International Scholarly Research Network
ISRN Mathematical Physics
Volume 2012, Article ID 209678, 15 pages
doi:10.5402/2012/209678

Research Article

Exact Solutions for the Axial Couette Flow of a Fractional Maxwell Fluid in an Annulus

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Received 10 October 2011; Accepted 3 November 2011

Academic Editors: M. Rasetti, A. Sanyal, and G. F. Torres del Castillo

The velocity field and the adequate shear stress corresponding to the rotational flow of a fractional Maxwell fluid, between two infinite coaxial circular cylinders, are determined by applying the Laplace and finite Hankel transforms. The solutions that have been obtained are presented in terms of generalized $G_{a,b,c}(\cdot,t)$ and $R_{a,b}(\cdot,t)$ functions. Moreover, these solutions satisfy both the governing differential equations and all imposed initial and boundary conditions. The corresponding solutions for ordinary Maxwell and Newtonian fluids are obtained as limiting cases of our general solutions. Finally, the influence of the material parameters on the velocity and shear stress of the fluid is analyzed by graphical illustrations.

1. Introduction

Due to the several technological applications, the flow analysis of non-Newtonian fluids is very important in the fields of fluid mechanics. Many investigators have not studied the flow behavior of non-Newtonian fluids in various flow fields due to the complex stress-strain relationship [1]. The study of non-Newtonian fluids has got much attention because of their practical applications. Non-Newtonian characteristics are displayed by a number of industrially important fluids including polymers, molten plastic, pulps, microfluids, and food stuff display. Exact analytic solutions for the flows of non-Newtonian fluids are important provided they correspond to physically realistic problems, and they can be used as checks against complicated numerical codes that have been developed for much more complex flows. Many non-Newtonian models such as differential type, rate type, and integral type fluids have been proposed in recent years. Among them, the rate type fluid models have received special attention. The differential type fluids do not predict stress relaxation, and they are not successful for describing the flows of some polymers.

The flow between rotating cylinders or through a rotating cylinder has applications in the food industry, it being one of the most important and interesting problems of motion near rotating bodies. As early as 1886, Stokes [2] established an exact solution for the rotational oscillations of an infinite rod immersed in a linearly viscous fluid. However, such motions have been intensively studied since G. I. Taylor (1923) reported the results of his famous investigations [3]. For Newtonian fluids, the velocity distribution for a fluid contained in a circular cylinder can be found in [4]. The first exact solutions corresponding to different motions of non-Newtonian fluids, in cylindrical domains, seem to be those of Ting [5], Srivastava [6], and Waters and King [7]. A lot of interests and studies were also given to the unidirectional start-up pipe flows, which has a significant practical and mathematical meaning. Zhu et al. [8] studied the characteristics of the velocity filed and the shear stress field for an ordinary Maxwell fluid, and Yang and Zhu [9] studied it for a fractional Maxwell fluid. In the last decade, the unidirectional flow of viscoelastic fluid with the fractional Maxwell model was studied by Tan et al. [10, 11] and Hayat et al. [12]. Tong et al. [13, 14] discussed the unsteady flow with a generalized Jeffrey's model in an annular pipe. In the meantime, a lot of papers regarding such motions have been published. The interested readers can see for instance the papers [15–25] and their related references.

The purpose of this paper is to provide exact solutions of the velocity field and the shear stress corresponding to the motion of a fractional Maxwell fluid between two infinite circular cylinders. The Laplace and finite Hankel transforms are used to solve the problem, and the solutions obtained are presented in terms of generalized $G_{a,b,c}(\cdot,t)$ and $R_{a,b}(\cdot,t)$ functions. The solutions for ordinary Maxwell and Newtonian fluids are obtained as limiting cases of our general solutions. Furthermore, the solutions for the motion between the cylinders, when one of them is at rest, are also obtained as special cases from our general results. At the end, obtained solutions are discussed graphically for different values of time and material parameters.

2. Basic Governing Equations

The constitutive equations of an incompressible Maxwell fluid with fractional calculus are given by [14]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad \mathbf{S} + \lambda\frac{D\mathbf{S}}{Dt} = \mu\mathbf{A}, \tag{2.1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $-p\mathbf{I}$ denotes the indeterminate spherical stress, $\mathbf{S}$ is the extrastress tensor, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ with $\mathbf{L}$ the velocity gradient, μ is the dynamic viscosity of the fluid, λ is the material constant called relaxation time, and $D\mathbf{S}/Dt$ is defined by

$$\frac{D\mathbf{S}}{Dt} = D_t^{\beta}\mathbf{S} + \mathbf{w}\cdot\nabla\mathbf{S} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T. \tag{2.2}$$

Here, $\mathbf{w}$ is the velocity vector, ∇ is the gradient operator, the superscript T denotes the transpose operation, and the Caputo fractional derivative operator D_t^{β} is defined as [26]

$$D_t^{\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\beta}} d\tau; \quad 0 \leq \beta < 1, \tag{2.3}$$

where $\Gamma(\cdot)$ is the Gamma function which is defined as

$$\Gamma(x) = \int_0^{\infty} s^{x-1} e^{-s} ds; \quad x > 0. \tag{2.4}$$

This model can be reduced to ordinary Maxwell model when $\beta \to 1$, because in this case $D_t^{\beta} f(t) \to df(t)/dt$. Furthermore, this model reduces to the classical Newtonian model for $\beta \to 1$ and $\lambda \to 0$.

