, \tag{2.4}$$

$$8\pi p = \frac{6mY_0{}^2(\mu_0{}^2-\mu^2)}{r_0{}^3\alpha Z F^{2\alpha+3}}, \tag{2.5}$$

$$v^m = (0, 0, 0, YZ^{-1}), \tag{2.6}$$

where

$$\begin{aligned} Y &= F^\alpha - 1+\alpha(1+\mu^2),\\ Z &= F^\alpha - 1+\alpha^2+\alpha(1+\alpha)\mu^2,\\ \mu^2 &= r^2/4R^2, \end{aligned} \tag{2.7}$$

and a suffix zero means that r is to be put equal to r_0, e.g. $\mu_0{}^2 = r_0{}^2/4R^2$; m, r_0, R^2 and $\alpha(\neq 0)$ are real constants and $F(t)$ satisfies the two equations

$$\dot{F}^2 = \frac{2mY_0{}^3}{r_0{}^3\alpha^3F^{3\alpha+1}}-\frac{1}{\alpha R^2}\left[\frac{1}{F^\alpha}-\frac{(1-\alpha)}{F^{2\alpha}}\right], \tag{2.8}$$

$$\frac{Y_0}{Z_0}\left(\frac{2\dot{F}^2}{F^2}-\frac{2\ddot{F}}{F}\right)-\frac{3\dot{F}^2}{F^2} + \frac{(1-\alpha)\{\alpha^2-(F^\alpha-1)^2+\alpha\mu_0{}^2[1+\alpha-(1+2\alpha)(1-\alpha)^{-1}F^\alpha]\}}{\alpha R^2F^{2(\alpha+1)}Z_0} = 0, \tag{2.9}$$

where a dot means d/dt. Equation (2.9) is a consequence of (2.8) *unless* $\dot{F} \equiv 0$. The latter case, which gives a static solution, will be considered in Section 3. Elsewhere it will be sufficient to take account of equation (2.8) only.

Exterior region $\mathscr{R}_e$. The line element is that of Schwarzschild

$$ds^2 = -\left(1-\frac{2m}{\bar{r}}\right)^{-1} d\bar{r}^2 - \bar{r}^2(d\bar{\theta}^2 + \sin^2\bar{\theta}\, d\bar{\phi}^2) + \left(1-\frac{2m}{\bar{r}}\right) d\bar{t}^2 \tag{2.10}$$

$$(\zeta \leq \bar{r} \leq \infty,\ 0 \leq \bar{\theta} \leq \pi,\ 0 \leq \bar{\phi} \leq 2\pi,\ -\infty \leq \bar{t} \leq \infty)$$

and

$$p = \rho = 0; \tag{2.11}$$

ζ is defined by equation (2.13) below.

Bounding hypersurface S. This is described in the following way. S has two representations, $x_{(i)}{}^m = h_{(i)}{}^m(u^2, u^3, u^4)$ in $\mathscr{R}_i$ and $x_{(e)}{}^m = h_{(e)}{}^m(u^2, u^3, u^4)$ in $\mathscr{R}_e$, such that S is covered by the same domain of u^α ($\alpha = 2, 3, 4$) in both; and these representations are

$$x_{(i)}{}^1 = r_0,\ x_{(i)}{}^2 = u^2,\ x_{(i)}{}^3 = u^3,\ x_{(i)}{}^4 = u^4, \tag{2.12}$$

and

$$\left.\begin{aligned} x_{(e)}{}^1 &= \frac{\alpha r_0 F^{\alpha+1}}{Y_0} \overset{\text{def}}{=} \zeta,\ x_{(e)}{}^2 = u^2,\ x_{(e)}{}^3 = u^3, \\ x_{(e)}{}^4 &= \int \frac{[F^\alpha - 1 + \alpha(1-\mu_0{}^2)]\, Z_0}{\left(1 - \dfrac{2m}{r_0 \alpha F^{\alpha+1}}\, Y_0\right) Y_0{}^2}\, du^4, \end{aligned}\right\} \tag{2.13}$$

the argument of F being u^4 throughout. We now identify points on S which have $\theta = \bar{\theta}$, $\phi = \bar{\phi}$, and so take for the parameters

$$u^2 = \theta = \bar{\theta},\quad u^3 = \phi = \bar{\phi},\quad u^4 = t. \tag{2.14}$$

The boundary conditions to be satisfied on S are that the first and second fundamental forms of S, calculated from the metrics on the two sides of S, shall be identical (Cocke 1966). We show in the Appendix that these conditions are satisfied.

We may regard (2.13) as giving a parametric relation between $x_{(e)}{}^1$ and $x_{(e)}{}^4$ on S, that is to say between $\bar{r}$ and $\bar{t}$ on S. We may write this formally as

$$S : \bar{\bar{r}} = \zeta(\bar{t}), \tag{2.15}$$

which determines the lowest value assumed by the coordinate $\bar{r}$ in equation (2.10).

