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Exact solutions for the static bending of Euler-Bernoulli beams using Eringen's two-phase local/nonlocal model

Y. B. Wang,[1] X. W. Zhu,[2,a] and H. H. Dai[3]
[1]*Department of Mathematics, ShaoXing University, No.900, ChengNan Avenue 312000, ShaoXing, Zhejiang, China*
[2]*School of Statistics and Mathematics, Zhongnan University of Economics and Law, Wuhan 430073, China*
[3]*Department of Mathematics, City University of HongKong, 83 Tat Chee Avenue, Kowloon Tong, HongKong*

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Though widely used in modelling nano- and micro- structures, Eringen's differential model shows some inconsistencies and recent study has demonstrated its differences between the integral model, which then implies the necessity of using the latter model. In this paper, an analytical study is taken to analyze static bending of nonlocal Euler-Bernoulli beams using Eringen's two-phase local/nonlocal model. Firstly, a reduction method is proved rigorously, with which the integral equation in consideration can be reduced to a differential equation with mixed boundary value conditions. Then, the static bending problem is formulated and four types of boundary conditions with various loadings are considered. By solving the corresponding differential equations, exact solutions are obtained explicitly in all of the cases, especially for the paradoxical cantilever beam problem. Finally, asymptotic analysis of the exact solutions reveals clearly that, unlike the differential model, the integral model adopted herein has a consistent softening effect. Comparisons are also made with existing analytical and numerical results, which further shows the advantages of the analytical results obtained. Additionally, it seems that the once controversial nonlocal bar problem in the literature is well resolved by the reduction method. © *2016 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).* [http://dx.doi.org/10.1063/1.4961695]

I. INTRODUCTION

The nonlocal elasticity advanced by Eringen and his co-workers[1–4] is based on the hypothesis that the nonlocal stress at a reference point depends not only on the strain at that point but also on all other points of the body. Consequently, in the constitutive equation of nonlocal elasticity Hooke's law (for the classical elasticity) is replaced by an integration. Such integral form constitutive relations account for the forces between atoms and internal length scale, and have been applied to many problems including wave propagation, fracture mechanics, dislocations, etc.

In recent years, with the growing need in designing or analysing size-dependent materials and structures, Eringen's nonlocal elasticity received much attention as reviewed by Arash and Wang.[5] Since it is usually difficult to deal with integro-differential equations, in the literature an approximate differential model once proposed in Eringen[6] for a specific integral kernel is widely used to incorporate nonlocal effects. However, it is recognized that, except some ad hoc approaches, there are some inconsistences regarding this differential model.[5,7–11] To be specific, for all boundary conditions except the cantilever, the model has a softening effect (i.e., large deflections and lower fundamental frequencies) as the nonlocal parameter increases. While for a cantilever beam with concentrated load, there is no nonlocal effect. As far as the authors are

[a]Corresponding author; electronic mail: xiaowuzhu1026@znufe.edu.cn (Xiaowu Zhu)

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aware of, such inconsistences were further clarified most recently in three important pieces of work[12–14] which resorted to Eringen's integral models. In the literature, the integral models mainly include the pure nonlocal model[6] and the two-phase local/nonlocal model,[15] and the former (with a specific kernel) can be viewed as the original integral model of the aforementioned differential model.[6] In Khodabakhshi and Reddy,[12] a unified integro-differential nonlocal model, which resembles the two-phase local/nonlocal model, was presented. Numerical study of the nonlocal Euler-Bernoulli beam showed that, other than the simply supported case which showed a stiffening effect, the nonlocal beam has a softening effect for most boundary conditions. More importantly, with this model the paradoxical cantilever beam problem was resolved. Further, it was shown in Fernández-Sáez *et al.*[13] that, in general, the differential model is not equivalent to its original integral model (i.e., the corresponding pure nonlocal model): the solution of the original integral model actually is obtained by adding those of the differential model and of other two integral equations. The contrasts were illustrated through the static bending analysis of Euler-Bernoulli beam, and the paradox that appears when solving the cantilever beam problem with differential model was also resolved numerically. In Tuna and Kirca,[14] using the pure nonlocal model as in Fernández-Sáez *et al.*,[13] by Laplace transformation exact solutions were claimed to be obtained for both Euler-Bernoulli and Timoshenko beams, and the analytical results showed that the integral model has a consistent softening effect.

Because of the inconsistences in the differential model and some promising results reported by the integral models, especially for the cantilever beam problem which would have many applications in science, there is a great need to investigate the integral models. For structures with Eringen's integral models, numerical methods are usually applied in the literature to deal with the resulting integro-differential equations,[12,13,16–18] which are shown to be time consuming and the numerical errors are sometimes hard to detect for some specific nonlocal kernels.[12] As to analytical results, the existence and uniqueness of the field equations in nonlocal elasticity (boundary-value problem and initial-boundary-value problem) were established almost decades ago.[19–21] Due to the difficulties in dealing with integro-differential equations, analytical solutions of problems in nonlocal elasticity are very few. Most of the analytical solutions were concerned with a nonlocal bar problem firstly proposed in Pisano and Fuschi.[22] The closed form solution there were shown to have considerable error, and more accurate analytical solutions were then presented for this problem.[23,24] It should be noted that, before any progress can be made in this direction, the aforementioned analytical solutions are only valid for some specific loadings, and thus can not be applied to more complicated cases (say, boundary conditions and loadings). The exact solutions for the pure nonlocal model in Tuna and Kirca[14] as they claimed, however, would incur some unknown errors, and we shall discuss it in Section V. To summarize, there are very few exact solutions neither for the two-phase local/nonlocal model, nor for the pure nonlocal model. So, more comprehensive analytical studies of Eringen's integral models as applied to nano- and micro- structures would be desirable. These have become the motivations of the work.

In this paper, we adopt the two-phase local/nonlocal model[15,25] and aim at searching for the analytical solutions to the bending problem of nonlocal Euler-Bernoulli beams. By reducing the integral equation to a differential equation with mixed boundary conditions, we manage to obtain exact solutions of the beam problem under several typical boundary conditions, especially for the paradoxical cantilever case. Analysis of the solutions shows that, as far as the examples examined, the two-phase local/nonlocal model has a consistent softening effect. Also, with the analytical results obtained, we find that the numerical solutions in Khodabakhshi and Reddy[12] encountered some numerical errors for simply supported beam, the solutions in Tuna and Kirca[14] can be viewed as "limit solutions" of our exact solutions and the possible error in those solutions can be discussed, and as a byproduct, the once controversial nonlocal bar problem[22] is well resolved by the reduction method proved here. The layout of the remaining part is as follows. In Section II, we review briefly Eringen's nonlocal theory and provide the key method in our latter derivations. In Section III, we set up the governing equations for static bending of nonlocal Euler-Bernoulli beam using Eringen's two-phase local/nonlocal model and discuss the applicability of the method. Exact solutions are obtained for various types of boundary conditions and loadings in Section IV. In Section V, some analysis will be taken to the exact solutions obtained, and comparisons will be made with existing analytical and numerical results. In Section VI, we draw conclusions and discuss the future work.

II. ERINGEN'S NONLOCAL ELASTICITY AND METHOD DESCRIPTION

According to Eringen,[25] the long rang force in a linear homogeneous and isotropic elastic material as a consequence of a strain field, is expressed by the following constitutive relation

$$\mathbf{t}(\mathbf{x}) = \int_V \alpha(|\mathbf{x}\text{-}\mathbf{x}'|,\tau)\sigma_{kl}(x')d\mathbf{x}', \tag{1}$$

where $\mathbf{t}(\mathbf{x})$ is the nonlocal stress tensor, σ_{kl} is the classical stress tensor at $\mathbf{x}'$, V is the region occupied by the body. The constant $\tau = e_0 a/l$, where a is an internal characteristic length (e.g., lattice parameter, C-C bond length), l is an external characteristic length (e.g., crack length, wave length), and e_0 is a material constant which can be determined experimentally or approximated by matching some reliable results. The scalar function $\alpha(|\mathbf{x}\text{-}\mathbf{x}'|,\tau)$ is called *kernel function* or *influence function*, which is positive and decays rapidly with the increase of $|\mathbf{x}\text{-}\mathbf{x}'|$. One additional constraint on the kernel function is that for $\tau \to 0$ we obtain Dirac delta function. Due to this property, nonlocal elasticity reverts to classical elasticity in the limit $\tau \to 0$. The former constitutive equation is usually referred to as pure nonlocal model. Another constitutive model in the literature[15,20,21,25] takes the following form

$$\mathbf{t}(\mathbf{x}) = \xi_1\sigma_{kl}(x) + \xi_2\int_V \alpha(|\mathbf{x}\text{-}\mathbf{x}'|,\tau)\sigma_{kl}(\mathbf{x}')d\mathbf{x}', \tag{2}$$

where $\xi_1 > 0, \xi_2 > 0$ and satisfy $\xi_1 + \xi_2 = 1$. Equation (2) can be viewed as a constitutive relation of a two-phase elastic material, in which phase 1 (of volume fraction ξ_1) complies with local elasticity, phase 2 (of volume fraction ξ_2) complies with nonlocal elasticity. It is usually typed as two-phase local/nonlocal model.

As a first attempt, in this paper we adopt the two-phase local/nonlocal model (2) and devote to deriving analytical solutions to bending problems of nonlocal Euler-Bernoulli beam. We consider a beam of length L, with uniform cross section S, Young modulus E and subjected to distributed transverse load $q(x)$ at the top. The coordinate system is introduced as: x coordinate is taken along the length of the beam and z coordinate is taken along the height of the beam, while w denotes the transverse displacement of the middle plane (i.e., $z = 0$). For a beam problem, we make the assumptions that the nonlocal behavior in the thickness direction is negligible. As commonly used in the literature, by taking the normalized bi-exponential kernel in Eringen,[6] the nonlocal stress along x direction then can be expressed by its local counterpart as

$$t_{xx}(x) = \xi_1\sigma(x) + \frac{\xi_2}{2\tau}\int_0^L e^{-\frac{|x-s|}{\tau}}\sigma(s)ds, \tag{3}$$

where L denotes the length of the beam.

Remark 2.1. In the literature, both of the models (1)(2) have been investigated either numerically[12,13,16–18,21] or analytically for a nonlocal bar problem[22–24] and for nonlocal beam problems.[14] It should be noted that for the bi-exponential kernel adopted here, by choosing $\xi_1 = 0$ constitutive equation (3) then reduces to the corresponding pure nonlocal model

$$t(x) = \frac{1}{2\tau}\int_0^L e^{-\frac{|x-s|}{\tau}}\sigma(s)ds. \tag{4}$$

It has been demonstrated (for the case of axial behavior of bars[24] and bending of Euler-Bernoulli beams[13]) that in the literature there is an improper transformation from this pure nonlocal model into the following differential model

$$(1 - \tau^2\frac{d^2}{dx^2})t(x) = \sigma(x). \tag{5}$$

From a mathematical point of view, models (4)(3) (also generally for (1)(2)) can be typed as Fredholm integral equations of the first kind and the second kind respectively, whose solutions are usually very different. In our latter derivations, it will be further shown that the exact solutions in Tuna and Kirca[14] *with model (4) can actually be viewed as "limit solution" (i.e.,* $\xi_1 \to 0$*) from our exact solutions. Unfortunately, such solutions may not be the exact solutions to model (4).*

To begin with, we present the following proposition, which is the key in our method.