In cylindrical coordinates (r, θ, z), the rotational flow velocity is given by

$$\mathbf{w} = \mathbf{w}(r,t) = w(r,t)\mathbf{e}_{\theta}, \tag{2.5}$$

where $\mathbf{e}_{\theta}$ is the unit vector in the θ-direction. For such flows, the constraint of incompressibility is automatically satisfied. Since the velocity field $\mathbf{w}$ is independent of θ and z, we also assume that $\mathbf{S}$ depends only on r and t. Furthermore, if the fluid is assumed to be at rest at the moment $t = 0$, then

$$\mathbf{w}(r,0) = \mathbf{0}, \qquad \mathbf{S}(r,0) = \mathbf{0}. \tag{2.6}$$

Equations (2.1), (2.5), and (2.6) imply $S_{rr} = S_{zz} = S_{\theta z} = 0$ [18],

$$\left(1 + \lambda D_t^{\beta}\right) \tau(r,t) = \mu \left(\frac{\partial}{\partial r} - \frac{1}{r}\right) w(r,t), \tag{2.7}$$

where $\tau(r,t) = S_{r\theta}(r,t)$ is the nontrivial shear stress. In the absence of body forces and a pressure gradient in the axial direction, the equations of motion lead to the relevant equation

$$\rho \frac{\partial w(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right) \tau(r,t), \tag{2.8}$$

where ρ is the constant density of the fluid. Eliminating τ between (2.7) and (2.8), we attain to the governing equation

$$\left(1 + \lambda D_t^{\beta}\right) \frac{\partial w(r,t)}{\partial t} = \nu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) w(r,t), \tag{2.9}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid. In the following, the fractional partial differential equations (2.9) and (2.7), with appropriate initial and boundary conditions, will be solved by means of Laplace and finite Hankel transforms. In order to avoid lengthy calculations of residues and contours integrals, the discrete inverse Laplace method will be used [13, 14].

3. Axial Couette Flow between Two Infinite Circular Cylinders

Let us consider an incompressible fractional Maxwell fluid at rest in an annular region between two coaxial circular cylinders of radii R_1 and $R_2(> R_1)$. At time $t = 0^+$, both cylinders with radii R_1 and R_2 begin to rotate along their common axis. Owing to the shear, the fluid is gradually moved, its velocity being of the form (2.5). The governing equations are given by (2.9), while the appropriate initial and boundary conditions are

$$w(r,0) = \frac{\partial w(r,0)}{\partial t} = 0, \quad \tau(r,0) = 0; \quad r \in [R_1, R_2], \tag{3.1}$$

$$w(R_1,t) = \Omega_1 R_1 t, \quad w(R_2,t) = \Omega_2 R_2 t \quad \text{for } t \geq 0, \tag{3.2}$$

where Ω_1 and Ω_2 are constants with dimensions T^{-2}.

3.1. *Calculation of the Velocity Field*

Applying the Laplace transform to (2.9), using the Laplace transform formula for sequential fractional derivatives [26], and having the initial and boundary conditions (3.1) and (3.2) in mind, we find that

$$\left(q + \lambda q^{\beta+1}\right)\overline{w}(r,q) = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\overline{w}(r,q); \quad r \in [R_1, R_2], \tag{3.3}$$

where $\overline{w}(r,q)$ is the Laplace transform of the function $w(r,t)$ which is defined as

$$\overline{w}(r,q) = \mathcal{L}\{w(r,t)\} = \int_0^\infty e^{-qt} w(r,t)dt, \tag{3.4}$$

and the image function $\overline{w}(r,q)$ has to satisfy the conditions

$$\overline{w}(R_1,q) = \frac{\Omega_1 R_1}{q^2}, \qquad \overline{w}(R_2,q) = \frac{\Omega_2 R_2}{q^2}. \tag{3.5}$$

In the following, we denote by [27]

$$\overline{w}_H(r_n,q) = \int_{R_1}^{R_2} r\overline{w}(r,q)B(r,r_n)dr, \tag{3.6}$$

and the Hankel transform of $\overline{w}(r,q)$, where

$$B(r,r_n) = J_1(rr_n)Y_1(R_2r_n) - J_1(R_2r_n)Y_1(rr_n), \tag{3.7}$$

and r_n are the positive roots of the transcendental equation $B(R_1, r) = 0$, while $J_1(\cdot)$ and $Y_1(\cdot)$ are Bessel functions of the first and second kind of order one.

Multiplying both sides of (3.3) by $rB(r, r_n)$, integrating with respect to r from R_1 to R_2, and taking into account the conditions (3.5) and the identity

$$\int_{R_1}^{R_2} r\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right]\overline{w}(r,q)B(r,r_n)dr = \frac{2}{\pi q^2}\left[\frac{\Omega_2 R_2 J_1(R_1 r_n)-\Omega_1 R_1 J_1(R_2 r_n)}{J_1(R_1 r_n)}\right] - r_n^2\overline{w}_H(r_n,q), \tag{3.8}$$

we find that

$$\overline{w}_H(r_n,q) = \frac{2\nu[\Omega_2 R_2 J_1(R_1 r_n)-\Omega_1 R_1 J_1(R_2 r_n)]}{\pi J_1(R_1 r_n)}\frac{1}{q^2(\lambda q^{\beta+1}+q+\nu r_n^2)}. \tag{3.9}$$

