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EXACT SOLUTIONS TO THE RIEMANN PROBLEM FOR COMPRESSIBLE ISOTHERMAL EULER EQUATIONS FOR TWO-PHASE FLOWS WITH AND WITHOUT PHASE TRANSITION

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Abstract. We consider the isothermal Euler equations with phase transition between a liquid and a vapor phase. The mass transfer is modeled by a kinetic relation. We prove existence and uniqueness results. Further, we construct the exact solution for Riemann problems. We derive analogous results for the cases of initially one phase with resulting condensation by compression or evaporation by expansion. Further we present numerical results for these cases. We compare the results to similar problems without phase transition.

1. Introduction. We study compressible multi-phase flows without and with phase transitions relying on the isothermal Euler equations with a nonmonotone pressure-density function. Our main objective is a detailed discussion of a thermodynamically based kinetic relation that controls the mass transfer across a sharp interface between two coexisting phases. The derivation of the kinetic relation is based on thermodynamics, especially on classical Hertz-Knudsen theory; see Bond and Struchtrup [4]. To this end we study Riemann problems and show for various classes of initial data the existence and

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uniqueness of solutions. We consider single-phase initial data describing condensation by compression or evaporation by expansion, as well as initial data describing two differing adjacent phases. The case of multi-phase flows without phase transition mainly serves as an illustration and as a comparison with other treatments of the same subject in the literature.

Phase transitions can be treated either by sharp interface models or by models that describe the interface between two adjacent phases by a smooth transition within the setting of phase field models. Sharp interface models are physically better founded while phase field models may have numerical advantages. The available sharp interface models are surveyed in Zein [23].

The phase field model of Euler-Korteweg type by Dreyer et al. [9] establishes a sharp interface limit that produces our kinetic relation, whereupon the mass flux across the interface is proportional to the jump of the Gibbs free energy. A similar study of the same model by Benzoni-Gavage et al. [3] ends up with a kinetic relation describing local equilibrium at the interface; i.e., the Gibbs free energy is continuous.

The seminal paper by Abeyaratne and Knowles [1] considers a solid-solid phase transition and describes the Riemann problem of the corresponding Euler system in Lagrangian coordinates. For this reason the nonlinearities appearing there are different from the current study. The kinetic relation in [1] relies on the same driving force as we use here. However, Abeyaratne and Knowles relate the mass flux to the jump of the Gibbs free energy in a nonlinear manner.

A very interesting review on the Riemann problem for a large class of thermodynamically consistent constitutive models in the setting of Euler equation models by Menikoff and Plohr [14] is restricted to a simple kinetic relation that results from the assumption of a local equilibrium at the interface. For isothermal processes a local interfacial equilibrium is guaranteed by the continuity of the Gibbs free energy.

Merkle [15] also considered the Riemann problem for the isothermal Euler system. Differences to the current work are: he used the van der Waals equation to model the nonmonotone pressure-density dependence. We observed that it is better to model the pressure-density function by pieces of three linear functions. This leads to a closer agreement with measured data, e.g. for a substance such as water. The kinetic relation introduced by Merkle does arise from thermodynamic motivations. But there are initial data for which it must be supplemented by further assumptions in order to pick up a unique solution. Furthermore the structure of the solutions is essentially different from those that we obtain here. Our solutions consist exclusively of three types of elementary waves, namely classical shocks, rarefaction waves and phase transitions, that separate a certain number of constant states. Merkle needs composite waves to construct the solution.

The isothermal Euler system was also studied by Müller and Voss [18], [21]. They modeled the fluid by a van der Waals equation; however, instead of a kinetic relation they exclusively applied the Liu entropy condition in order to establish uniqueness. Consequently Müller and Voss also need composite waves.

There are also studies of the same subject that use the Euler equations in a different manner than they are used here. Despite the fact that in those studies the nonisothermal case is considered, the main difference to our study concerns the application of a full

Euler system to each phase everywhere in space. Thus the number of balance equations is doubled. Additionally there is an equation determining the local phase fraction. The basic paper is that of Baer and Nunziato [2]. However, it is restricted to 2-phase flows without phase transition. The main aim of those models is to study phase mixtures such as e.g. bubbly flows or sprays. Zein et al. [24] started from this approach and added the continuity of the Gibbs free energy across the interface in order to allow for a phase transition.