Proposition 2.1. Consider the following linear integral equation of the second kind

$$y(x) + A\int_a^b e^{\lambda|x-s|} y(s)ds = f(x). \tag{6}$$

If $f(x) \in C^2[a,b]$ *(i.e., twice differentiable) and the parameters* $\lambda < 0, A > 0$*, then the solution* $y(x)$ *to the integral equation also belongs to* $C^2[a,b]$ *and can be uniquely determined by the following differential equation*

$$y''(x) + \lambda(2A - \lambda)y(x) = f''(x) - \lambda^2 f(x) \tag{7}$$

with mixed boundary value conditions

$$y'(a) + \lambda y(a) = f'(a) + \lambda f(a), \quad y'(b) - \lambda y(b) = f'(b) - \lambda f(b). \tag{8}$$

The proof is reported in Appendix A. With this "reduction method", for any given initial data that satisfy the assumptions, one can solve the differential equation instead of the integral equation. It should be noted that the method is brought from Polyanin and Manzhirov,[26] and suits for the problem in consideration. Such a procedure was also applied to get the analytical solution of a nonlocal bar problem previously,[24] while without further justifications of its applicability. Regarding the proposition, we make the following comments according to nonlocal elasticity problems.

1. The reduction method requires $f(x) \in C^2[a,b]$. So, for cases in which $f(x)$ are not $C^2[a,b]$, it seems that the method can not be applied directly, and some related numerical studies have been done.[17] As it is reported in the appendix, $f(x) \in C^2[a,b]$ would imply $y(x) \in C^2[a,b]$, and one can even deduce that $y(x)$ shares the same regularity with $f(x)$. This is true for the adopted kernel, but it is not always the case in nonlocal elasticity, especially for the pure nonlocal model (1). For instance, the constitutive equation with Gaussian kernel

$$t(x) = \frac{1}{2\tau}\int_a^b e^{-\frac{(x-s)^2}{\tau}}\sigma(s)ds. \tag{9}$$

As the integral kernel has the effect of mollification, then for any integrable function $\sigma(x)$, the nonlocal stress is smooth. Consequently, for any given function $\mathbf{t}(\mathbf{x})$ which is smooth, we can not claim that $\sigma(x)$ is smooth. Or, for the problem in which $\mathbf{t}(\mathbf{x})$ is not smooth enough, there would be no solution for $\sigma(x)$. This issue is also addressed before.[21]

2. The result that $y(x)$ is unique depends on the choice of the parameters. Though it is sufficient for the problem in hand, the condition $\lambda < 0, A > 0$ can be made stronger mathematically. As it had been investigated for general theory of well-posedness in nonlocal theory,[19,20] the uniqueness of $y(x)$ here essentially relies on the relations between $\lambda < 0, A > 0$ and eigenvalues κ of the corresponding integral operator

$$\int_a^b e^{\lambda|x-s|} y(s)ds = \kappa y(x). \tag{10}$$

3. In some of the cases as considered in the literature numerically,[12,16,17,22] the differential equation can be solved explicitly for given functions $f(x)$. Thus, we can get the analytical solutions, which once is a very difficult task. As to the mixed boundary conditions (8), we can consider it as consistent requirements. While from another point of view, by constructing Green function (Tricomi[27] can be consulted) of the differential operator in equation (7) with boundary conditions (8), integral equation (6) can be recovered. Thus, the boundary conditions can be viewed as an

intrinsic condition in the integral equation, and which, in turn tells us that boundary conditions (8) are requisites for the unknown solution $y(x)$ in the problem.

III. GOVERNING EQUATIONS

In this section, we adopt constitutive equation (3) and set up the governing equation for the nonlocal Euler-Bernoulli beam. Also, we make some discussions about the applicability of the method described in last section.

The strain-displacement relation of Euler-Bernoulli beam can be given as

$$\varepsilon_{xx}(x) = -z\frac{d^2w(x)}{dx^2}. \tag{11}$$

By constitutive equation (3), nonlocal stress t_{xx} can be expressed by

$$t_{xx}(x) = E\Big(\xi_1\varepsilon_{xx}(x) + \frac{\xi_2}{2\tau}\int_0^L e^{-\frac{|x-s|}{\tau}}\varepsilon_{xx}(s)ds\Big), \tag{12}$$

and the corresponding bending moment is

$$M(x) = \int_S t_{xx}(x)zdS = -EI\Big(\xi_1\frac{d^2w(x)}{dx^2} + \frac{\xi_2}{2\tau}\int_0^L e^{-\frac{|x-s|}{\tau}}\frac{d^2w(s)}{ds^2}ds\Big), \tag{13}$$

where $I = \int_S z^2 dS$.

Note that the principle of virtual displacements[28,29] is independent of constitutive models, the equations expressed in terms of stress resultants are valid for local or nonlocal models. So, we can apply it to derive the governing equation and boundary conditions for the unknown transverse displacement $w(x)$.

The internal virtual work due to $\delta w(x)$ is

$$\delta W_I = \int_0^L\int_S \sigma_{xx}(x)\delta\varepsilon_{xx}(x)dxdS = -\int_0^L M(x)\frac{d^2\delta w(x)}{dx^2}dx. \tag{14}$$

The external virtual work is

$$\delta W_E = -\int_0^L q(x)\delta w(x)dx. \tag{15}$$

The principle of virtual displacements then gives

$$\delta W_I + \delta W_E = 0, \tag{16}$$

which can be expressed as

$$-\int_0^L M(x)\frac{d^2\delta w(x)}{dx^2}dx - \int_0^L q(x)\delta w(x)dx = 0. \tag{17}$$

Integrating the first term twice by parts, we arrive at

$$-M(x)\frac{\delta dw(x)}{dx}\Big|_0^L + \frac{dM(x)}{dx}\delta w(x)\Big|_0^L - \int_0^L\Big(\frac{d^2}{dx^2}M(x) + q(x)\Big)\delta w(x)dx = 0. \tag{18}$$

By the Fundamental Lemma of Variational Calculus, the Euler equation is

$$\frac{d^2M(x)}{dx^2} + q(x) = 0, \tag{19}$$

or, in terms of displacement (cf. (13))

$$-EI\frac{d^2}{dx^2}\Big[\xi_1\frac{d^2w(x)}{dx^2} + \frac{\xi_2}{2\tau}\int_0^L e^{-\frac{|x-s|}{\tau}}\frac{d^2w(s)}{ds^2}ds\Big] + q(x) = 0. \tag{20}$$

The boundary conditions involve specifying one element of each of the following two pairs at $x = 0$ and $x = L$:

$$w \quad \text{or} \quad \frac{dM(x)}{dx}, \tag{21}$$

$$\frac{dw(x)}{dx} \quad \text{or} \quad M(x). \tag{22}$$

For given external load $q(x)$, after integrating both sides of governing equation (20), we can get the following integral equation for $y(x)(y(x) = w''(x))$ as

$$y(x) + \frac{\xi_2}{2\tau\xi_1}\int_0^L e^{-\frac{|x-x'|}{\tau}} y(s)ds = \frac{-1}{EI\xi_1}\Big(C_1 + C_2 x - \int_0^x (x-s)q(s)ds\Big), \tag{23}$$

where C_1, C_2 are integration constants to be determined. Once $y(x)$ is obtained, the unknown displacement $w(x)$ can be obtained as

$$w(x) = C_3 + C_4 x + \int_0^x (x-s)y(s)ds, \tag{24}$$

where C_3, C_4 are actually the values for $w(0), w'(0)$ respectively.

Now, we examine the applicability of the reduction method to our problem. Comparing equation (23) with integral equation (6), we have in our case $\lambda = -\frac{1}{\tau} < 0$, $A = \frac{\xi_2}{2\tau\xi_1} > 0$. Thus, it only requires the right hand side function

$$f(x) = \frac{-1}{EI\xi_1}\Big(C_1 + C_2 x - \int_0^x (x-s)q(s)ds\Big) \tag{25}$$

to be in $C^2[0, L]$, and hence $q(x)$ should belong to $C[0, L]$ (i.e., continuous), which can be generally satisfied. So, if $q(x)$ is continuous, then according to proposition 2.1 equation (23) can be uniquely solved by the corresponding differential equation

$$y''(x) - k^2 y(x) = f''(x) - \frac{1}{\tau^2} f(x), \tag{26}$$

with the mixed boundary value conditions

$$y'(0) - \frac{1}{\tau} y(0) = f'(0) - \frac{1}{\tau} f(0), \quad y'(L) + \frac{1}{\tau} y(L) = f'(L) + \frac{1}{\tau} f(L), \tag{27}$$

where $k = \frac{1}{\sqrt{\xi_1}\tau}$. The general solution to equation (26) can be expressed as

$$y(x) = C_5 \cosh(kx) + C_6 \sinh(kx) + f(x) + k\big(1 - \frac{1}{k^2\tau^2}\big)\int_0^x \sinh(k(x-s)) f(s)ds, \tag{28}$$

with C_5, C_6 being uniquely determined from boundary value conditions (27). That is, if C_1, C_2 are known from the boundary conditions, then we can get a unique pair C_5, C_6, or vise versa. So, we actually have four unknowns C_1, C_2, C_3, C_4 as usual, and all of which can be determined from the prescribed boundary conditions.

Remark 3.1. Note that in nonlocal beam theory we usually have integral ($nonlocal$) *boundary conditions (e.g., for clamped-clamped and clamped-pinned ends), however, as the solution to the integral equation can be expressed explicitly, the unknown coefficients can be uniquely determined by solving a linear system as before. The lengthy calculations can be performed with the help of mathematical tools, such as Mathematica.*[30]

IV. BENDING SOLUTIONS

In this section, we shall derive analytical solutions for static bending problem of nonlocal Euler-Bernoulli beam with various boundary conditions. In all of the examples, the external loads $q(x) \in C[0, L]$.

(1). (S-S) Simply supported beam with uniformly distributed load q_0

For a beam simply supported at $x = 0$ and $x = L$, the boundary conditions are

$$w(0) = w(L) = 0, \quad M(0) = M(L) = 0. \tag{29}$$

Using the boundary conditions on the moment, we firstly obtain

$$C_1 = 0, \; C_2 = \frac{q_0 L}{2}. \tag{30}$$

On substitution of the values for C_1, C_2 into equation (25) and using equations (26)(27), we can get C_5, C_6 as

$$C_5 = \frac{q_0(\xi_1 - 1)\tau\left(-L\cosh(\frac{L}{2\sqrt{\xi_1}\tau}) + 2\tau\sqrt{\xi_1}\sinh(\frac{L}{2\sqrt{\xi_1}\tau})\right)}{2EI\left(\xi_1\cosh(\frac{L}{2\sqrt{\xi_1}\tau}) + \sqrt{\xi_1}\sinh(\frac{L}{2\sqrt{\xi_1}\tau})\right)}, \quad C_6 = \sqrt{\xi_1}C_5. \tag{31}$$

With the obtained solution for $y(x) = w''(x)$ and the boundary conditions on the displacement, we can get

$$\begin{aligned} C_3 &= 0, \\ C_4 &= \frac{q_0\left(\cosh(\frac{L}{2\sqrt{\xi_1}\tau}) + \sqrt{\xi_1}\sinh(\frac{L}{2\sqrt{\xi_1}\tau})\right)}{\left(2\sqrt{\xi_1}\cosh(\frac{L}{\sqrt{\xi_1}\tau}) + (1+\xi_1)\sinh(\frac{L}{\sqrt{\xi_1}\tau})\right)} \\ &\quad \frac{\left(L\sqrt{\xi_1}\left(L^2 - 12(\xi_1 - 1)\tau^2\right)\cosh(\frac{L}{2\sqrt{\xi_1}\tau}) + \left(L^3 + 24(\xi_1 - 1)\xi_1\tau^3\right)\sinh(\frac{L}{2\sqrt{\xi_1}\tau})\right)}{12EI}. \end{aligned} \tag{32}$$