Now, for a suitable presentation of the final results, we rewrite (3.9) in the following equivalent form:

$$\overline{w}_H(r_n,q) = \frac{2[\Omega_2 R_2 J_1(R_1 r_n)-\Omega_1 R_1 J_1(R_2 r_n)]}{\pi r_n^2 J_1(R_1 r_n)}\left[\frac{1}{q^2}-\frac{1+\lambda q^\beta}{q(\lambda q^{\beta+1}+q+\nu r_n^2)}\right]. \tag{3.10}$$

Now, applying the inverse Hankel transform formula [27]

$$\overline{w}(r,q) = \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2 J_1^2(R_1 r_n)B(r,r_n)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}\overline{w}_H(r_n,q), \tag{3.11}$$

we obtain the Laplace transform of the velocity field $\overline{w}(r,q)$ under the form

$$\overline{w}(r,q) = \frac{\Omega_1 R_1^2(R_2^2-r^2)+\Omega_2 R_2^2(r^2-R_1^2)}{r(R_2^2-R_1^2)}\frac{1}{q^2} - \pi\sum_{n=1}^{\infty}\frac{J_1(R_1 r_n)B(r,r_n)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)} \times\{\Omega_2 R_2 J_1(R_1 r_n)-\Omega_1 R_1 J_1(R_2 r_n)\}\frac{1+\lambda q^\beta}{q(\lambda q^{\beta+1}+q+\nu r_n^2)}, \tag{3.12}$$

writing the last factor of (3.12) in the following equivalent form:

$$\frac{1+\lambda q^\beta}{q(\lambda q^{\beta+1}+q+\nu r_n^2)} = \frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\left[\frac{q^{-k-2}}{(q^\beta+\lambda^{-1})^{k+1}}+\lambda\frac{q^{\beta-k-2}}{(q^\beta+\lambda^{-1})^{k+1}}\right]. \tag{3.13}$$

Introducing (3.13) into (3.12), applying the discrete inverse Laplace transform, and using the known result [28, equation (97)],

$$\mathcal{L}^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\} = G_{a,b,c}(d,t); \quad \mathrm{Re}(ac-b), \mathrm{Re}(q) > 0, \ \left|\frac{d}{q^a}\right| < 1, \tag{3.14}$$

where the generalized $G_{a,b,c}(\cdot,\cdot)$ function is defined by

$$G_{a,b,c}(d,t) = \sum_{j=0}^{\infty} \frac{d^j \Gamma(c+j)}{\Gamma(c)\Gamma(j+1)} \frac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]}, \tag{3.15}$$

and we find the velocity field under the form

$$\begin{aligned} w(r,t) &= \frac{\Omega_1 R_1^2 (R_2^2 - r^2) + \Omega_2 R_2^2 (r^2 - R_1^2)}{r(R_2^2 - R_1^2)} t \\ &- \frac{\pi}{\lambda} \sum_{n=1}^{\infty} \frac{J_1(R_1 r_n) B(r, r_n)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} \{\Omega_2 R_2 J_1(R_1 r_n) - \Omega_1 R_1 J_1(R_2 r_n)\} \\ &\times \sum_{k=0}^{\infty} \left(\frac{-\nu r_n^2}{\lambda} \right)^k \left\{ G_{\beta,-k-2,k+1}\left(-\lambda^{-1}, t\right) + \lambda G_{\beta,\beta-k-2,k+1}\left(-\lambda^{-1}, t\right) \right\}. \end{aligned} \tag{3.16}$$

3.2. Calculation of the Shear Stress

Applying the Laplace transform to (2.7), we find that

$$\overline{\tau}(r,q) = \frac{\mu}{1 + \lambda q^\beta} \left(\frac{\partial \overline{w}(r,q)}{\partial r} - \frac{\overline{w}(r,q)}{r} \right), \tag{3.17}$$

where

$$\begin{aligned} \frac{\partial \overline{w}(r,q)}{\partial r} - \frac{\overline{w}(r,q)}{r} &= \frac{2R_1^2 R_2^2 (\Omega_2 - \Omega_1)}{r^2 (R_2^2 - R_1^2)} \frac{1}{q^2} + \pi \sum_{n=1}^{\infty} \frac{J_1(R_1 r_n) \left(2/r B(r, r_n) - r_n \widetilde{B}(r, r_n) \right)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} \\ &\times \{\Omega_2 R_2 J_1(R_1 r_n) - \Omega_1 R_1 J_1(R_2 r_n)\} \frac{1 + \lambda q^\beta}{q(\lambda q^{\beta+1} + q + \nu r_n^2)} \end{aligned} \tag{3.18}$$

is obtained from (3.12) and

$$\widetilde{B}(r, r_n) = J_0(r r_n) Y_1(R_2 r_n) - J_1(R_2 r_n) Y_0(r r_n). \tag{3.19}$$