For basics on conservation laws, see the books of Toro [20], Lax [12], LeVeque [13], Smoller [19], Kröner [11], Dafermos [5] and others. For thermodynamics, see for instance Müller and Müller [16] as well as Müller [17].

Next we describe the main results of the current study. Our kinetic relation can be obtained in two different ways. It follows in the sharp interface limit that starts with the isothermal Navier-Stokes-Korteweg model and ends up with the corresponding isothermal Euler equations; see Dreyer et al. [9]. In this case the kinetic relation gives the mass flux across the interface as a linear function of the jump of the Gibbs free energy and it is proportional to the Navier-Stokes viscosities. A more physical derivation of the kinetic relation can be given in the setting of the Hertz-Knudsen theory; its nonisothermal version is described in Bond and Struchtrup [4]. Here the only difference between the two derivations is the factor of proportionality that is related to the sound velocity at the gas side of the interface.

A main consequence of this kinetic equation is the absence of composite waves in the solution to Riemann problems. If we consider a Riemann problem where the left and right state correspond to two different phases, our kinetic relation implies a solution that exclusively consists of two classical waves and a phase transition in between. This construction is unique and generates classes of initial data, for which the existence of solutions is guaranteed.

If we consider a Riemann problem where the left and right state correspond to the same phase, two cases may occur. Either the two states can be connected by only classical waves or, if this is not possible, nucleation of the other phase is enforced by the kinetic relation. Also here we prove existence and uniqueness.

The paper is organized as follows. In Section 2 we introduce the system of balances in the bulk and across the interface. Details of the equations of state are given in Section 3, whereas the entropy inequality is discussed in the following section. In Section 5 we obtain mathematical properties of the system considered. Moreover we discuss rarefactions and shocks for the isothermal case. The main part of this section is Subsection 5.3. Here we introduce the kinetic relation and prove a uniqueness result for the pressures at the phase interface. Moreover, we derive monotonicity results for interface quantities. Based on these results we construct the exact solution for the isothermal Euler euqations with phase transition, presented in Subsection 6.2. We prove uniqueness results within the class of Riemann problems as well as sufficient conditions for solvability. In Section 7 we discuss the cases of condensation by compression as well as evaporation by expansion. As before we prove several existence and uniqueness results. Also we present the exact solution for the Riemann problems considered. Finally we give numerical examples for all cases considered. These are presented in Section 8.

2. Isothermal Euler equations. In our study we consider inviscid fluids under the isothermality assumption. This means that the temperature T_0 is fixed. The phases are indicated by the value of the mass density ρ, and we have the velocity v as a variable. The physical fields are assumed to depend on time $t \in \mathbb{R}_{\geq 0}$ and space $x \in \mathbb{R}$. In regular points of the bulk phases we have the local mass conservation law (2.1) and the balance law for momentum (2.2). These are

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho v^2 + p)}{\partial x} = 0\,. \tag{2.2}$$

In the momentum balance equation (2.2) there is a further quantity, the pressure p. It is not among the basic variables and is therefore called a constitutive quantity. This quantity is related to the variable ρ in a material dependent manner by an equation of state. This will be given in Section 3. The system (2.1)-(2.2) is called a system of isothermal Euler equations.

Across any discontinuity we have the jump conditions

$$[\![\rho(v - W)]\!] = 0, \tag{2.3}$$

$$\rho(v - W)[\![v]\!] + [\![p]\!] = 0\,. \tag{2.4}$$

Here we use the jump brackets $[\![\Psi]\!] = \Psi'' - \Psi'$ for any physical quantity Ψ, where $'$ and $''$ denote the one-sided limits to the left and right of the discontinuity, respectively, on the horizontal x-axis. Further, W denotes the propagation speed of the discontinuity. The mass flux Z across the discontinuity is given by

$$Z = -\rho(v - W) \tag{2.5}$$

with

$$Z = \begin{cases} Q & \text{shock wave} \\ z & \text{phase boundary} \end{cases} \quad \text{and} \quad W = \begin{cases} S & \text{shock wave} \\ w & \text{phase boundary.} \end{cases} \tag{2.6}$$

For more details on interface relations, see Dreyer [7, Sections 5-14] and Müller [17, Section 2.2.2, Chapter 3].