3. *Physical interpretation*. The exterior metric equation (2.10) is just the Schwarzschild line-element, so the complete solutions represent a finite spherically symmetric distribution. From the form of v^m in equation (2.6) it is clear that the coordinates $x_{(i)}{}^m$ are comoving; this accounts for the equation of the boundary S which from equation (2.12) may be written

$$S : r = r_0\ (\text{const.}) \tag{3.1}$$

Equation (2.5) shows that the pressure vanishes on S, as of course it must, physically. In Schwarzschild coordinates S does not generally reduce to $\bar{r}_0 = \text{const.}$, as is apparent from equation (2.13).

The density is independent of r but depends on t; the pressure depends on both r and t. For this reason the matter of the solution has in general no barytropic equation of state $p = f(\rho)$.

The solution reduces to the Schwarzschild interior and exterior solution if we make the specialization $F \equiv 1$. In this case both equations (2.8) and (2.9) must be satisfied separately and this leads to two relations between constants:

$$\alpha = \frac{1-2\mu_0^2}{1+\mu_0^2}, \qquad \frac{1}{R^2} = \frac{2m}{r_0^3}(1+\mu_0^2)^3. \tag{3.2}$$

The solution (2.3)–(2.5) now reduces to

$$ds^2 = -(1+\mu^2)^{-2}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2) + \frac{[(1-2\mu_0^2)+(2-\mu_0^2)\mu^2]^2}{(1+\mu^2)^2(1+\mu_0^2)^2}dt^2, \tag{3.3}$$

$$8\pi\rho = 3R^{-2}, \tag{3.4}$$

$$8\pi p = \frac{3(\mu_0^2-\mu^2)}{R^2[(1-2\mu_0^2)+(2-\mu_0^2)\mu^2]}, \tag{3.5}$$

which is the Schwarzschild interior solution in isotropic coordinates, the boundary being $r = r_0$, $\bar{r} = r_0(1+\mu_0^2)^{-1}$. We notice that if the pressure is to be positive and finite at $r = 0$ it follows from equation (3.5) that $2\mu_0^2 < 1$. Using this and the fact that μ_0^2 must be positive we obtain limits for α

$$0 < \alpha < 1. \tag{3.6}$$

It must be emphasized, however, that *these apply only in the static case.*

If we impose the more stringent physical condition (1.3) we find $\mu_0^2 < \frac{1}{5}$, and

$$\tfrac{1}{2} < \alpha < 1. \tag{3.7}$$

If F is not constant, the time dependence of the field is determined by equation (2.8). This equation has both oscillatory solutions and solutions corresponding to gravitational collapse, as we shall see. The constants m, r_0, α and R^2 are all arbitrary in this case, but m and r_0 must be positive for physical reasons. We shall also take α and R^2 positive, though possibly there may be some physically interesting solutions with one or both these quantities negative. The metric (2.3) is singular when $F = 0$ so this value of F must be excluded. Hence F must either be positive for all t or negative for all t. We shall investigate here only the case when it is positive. Hence we assume

$$m,\ r_0,\ \alpha,\ R^2,\ F(t) > 0. \tag{3.8}$$

Some inequalities arise when we try to satisfy the physical requirements (1.1)–(1.3). From equations (1.2) and (2.4) we see that Z must be positive, and applying this condition to equation (2.7) at $r = 0$ we require

$$F^\alpha > 1-\alpha^2, \text{ (all } t). \tag{3.9}$$

Inequality (1.1) requires $Y > 0$, which at $r = 0$ gives

$$F^\alpha > 1-\alpha, \text{ (all } t). \tag{3.10}$$

The third inequality (1.3) requires that F^α shall not take values between the roots of the equation

$$F^{2\alpha}+F^\alpha[\alpha^2+\alpha-2-\alpha\mu_0^2(3\alpha-1)]+(1-\alpha^2)[1-\alpha(1+\mu_0^2)] = 0. \tag{3.11}$$

We notice from equation (2.13) that the sign of $dx_{(e)}{}^4/du^4$ may change. It may be desirable to impose conditions on F in order to prevent this, but the case does not seem to us overwhelming. One can however say from equation (2.13) that if Y_0 or the Schwarzschild factor $1-(2m/\bar{r}_0)$ vanishes the parametrization of S becomes improper, and this constitutes a singularity. In the following sections we shall give some particular solutions which are physically reasonable in that their metric is non-singular and in that they satisfy equations (1.1)–(1.3): it is easy to show—though we do not do so here—that in these solutions $dx_{(e)}{}^4/du^4$ is positive and bounded for all u^4.

4. *Exact oscillatory solutions.* The character of the exact solutions can be obtained by examining equation (2.8). This may conveniently be done by studying the graph of

$$\dot{F}^2R^2 \overset{\text{def}}{=} Q = \frac{2mR^2}{r_0{}^3}\frac{Y_0{}^3}{F^{3\alpha+1}\alpha^3} - \frac{1}{\alpha}\left[\frac{1}{F^\alpha} - \frac{(1-\alpha)}{F^{2\alpha}}\right] \tag{4.1}$$

against F, for some definite values of the constants. To illustrate this we first insert the constants of a particular Schwarzschild interior solution, namely

$$\alpha = \frac{1}{2}, \quad \frac{r_0{}^2}{R^2} = \frac{4}{5}, \quad m = \frac{\sqrt{125}}{54}R\ (\doteq 0\cdot 2070\, R), \tag{4.2}$$

and study the corresponding graph of

$$Q_1 = \frac{125}{27\,F^{5/2}}\left(F^{1/2} - \frac{2}{5}\right)^3 - \left(\frac{2}{F^{1/2}} - \frac{1}{F}\right) \tag{4.3}$$

against F. Confining attention to $F > 0$ in accordance with equation (3.8) we find Fig. 1.