Thus, (3.17) becomes

$$\begin{aligned} \overline{\tau}(r,q) &= \frac{2\mu R_1^2 R_2^2 (\Omega_2 - \Omega_1)}{r^2 (R_2^2 - R_1^2)} \frac{1}{q^2 (1 + \lambda q^\beta)} + \pi\mu \sum_{n=1}^{\infty} \frac{J_1(R_1 r_n) \left(2/r B(r, r_n) - r_n \widetilde{B}(r, r_n) \right)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} \\ &\times \{\Omega_2 R_2 J_1(R_1 r_n) - \Omega_1 R_1 J_1(R_2 r_n)\} \frac{1}{q(\lambda q^{\beta+1} + q + \nu r_n^2)}, \end{aligned} \tag{3.20}$$

applying again the discrete inverse Laplace transform as well as using the known relation [28, equation (21)],

$$\mathcal{L}^{-1}\left\{\frac{q^b}{q^a - c}\right\} = R_{a,b}(c,t); \quad \operatorname{Re}(a-b) > 0, \ \operatorname{Re}(q) > 0, \tag{3.21}$$

where the generalized $R_{a,b}(c,t)$ functions are defined by [28]

$$R_{a,b}(c,t) = \sum_{n=0}^{\infty} \frac{c^n t^{(n+1)a-b-1}}{\Gamma[(n+1)a-b]} \tag{3.22}$$

and the expansion

$$\frac{1}{q(\lambda q^{\beta+1} + q + \nu r_n^2)} = \frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \frac{q^{-k-2}}{(q^\beta + \lambda^{-1})^{k+1}}, \tag{3.23}$$

and we obtain the shear stress $\tau(r,t)$ under the form

$$\begin{aligned}
\tau(r,t) = {} & \frac{2\mu R_1^2 R_2^2 (\Omega_2 - \Omega_1)}{\lambda r^2 (R_2^2 - R_1^2)} R_{\beta,-2}\left(-\lambda^{-1}, t\right) + \frac{\pi\mu}{\lambda}\sum_{n=1}^{\infty} \frac{J_1(R_1 r_n)\left(2/rB(r,r_n) - r_n \widetilde{B}(r,r_n)\right)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} \\
& \times \{\Omega_2 R_2 J_1(R_1 r_n) - \Omega_1 R_1 J_1(R_2 r_n)\} \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{\beta,-k-2,k+1}\left(-\lambda^{-1}, t\right).
\end{aligned} \tag{3.24}$$

4. Limiting Cases

4.1. Classical Maxwell Fluid

Making $\beta \to 1$ into (3.16) and (3.24), we obtain the velocity field

$$\begin{aligned}
w_M(r,t) = {} & \frac{\Omega_1 R_1^2 (R_2^2 - r^2) + \Omega_2 R_2^2 (r^2 - R_1^2)}{r(R_2^2 - R_1^2)} t \\
& - \frac{\pi}{\lambda}\sum_{n=1}^{\infty} \frac{J_1(R_1 r_n) B(r,r_n)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} \{\Omega_2 R_2 J_1(R_1 r_n) - \Omega_1 R_1 J_1(R_2 r_n)\} \\
& \times \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \left\{G_{1,-k-2,k+1}\left(-\lambda^{-1}, t\right) + \lambda G_{1,-k-1,k+1}\left(-\lambda^{-1}, t\right)\right\}
\end{aligned} \tag{4.1}$$

and the shear stress

$$\tau_M(r,t) = \frac{2\mu R_1^2 R_2^2(\Omega_2-\Omega_1)}{\lambda r^2\left(R_2^2-R_1^2\right)} R_{1,-2}\left(-\lambda^{-1},t\right) + \frac{\pi\mu}{\lambda}\sum_{n=1}^{\infty}\frac{J_1(R_1 r_n)\left(2/rB(r,r_n) - r_n\widetilde{B}(r,r_n)\right)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)}$$
$$\times\left\{\Omega_2 R_2 J_1(R_1 r_n) - \Omega_1 R_1 J_1(R_2 r_n)\right\}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{1,-k-2,k+1}\left(-\lambda^{-1},t\right), \tag{4.2}$$

corresponding to an ordinary Maxwell fluid, performing the same motion. Of course, in view of the identities

$$\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{1,-k-1,k+1}\left(-\lambda^{-1},t\right) = \frac{e^{q_{2n}t} - e^{q_{1n}t}}{q_{2n}-q_{1n}},$$
$$\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{1,-k-2,k+1}\left(-\lambda^{-1},t\right) = \frac{\lambda}{\nu r_n^2}\left(1 + \frac{q_{1n}e^{q_{2n}t} - q_{2n}e^{q_{1n}t}}{q_{2n}-q_{1n}}\right), \tag{4.3}$$
$$R_{1,-2}\left(-\lambda^{-1},t\right) = \lambda t - \lambda^2\left(1 - e^{-t/\lambda}\right); \qquad q_{1n},\ q_{2n} = \frac{-1 \pm \sqrt{1 - 4\nu\lambda r_n^2}}{2\lambda},$$

the expressions (4.1) and (4.2) can be written in the simplified form

$$w_M(r,t) = \frac{\Omega_1 R_1^2\left(R_2^2 - r^2\right) + \Omega_2 R_2^2\left(r^2 - R_1^2\right)}{r\left(R_2^2 - R_1^2\right)} t - \frac{\pi}{\nu}\sum_{n=1}^{\infty}\frac{J_1(R_1 r_n)B(r,r_n)}{r_n^2\left[J_1^2(R_1 r_n) - J_1^2(R_2 r_n)\right]}$$
$$\times\left\{\Omega_2 R_2 J_1(R_1 r_n) - \Omega_1 R_1 J_1(R_2 r_n)\right\}\left\{1 - \lambda\frac{q_{1n}^2 e^{q_{2n}t} - q_{2n}^2 e^{q_{1n}t}}{q_{2n}-q_{1n}}\right\},$$