3. Equations of state. The pressure is related to the density by the equation of state

$$p = p(\rho) \quad \text{with} \quad p'(\rho) = a^2 = \text{const}\,, \tag{3.1}$$

where a denotes the speed of sound.

In particular, for the vapor phase V we use the ideal gas law

$$p_V = \rho_V \frac{kT_0}{m} \tag{3.2}$$

for given temperature T_0. Here k denotes the Boltzmann constant and m is the mass of a single water molecule.

The liquid phase L is modeled as a compressible fluid whose pressure is related to the liquid density by

$$p_L = p_0 + K_0 \left(\frac{\rho_L}{\rho_0} - 1 \right), \tag{3.3}$$

where the pressure p_0 and the density ρ_0 denote arbitrary reference values. The constant K_0 is the modulus of compression, which is temperature dependent. For convenience we choose p_0, ρ_0, K_0 at the saturation state; see the table in [22]. The data can also be found in [10].

In order to characterize the two phases we introduce two constant parameters that will be properly defined later on. Within a range $0 \leq \rho_V \leq \tilde{\rho}$ the fluid is assumed to be in the vapor state. For $\rho_L \geq \rho_m$ the liquid phase is present. Between the pure phases there are intermediate states, whose pressure is defined by a linear function of negative slope. For more details, see Section 5.3.

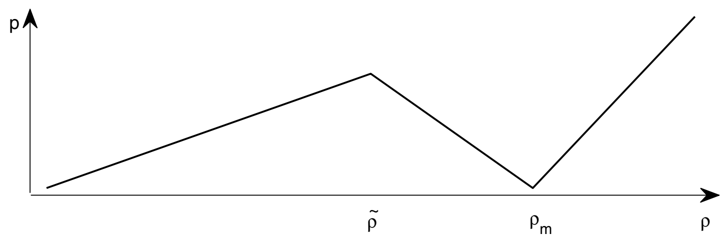

FIG. 1. Equation of state: $p(\rho)$

According to the second law of thermodynamics the pressure is the derivative of the Helmholtz free energy with respect to $1/\rho$,

$$p = -\frac{\partial \psi}{\partial (1/\rho)} .$$

The Gibbs free energy is defined by

$$g = \psi + \frac{p}{\rho} .$$

This quantity occurs in the entropy inequality for isothermal processes

$$Z[\![g + \frac{1}{2}(v - W)^2]\!] \leq 0 . \tag{3.4}$$

For details, see Dafermos [5], Merkle [15], Müller and Voss [18].

4. Riemann problem. In our study we consider the Riemann problem for the isothermal Euler equations. This is given by the balances (2.1)-(2.2), the equation of state (3.1) and the corresponding Riemann initial data

$$\rho(x,0) = \begin{cases} \rho_- & \text{for} \quad x < 0 \\ \rho_+ & \text{for} \quad x > 0 \end{cases} \qquad \text{and} \qquad v(x,0) = \begin{cases} v_- & \text{for} \quad x < 0 \\ v_+ & \text{for} \quad x > 0 . \end{cases} \tag{4.1}$$

We denote the solution to the Riemann problem by $\mathbf{W}$. The solution consists of constant states $\mathbf{W} = const$, that are separated by waves or phase boundaries. We will denote neighboring states by $'$ and $''$, as done in Section 2. The Riemann problem is solved by self-similar solutions of type $\mathbf{W}(t,x) = \hat{\mathbf{W}}(x/t)$.

5. Generic solution. In order to give the mathematical properties of the Euler system (2.1)-(2.2), we rewrite the system in quasilinear form in terms of ρ and v:

$$\begin{pmatrix} \rho \\ v \end{pmatrix}_t + \begin{pmatrix} v & \rho \\ \frac{a^2}{\rho} & v \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The Jacobian matrix is

$$\mathbf{A} = \begin{pmatrix} v & \rho \\ \frac{a^2}{\rho} & v \end{pmatrix}$$

with the eigenvalues

$$\lambda_1 = v - a \qquad \text{and} \qquad \lambda_2 = v + a$$

as well as the corresponding right eigenvectors

$$\mathbf{r}_1 = \begin{pmatrix} \rho \\ -a \end{pmatrix} \qquad \text{and} \qquad \mathbf{r}_2 = \begin{pmatrix} \rho \\ a \end{pmatrix}.$$

The system is strictly hyperbolic. Finally we give the Riemann invariants

$$I_1 = v + a \ln \rho = const \qquad \text{and} \qquad I_2 = v - a \ln \rho = const \tag{5.1}$$

across the left and right wave, respectively.