By equation (24), the exact solution for transverse displacement $w(x)$ can be expressed as

$$\begin{aligned} w(x) = {} & \frac{q_0 x^4}{24EI} - \frac{q_0 L x^3}{12EI} + \frac{q_0(\xi_1 - 1)\tau^2 x^2}{2EI} + \frac{q_0 L\left(L^2 - 12(\xi_1 - 1)\tau^2\right)x}{24EI} \\ & + \frac{q_0\tau^3\sqrt{\xi_1}(\xi_1 - 1)(L + 2\xi_1\tau)\sinh(\frac{L-x}{2\sqrt{\xi_1}\tau})\sinh(\frac{x}{2\sqrt{\xi_1}\tau})}{EI\left(\sqrt{\xi_1}\cosh(\frac{L}{2\sqrt{\xi_1}\tau}) + \sinh(\frac{L}{2\sqrt{\xi_1}\tau})\right)}. \end{aligned} \tag{33}$$

By differentiating with respect to x (or symmetry of the solution), we can find that the maximum deflection occurs at $x = L/2$, and it is given by

$$\begin{aligned} w_{\max} = {} & \frac{q_0 L^4\left[5 + 48(1 - \xi_1)(\frac{\tau}{L})^2\right]}{384EI} \\ & - \frac{q_0 L^4(1 - \xi_1)\sqrt{\xi_1}\left[(\frac{\tau}{L})^3 + 2\xi_1(\frac{\tau}{L})^4\right]\sinh^2(\frac{L}{4\sqrt{\xi_1}\tau})}{EI\left(\sqrt{\xi_1}\cosh(\frac{L}{2\sqrt{\xi_1}\tau}) + \sinh(\frac{L}{2\sqrt{\xi_1}\tau})\right)}. \end{aligned} \tag{34}$$

In order to reveal the nonlocal effect more clearly, we can take some asymptotic analysis to the exact solutions for $w(x)$, $w_{\max}$. Since L/τ is relatively large and $0 < \xi_1 < 1$, the $O(e^{-L/(\sqrt{\xi_1}\tau)})$ terms can be neglected. Asymptotic analysis then gives

$$\begin{aligned} w(x) = {} & \frac{q_0 x^4}{24EI} - \frac{q_0 L x^3}{12EI} + \frac{q_0(\xi_1 - 1)\tau^2 x^2}{2EI} + \frac{q_0 L\left(L^2 - 12(\xi_1 - 1)\tau^2\right)x}{24EI} \\ & - \frac{q_0\sqrt{\xi_1}(1 - \sqrt{\xi_1})(L + 2\xi_1\tau)\tau^3}{2EI}(1 - e^{-\frac{x}{\sqrt{\xi_1}\tau}} - e^{-\frac{L-x}{\sqrt{\xi_1}\tau}}) \\ & + E.S.T., \end{aligned} \tag{35}$$

$$w_{\max} = \frac{q_0 L^4 \left[5 + 48(1-\xi_1)\left(\frac{\tau}{L}\right)^2 - 384\sqrt{\xi_1}(1-\sqrt{\xi_1})\left(\left(\frac{\tau}{L}\right)^3 + 2\xi_1\left(\frac{\tau}{L}\right)^4\right)\right]}{384EI} + E.S.T. \tag{36}$$

Hereafter, "$E.S.T.$" denotes the $O(e^{-L/(\sqrt{\xi_1}\tau)})$ terms. It can be easily checked by setting $\xi_1 = 1$ in the exact solution that the classical solution can be fully recovered.

(2). (C-C) Clamped beam with uniformly distributed load q_0

For a beam with clamped ends at $x = 0$ and $x = L$, the boundary conditions are

$$w(0) = w'(0) = 0; \quad w(L) = w'(L) = 0. \tag{37}$$

The determination of the unknown coefficients are omitted for brevity, expressions for these coefficients will not be presented hereafter and we just give the solutions. The exact solution for transverse displacement $w(x)$ is expressed as

$$\begin{aligned} w(x) &= \frac{q_0 x^4}{24EI} - \frac{q_0 L x^3}{12EI} \\ &+ \frac{q_0 x^2\left(\left(L^3 + 12L(\xi_1-1)\tau^2 + 24(\xi_1-1)\tau^3\right)\sinh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right) + L^3\sqrt{\xi_1}\cosh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right)\right)}{24EI\left((L - 2(\xi_1-1)\tau)\sinh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right) + L\sqrt{\xi_1}\cosh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right)\right)} \\ &- \frac{q_0 L(\xi_1-1)\tau\left(L^2 + 6L\tau + 12\tau^2\right)\sinh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right)x}{12EI\left((L - 2(\xi_1-1)\tau)\sinh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right) + L\sqrt{\xi_1}\cosh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right)\right)} \\ &+ \frac{q_0 L(\xi_1-1)\sqrt{\xi_1}\tau^2\left(L^2 + 6L\tau + 12\tau^2\right)\sinh\left(\frac{x}{2\sqrt{\xi_1}\tau}\right)\sinh\left(\frac{L-x}{2\sqrt{\xi_1}\tau}\right)}{6EI\left((L - 2(\xi_1-1)\tau)\sinh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right) + L\sqrt{\xi_1}\cosh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right)\right)}. \end{aligned} \tag{38}$$

The maximum deflection occurs at $x = L/2$, which can be expressed as

$$\begin{aligned} w_{\max} &= \\ &q_0 L\left(L^4\sqrt{\xi_1}\cosh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right) + 64(\xi_1-1)\sqrt{\xi_1}\tau^2\left(L^2 + 6L\tau + 12\tau^2\right)\sinh^2\left(\frac{L}{4\sqrt{\xi_1}\tau}\right)\right. \\ &\left.+L\left(L^3 - 10L^2(\xi_1-1)\tau - 48L(\xi_1-1)\tau^2 - 96(\xi_1-1)\tau^3\right)\sinh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right)\right)\Big/ \\ &384EI\left((L - 2(\xi_1-1)\tau)\sinh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right) + L\sqrt{\xi_1}\cosh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right)\right). \end{aligned} \tag{39}$$

After taking some asymptotic analysis to $w(x)$ and $w_{\max}$, we get

$$\begin{aligned} w(x) &= \frac{q_0 x^4}{24EI} - \frac{q_0 L x^3}{12EI} \\ &+ \frac{q_0 x^2\left(L^3 + 12L(\sqrt{\xi_1}-1)\tau^2 + 24(\sqrt{\xi_1}-1)\tau^3\right)}{24EI\left(L + 2(1-\sqrt{\xi_1})\tau\right)} \\ &+ \frac{q_0 L(1-\sqrt{\xi_1})\left(L^2 + 6L\tau + 12\tau^2\right)\tau x}{12EI\left(L + 2(1-\sqrt{\xi_1})\tau\right)} \\ &- \frac{q_0 L\sqrt{\xi_1}(1-\sqrt{\xi_1})\left(L^2 + 6L\tau + 12\tau^2\right)\tau^2}{12EI\left(L + 2(1-\sqrt{\xi_1})\tau\right)}\left(1 - e^{-\frac{x}{\sqrt{\xi_1}\tau}} - e^{-\frac{L-x}{\sqrt{\xi_1}\tau}}\right) + E.S.T., \end{aligned} \tag{40}$$

$$w_{\max} = \frac{q_0 L^4\Big(1 + 10\left(1-\sqrt{\xi_1}\right)\frac{\tau}{L} + 16\left(2\xi_1 - 5\sqrt{\xi_1}+3\right)\left(\frac{\tau}{L}\right)^2\Big)}{384EI\left(1-2\left(\sqrt{\xi_1}-1\right)\frac{\tau}{L}\right)} + \frac{q_0 L^4\Big(96\left(2\xi_1-3\sqrt{\xi_1}+1\right)\left(\frac{\tau}{L}\right)^3 + 384\left(\xi_1-\sqrt{\xi_1}\right)\left(\frac{\tau}{L}\right)^4\Big)}{384EI\left(1-2\left(\sqrt{\xi_1}-1\right)\frac{\tau}{L}\right)} + E.S.T. \tag{41}$$

(3). (C-P) Clamped-Pinned beam with uniformly distributed load q_0

For this case, the boundary conditions are

$$w(0) = w'(0) = 0, \quad w(L) = M(L) = 0. \tag{42}$$

The lengthy expression for the exact solution of $w(x)$ is reported in Appendix B, and the asymptotic solution can be expressed as

$$\begin{aligned}
w(x) =\ & \frac{q_0 x^4}{24EI} \\
&+ \frac{q_0 L\left(-5L^3 + 12L^2\left(\sqrt{\xi_1}-1\right)\tau + 12L\left(\sqrt{\xi_1}-1\right)\tau^2 - 24\left(\xi_1-1\right)\sqrt{\xi_1}\tau^3\right)x^3}{48EI\left(L^3 - 3L^2\left(\sqrt{\xi_1}-1\right)\tau + 3L\left(\sqrt{\xi_1}-1\right)^2\tau^2 + 6\left(\xi_1-\sqrt{\xi_1}\right)\tau^3\right)} \\
&+ q_0\Big(L^5 - 4L^3\left(\xi_1 - 3\sqrt{\xi_1}+2\right)\tau^2 + 24L^2\left(\sqrt{\xi_1}-1\right)\tau^3 \\
&\quad +24L\left(\sqrt{\xi_1}-1\right)^3\left(\sqrt{\xi_1}+1\right)\tau^4 + 48\left(\sqrt{\xi_1}-1\right)^2\left(\xi_1+\sqrt{\xi_1}\right)\tau^5\Big)x^2 \Big/ \\
&16EI\Big(L^3 - 3L^2\left(\sqrt{\xi_1}-1\right)\tau + 3L\left(\sqrt{\xi_1}-1\right)^2\tau^2 + 6\left(\xi_1-\sqrt{\xi_1}\right)\tau^3\Big) \\
&- q_0\tau\left(\sqrt{\xi_1}-1\right)\Big(L^5 + 5L^4\tau - 4L^3\left(\xi_1-3\right)\tau^2 - 12L^2\left(\sqrt{\xi_1}-1\right)\tau^3 \\
&\quad +24L\left(\sqrt{\xi_1}-1\right)\sqrt{\xi_1}\left(\xi_1+1\right)\tau^4 + 48\left(\sqrt{\xi_1}-1\right){\xi_1}^{3/2}\tau^5\Big)x \Big/ \\
&8EI\Big(L^3 - 3L^2\left(\sqrt{\xi_1}-1\right)\tau + 3L\left(\sqrt{\xi_1}-1\right)^2\tau^2 + 6\left(\xi_1-\sqrt{\xi_1}\right)\tau^3\Big) \\
&+ q_0\tau^2\left(\xi_1-1\right)\sqrt{\xi_1}\Big(L^2\left(L^3 + 5L^2\tau - 4L\left(\xi_1-3\right)\tau^2 - 12\left(\xi_1-1\right)\tau^3\right) \\
&+\sqrt{\xi_1}\left(L^5 + 5L^4\tau - 4L^3\left(\xi_1-3\right)\tau^2 + 24L\left({\xi_1}^2-1\right)\tau^4 + 48\left(\xi_1-1\right)\xi_1\tau^5\right) \\
&-\Big(L^2\left(L^3 + 5L^2\tau - 4L\left(\xi_1-3\right)\tau^2 - 12\left(\xi_1-1\right)\tau^3\right) \\
&\quad +\sqrt{\xi_1}\left(L^5 + 5L^4\tau - 4L^3\left(\xi_1-3\right)\tau^2 + 24L\left({\xi_1}^2-1\right)\tau^4 + 48\left(\xi_1-1\right)\xi_1\tau^5\right)\Big)e^{\frac{-x}{\sqrt{\xi_1}\tau}} \\
&+\Big(L\tau\left(-3L^3 + 4L^2\left(\xi_1-3\right)\tau + 12L\left(\xi_1-1\right)\left(2\xi_1+1\right)\tau^2 + 48\left(\xi_1-1\right)\xi_1\tau^3\right) \\
&\quad +\sqrt{\xi_1}\tau\big(-3L^4 - 8L^3\xi_1\tau - 24L^2\left(\xi_1-1\right)\tau^2 \\
&\quad -24L\left({\xi_1}^2-1\right)\tau^3 - 48\left(\xi_1-1\right)\xi_1\tau^4\big)\Big)e^{\frac{x-L}{\sqrt{\xi_1}\tau}}\Big) \Big/ \\
&\Big(8EI\Big(\sqrt{\xi_1}\left(2L^3 - 3L^2\left(\xi_1-1\right)\tau + 6\left(\xi_1-1\right)\tau^3\right) \\
&+\left(L^3\left(\xi_1+1\right) - 3L^2\left(\xi_1-1\right)\tau + 3L\left(\xi_1-1\right)^2\tau^2 + 6\left(\xi_1-1\right)\xi_1\tau^3\right)\Big)\Big) + E.S.T.
\end{aligned} \tag{43}$$