The Schwarzschild interior solution corresponds to $F \equiv 1$, and is seen to be the point at which the curve touches the F axis. For points below the F axis the curve has no physical meaning because $\dot{F}$ would be imaginary at them. Oscillation is possible between the values of F at A and B. In fact however the sphere would be non-physical in these oscillations because equation (3.9) is violated, so that there would be a region of negative pressure.

Oscillations about the Schwarzschild equilibrium position are impossible because $R^2\dot{F}^2$ is negative in the neighbourhood $F \equiv 1$, as illustrated in Fig. 1. This result is generally true: that is to say, if one takes equation (2.8), writes

$$F = 1 + \epsilon(t), \tag{4.4}$$

and chooses any set of constants satisfying equations (3.2) and (3.8), then one finds

$$R^2\dot{F}^2 = -\frac{(1-\alpha)(1+2\alpha)}{6}\,\epsilon^2 + o(\epsilon^3). \tag{4.5}$$

Since for any Schwarzschild interior solution we must have equation (3.6), the statement is proved.

To obtain oscillating solutions it is necessary to choose the constants so that they no longer satisfy the Schwarzschild relations (3.2). One such solution is

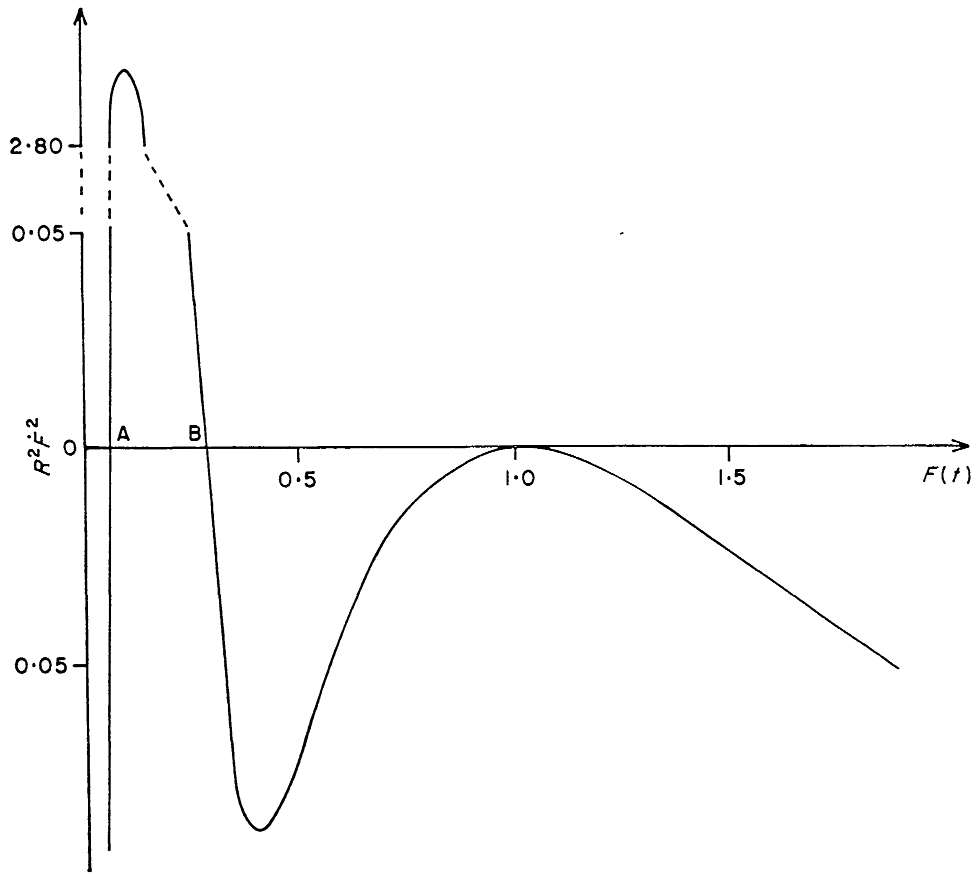

FIG. 1. *Plot of $R^2\dot{F}^2$ against F when α, r_0, R and m are the constants of a particular Schwarzschild solution. For larger values of F than those shown $R^2\dot{F}^2$ reaches a minimum, and tends to zero from below as $F \to \infty$.*

illustrated in Fig. 2 which plots the relation (4.1) for a case quite close to the Schwarzschild values in equation (4.2), namely

$$\alpha = \frac{1}{2}, \quad \frac{r_0^2}{R^2} = \frac{4}{5}, \quad m = 0\cdot 2075\, R. \tag{4.6}$$

An oscillation of F between A and B is non-physical because equation (3.9) is violated, but in the oscillation between C and D both equations (1.1) and (1.2) are satisfied. Equation (1.3) is not satisfied throughout the oscillations because one root of equation (3.11) is in fact $F^\alpha = 1$.