5.1. *Rarefaction wave fans.* Assume that the wave corresponding to λ_1 is a (left) 1-rarefaction. Then we use the Riemann invariant given in $(5.1)_1$ to obtain

$$v' + a \ln \rho' = v'' + a \ln \rho''. \tag{5.2}$$

For a left rarefaction the head speed is given by $v' - a$ whereas the tail speed is given by $v'' - a$. The slope inside the rarefaction fan is given by

$$\frac{dx}{dt} = \frac{x}{t} = v - a.$$

Using (5.2) we obtain that the solution $\mathbf{W}$ inside the fan is given by

$$\mathbf{W}_{1fan} = \begin{cases} v &= a + \frac{x}{t} \\ \rho &= \exp\left(\frac{v'-v}{a} + \ln \rho'\right). \end{cases} \tag{5.3}$$

On the other hand, using $(5.1)_2$ for a (right) 2-rarefaction we get

$$v' - a \ln \rho' = v'' - a \ln \rho''. \tag{5.4}$$

Analogously to the above calculations for a 2-rarefaction wave we have the head speed $v'' + a$ and the tail speed $v' + a$. The solution inside the fan is then given by

$$\mathbf{W}_{2fan} = \begin{cases} v &= -a + \frac{x}{t} \\ \rho &= \exp\left(\frac{v-v''}{a} + \ln \rho''\right). \end{cases} \tag{5.5}$$

5.2. *Shocks.*

5.2.1. *Entropy inequality across a shock wave.* In this section we want to prove that the Lax condition is equivalent to the entropy condition for the system considered. We take the case where the states

$$\begin{pmatrix} \rho' \\ v' \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \rho'' \\ v'' \end{pmatrix}$$

are separated by a shock wave that propagates with speed S. W.l.o.g. we assume that $v' = 0$. This assumption is used to simplify the following calculations and is only used in Section 5.2.1. Due to $v' = 0$ we have $v'' < 0$. Then from the Rankine-Hugoniot conditions we obtain for S,

$$S = -\frac{\rho'' v''}{\rho' - \rho''} \tag{5.6}$$

and

$$S = \frac{-\rho'' v''^2 + a^2(\rho' - \rho'')}{-\rho'' v''}\,.$$

This gives

$$v''^2 = a^2 \frac{(\rho' - \rho'')^2}{\rho' \rho''}\,. \tag{5.7}$$

Further, the entropy inequality is given by

$$\rho' S \left(a^2 \ln \frac{\rho'}{\rho''} + \frac{1}{2} S^2 - \frac{1}{2} (v'' - S)^2 \right) \leq 0\,.$$

For the second factor we obtain, using (5.1) twice, then (5.6) and (5.7),

$$\begin{aligned}
a^2 \ln \frac{\rho'}{\rho''} + \frac{1}{2} S^2 - \frac{1}{2}(v'' - S)^2 &= a^2 \ln \frac{\rho'}{\rho''} + \frac{Q^2}{2} \left(\frac{1}{\rho'^2} - \frac{1}{\rho''^2} \right) \\
&= a^2 \ln \frac{\rho'}{\rho''} + \frac{\rho'^2 S^2}{2} \left(\frac{1}{\rho'^2} - \frac{1}{\rho''^2} \right) \\
&= a^2 \ln \frac{\rho'}{\rho''} + \frac{\rho'^2 \rho''^2 v''^2}{2(\rho' - \rho'')^2} \left(\frac{1}{\rho'^2} - \frac{1}{\rho''^2} \right) \\
&= a^2 \ln \frac{\rho'}{\rho''} + a^2 \frac{\rho' \rho''}{2} \left(\frac{1}{\rho'^2} - \frac{1}{\rho''^2} \right) \\
&= a^2 \left(\ln \frac{\rho'}{\rho''} + \frac{\rho' \rho''}{2} \left(\frac{1}{\rho'^2} - \frac{1}{\rho''^2} \right) \right) \quad \begin{matrix} = 0, & \rho' = \rho'', \\ > 0, & \rho' < \rho'', \\ < 0, & \rho' > \rho''. \end{matrix}
\end{aligned}$$