The deflection of w_m at $x = (15-\sqrt{33})L/16$ (which is the maximum position in local theory) can be expressed by

$$\begin{aligned}
w_m =\ & \frac{q_0 L^4}{\left(65536EI\left(1 - 3\left(\sqrt{\xi_1}-1\right)\frac{\tau}{L} + 3\left(\sqrt{\xi_1}-1\right)^2\left(\frac{\tau}{L}\right)^2 + 6\left(\xi_1-\sqrt{\xi_1}\right)\left(\frac{\tau}{L}\right)^3\right)\right)} \\
&\Big(\left(39 + 55\sqrt{33}\right) + 2484.67\left(1-\sqrt{\xi_1}\right)\frac{\tau}{L} \\
&-\left(\sqrt{\xi_1}-1\right)\Big(\left(-3626.83\sqrt{\xi_1} + 10474.91\right)\left(\frac{\tau}{L}\right)^2
\end{aligned}$$

$$+\left(\sqrt{\xi_1}-1\right)\left(12612.34\xi_1+36451.74\sqrt{\xi_1}-23970.76\right)\left(\frac{\tau}{L}\right)^3$$
$$-\left(\sqrt{\xi_1}-1\right)\left(-126.64{\xi_1}^{3/2}+32894.64\xi_1-122274.77\sqrt{\xi_1}+23970.77\right)\left(\frac{\tau}{L}\right)^4$$
$$+\left(\sqrt{\xi_1}-1\right)^2\sqrt{\xi_1}\left(-113730.81\xi_1+65789.28\sqrt{\xi_1}-146245.54\right)\left(\frac{\tau}{L}\right)^5$$
$$+\left(\sqrt{\xi_1}-1\right)^2\xi_1\left(196608\xi_1-227461.63\xi_1\sqrt{\xi_1}+196608\right)\left(\frac{\tau}{L}\right)^6$$
$$+393216{\xi_1}^2\left(\sqrt{\xi_1}-1\right)^2\left(\frac{\tau}{L}\right)^7\Bigg)+E.S.T., \tag{44}$$

where the fractional representation is used.

(4). (C-F) Cantilever beam with concentrated load or distributed load

Case A (CFP): For a cantilever beam with concentrated load P at the free end, the boundary conditions are

$$w(0)=w'(0)=0;\quad M(L)=0,\ \left.\frac{dM(x)}{dx}\right|_{x=L}=P. \tag{45}$$

The exact solution for transverse displacement is

$$w(x)=-\frac{Px^3}{6EI}+\frac{PLx^2}{2EI}$$
$$+\frac{(1-\xi_1)P\tau\left((L+\tau)\left(\sinh(\frac{L}{\sqrt{\xi_1}\tau})+\sqrt{\xi_1}\cosh(\frac{L}{\sqrt{\xi_1}\tau})\right)+\sqrt{\xi_1}\tau\right)x}{EI\left((\xi_1+1)\sinh(\frac{L}{\sqrt{\xi_1}\tau})+2\sqrt{\xi_1}\cosh(\frac{L}{\sqrt{\xi_1}\tau})\right)}$$
$$-\frac{2(1-\xi_1)\sqrt{\xi_1}P\tau^2\sinh(\frac{x}{2\sqrt{\xi_1}\tau})}{EI\left((\xi_1+1)\sinh(\frac{L}{\sqrt{\xi_1}\tau})+2\sqrt{\xi_1}\cosh(\frac{L}{\sqrt{\xi_1}\tau})\right)}$$
$$\left((L+\tau)\cosh(\frac{2L-x}{2\sqrt{\xi_1}\tau})+\tau\sinh(\frac{x}{2\sqrt{\xi_1}\tau})+\sqrt{\xi_1}\tau\cosh(\frac{x}{2\sqrt{\xi_1}\tau})\right). \tag{46}$$

The maximum deflection occurs at $x=L$, which can be expressed as

$$w_{\max}=P\left(\sqrt{\xi_1}\left(2L^3-3L^2\left(\xi_1-1\right)\tau+6\left(\xi_1-1\right)\tau^3\right)\cosh(\frac{L}{\sqrt{\xi_1}\tau})\right.$$
$$-6\left(\xi_1-1\right)\sqrt{\xi_1}\tau^2(L+\tau)+\left(L^3\left(\xi_1+1\right)-3L^2\left(\xi_1-1\right)\tau\right.$$
$$\left.\left.+3L(\xi_1-1)^2\tau^2+6\left(\xi_1-1\right)\xi_1\tau^3\right)\sinh(\frac{L}{\sqrt{\xi_1}\tau})\right)\Bigg/$$
$$3EI\left((\xi_1+1)\sinh(\frac{L}{\sqrt{\xi_1}\tau})+2\sqrt{\xi_1}\cosh(\frac{L}{\sqrt{\xi_1}\tau})\right). \tag{47}$$

The asymptotic solutions for $w(x)$ and $w_{\max}$ are

$$w(x)=-\frac{Px^3}{6EI}+\frac{PLx^2}{2EI}+\frac{P(1-\sqrt{\xi_1})(L+\tau)\tau x}{EI}$$
$$-\frac{P(1-\sqrt{\xi_1})\sqrt{\xi_1}\tau^2}{EI(1+\sqrt{\xi_1})}\left(L+\tau-(L+\tau)e^{-\frac{x}{\sqrt{\xi_1}\tau}}+(1+\sqrt{\xi_1})\tau e^{-\frac{L-x}{\sqrt{\xi_1}\tau}}\right)$$
$$+E.S.T., \tag{48}$$

$$w_{\max}=\frac{PL^3\left(1+3\left(1-\sqrt{\xi_1}\right)\frac{\tau}{L}+3\left(\sqrt{\xi_1}-1\right)^2\left(\frac{\tau}{L}\right)^2+6\left(\xi_1-\sqrt{\xi_1}\right)\left(\frac{\tau}{L}\right)^3\right)}{3EI}$$
$$+E.S.T. \tag{49}$$

To the best of our knowledge, it is the first time that exact solutions to a cantilever beam problem is obtained with Eringen's integral model.

Case B (CFT): A cantilever beam with triangularly distributed load $(1 - x/L)q_0$. The load is obviously nonuniform,[12] but its second derivative is zero (for the purpose of comparing with Case C). The boundary conditions are

$$w(0) = w'(0) = 0; \quad M(L) = 0, \ \frac{dM(x)}{dx}\Big|_{x=L} = 0. \tag{50}$$

The exact solution for the transverse displacement is

$$\begin{aligned} w(x) &= -\frac{q_0 x^5}{120EIL} + \frac{q_0 x^4}{24EI} \\ &-\frac{q_0\left(L^2+2\left(\xi_1-1\right)\tau^2\right)x^3}{12EIL} + \frac{q_0\left(L^2+6\left(\xi_1-1\right)\tau^2\right)x^2}{12EI} \\ &+\frac{q_0\tau\left(\xi_1-1\right)\left(-c\left(\sinh(\frac{L}{\sqrt{\xi_1}\tau})+\sqrt{\xi_1}\cosh(\frac{L}{\sqrt{\xi_1}\tau})\right)-6{\xi_1}^{3/2}\tau^3\right)x}{6EIL\left(\left(\xi_1+1\right)\sinh(\frac{L}{\sqrt{\xi_1}\tau})+2\sqrt{\xi_1}\cosh(\frac{L}{\sqrt{\xi_1}\tau})\right)} \\ &+\frac{q_0\left(\xi_1-1\right)\sqrt{\xi_1}\tau^2}{6EIL\left(\left(\xi_1+1\right)\sinh(\frac{L}{\sqrt{\xi_1}\tau})+2\sqrt{\xi_1}\cosh(\frac{L}{\sqrt{\xi_1}\tau})\right)} \\ &\left(-c\ \sinh(\frac{L}{\sqrt{\xi_1}\tau})\left(\sqrt{\xi_1}\left(\cosh(\frac{x}{\sqrt{\xi_1}\tau})-1\right)-\sinh(\frac{x}{\sqrt{\xi_1}\tau})\right)\right. \\ &+c\cosh(\frac{L}{\sqrt{\xi_1}\tau})\left(\sqrt{\xi_1}\sinh(\frac{x}{\sqrt{\xi_1}\tau})-\cosh(\frac{x}{\sqrt{\xi_1}\tau})+1\right) \\ &\left.+6\xi_1\tau^3\left(\sqrt{\xi_1}\sinh(\frac{x}{\sqrt{\xi_1}\tau})+\cosh(\frac{x}{\sqrt{\xi_1}\tau})-1\right)\right), \end{aligned} \tag{51}$$

Hereafter, c denotes $L^3 + 3L^2\tau + 6L\xi_1\tau^2 + 6\xi_1\tau^3$.

The maximum deflection occurs at $x = L$, which can be expressed as

$$\begin{aligned} w_{\max} &= \frac{q_0\left(L^4+10L^2\left(\xi_1-1\right)\tau^2\right)}{30EI} \\ &+\frac{q_0\left(\xi_1-1\right)\tau\left(-c\left(\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)+\sqrt{\xi_1}\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\right)-6{\xi_1}^{3/2}\tau^3\right)}{6HT\left(\left(\xi_1+1\right)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)+2\sqrt{\xi_1}\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\right)} \\ &+\frac{q_0\left(\xi_1-1\right)\sqrt{\xi_1}\tau^2 c\ \sinh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right)}{6EIL\left(\sqrt{\xi_1}\sinh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right)+\cosh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right)\right)}. \end{aligned} \tag{52}$$

After taking some asymptotic analysis to $w(x)$ and $w_{\max}$, we get

$$\begin{aligned} w(x) &= -\frac{q_0 x^5}{120EIL} + \frac{q_0 x^4}{24EI} - \frac{q_0\left(L^2+2\left(\xi_1-1\right)\tau^2\right)x^3}{12EIL} \\ &+\frac{q_0\left(L^2+6\left(\xi_1-1\right)\tau^2\right)x^2}{12EI} + \frac{q_0\left(1-\sqrt{\xi_1}\right)c\,\tau x}{6EIL} \\ &-\frac{q_0\left(1-\sqrt{\xi_1}\right)\sqrt{\xi_1}\tau^2}{6EIL}\left(c-ce^{-\frac{x}{\sqrt{\xi_1}\tau}}+12\xi_1\tau^3\left(\sqrt{\xi_1}+1\right)e^{-\frac{L-x}{\sqrt{\xi_1}\tau}}\right)+E.S.T., \end{aligned} \tag{53}$$

$$w_{\max} = \frac{q_0 L^4}{30EI}\left(1 + 5\left(1-\sqrt{\xi_1}\right)\frac{\tau}{L} + 5L^3\left(3\xi_1 - 4\sqrt{\xi_1}+1\right)\left(\frac{\tau}{L}\right)^2\right.$$
$$-15L^2\left(2\xi_1^{3/2} - 3\xi_1 + \sqrt{\xi_1}\right)\left(\frac{\tau}{L}\right)^3 + 30L\left(\sqrt{\xi_1}-1\right)^2\xi_1\left(\frac{\tau}{L}\right)^4$$
$$\left.+60\left(\sqrt{\xi_1}-1\right)\xi_1^{3/2}\left(\frac{\tau}{L}\right)^5\right) + E.S.T. \tag{54}$$

Case C (CFE): A cantilever beam with exponentially distributed load $q_0 e^{x/L}$. The load is nonuniform, and its second derivative is nonzero.[11] The boundary conditions are the same with those in Case B.