$$\tau_M(r,t) = \frac{2\mu R_1^2 R_2^2(\Omega_2-\Omega_1)}{r^2\left(R_2^2-R_1^2\right)}\left\{t - \lambda\left(1 - e^{-t/\lambda}\right)\right\} + \pi\rho\sum_{n=1}^{\infty}\frac{J_1(R_1 r_n)\left(2/rB(r,r_n) - r_n\widetilde{B}(r,r_n)\right)}{r_n^2\left[J_1^2(R_1 r_n) - J_1^2(R_2 r_n)\right]}$$
$$\times\left\{\Omega_2 R_2 J_1(R_1 r_n) - \Omega_1 R_1 J_1(R_2 r_n)\right\}\left\{1 + \frac{q_{1n}e^{q_{2n}t} - q_{2n}e^{q_{1n}t}}{q_{2n}-q_{1n}}\right\}. \tag{4.4}$$

4.2. Newtonian Fluid

By now letting $\lambda \to 0$ into (4.4) or $\beta \to 1$ and $\lambda \to 0$ into (3.16) and (3.24), using

$$\lim_{\lambda\to 0}\frac{1}{\lambda^k}G_{1,b,k}\left(\frac{-1}{\lambda},t\right) = \frac{t^{-b-1}}{\Gamma(-b)}, \quad b<0, \tag{4.5}$$

we obtain the velocity field

$$w_N(r,t) = \frac{\Omega_1 R_1^2\left(R_2^2 - r^2\right) + \Omega_2 R_2^2\left(r^2 - R_1^2\right)}{r\left(R_2^2 - R_1^2\right)} t - \frac{\pi}{\nu}\sum_{n=1}^{\infty} \frac{J_1(R_1 r_n) B(r, r_n)}{r_n^2\left[J_1^2(R_1 r_n) - J_1^2(R_2 r_n)\right]} \times \{\Omega_2 R_2 J_1(R_1 r_n) - \Omega_1 R_1 J_1(R_2 r_n)\}\left\{1 - e^{-\nu r_n^2 t}\right\} \tag{4.6}$$

and the associated shear stress

$$\tau_N(r,t) = \frac{2\mu R_1^2 R_2^2(\Omega_2 - \Omega_1)}{r^2\left(R_2^2 - R_1^2\right)} t + \pi\rho\sum_{n=1}^{\infty} \frac{J_1(R_1 r_n)\left(2/rB(r, r_n) - r_n \widetilde{B}(r, r_n)\right)}{r_n^2\left[J_1^2(R_1 r_n) - J_1^2(R_2 r_n)\right]} \times \{\Omega_2 R_2 J_1(R_1 r_n) - \Omega_1 R_1 J_1(R_2 r_n)\}\left\{1 - e^{-\nu r_n^2 t}\right\}, \tag{4.7}$$

corresponding to a Newtonian fluid, performing the same motion.

5. Special Cases

5.1. When the Inner Cylinder Is at Rest

Making $\Omega_1 = 0$ and $\Omega_2 = \Omega$ into (3.16) and (3.24), for instance, we obtain the velocity field

$$w_1(r,t) = \frac{\Omega R_2^2\left(r^2 - R_1^2\right)}{r\left(R_2^2 - R_1^2\right)} t - \frac{\pi\Omega R_2}{\lambda}\sum_{n=1}^{\infty} \frac{J_1^2(R_1 r_n) B(r, r_n)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} \times \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \left\{G_{\beta,-k-2,k+1}\left(-\lambda^{-1}, t\right) + \lambda G_{\beta,\beta-k-2,k+1}\left(-\lambda^{-1}, t\right)\right\} \tag{5.1}$$

and the shear stress

$$\tau_1(r,t) = \frac{2\mu\Omega R_1^2 R_2^2}{\lambda r^2\left(R_2^2 - R_1^2\right)} R_{\beta,-2}\left(-\lambda^{-1}, t\right) + \frac{\pi\mu\Omega R_2}{\lambda}\sum_{n=1}^{\infty} \frac{J_1^2(R_1 r_n)\left(2/rB(r, r_n) - r_n \widetilde{B}(r, r_n)\right)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{\beta,-k-2,k+1}\left(-\lambda^{-1}, t\right), \tag{5.2}$$

corresponding to a fractional Maxwell fluid when the inner cylinder is at rest. Figure 1(a) shows velocity profile corresponding to (5.1) for different values of time, when the inner cylinder is at rest. It shows that velocity is an increasing function with regard to t and r on the whole flow domain.