For the case $\rho' < \rho''$ we have from (5.7) that $S < 0$, whereas for the second case $\rho' > \rho''$ this leads to $S > 0$. In the first case we thus have from (5.6) and (5.7) that

$$S = \frac{\rho''}{\rho'' - \rho'} v'' > v'' \qquad \text{and} \qquad S = -a \frac{\rho''}{\sqrt{\rho' \rho''}} < -a\,.$$

This implies the Lax condition $a > -a > S > v'' - a$, which in general notation is given by

$$v' + a > v' - a > S > v'' - a;$$

see Lax [12]. Obviously in that case we have a left or 1-shock. Similarly in the second case we have a right or 2-shock and we obtain the corresponding Lax condition

$$v' + a > S > v'' + a > v'' - a\,.$$

In summary, for the isothermal Euler equations, the entropy condition and the Lax condition are equivalent. For this special system this is a more general result than that given in Dafermos [5]. Based on the explicit constitutive functions used here this statement is true for arbitrarily strong shocks.

5.2.2. *Shock relations.* Let us assume that the left wave is a shock wave, propagating with speed S_1. As was done in Toro [20] we define the relative velocities

$$\hat{v}' = v' - S_1 \qquad \text{and} \qquad \hat{v}'' = v'' - S_1\,. \tag{5.8}$$

We obtain the corresponding Rankine-Hugoniot conditions

$$\rho'\hat{v}' = \rho''\hat{v}'', \tag{5.9}$$

$$\rho'\hat{v}'^2 + p' = \rho''\hat{v}''^2 + p''\,. \tag{5.10}$$

For the mass flux Q_1 we have

$$-Q_1 = \rho'(v' - S_1) = \rho''(v'' - S_1) = \rho'\hat{v}' = \rho''\hat{v}''\,. \tag{5.11}$$

We substitute Q_1 into (5.10) to obtain

$$-Q_1\hat{v}' + a^2\rho' = -Q_1\hat{v}'' + a^2\rho''\,.$$

Solving for $-Q_1$ and using the entropy condition discussed above, this leads to

$$-Q_1 = -\frac{a^2(\rho'' - \rho')}{\hat{v}'' - \hat{v}'} = -\frac{a^2(\rho'' - \rho')}{v'' - v'} > 0\,, \tag{5.12}$$

which gives us

$$v'' = v' + \frac{a^2(\rho'' - \rho')}{Q_1}\,. \tag{5.13}$$

On the other hand, using (5.11) to substitute $\hat{v}'$ and $\hat{v}''$ in (5.12), we derive the relation

$$-Q_1 = -\frac{a^2(\rho'' - \rho')}{-\frac{Q_1}{\rho''} + \frac{Q_1}{\rho'}} \tag{5.14}$$

and get

$$Q_1^2 = a^2\rho'\rho''\,. \tag{5.15}$$

In combination with (5.13) and $Q_1 < 0$ this gives us across a left shock,

$$v'' = v' - \frac{a^2(\rho'' - \rho')}{\sqrt{a^2\rho'\rho''}}\,.$$

Finally, from (5.11) and (5.15) we obtain the speed of a left shock,

$$S_1 = v' + \frac{Q_1}{\rho'} = v' - \frac{\sqrt{a^2\rho'\rho''}}{\rho'}\,.$$

For a right shock the calculations are very similar. We obtain $Q_2 > 0$ and

$$v'' = v' + \frac{a^2(\rho'' - \rho')}{\sqrt{a^2\rho'\rho''}}$$

as well as

$$S_2 = v' + \frac{Q_2}{\rho'} = v' + \frac{\sqrt{a^2\rho'\rho''}}{\rho'}\,.$$

In general terms the result is given by

$$v'' = v' - \frac{a^2|\rho'' - \rho'|}{\sqrt{a^2\rho'\rho''}} \qquad \text{and} \qquad S = v' + \frac{Q}{\rho'}\,. \tag{5.16}$$

REMARK 5.1. Note that our notation is similar to, but slightly different from, the notation in the book of Toro [20].