A model satisfying all three physical requirements (1.1)–(1.3) *is obtained by choosing*

$$\alpha = 0\cdot 7, \quad \frac{r_0^2}{R^2} = \frac{4}{9}, \quad m = 0\cdot 1084\, R. \tag{4.7}$$

This is illustrated in Fig. 3. The period T of the physical oscillation CD can be calculated by the numerical integration of equation (2.8) with constants (4.7) inserted. The result to three significant figures is

$$T = 17\cdot 4\, R. \tag{4.8}$$

The physical constants m, r_0 and T are best expressed in terms of the mean

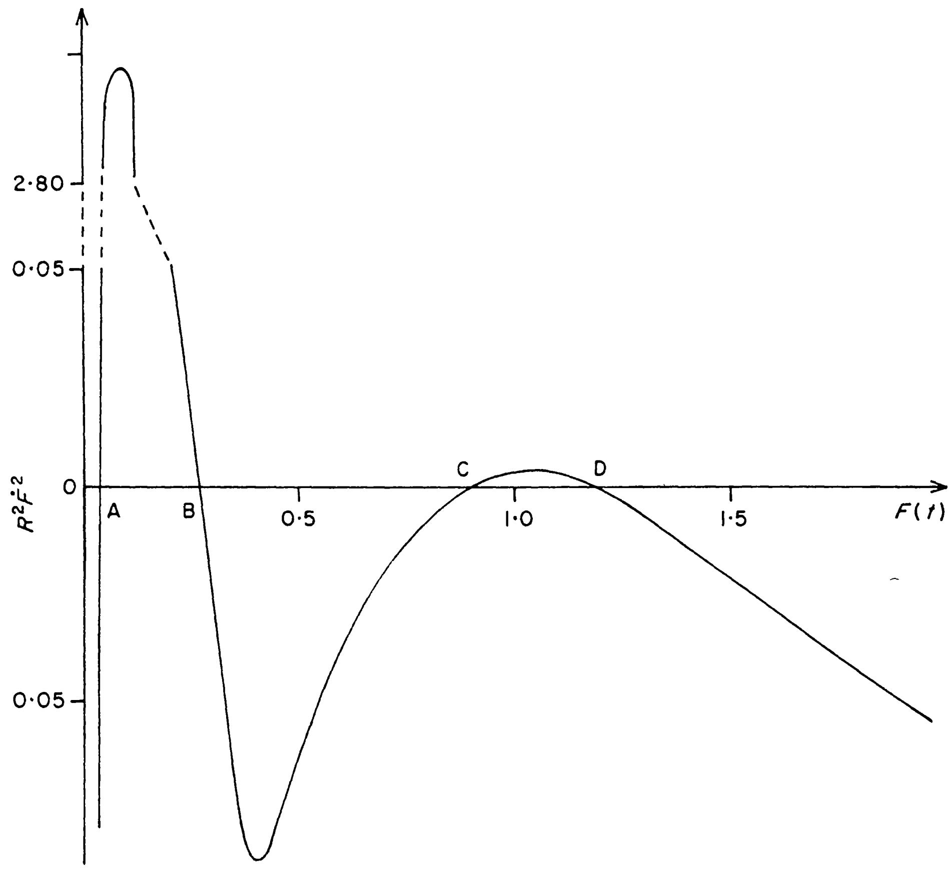

FIG. 2. *The constants* α, r_0, *R and m assume the values* (4.6). *As in Fig.* 1, $R^2\dot{F}^2$ *achieves a second minimum and tends to zero from below as* $F \to \infty$. *The region CD represents an oscillation satisfying equations* (1.1) *and* (1.2).

density over a cycle. Substituting the figures (4.7) into (2.4) and using equation (2.7) we find to three decimal places

$$8\pi\rho = \frac{6\cdot400}{F^{5\cdot1}R^2}(F^{0\cdot7}-0\cdot222)^3. \tag{4.9}$$

To save calculations we estimate the mean density by taking the average of the extreme values which it assumes when F assumes its greatest and least values. The latter are, to two decimal places

$$F_{\text{max}} = 1\cdot20 \qquad F_{\text{min}} = 0\cdot84,$$

and the corresponding values for ρ are

$$\rho_{\text{min}} = 0\cdot077\,R^{-2} \qquad \rho_{\text{max}} = 0\cdot18\,R^{-2},$$

so

$$\bar{\rho} \overset{\text{def}}{=} \tfrac{1}{2}(\rho_{\text{max}}+\rho_{\text{min}}) = 0\cdot13\,R^{-2}. \tag{4.10}$$

We now have from equation (4.7) and (4.10)

$$r_0 = \frac{0\cdot24}{\sqrt{\bar{\rho}}}, \qquad m = \frac{0\cdot039}{\sqrt{\bar{\rho}}}.$$