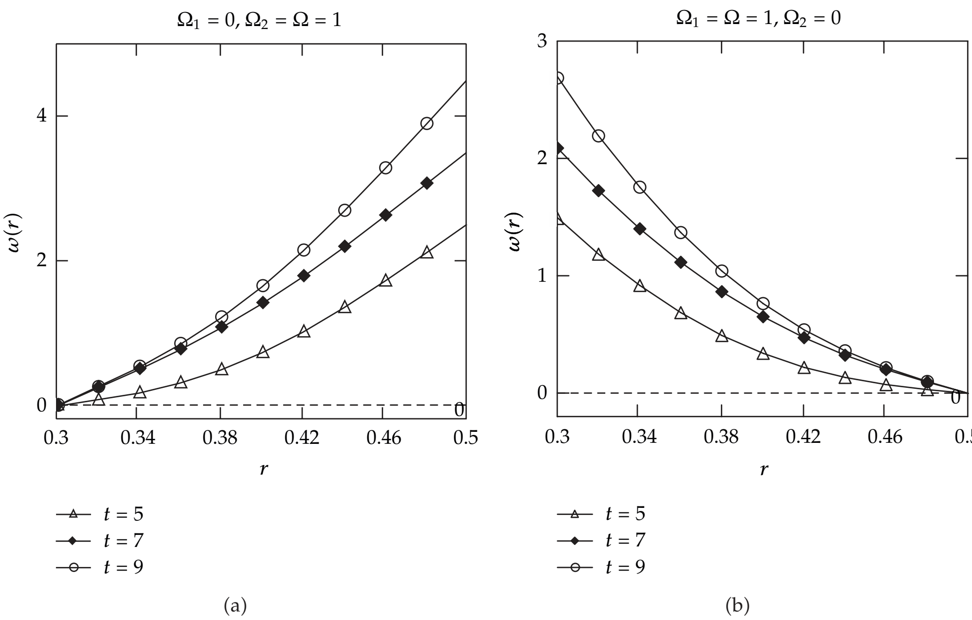


Figure 1: Profiles of the velocity $w(r,t)$ given by (5.1) and (5.3) for $R_1 = 0.3$, $R_2 = 0.5$, $\nu = 0.004$, $\mu = 2.916$, $\lambda = 3$, $\beta = 0.8$, and different values of t.

5.2. When the Outer Cylinder Is at Rest

Making $\Omega_1 = \Omega$ and $\Omega_2 = 0$ into (3.16) and (3.24), we obtain the velocity field

$$w_2(r,t) = \frac{\Omega R_1^2(R_2^2 - r^2)}{r(R_2^2 - R_1^2)}t + \frac{\pi\Omega R_1}{\lambda}\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)J_1(R_2r_n)B(r,r_n)}{J_1^2(R_1r_n) - J_1^2(R_2r_n)} \times \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\left\{G_{\beta,-k-2,k+1}\left(-\lambda^{-1},t\right) + \lambda G_{\beta,\beta-k-2,k+1}\left(-\lambda^{-1},t\right)\right\} \tag{5.3}$$

and the associated shear stress

$$\tau_2(r,t) = \frac{2\mu\Omega R_1^2R_2^2}{\lambda r^2(R_2^2 - R_1^2)}R_{\beta,-2}\left(-\lambda^{-1},t\right) - \frac{\pi\mu\Omega R_1}{\lambda} \times \sum_{n=1}^{\infty}\frac{J_1(R_1r_n)J_1(R_2r_n)\left(2/rB(r,r_n) - r_n\widetilde{B}(r,r_n)\right)}{J_1^2(R_1r_n) - J_1^2(R_2r_n)}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{\beta,-k-2,k+1}\left(-\lambda^{-1},t\right), \tag{5.4}$$

corresponding to a fractional Maxwell fluid when the outer cylinder is at rest. Figure 1(b) shows the profile of the velocity field corresponding to (5.3) for different values of time, when the outer cylinder is at rest, respectively. It shows that velocity is an increasing function with

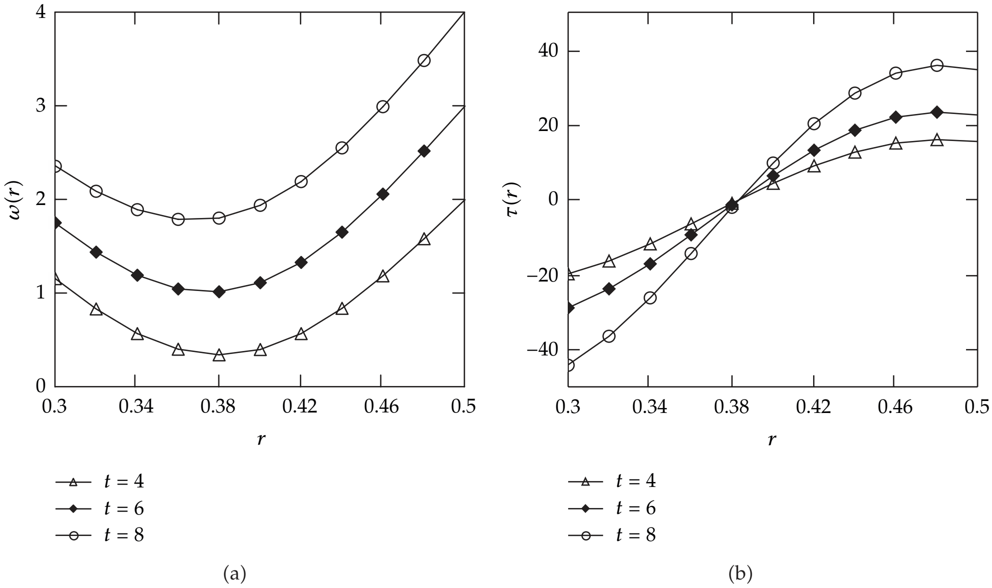


Figure 2: Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ given by (3.16) and (3.24) for $R_1 = 0.3$, $R_2 = 0.5$, $\Omega_1 = 1$, $\Omega_2 = 1$, $\nu = 0.004$, $\mu = 2.916$, $\lambda = 4$, $\beta = 0.5$, and different values of t.