5.3. *Phase transition.*

5.3.1. *Definition of the phases.* In the case that the discontinuity represents a boundary between two phases we always have

$$\rho_V < \rho_L\,. \tag{5.17}$$

Furthermore, from the mass and momentum balances (2.3)-(2.4) across the phase boundary together with (2.5) and (2.6) we obtain

$$z^2 = -\frac{p' - p''}{\frac{1}{\rho'} - \frac{1}{\rho''}}\,.$$

With the above relation for the densities (5.17) we conclude that

$$p_L \geq p_V \qquad \text{and with} \qquad p_V \geq 0 \qquad \text{we have} \qquad p_L \geq 0\,. \tag{5.18}$$

The second statement is due to the fact that we ignore surface tension. We define, see (3.3),

$$\rho_m = \frac{\rho_0}{K_0}(K_0 - p_0)\,, \tag{5.19}$$

which gives $p_L(\rho_m) = 0$. Corresponding to ρ_m we have to find $\tilde{\rho}$. This value is uniquely defined by the equation of state (3.1), equation (5.19) and the Maxwell condition

$$\int_{\frac{1}{\rho_0}}^{\frac{1}{\rho_V(p_0)}} p(\rho)\, d\frac{1}{\rho} = \left(\frac{1}{\rho_V(p_0)} - \frac{1}{\rho_0}\right) \cdot p_0\,.$$

After some calculations we obtain

$$\frac{K_0}{\rho_0}\ln\frac{\rho_0}{\rho_m} + \frac{kT_0}{m}\ln\frac{\rho_m}{\rho_V(p_0)} - \frac{\rho_m}{\rho_m - \tilde{\rho}}\frac{kT_0}{m}\ln\frac{\rho_m}{\tilde{\rho}} = 0\,. \tag{5.20}$$

This relation defines $\tilde{\rho}$ uniquely for sufficiently low temperatures $T \leq 633.15$K. For higher temperatures the definition of ρ_m gives a negative value. The critical temperature T_c for water is given by $T_c = 647.096$K. For $T_0 = 573.15$K we obtain $\tilde{\rho} = 36.515$ kg/m^3; see Figure 2. The corresponding reference values are given by $p_0 = 18.6664$mPa, $\rho_0 = 1/0.00189451$kg/m^3 and $K_0 = 1/36.627 \cdot 10^9$pa.

Furthermore we give the curves $\rho_m(T)$ and $\tilde{\rho}(T)$, see Figure 3*a*, and the quotient $\tilde{\rho}(T)/\rho_m(T)$, see Figure 3*b*. Obviously one has

$$\tilde{\rho}(T)/\rho_m(T) < 1/4 \tag{5.21}$$

for all temperatures 273.15K$\leq T_0 \leq 623.15$K.

REMARK 5.2. In our notation all temperature-dependent constants have index 0. If we choose T_0 we have to use the corresponding reference values ρ_0, p_0, K_0.

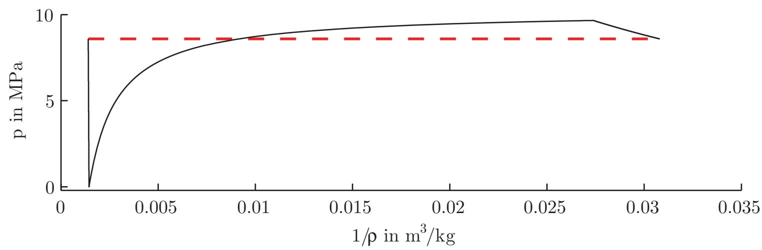

FIG. 2. Equation of state: $p(1/\rho)$ for $T_0 = 573.15\mathrm{K}$, dashed: Maxwell line

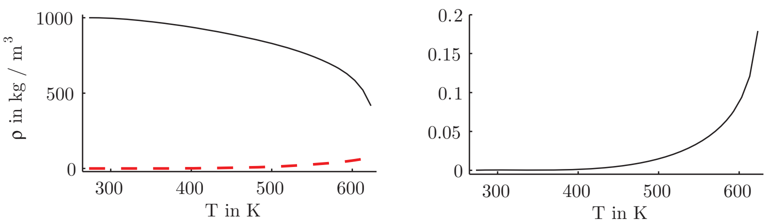

FIG. 3. *a*) dashed: $\tilde{\rho}(T)$, solid: $\rho_m(T)$ *b*) $\tilde{\rho}(T)/\rho_m(T)$

REMARK 5.3. Most estimations in this paper are based on the data found in [22]. Accordingly *for all temperatures* usually means the finite number of discrete temperature values in the table in [22]. For the intermediate temperatures not included in the table we have: If for monotonic temperature changes the temperature-dependent constants change monotonically, the estimations are also valid for the intermediate temperatures.