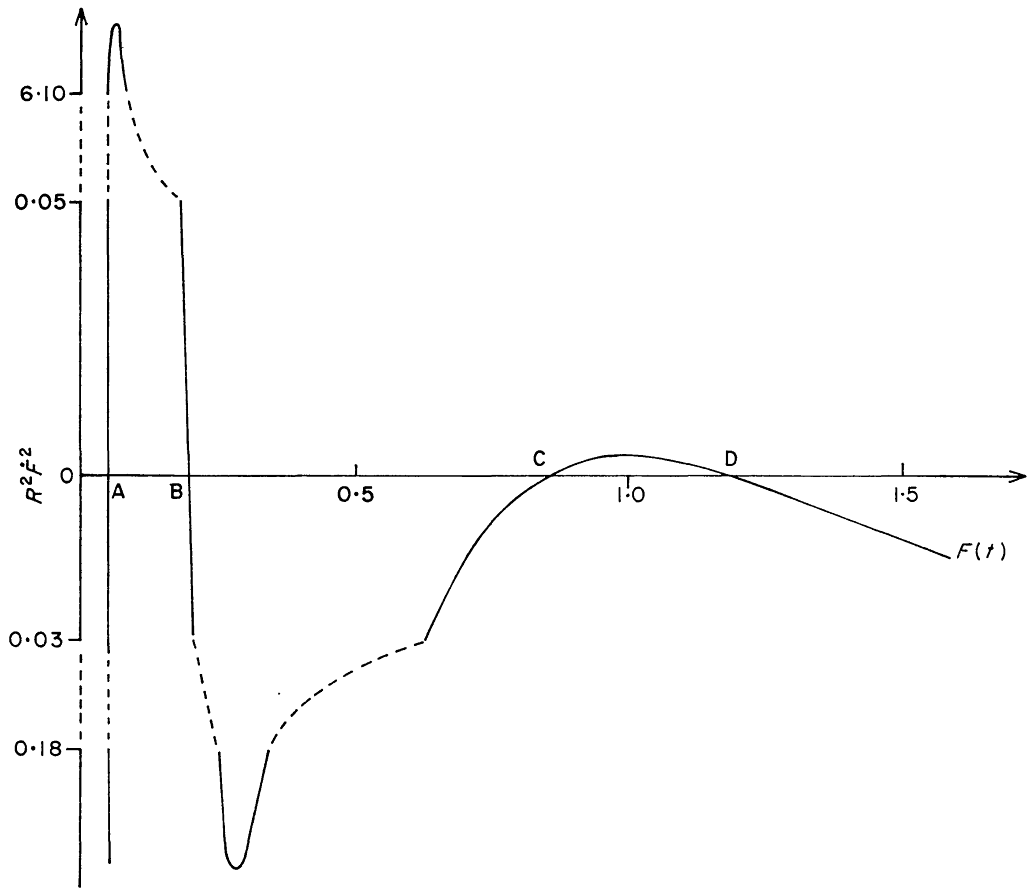

FIG. 3. *The values of the constants are as in equation* (4.7). *AB represents a non-physical oscillation. CD represents an oscillation satisfying equation* (1.1)–(1.3).

These expressions are in relativistic units; if we now write them in units with customary dimensions we have

$$r_0 = \frac{0 \cdot 24\, c}{\sqrt{(G\bar{\rho})}}, \qquad m = \frac{0 \cdot 039\, c^3}{\sqrt{(G^3\bar{\rho})}} \tag{4.11}$$

c being the velocity of light and G the gravitational constant. In dimensional units the period (4.8) is (to 1 decimal place)

$$T = \frac{6 \cdot 2}{\sqrt{(G\bar{\rho})}}. \tag{4.12}$$

This is about the same as the period for the fundamental vibration of a homogeneous star of density $\bar{\rho}$ in classical theory (Rosseland 1949, Chap 3). The solution is not of much use as a model of a star because if we take $\bar{\rho} = 1$ g cm^{-3}, we find

$$r_0 = 2 \cdot 8 \times 10^{13}\,\text{cm}, \quad m = 6 \cdot 1 \times 10^{40}\,\text{g},$$

which are much too large. On the other hand if we put $\bar{\rho} = 10^{-8}$ g cm^{-3} we find

$$r_0 = 2 \cdot 8 \times 10^{17}\,\text{cm}, \quad m = 6 \cdot 1 \times 10^{44}\,\text{g}, \quad T = 7 \cdot 6\,\text{yr}. \tag{4.13}$$

This could represent the oscillations of a very dense and massive quasar, provided it is permissible to neglect the quasar's energy flux. Another model is given in the next section.

5. *Small oscillations.* As pointed out in the previous section, oscillations about the Schwarzschild equilibrium position are impossible. However, as we have seen, oscillating solutions may be obtained by giving the constants α, μ_0 and m/R values slightly different from their Schwarzschild values, i.e. from the values they have when equation (3.2) are satisfied. If one chooses values infinitesimally different from the Schwarzschild values, one obtains small oscillations with a period independent of the infinitesimal variations, but dependent on the Schwarzschild constants α and R. We shall call these solutions *perturbed Schwarzschild solutions.*

To investigate perturbed Schwarzschild solutions we write in place of equation (3.2)

$$F(t) = 1+\epsilon(t), \quad \frac{2m}{r_0^3}(1+\mu_0^2)^3 = \frac{1+\eta}{R^2}, \quad \alpha = \frac{1-2\mu_0^2}{1+\mu_0^2}+\eta^*, \tag{5.1}$$

where η and η^* are small constants and ϵ is a small function of t. We now substitute equations (5.1) into equation (2.8). Neglecting cubic terms in small quantities we obtain

$$R^2\dot{\epsilon}^2 = \eta+[\epsilon\eta(1-2\alpha)-\epsilon\eta^*]-\tfrac{1}{6}\epsilon^2(1-\alpha)(1+2\alpha). \tag{5.2}$$