The exact solution for transverse displacement can be obtained and the lengthy expressions are listed in Appendix B. The asymptotic expressions can be expressed as

$$w(x) = -\frac{q_0 L e x^3}{6EI} + \frac{q_0 L\left(-L^3 + L^2\tau - eL\left(\sqrt{\xi_1}-1\right)\tau^2 + e\left(\xi_1 - \sqrt{\xi_1}\right)\tau^3\right)x}{EI\left(L-\sqrt{\xi_1}\tau\right)}$$
$$+\frac{q_0 L}{EI\left(L^2 - \xi_1\tau^2\right)\left(1+\sqrt{\xi_1}\right)^2}\Big(\left(L^5\left(\xi_1+1\right)\left(e^{x/L}-1\right)\right.$$
$$+L^3\tau^2\left(-\left(\xi_1+1\right)e^{x/L} + \xi_1{}^2 + 1\right) + (e-1)L^2\left(\xi_1-1\right)\xi_1\tau^3$$
$$\left.-e\left(\xi_1-1\right)\xi_1{}^2\tau^5\right) + \sqrt{\xi_1}\left(2L^5\left(e^{x/L}-1\right) - L^3\tau^2\left(2e^{x/L} - \xi_1 - 1\right)\right.$$
$$\left.+(e-1)L^2\left(\xi_1-1\right)\tau^3 - e\left(\xi_1-1\right)\xi_1\tau^5\right)$$
$$-\left(\xi_1-1\right)\xi_1\sqrt{\xi_1}e\tau^4 e(L+\tau)\left(1+\sqrt{\xi_1}\right)e^{\frac{x-L}{\sqrt{\xi_1}\tau}}$$
$$-\left(\xi_1-1\right)\sqrt{\xi_1}\tau^2\left(L^3 + (e-1)L^2\tau - e\xi_1\tau^3\right)\left(1+\sqrt{\xi_1}\right)e^{\frac{-x}{\sqrt{\xi_1}\tau}}\Big) + E.S.T., \tag{55}$$

and the maximum deflection occurs at $x = L$, which can be expressed as

$$w_{\max} = \frac{q_0 L^4}{6EI\left(1-\xi_1\left(\frac{\tau}{L}\right)^2\right)}\left((5e-12) + 6\left(1-\sqrt{\xi_1}\right)\frac{\tau}{L}\right.$$
$$+\left((6+e)\xi_1 - 6e\sqrt{\xi_1} + 6\right)\left(\frac{\tau}{L}\right)^2 + 6(e-1)\left(\xi_1 - \sqrt{\xi_1}\right)\left(\frac{\tau}{L}\right)^3$$
$$\left.-6e\left(\sqrt{\xi_1}-1\right)^2\xi_1\left(\frac{\tau}{L}\right)^4 - 12e\left(\sqrt{\xi_1}-1\right)\xi_1^{3/2}\left(\frac{\tau}{L}\right)^5\right) + E.S.T. \tag{56}$$

V. ANALYSIS AND COMPARISONS OF SOLUTIONS

In this section, we shall examine the exact solutions obtained in last section more closely, which mainly includes nonlocal effect of the model, the effect of the boundary conditions, the relations with existing numerical and analytical solutions. The normalized deflections $\overline{w}$ (with respect to the maximum deflection of the local beam) of the beam for $\tau/L = 0.01, 0.03, 0.05$ with $\xi_1 = 0.1$ in each case are shown in Figures 1,2,3,4,5,6, respectively (denoted by *Group I*). The normalized maximum deflections $\overline{w}_{\max}$ (with respect to the maximum deflection of the local beam) for different values of τ/L and ξ_1 in each case are shown in Figures 7,8,9,10,11,12, respectively (denoted by *Group II*). Note that the maximum deflection of the local beam in each case is as follows[10–12]

$$\text{S-S:}\ \ w_{\max} = \frac{5q_0L^4}{384EI};\quad \text{C-C:}\ \ w_{\max} = \frac{q_0L^4}{384EI};$$
$$\text{C-P:}\ \ w_{\max} = \frac{q_0L^4}{65536EI}\left(39+55\sqrt{33}\right);\quad \text{CFP:}\ \ w_{\max} = \frac{PL^3}{3EI};$$
$$\text{CFT:}\ \ w_{\max} = \frac{q_0L^4}{30EI};\quad \text{CFE:}\ \ w_{\max} = \frac{(5e-12)q_0L^4}{6EI}. \tag{57}$$

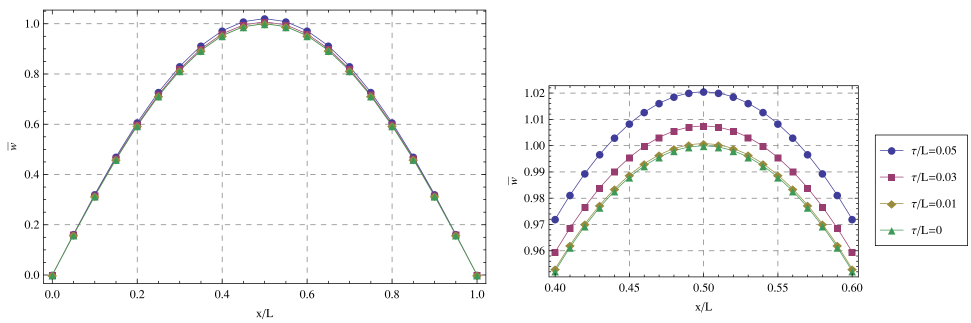

FIG. 1. $\overline{w}$ of a simply supported beam with uniformly distributed load for different values of τ/L.

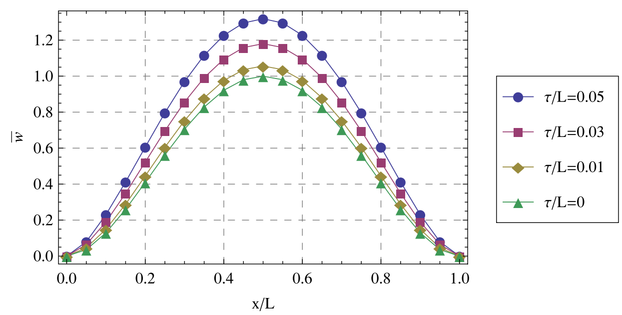

FIG. 2. $\overline{w}$ of a clamped beam with uniformly distributed load for different values of τ/L.

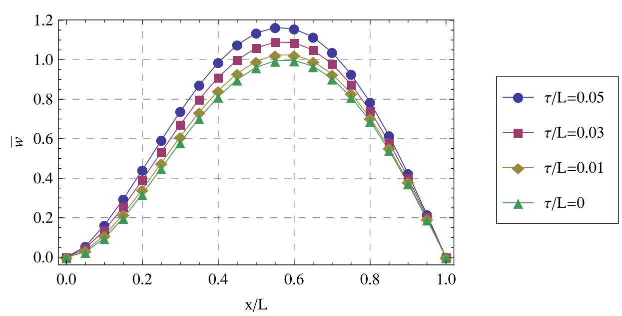

FIG. 3. $\overline{w}$ of a clamped-pinned beam with uniformly distributed load for different values of τ/L.

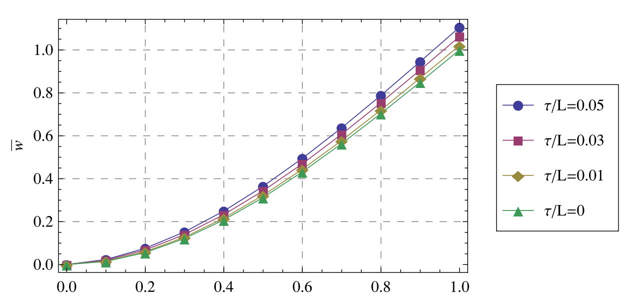

FIG. 4. $\overline{w}$ of a cantilever beam with a concentrated load for different values of τ/L.

1. nonlocal effects and boundary conditions. From the figures in *Group I*, it is clear that in all of the cases the deflection of the nonlocal beam is larger than that of the local one, and an increase in τ/L would increase the deflection of the beam, i.e., a softening effect. This can be also found from the solutions, especially the asymptotic one: the additional terms incorporated with

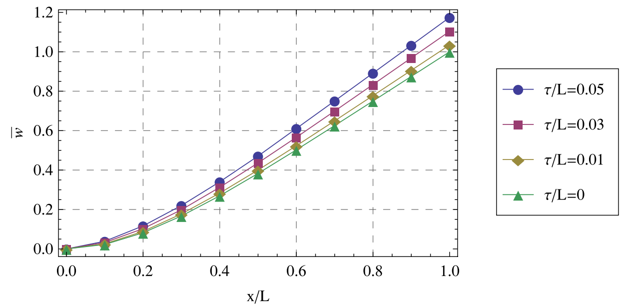

FIG. 5. $\overline{w}$ of a cantilever beam with triangularly distributed load for different values of τ/L.

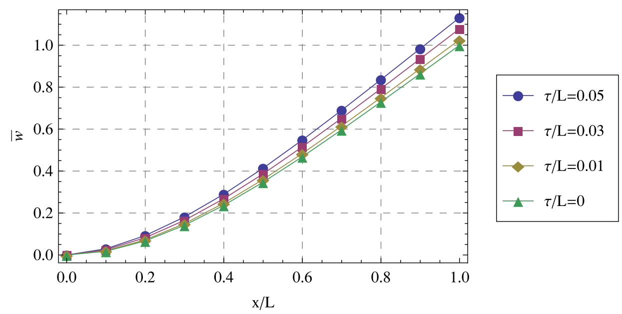

FIG. 6. $\overline{w}$ of a cantilever beam with exponentially distributed load for different values of τ/L.

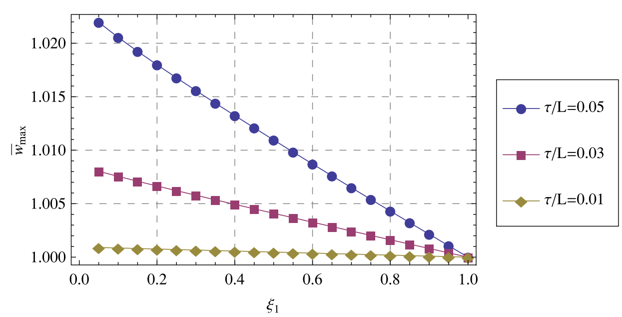

FIG. 7. $\overline{w}_{\max}$ of a simply supported beam with uniformly distributed load for different values of ξ_1 and τ/L.