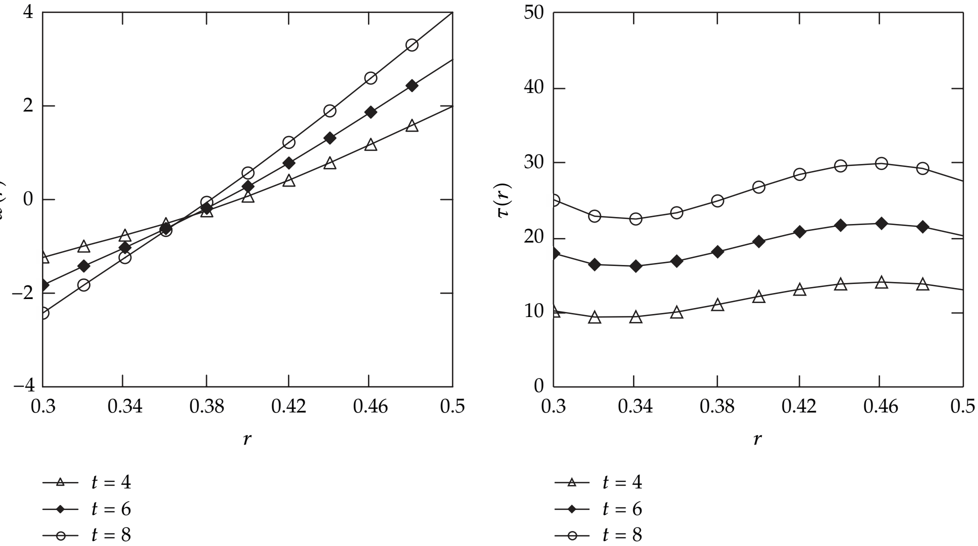
regard to t like Figure 1(a), but it has opposite effect for r, more exact velocity is decreasing with regard to r on the whole flow domain.

6. Conclusions

In this paper, the velocity $w(r,t)$ and the shear stress $\tau(r,t)$ corresponding to the flow of an incompressible Maxwell fluid with fractional derivatives, in the annular region between two infinite coaxial circular cylinders, have been determined using the Laplace and finite Hankel transforms. The solutions that have been obtained, written under a series form in terms of generalized $G_{a,b,c}(\cdot,t)$- and $R_{a,b}(\cdot,t)$-functions, satisfy the governing equations and all imposed initial and boundary conditions. In the limiting cases, when $\beta \to 1$ or $\beta \to 1$ and $\lambda \to 0$, the corresponding solutions for the ordinary Maxwell and Newtonian fluids are obtained. These solutions also satisfy the associated initial and boundary conditions (3.1) and (3.2), respectively. Moreover, the solutions for the motion between the cylinders, when one of them is at rest, are also obtained from our general results.

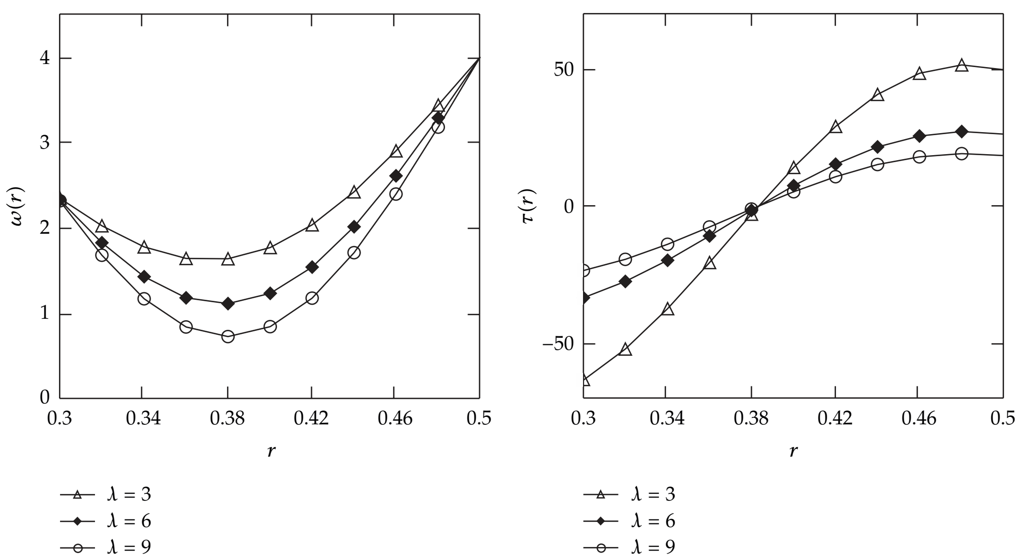
In order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity field $w(r,t)$ and the shear stress $\tau(r,t)$ given by (3.16) and (3.24) have been drawn against r for different values of the time t and the material parameters. Figures 2 and 3 show the profile of the fluid motion at different values of time when both inner and outer are rotating with the same angular velocity in the same direction and in the opposite direction, respectively. From these figures, one can clearly see that both velocity and shear stress in absolute values are increasing function of t. From Figure 3(a), one can also observe that fluid has zero velocity nearer to inner cylinder.