Taking ϵ and $\dot{\epsilon}$ to be small quantities of the first order, we see that η must be of the second order so the first term in the square bracket can be neglected as it is of the third order. The equation for small motions is therefore

$$R^2\dot{\epsilon}^2 = \eta-\epsilon\eta^*-\tfrac{1}{6}\,\epsilon^2(1-\alpha)(1+2\alpha). \tag{5.3}$$

Introducing a new dependent variable $\bar{\epsilon}(t)$ by

$$\bar{\epsilon} = \epsilon+\frac{3\eta^*}{(1-\alpha)(1+2\alpha)} \tag{5.4}$$

we obtain from equation (5.3)

$$R^2\dot{\bar{\epsilon}}^2 = \eta+\frac{(1-\alpha)(1+2\alpha)}{6}\left(\frac{9\eta^{*2}}{(1-\alpha)^2(1+2\alpha)^2}-\bar{\epsilon}^2\right). \tag{5.5}$$

Suppose that the perturbations η and η^* to the Schwarzschild constants satisfy

$$\eta+\tfrac{3}{2}(1-\alpha)^{-1}(1+2\alpha)^{-1}\eta^{*2}>0,$$

and recall from equation (3.6) that for a Schwarzschild solution $0<\alpha<1$. Then *it follows that the Schwarzschild sphere is stable to perturbations* (5.1), *and small vibrations have period*

$$T = \frac{2\pi\sqrt{6}\,R}{\sqrt{(1-\alpha)(1+2\alpha)}}. \tag{5.6}$$

In dimensional units this may be written

$$T = \sqrt{\frac{9\pi}{(1-\alpha)(1+2\alpha)G\,\rho_e}}; \tag{5.7}$$

here ρ_e is the equilibrium density of the corresponding Schwarzschild solution, given by equation (3.4). We can use equations (3.2) and (3.4) to express m and r_0 in terms of α and ρ_e:

$$m = \frac{\sqrt{2}c^3(1-\alpha)^{3/2}(2+\alpha)^{3/2}}{9\sqrt{3\pi G^3\rho_e}}, \qquad r_0 = \sqrt{\frac{3c^2(1-\alpha)}{2\pi G\,\rho_e(2+\alpha)}}. \tag{5.8}$$

A model for the oscillations of a quasar (once again neglecting flux of energy), rather similar to that of Section 4, is obtained by taking $\alpha = 0{\cdot}9$ and $\rho_e = 10^{-8}$ g cm^{-3}, which gives

$$m = 1{\cdot}2 \times 10^{44}\,\mathrm{g}, \quad r_0 = 1{\cdot}5 \times 10^{17}\,\mathrm{cm}, \quad T = 12\,\mathrm{yr}. \tag{5.9}$$

Stability to the perturbations (5.1) has been proved for $0 < \alpha < 1$. These limits correspond with

$$0 < \frac{2m}{r_0} < \frac{16}{27}, \text{ or } 0 < \frac{2m}{\zeta} < \frac{8}{9},$$

ζ being the lower limit of the exterior Schwarzschild coordinate $\bar{r}$ in (2.10). This result is consistent with that of Chandrasekhar (1964) who found instability for $\zeta < 9m/4$.

6. *A case of collapse to zero volume.* If we take

$$\alpha = 1, \qquad \frac{r_0^2}{R^2} = 2, \qquad m = \tfrac{1}{2}\sqrt{2}\,R, \tag{6.1}$$

equation (2.8) becomes

$$R^2\dot{F}^2 = \frac{(F+\frac{1}{2})^3 - 2F^3}{2F^4}. \tag{6.2}$$

The graph of $R^2\dot{F}^2$ against F is shown in Fig. 4. From the graph it is clear that F will increase from zero to a maximum given by the point A, and then decrease