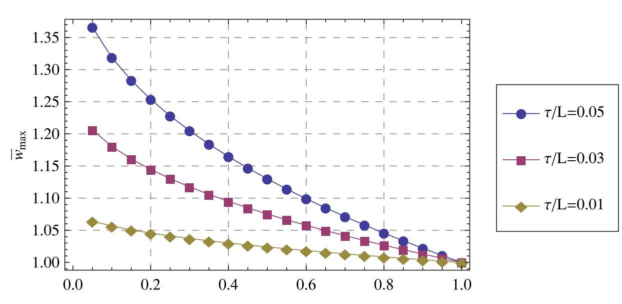

FIG. 8. $\overline{w}_{\max}$ of a clamped beam with uniformly distributed load for different values of ξ_1 and τ/L.

parameter τ/L contribute to nonlocal effect, especially the positive leading order term concerned with $O((\tau/L)^2)$ in the maximum deflection strongly shows the softening effect. The corresponding curve for $\tau/L = 0.01$ for S-S case is very close to the local one, and we provide a partial enlarged view to show the behavior near the mid-span. It should be pointed out that the softening effect predicted by the analytical solutions in this case is opposite to that in Khodabakhshi and Reddy.[12]

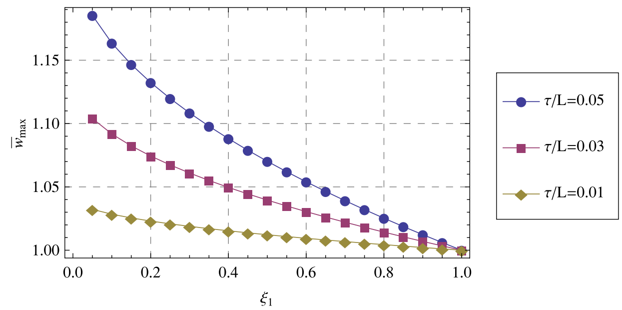

FIG. 9. $\overline{w}_{\max}$ of a clamped-pinned beam with uniformly distributed load for different values of ξ_1 and τ/L.

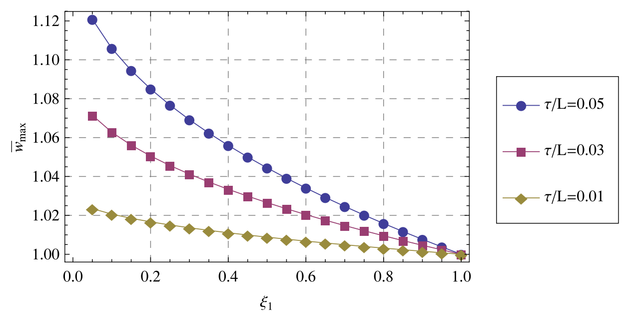

FIG. 10. $\overline{w}_{\max}$ of a cantilever beam with a concentrated load for different values of ξ_1 and τ/L.

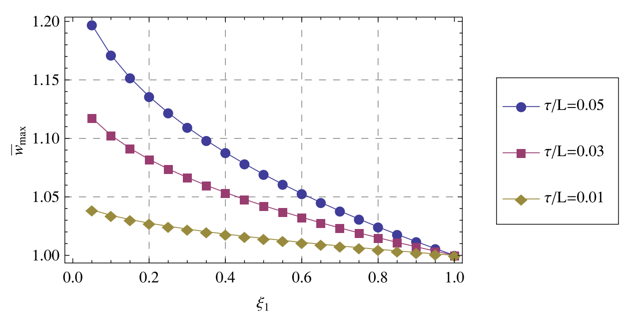

FIG. 11. $\overline{w}_{\max}$ of a cantilever beam with triangularly distributed load for different values of ξ_1 and τ/L.

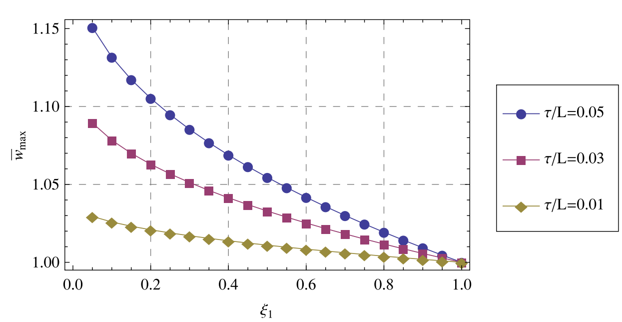

FIG. 12. $\overline{w}_{\max}$ of a cantilever beam with exponentially distributed load for different values of ξ_1 and τ/L.

In that paper, a FEM analysis is taken and the nonlocal beam is found to be slightly stiffer than the local one. This may due to the numerical error (some other comparisons can also be made[13,14]). A comparison among figures in *Group I* then tells that nonlocal effect for each boundary condition from largest to least generally would be: C-C, C-P, C-F, S-S. From the figures in *Group II*, it is clear that a decrease in ξ_1 would lead to an increase in the deflection of the beam. Moreover, the dependence of the maximum deflection on ξ_1 for S-S is nearly linear, while for the other cases it is nonlinear. This can also be seen from the leading order terms in the maximum deflections. Thus, in general, higher τ/L and lower ξ_1 both denote stronger softening effect in the nonlocal beam for all of the boundary conditions.

2. Cantilever beam: effect of loadings. Of interest here we have considered the cantilever beam with three types of loadings: concentrated load at the free end, nonuniformly distributed load whose second derivative is either zero or nonzero. In all of the cases, the model shows a consistent softening effect. It should be mentioned that, though the results are not presented here, for a cantilever beam with uniformly distributed load there is also a softening effect. The nonlocal effects in the cases CFP and CFT are consistent with the numerical results in Khodabakhshi and Reddy,[12] where a similar model is adopted. The nonlocal effect for CFE is opposite to that in Cheng *et al.*[11] (latter discussed by Preethi *et al.*[31] combined with surface stress effects), where the differential model (5) is used and a stiffening effect is found. It may be partially due to the integral kernel chosen here.

3. Limit solutions when $\xi_1 = 1$, $\xi_1 \to 0$. By setting $\xi_1 = 1$ in the nonlocal solutions obtained in last section, it can be easily checked that local solutions can be recovered in all of the cases. By letting $\xi_1 \to 0$ in the solutions, we can get a set of "limit solutions". For example, for cases S-S and CFP, if we let $\xi_1 \to 0$ in the solutions (cf. equations (35)(48)), then we obtain

$$\begin{aligned} w_0(x) &= \frac{q_0 x^4}{24EI} - \frac{q_0 L x^3}{12EI} - \frac{q_0 \tau^2 x^2}{2EI} + \frac{q_0 L\left(L^2 + 12\tau^2\right)x}{24EI}, \\ w_0(x) &= -\frac{Px^3}{6EI} + \frac{PLx^2}{2EI} + \frac{P(L+\tau)\tau x}{EI}. \end{aligned} \tag{58}$$

The above "limit solutions" coincide with the solutions in Tuna and Kirca[14] (equations (29) and (34) in that paper). For S-S case, equation (58a) also coincides with the solution by Reddy and Pang[10] using differential model (5). In order to check whether such "limit solutions" $w_0(x)$ are exact ones for pure nonlocal model (4), we can simply calculate the resulting nonlocal bending moment. That is, we need to check whether the corresponding version for pure nonlocal model (4) of equation (23) (i.e., by setting $\xi_1 = 0$)

$$\frac{1}{2\tau}\int_0^L e^{-\frac{|x-x'|}{\tau}} y(s)ds = \frac{-1}{EI}\left(C_1 + C_2 x - \int_0^x (x-s)q(s)ds\right), \tag{59}$$

is satisfied by $y(x) = w_0''(x)$. We can take the S-S case as an example. Note that, from equation (29), the right handside of (23)(59) should be the same, i.e., the bending moment calculated directly from the boundary conditions are both $M(x) = q_0(Lx - x^2)/2$. After substituting $y(x) = w_0''(x)$ (i.e., equation (58a)) into the left handside of equation (59), the resulting nonlocal bending moment $M_0(x)$ can be expressed as

$$M_0(x) = -EI\frac{1}{2\tau}\int_0^L e^{-\frac{|x-s|}{\tau}}\frac{d^2 w_0(s)}{ds^2}ds = \frac{q_0}{2}(Lx - x^2) + \frac{q_0 L\tau}{4}\left(e^{-\frac{x}{\tau}} + e^{-\frac{L-x}{\tau}}\right). \tag{60}$$

Obviously, $M_0(x) \neq M(x)$, especially at the boundaries $x = 0, L$, and the non-dimensional error is $O(\tau/L)$. That is, the boundary conditions are not exactly satisfied, hence the exact solutions for pure nonlocal model (4) can not be recovered by the "limit solutions". As $w_0''(x) = \lim_{\xi_1\to 0} w''(x,\xi_1)$ and $w''(x,\xi_1)$ satisfies equation (23) for any $\xi_1 > 0$, using equations (60) and (23) we can finally express "$M_0(x) \neq M(x)$" as

$$\int_0^L e^{-\frac{|x-s|}{\tau}}\frac{d^2 w_0(s)}{ds^2}ds = \int_0^L \lim_{\xi_1\to 0} e^{-\frac{|x-s|}{\tau}}\frac{d^2 w(s,\xi_1)}{ds^2}ds \neq \lim_{\xi_1\to 0}\int_0^L e^{-\frac{|x-s|}{\tau}}\frac{d^2 w(s,\xi_1)}{ds^2}ds, \tag{61}$$

which implies the underlying reason is that the bi-exponential integral kernel here does not allow an exchange between the limit (of ξ_1) and the integration. From an integral equation point of view, it means that the limit of the solution to the second kind integral equation (23) is not the solution to its first kind version (59), a phenomenon usually encountered in the field of integral equations. It is the same situation for CFP case (probably for all of the other cases).

As the analytical solutions by Tuna and Kirca[14] share the same expressions with the "limit solutions", they are not exact solutions to pure nonlocal model (4), neither. For example, for S-S case in that paper, although the expression in equation (25) for the bending moment meets the boundary conditions, however, if one substitutes equation (29) into equation (7) to compute the resulting bending moment at either $x = 0$ or $x = L$, the values are then nonzero and the error is $O(\tau/L)$ (κ in that paper has the same meaning as τ). Hence the boundary conditions are not exactly satisfied. Actually, as the numerical solution in Fernández-Sáez *et al.*[13] showed that, for S-S case, the relative error of such analytical solution at $x = L/4$ is about 0.1 percent. Moreover, we think some exponential-like terms, which would affect the behavior near the boundaries, may be neglected in such analytical solutions. Thus, the relative error near the boundaries would be even larger. To summarize, we conclude that the "limit solutions" (or the solutions in Tuna and Kirca[14]) are not exact solutions for pure nonlocal model (4). However, for relatively small τ/L, they may be good approximations to the inner part of the beam, and it has been shown in Tuna and Kirca.[14]

VI. CONCLUSIONS AND FUTURE WORK

In this work, a reduction method in the literature is proved rigorously and applied to study bending problem of nonlocal Euler-Bernoulli beam with various boundary conditions. Eringen's two-phase local/nonlocal model is adopted. As far the examples examined, the following conclusions can be made:

1. Exact solutions are obtained for nonlocal beams with the integral model, especially for the paradoxical cantilever beam problem.

2. For the local/nonlocal model adopted here, through examining several typical boundary conditions and loadings, there is a consistent softening effect.

The analytical solutions are validated through comparisons with existing analytical and numerical solutions. It appears that the integral model considered here has some advantages as compared with differential model (5). That is, it has a consistent softening effect for bending, and there is no paradox when solving a cantilever beam problem. So, it would be interesting to investigate the buckling and vibration response of the model. Also, the reduction method proved here can be applied to related analytical studies of this model.