In Figure 4, the influence of the relaxation time λ on the fluid motion is shown. As expected, both the velocity and the shear stress (in absolute value) are decreasing functions



(a) (b)

Figure 3: Profiles of the velocity $w(r, t)$ and shear stress $\tau(r, t)$ given by (3.16) and (3.24) for $R_1 = 0.3$, $R_2 = 0.5$, $\Omega_1 = -1$, $\Omega_2 = \Omega = 1$, $\nu = 0.003$, $\mu = 2.916$, $\lambda = 4$, $\beta = 0.5$, and different values of t.



(a) (b)

Figure 4: Profiles of the velocity $w(r, t)$ and shear stress $\tau(r, t)$ given by (3.16) and (3.24) for $R_1 = 0.3$, $R_2 = 0.5$, $\Omega_1 = 1$, $\Omega_2 = 1$, $t = 8$, $\nu = 0.003$, $\mu = 2.916$, $\beta = 0.5$, and different values of λ.

with respect to λ. Effect of fractional parameter β on the fluid motion is represented in Figure 5, and it is clearly seen that both velocity and shear stress (in absolute value) are increasing with respect to β.

Finally, for comparison, the diagrams of $w(r, t)$ and $\tau(r, t)$ corresponding to the three models (fractional Maxwell, ordinary Maxwell, and Newtonian) are together depicted in

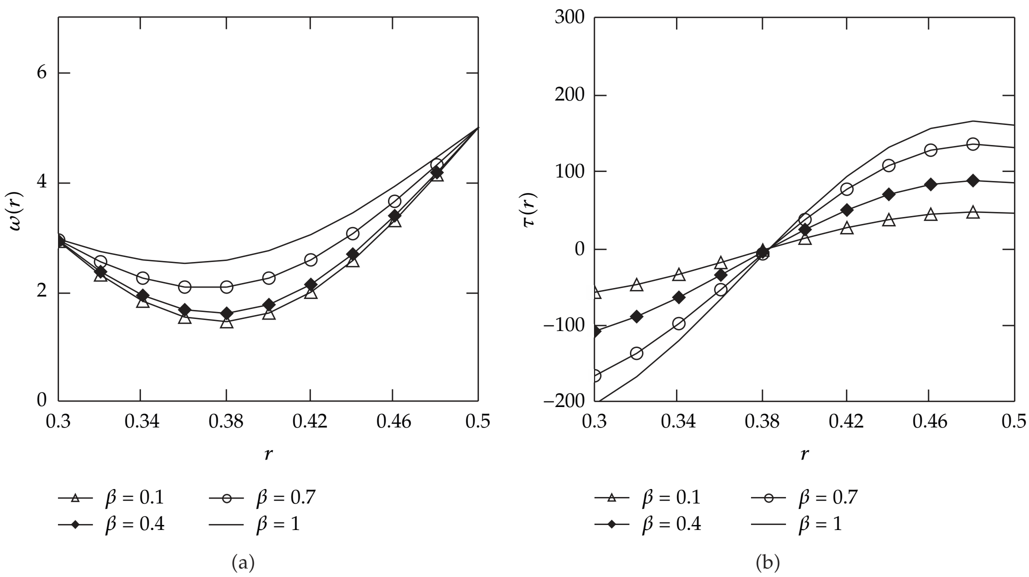


Figure 5: Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ given by (3.16) and (3.24) for $R_1 = 0.3$, $R_2 = 0.5$, $\Omega_1 = 1$, $\Omega_2 = 1$, $t = 10$, $\nu = 0.005$, $\mu = 2.916$, $\lambda = 4$, and different values of β.

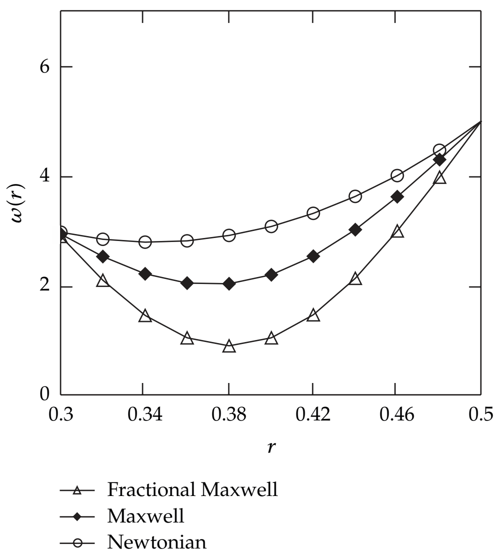


Figure 6: Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ corresponding to the Newtonian, Maxwell, and fractional Maxwell fluids, for $R_1 = 0.3$, $R_2 = 0.5$, $\Omega_1 = 1$, $\Omega_2 = 1$, $t = 7$, $\nu = 0.002$, $\mu = 5$, $\lambda = 2$, and $\beta = 0.4$.

Figure 6 for the same values of the common material constants and time t. The Newtonian fluid is the swiftest, while the fractional Maxwell fluid is the slowest on the whole flow domain. One thing is worth of mentioning that the units of the material constants are SI units in all the figures, and the roots r_n have been approximated by $n\pi/(R_2 - R_1)$.

Acknowledgments

The authors are very grateful to the referee for suggestions and constructive comments. The authors M. Imran, A. U. Awan, and M. Rana are thankful to Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan and the Higher Education Commission, Pakistan for supporting and facilitating this research work.

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