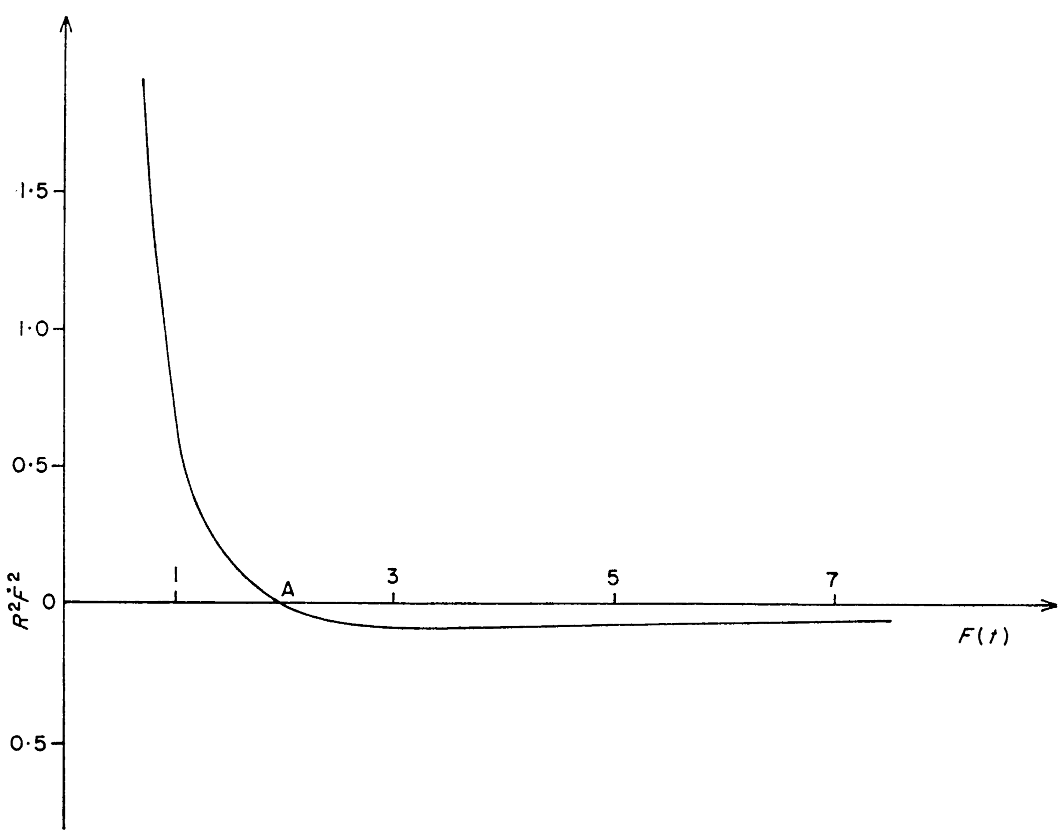

FIG. 4. *A case of collapse.*

to zero again. This corresponds to a sphere which expands from zero proper volume and infinite density to a maximum proper radius, and then collapses to zero proper volume again. From the form of $\dot{F}$ for small F, namely $|\dot{F}| \sim \frac{1}{4}F^{-2}$, it is clear that the entire process occupies a finite time. The model satisfies equations (1.1) and (1.2) throughout its life, but equation (1.3) is satisfied for only part of the time. Of course it becomes non-physical also as F tends to zero. The parametrization of the boundary S becomes singular (for the reason described at the end of Section 3) when $F = \frac{1}{2}(1+\sqrt{3})$.

7. *Conclusion.* The main result of the paper is to exhibit exact solutions of the field equations referring to oscillating spheres of perfect fluid. These are physically realistic in that they can be chosen to satisfy equations (1.1)–(1.3), though the actual material of the spheres is unsatisfactory in that ρ is uniform but p is not. Since we are here interested more in a mechanical point of view than a thermodynamic one, the conditions (1.1)–(1.2) seem more important than the precise fulfilment of a realistic equation of state. They ensure that neither the gravitational nor the inertial mass density of the matter present is negative.

The periods calculated in Sections 4 and 5 offer some hope of explaining the oscillations of quasars within existing theory.

Since the bulk of this paper was written, a most interesting generalization of the interior solutions in Section 2 has been published by Thompson & Whitrow (1967). These authors are interested only in collapsing spheres, however.

To get our interior solution from that of Thompson & Whitrow we first put their $f = r^{-1}$, and write their solution in the slightly different notation:

$$ds^2 = -(B+Cr^2)^{-2}[dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2]+\left[\frac{1+r^2\dfrac{dC}{d\tau}\left(\dfrac{dB}{d\tau}\right)^{-1}}{1+r^2\dfrac{C}{B}}\right]^2 d\tau^2. \quad (7.1)$$

We now submit our interior solution (2.3) to the coordinate transformation

$$t = \phi(\tau) \text{ where } \frac{d\phi}{d\tau} = \frac{[F(t)]^\alpha - 1+\alpha}{[F(t)]^\alpha - 1+\alpha^2} \quad (7.2)$$

which brings it into the form

$$ds^2 = -\left\{\frac{\{F[\phi(\tau)]\}^\alpha - 1+\alpha}{\alpha\{F[\phi(\tau)]\}^{\alpha+1}}+\frac{r^2}{4R^2\{F[\phi(\tau)]\}^{\alpha+1}}\right\}^{-2} \times\{dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2\}$$
$$+\left[\frac{1+\dfrac{r^2}{4R^2}\dfrac{\alpha(1+\alpha)}{\{F[\phi(\tau)]\}^\alpha-1+\alpha^2}}{1+\dfrac{r^2}{4R^2}\dfrac{\alpha}{\{F[\phi(\tau)]\}^\alpha-1+\alpha}}\right]^2 d\tau^2. \quad (7.3)$$

In this form our solution satisfies the condition

$$g_{44}(0, \tau) = 1$$

imposed by Thompson & Whitrow. The two solutions (7.1) and (7.3) now become identical if we put

$$B(\tau) = \frac{\{F[\phi(\tau)]\}^{\alpha}-1+\alpha}{\alpha\{F[\phi(\tau)]\}^{\alpha+1}}, \qquad C(\tau) = \frac{1}{4R^2\{F[\phi(\tau)]\}^{\alpha+1}}.$$

Our equation (2.8) then corresponds with their equation (34) with $f(r_s) = r_0^{-1}$ and $K^2 = 2m\,r_0^3$.