ACKNOWLEDGEMENTS

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APPENDIX A

Consider the integral equation

$$y(x) + A\int_a^b e^{\lambda|x-s|}y(s)ds = f(x), \tag{A1}$$

where $f(x) \in C^2[a,b],\ A > 0,\ \lambda < 0$.

In nonlocal theory, for given loadings, the unknown local stress field is not necessarily to be smooth enough (say, twice differentiable). So, an assumption is made on the loadings (i.e., $f(x) \in C^2[a,b]$), then we can deduce the smoothness of the unknown local stress and differentiate the two sides of the integral equation. The method is mainly based on the fact that for a twice differentiable function $Y(x) = y(x) + A\int_a^b e^{\lambda|x-s|}y(s)ds$ (or, we can view it as a functional of the unknown function $y(x)$), the equation

$$Y(x) = f(x),\quad x \in [a,b] \tag{A2}$$

can be transformed to an equivalent one

$$Y''(x) = f''(x), \;\; Y(a) = f(a), Y(b) = f(b), \tag{A3}$$

as long as the latter equation can be satisfied for given function $f(x)$. As to the existence and uniqueness of the solutions in nonlocal theory, though its general conditions has been well established,[19–21] for specific problems we still need to check such issues. Since it is usually difficult to determine the eigenvalues of the integral operator (crucial for the existence and uniqueness of the solutions), we prove the uniqueness of its equivalent differential equation instead, and it can serve as a theoretic basis for the unique determination of the related constants (c.f. (28)) in a beam theory. Here, we assume the existence of the solution in $L^2[a,b]$ (i.e., square integrable) and one can refer to Tricomi[27] for the Fredholm theory for integral equation of the second kind. Base on the above-mentioned issues, the proof is divided into three parts.

Firstly, we show that $y(x) \in C^2[a,b]$. In order to show this, we can express $y(x)$ as

$$\begin{aligned} y(x) &= f(x) - A\int_a^b e^{\lambda|x-s|}y(s)ds \\ &= f(x) - A\left(e^{\lambda x}\int_a^x e^{-\lambda s}y(s)ds + e^{-\lambda x}\int_x^b e^{\lambda s}y(s)ds\right). \end{aligned} \tag{A4}$$

Note that $f(x) \in C^2[a,b]$ and the two integrals at the right hand side both belong to $C[a,b]$, it then implies that $y(x) \in C[a,b]$. Once we have $y(x) \in C[a,b]$, the aforementioned integrals then belong to $C^1[a,b]$, which then implies that $y(x) \in C^1[a,b]$. With the same deduction, we can finally get $y(x) \in C^2[a,b]$. Actually, it can be shown that $y(x)$ shares the same regularity with $f(x)$ for this specialized integral equation.

Secondly, we get the differential equation by differentiating both sides of the integral equation. As all terms of equation (A1) are now $C^2[a,b]$, we can differentiate both sides term by term twice. The proof in this step just follows those provided in Polyanin and Manzhirov.[26]

Differentiate both sides of (A1), we have

$$y'(x) + \lambda A e^{\lambda x}\int_a^x e^{-\lambda s}y(s)ds - \lambda A e^{-\lambda x}\int_x^b e^{\lambda s}y(s)ds = f'(x). \tag{A5}$$

Differentiate the resulting equation one more time, we then have

$$y''(x) + \lambda^2 A e^{\lambda x}\int_a^x e^{-\lambda s}y(s)ds + \lambda^2 A e^{-\lambda x}\int_x^b e^{\lambda s}y(s)ds + 2\lambda A y(x) = f''(x), \tag{A6}$$

by (A1) the above equation can be rewritten as

$$y''(x) + \lambda(2A - \lambda)y(x) = f''(x) - \lambda^2 f(x). \tag{A7}$$

Equation (A1) at $x = a, b$ can be expressed as

$$\begin{aligned} A\int_a^b e^{\lambda(s-a)}y(s)ds &= f(a) - y(a) \\ A\int_a^b e^{\lambda(b-s)}y(s)ds &= f(b) - y(b). \end{aligned} \tag{A8}$$

That is, the solution to equation (A7) should also satisfy equation (A8), if it is supposed to be a solution of equation (A1). This can be done by expressing $\lambda A y(x)$ from (A7) via $y''(x), f''(x)$ and substitute the results into (A8). Integration by parts then yields the conditions

$$\begin{aligned} e^{\lambda b}(\phi'(b) - \lambda\phi(b)) - e^{\lambda a}(\phi'(a) + \lambda\phi(a)) = 0, \\ e^{-\lambda b}(\phi'(b) - \lambda\phi(b)) - e^{-\lambda a}(\phi'(a) + \lambda\phi(a)) = 0, \end{aligned} \tag{A9}$$

where $\phi(x) = y(x) - f(x)$. It then requires that

$$\phi'(b) - \lambda\phi(b) = 0, \;\; \phi'(a) + \lambda\phi(a) = 0, \tag{A10}$$

which can be rewritten as the boundary conditions for $y(x)$

$$y'(a) + \lambda y(a) = f'(a) + \lambda f(a), \ y'(b) - \lambda y(b) = f'(b) - \lambda f(b). \tag{A11}$$

Equations (A7), (A11) together constitute the boundary value problem of a differential equation which the solution $y(x)$ of integral equation (A1) should satisfy.

Thirdly, we prove that equations (A7), (A11) admit a unique solution. To prove this, it is equivalent to demonstrate that the corresponding homogeneous equation

$$y''(x) + \lambda(2A - \lambda)y(x) = 0, \tag{A12}$$

$$y'(a) + \lambda y(a) = 0, \quad y'(b) - \lambda y(b) = 0 \tag{A13}$$

has trivial solution only. Since we assume that $A > 0, \lambda < 0$, it implies that $\mu = \lambda(2A - \lambda) < 0$. So, it is suffice to demonstrate that $\mu < 0$ is not an eigenvalue of the above differential operator with the boundary conditions. We can prove it by contradiction as follows.

If $\mu < 0$ is an eigenvalue of the above problem, then there exists constants $l_1, l_2(l_1 l_2 \neq 0)$ such that

$$y(x) = l_1 e^{-\sqrt{-\mu}x} + l_2 e^{\sqrt{-\mu}x} \tag{A14}$$

is a solution of the problem. To determine l_1, l_2, we substitute it into the boundary conditions to get

$$\begin{aligned}(\lambda - \sqrt{-\mu})e^{-\sqrt{-\mu}a} l_1 + (\lambda + \sqrt{-\mu})e^{\sqrt{-\mu}a} l_2 = 0,\\ (-\lambda - \sqrt{-\mu})e^{-\sqrt{-\mu}a} l_1 + (-\lambda + \sqrt{-\mu})e^{\sqrt{-\mu}a} l_2 = 0.\end{aligned} \tag{A15}$$

For the linear system to have nonzero solutions, its determinant D should be zero

$$D = -(\lambda - \sqrt{-\mu})^2 e^{\sqrt{-\mu}(b-a)} + (\lambda + \sqrt{-\mu})^2 e^{\sqrt{-\mu}(a-b)} = 0, \tag{A16}$$

which then requires that

$$e^{2\sqrt{-\mu}(a-b)} = \frac{(\lambda - \sqrt{-\mu})^2}{(\lambda + \sqrt{-\mu})^2}. \tag{A17}$$

As $a < b$, the left handside is always less than one. However, since $\lambda < 0, \sqrt{-\mu} > 0$, the right hand-side is larger than one. Thus, there is a contradiction. So, our assumption $l_1, l_2(l_1 l_2 \neq 0)$ is untrue. That is, equations (A7),(A11) have a unique solution. We have completed our proof.

APPENDIX B

1. Exact solution for the transverse displacement $w(x)$ of a clamped-pinned beam with transverse distributed load q_0.

$$\begin{aligned}
w(x) = {} & \frac{q_0 x^4}{24EI} - q_0 L x^3 \Big(2\sqrt{\xi_1}\left(5L^3 - 6L^2(\xi_1 - 1)\tau - 6L(\xi_1 - 1)\tau^2\right.\\
& \left.+12\left(\xi_1{}^2 - 1\right)\tau^3\right)\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right) + \left(5L^3(\xi_1 + 1) - 12L^2(\xi_1 - 1)\tau - 12L(\xi_1 - 1)\tau^2\right.\\
& \left.+48(\xi_1 - 1)\xi_1\tau^3\right)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right) - 12(\xi_1 - 1)\sqrt{\xi_1}\tau^2(3L + 2(\xi_1 + 1)\tau)\Big)\Big/\\
& 48EI\Big(\sqrt{\xi_1}\left(2L^3 - 3L^2(\xi_1 - 1)\tau + 6(\xi_1 - 1)\tau^3\right)\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\\
& + \left(L^3(\xi_1 + 1) - 3L^2(\xi_1 - 1)\tau + 3L(\xi_1 - 1)^2\tau^2 + 6(\xi_1 - 1)\xi_1\tau^3\right)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\\
& -6(\xi_1 - 1)\sqrt{\xi_1}\tau^2(L + \tau)\Big)\\
& +q_0 x^2\Big(-6(\xi_1 - 1)\xi_1\tau^2\left(L^3 + 2L^2\xi_1\tau + 4L(\xi_1 - 1)\tau^2 + 4(\xi_1 - 1)\tau^3\right)
\end{aligned}$$