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Appendix

The boundary conditions. We follow the procedure of Cocke (1966). The conditions to be satisfied are that the first and second fundamental forms of S, calculated from the metrics on the two sides of S, shall be identical.

The first fundamental form of S, calculated from the inside metric (2.3) by inserting (2.12) is

$$d\sigma^2 = -\alpha^2F^{2\alpha+2}Y_0^{-2}r_0^2((du^2)^2+\sin^2(u^2)(du^3)^2)+Z_0^2Y_0^{-2}(du^4)^2. \qquad (A.1)$$

To calculate $d\sigma^2$ from the outside metric (2.10) we must substitute equation (2.13), and remember that F and Y_0 are functions of u^4. A fairly long but straightforward calculation, in which one must use equation (2.8) with u^4 replacing t, then reduces equation (2.10) to equation (A.1). *The condition on the first fundamental form of S is therefore satisfied.*

The coefficients of the second fundamental form of S are given by

$$\Omega_{\mu\nu} = -n_{a;\,b}\frac{\partial x^a}{\partial u^\mu}\frac{\partial x^b}{\partial u^\nu}. \qquad (A.2)$$

where a, b run from 1 to 4 and μ, ν from 2 to 4; n_a is the unit normal to S and; denotes covariant differentiation with respect to the four dimensional space; x^a means either $x_{(i)}{}^a$ or $x_{(e)}{}^a$ given by equations (2.12) and (2.13). (Eisenhart 1949). The normal is calculated as follows. S may be written in either of the forms

$$f_{(i)}(x_{(i)}{}^a) \equiv x_{(i)}{}^1 - r_0 = 0, \qquad (A.3)$$

$$f_{(e)}(x_{(e)}{}^a) \equiv x_{(e)}{}^1 - \zeta(x_{(e)}{}^4) = 0, \qquad (A.4)$$

equation (A.4) being the relation (2.15). In either case the unit normal is got from the formula

$$n_a = f,_a |g^{bc} f,_b f,_c|^{-1/2}, \tag{A.5}$$

and we find

$$n_{(i)a} = [\alpha F^{(\alpha+1)} Y^{-1}, 0, 0, 0], \tag{A.6}$$

$$n_{(e)a} = \left[\frac{1, 0, 0, -\zeta_4}{|g_{(e)}{}^{11} + g_{(e)}{}^{44} \zeta_4{}^2|^{1/2}}\right] \tag{A.7}$$

where ζ_4 means $\partial\zeta/\partial x_{(e)}{}^4$. On S we may write (A.7) as a function of u^4 only, using (2.13):

$$n_{(e)a} = \left\{\frac{F^\alpha - 1 + \alpha(1-\mu_0{}^2)}{\left(1 - \dfrac{2mY_0}{r_0 \alpha F^{\alpha+1}}\right) Y_0}, 0, 0, -\frac{Y_0}{Z_0}\frac{d}{du^4}\left(\frac{\alpha r_0 F^{\alpha+1}}{Y_0}\right)\right\}. \tag{A.8}$$

However, in the partial differentiations of n_a contained in equation (A.2) the expressions (A.6) and (A.7) must be used.

We are now in a position to calculate the $\Omega_{\mu\nu}$. For the interior metric this is a straight-forward matter of substituting equation (A.6) into equation (A.2) and using equation (2.12). The non-zero $\Omega_{(i)\mu\nu}$ on S are

$$\left.\begin{aligned}\Omega_{(i)22} &= -\alpha F^{\alpha+1} Y_0{}^{-2} r_0 [F^\alpha - 1 + \alpha(1-\mu_0{}^2)] = \Omega_{(i)33} \operatorname{cosec}^2 \theta,\\ \Omega_{(i)44} &= \alpha r_0 Z/(2R^2 F Y_0{}^2),\end{aligned}\right\} \tag{A.9}$$

where the argument of F is u^4. For the exterior metric the calculation of the $\Omega_{(e)\mu\nu}$ is long and tedious. One starts with the components of n_i in form (A.7) and first calculate $\Omega_{(e)\mu\nu}$ in the coordinates $x_{(e)}{}^a$. It is then necessary to substitute from equation (2.13), and to express all derivatives with respect to $x_{(e)}{}^4$ in terms of derivatives with respect to u^4, e.g.

$$\frac{d\zeta}{dx_{(e)}{}^4} = \frac{d\zeta}{du^4}\Big/\frac{dx_{(e)}{}^4}{du^4} = \frac{[F^\alpha - 1 + \alpha(1-\mu_0{}^2)]\, Z_0}{\left(1 - \dfrac{2mY_0}{r\alpha F^{\alpha+1}}\right) Y_0{}^2}\frac{d\zeta}{du^4}.$$

Eventually, after use of equation (2.8), one finds that the $\Omega_{(e)\mu\nu}$ reduce to (A.9), so that *the second fundamental forms are identical on both sides of S.*

This completes the proof that the solution satisfies the boundary conditions.