$$
\begin{aligned}
&+\xi_1\left(L^2+2\left(\xi_1-1\right)\tau^2\right)\left(L^3+12\left(\xi_1-1\right)\tau^3\right)\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\\
&+\frac{1}{2}\sqrt{\xi_1}\left(L^5\left(\xi_1+1\right)-4L^3\left(\xi_1-2\right)\left(\xi_1-1\right)\tau^2+24L^2\left(\xi_1-1\right)\tau^3+24L\left(\xi_1-1\right)^3\tau^4\right.\\
&\quad\left.\left.\left.+48\left(\xi_1-1\right)^2\xi_1\tau^5\right)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\right)\right)\Big/\\
&8EI\sqrt{\xi_1}\left(\sqrt{\xi_1}\left(2L^3-3L^2\left(\xi_1-1\right)\tau+6\left(\xi_1-1\right)\tau^3\right)\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\right.\\
&+\left(L^3\left(\xi_1+1\right)-3L^2\left(\xi_1-1\right)\tau+3L\left(\xi_1-1\right)^2\tau^2+6\left(\xi_1-1\right)\xi_1\tau^3\right)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\\
&\quad\left.-6\left(\xi_1-1\right)\sqrt{\xi_1}\tau^2(L+\tau)\right)\\
&+q_0\left(\xi_1-1\right)\tau x\left(L^2\left(-\left(L^3+5L^2\tau-4L\left(\xi_1-3\right)\tau^2-12\left(\xi_1-1\right)\tau^3\right)\right)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\right.\\
&-\sqrt{\xi_1}\left(L^5+5L^4\tau-4L^3\left(\xi_1-3\right)\tau^2+24L\left({\xi_1}^2-1\right)\tau^4+48\left(\xi_1-1\right)\xi_1\tau^5\right)\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\\
&\quad\left.+\sqrt{\xi_1}\tau\left(3L^4+8L^3\xi_1\tau+24L^2\left(\xi_1-1\right)\tau^2+24L\left({\xi_1}^2-1\right)\tau^3+48\left(\xi_1-1\right)\xi_1\tau^4\right)\right)\Big/\\
&8EI\left(\sqrt{\xi_1}\left(2L^3-3L^2\left(\xi_1-1\right)\tau+6\left(\xi_1-1\right)\tau^3\right)\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\right.\\
&+\left(L^3\left(\xi_1+1\right)-3L^2\left(\xi_1-1\right)\tau+3L\left(\xi_1-1\right)^2\tau^2+6\left(\xi_1-1\right)\xi_1\tau^3\right)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\\
&\quad\left.-6\left(\xi_1-1\right)\sqrt{\xi_1}\tau^2(L+\tau)\right)\\
&+\left(\left(\xi_1-1\right)\sqrt{\xi_1}q_0\tau^2\left(L\tau\left(3L^3-4L^2\left(\xi_1-3\right)\tau+12L\left(-2{\xi_1}^2+\xi_1+1\right)\tau^2\right.\right.\right.\\
&\quad\left.-48\left(\xi_1-1\right)\xi_1\tau^3\right)-L^2\left(L^3+5L^2\tau-4L\left(\xi_1-3\right)\tau^2-12\left(\xi_1-1\right)\tau^3\right)\cosh\left(\frac{L-x}{\sqrt{\xi_1}\tau}\right)\\
&+L^2\left(L^3+5L^2\tau-4L\left(\xi_1-3\right)\tau^2-12\left(\xi_1-1\right)\tau^3\right)\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\\
&+\sqrt{\xi_1}\left(L^5+5L^4\tau-4L^3\left(\xi_1-3\right)\tau^2+24L\left({\xi_1}^2-1\right)\tau^4+48\left(\xi_1-1\right)\xi_1\tau^5\right)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\\
&+\left(L\tau\left(-3L^3+4L^2\left(\xi_1-3\right)\tau+12L\left(\xi_1-1\right)\left(2\xi_1+1\right)\tau^2+48\left(\xi_1-1\right)\xi_1\tau^3\right)\right.\\
&-\sqrt{\xi_1}\left(L^5+5L^4\tau-4L^3\left(\xi_1-3\right)\tau^2+24L\left({\xi_1}^2-1\right)\tau^4\right.\\
&\quad\left.\left.+48\left(\xi_1-1\right)\xi_1\tau^5\right)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\right)\cosh\left(\frac{x}{\sqrt{\xi_1}\tau}\right)+\sqrt{\xi_1}\left(\left(L^5+5L^4\tau-4L^3\left(\xi_1-3\right)\tau^2\right.\right.\\
&\quad\left.+24L\left({\xi_1}^2-1\right)\tau^4+48\left(\xi_1-1\right)\xi_1\tau^5\right)\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)+\tau\left(-3L^4-8L^3\xi_1\tau\right.\\
&\quad\left.\left.\left.\left.-24L^2\left(\xi_1-1\right)\tau^2-24L\left({\xi_1}^2-1\right)\tau^3-48\left(\xi_1-1\right)\xi_1\tau^4\right)\right)\sinh\left(\frac{x}{\sqrt{\xi_1}\tau}\right)\right)\right)\Big/\\
&\left(8EI\left(\sqrt{\xi_1}\left(2L^3-3L^2\left(\xi_1-1\right)\tau+6\left(\xi_1-1\right)\tau^3\right)\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\right.\right.\\
&+\left(L^3\left(\xi_1+1\right)-3L^2\left(\xi_1-1\right)\tau+3L\left(\xi_1-1\right)^2\tau^2+6\left(\xi_1-1\right)\xi_1\tau^3\right)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\\
&\quad\left.\left.-6\left(\xi_1-1\right)\sqrt{\xi_1}\tau^2(L+\tau)\right)\right).
\end{aligned}
\tag{B1}
$$

2. Exact solution for the transverse displacement $w(x)$ of a cantilever beam with transverse distributed load $q_0 e^{x/L}$.

$$
\begin{aligned}
w(x) = &-\frac{q_0 e L x^3}{6EI} \\
&-\frac{q_0 L x}{\left(EI\left(L^2-\xi_1\tau^2\right)\left((\xi_1+1)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)+2\sqrt{\xi_1}\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\right)\right)} \\
&\left(\left(-e(\xi_1-1)\xi_1{}^{3/2}\tau^3(L+\tau)+\left(L^4(\xi_1+1)+L^3(\xi_1-1)\tau\right.\right.\right. \\
&\left.+L^2((e-2)\xi_1-e)\tau^2-e(\xi_1-1)\xi_1\tau^4\right)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right) \\
&-\sqrt{\xi_1}\left(-2L^4+L^3(\tau-\xi_1\tau)+L^2(-e\xi_1+\xi_1+e+1)\tau^2\right. \\
&\left.\left.\left.+e(\xi_1-1)\xi_1\tau^4\right)\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\right)\right) \\
&+\frac{q_0 L}{EI\left(\left(L^2-\xi_1\tau^2\right)\left((\xi_1+1)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)+2\sqrt{\xi_1}\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\right)\right)} \\
&\left(-(\xi_1-1)\sqrt{\xi_1}\tau^2\left(\left(-L^3+L^2(\tau-e\tau)+e\xi_1\tau^3\right)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\right.\right. \\
&\left.+e\xi_1{}^{3/2}\tau^2(L+\tau)\right)\sinh\left(\frac{x}{\sqrt{\xi_1}\tau}\right)+(\xi_1-1)\xi_1\tau^2 \\
&\left(\left(-L^3+L^2(\tau-e\tau)+e\xi_1\tau^3\right)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)-e\sqrt{\xi_1}\tau^2(L+\tau)\right)\cosh\left(\frac{x}{\sqrt{\xi_1}\tau}\right) \\
&+\left(L^5(\xi_1+1)\left(e^{\frac{x}{L}}-1\right)+L^3\tau^2\left(-(\xi_1+1)e^{\frac{x}{L}}+\xi_1{}^2+1\right)\right. \\
&\left.+(e-1)L^2(\xi_1-1)\xi_1\tau^3-e(\xi_1-1)\xi_1{}^2\tau^5\right)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right) \\
&+\sqrt{\xi_1}\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\left(2L^5\left(e^{\frac{x}{L}}-1\right)-L^3\tau^2\left(2e^{\frac{x}{L}}-\xi_1-1\right)+(e-1)L^2(\xi_1-1)\tau^3\right. \\
&-(\xi_1-1)\tau^2\left(L^3+(e-1)L^2\tau-e\xi_1\tau^3\right)\left(\cosh\left(\frac{x}{\sqrt{\xi_1}\tau}\right)\right. \\
&\left.\left.\left.-\sqrt{\xi_1}\sinh\left(\frac{x}{\sqrt{\xi_1}\tau}\right)\right)-e(\xi_1-1)\xi_1\tau^5\right)+e(\xi_1-1)\xi_1{}^{3/2}\tau^4(L+\tau)\right).
\end{aligned}
\tag{B2}
$$

The maximum deflection occurs at $x = L$, which can be expressed as

$$
\begin{aligned}
w_{\max} = &\frac{q_0 L}{6EI}\left(-eL^3+\left(6(-1+e)L^3(L-\tau)(L+\tau)\cosh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right)\right.\right. \\
&-6\sqrt{\xi_1}\left(-(-1+e)L^5+L^3(e-\xi_1)\tau^2+L^2(-1+e+\xi_1-e\xi_1)\tau^3\right. \\
&\left.\left.+eL(-1+\xi_1)\xi_1\tau^4+2e(-1+\xi_1)\xi_1\tau^5\right)\sinh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right)\right)\Bigg/ \\
&\left(\left(L^2-\xi_1\tau^2\right)\left(\cosh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right)+\sqrt{\xi_1}\sinh\left(\frac{L}{2\sqrt{\xi_1}\tau}\right)\right)\right) \\
&-\left(6L\left(-e(-1+\xi_1)\xi_1{}^{3/2}\tau^3(L+\tau)-\sqrt{\xi_1}\left(-2L^4+L^2(1+e+\xi_1-e\xi_1)\tau^2\right.\right.\right. \\
&\left.+e(-1+\xi_1)\xi_1\tau^4+L^3(\tau-\xi_1\tau)\right)\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right) \\
&+\left(L^4(1+\xi_1)+L^3(-1+\xi_1)\tau+L^2(-e+(-2+e)\xi_1)\tau^2\right.
\end{aligned}
$$

$$-e\left(-1+\xi_1\right)\xi_1\tau^4\Big)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\Big)\Big)\Big)\Bigg/$$
$$\left(\left(L^2-\xi_1\tau^2\right)\left(2\sqrt{\xi_1}\cosh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)+\left(1+\xi_1\right)\sinh\left(\frac{L}{\sqrt{\xi_1}\tau}\right)\right)\right)\Bigg). \tag{B3}$$

[1] A. C. Eringen and B. S. Kim, Mech. Res. Commun. **1**, 233 (1974).
[2] A. C. Eringen, Int. J. Fract. **14**, 367 (1978).
[3] A. C. Eringen, Eng. Fract. Mech. **12**, 211 (1979).
[4] A. C. Eringen, C. G. Speziale, and B. S. Kim, J. Mech. Phys. Solids **25**, 339 (1977).
[5] B. Arash and Q. Wang, Comp. Mater. Sci. **51**, 303 (2012).
[6] A. C. Eringen, J. Appl. Phys. **54**, 4703 (1983).
[7] J. Peddieson, G. R. Buchanan, and R. P. McNitt, Int. J. Eng. Sci. **41**, 305 (2003).
[8] Q. Wang and K. M. Liew, Phys. Lett. A **363**, 236 (2007).
[9] N. Challamel and C. Wang, Nanotechnology **19**, 345703 (2008).
[10] J. N. Reddy and S. D. Pang, J. Appl. Phys. **103**, 023511 (2008).
[11] C. Li, L. Yao, W. Chen, and S. Li, Int. J. Eng. Sci. **87**, 47 (2015).
[12] P. Khodabakhshi and J. N. Reddy, Int. J. Eng. Sci. **95**, 60 (2015).
[13] J. Fernández-Sáez, R. Zaera, J. A. Loya, and J. N. Reddy, Int. J. Eng. Sci. **99**, 107 (2016).
[14] M. Tuna and M. Kirca, Int. J. Eng. Sci. **105**, 80 (2016).
[15] A. C. Eringen, Res Mechanica **21**, 313 (1987).
[16] A. A. Pisano and P. Fuschi, Int. J. Solids Struct. **46**, 3836 (2009).
[17] R. Abdollahi and B. Boroomand, Int. J. Solids Struct. **50**, 2758 (2013).
[18] R. Abdollahi and B. Boroomand, Int. J. Solids Struct. **51**, 1758 (2014).
[19] S. B. Altan, Bull. Tech. Univ. Istanbul **37**, 373 (1984).
[20] S. B. Altan, Int. J. Solids Struct. **25**, 1271 (1989).
[21] C. Polizzotto, Int. J. Solids Struct. **38**, 7359 (2001).
[22] A. A. Pisano and P. Fuschi, Int. J. Solids Struct. **40**, 13 (2003).
[23] X. W. Zhu and H. H. Dai, Sci. China Phys. Mech. Astron. **55**, 1059 (2012).
[24] E. Benvenuti and A. Simone, Mech. Res. Commun. **48**, 46 (2013).
[25] A. C. Eringen, *Nonlocal Continuum Field Theories* (Springer, New York, 2002).
[26] A. Polyanin and A. Manzhirov, *Handbook of integral equations* (CRC Press, New York, 2008).
[27] F. G. Tricomi, *Integral eqations* (Dove Publications, New York, 1985).
[28] J. N. Reddy, *Energy principles and variational methods in applied mechanics*, 2nd ed. (John Wiley and Sons, New York, 2002).
[29] J. N. Reddy, Int. J. Eng. Sci. **45**, 288 (2007).
[30] Mathematica version 9, Wolframe Research, Inc., Champaign, Illinois, 2013.
[31] K. Preethi, A. Rajagopal, and J. N. Reddy, Int. J. Nonlinear Mech. **76**, 100 (2015).