5.3.2. *A simple kinetic relation to describe phase transitions.* Besides the balances for mass (2.3) and momentum (2.4) at the phase boundary we need a further equation, which is called a kinetic relation. This equation describes the rate of change of mass across the interface. We choose

$$z = \frac{p_V}{\sqrt{2\pi}} \left(\frac{m}{kT_0}\right)^{\frac{3}{2}} [\![g + e_{kin}]\!] \,, \tag{5.22}$$

where V denotes the vapor phase. For details of the derivation, see Dreyer et al. [8]. If the vapor phase is to the left of the liquid phase, this results in

$$z = \frac{p_V}{\sqrt{2\pi}} \left(\frac{m}{kT_0}\right)^{\frac{3}{2}} \left[\frac{K_0}{\rho_0} \ln\frac{\rho_L}{\rho_0} - \frac{kT_0}{m} \ln\frac{p_V}{p_0} + \frac{1}{2}(v_L - w)^2 - \frac{1}{2}(v_V - w)^2\right] . \tag{5.23}$$

Here V and L denote the vapor and the liquid phase, respectively. Equation (5.24) gives the kinetic relation for the case that the vapor phase is to the right:

$$-z = \frac{p_V}{\sqrt{2\pi}} \left(\frac{m}{kT_0}\right)^{\frac{3}{2}} \left[\frac{K_0}{\rho_0} \ln\frac{\rho_L}{\rho_0} - \frac{kT_0}{m} \ln\frac{p_V}{p_0} + \frac{1}{2}(v_L - w)^2 - \frac{1}{2}(v_V - w)^2\right] . \tag{5.24}$$

For the moment we restrict ourselves to the case that the vapor phase is on the left side. Therefore in this section we identify $'$ (left state) with the vapor phase and $''$ (right state) with the liquid phase.

If condensation and evaporation are excluded, we replace (5.22) by the new kinetic relation

$$z = 0\,. \tag{5.25}$$

This implies that $v_V = v_L$ at the phase boundary; see (2.5) and (2.6).

5.3.3. *Uniqueness of p_L for given p_V.* If p_V is given, we have to determine 4 unknowns, namely p_L, v_L, v_V and z. At the interface we have 4 conditions: two mass flux conditions (2.5), the interface momentum balance (2.4) and furthermore the kinetic relation (5.23). Our goal is to determine an equation for p_L. The interface momentum balance can be written as

$$[\![p]\!] = -z^2[\![\frac{1}{\rho}]\!]\,. \tag{5.26}$$

Because $\rho_L > \rho_V$ we have

$$p_L = p_V \quad \Leftrightarrow \quad z = 0\,.$$

This is the equilibrium case $p_L = p_V = p_0$. Otherwise we have $p_V < p_L$.

In the following lemma we will make the assumption

$$-a_V\rho_V \le z \le a_L\rho_L\,. \tag{5.27}$$

It simplifies the calculations and later it turns out to be automatically satisfied due to physical considerations; see Remark 5.11.

Lemma 5.4. Consider the isothermal case with 273.15K$\le T_0 \le$ 623.15K. Then for a given interface pressure p_V of the vapor phase with $0 \le p_V \le \tilde{p}$, the conditions (5.27) and the corresponding equations of state (3.2), (3.3) define the liquid interface pressure p_L, uniquely. Furthermore by these relations the mass flux z is uniquely defined.

Proof. We replace z in (5.26) by the kinetic relation (5.23) and get

$$[\![p]\!] + \left(\frac{m}{kT_0}\right)^3 \frac{p_V^2}{2\pi}\left[\frac{K_0}{\rho_0}\ln\frac{\rho_L}{\rho_0} - \frac{kT_0}{m}\ln\frac{p_V}{p_0} - \frac{1}{2}[\![p]\!]\left(\frac{1}{\rho_L}+\frac{1}{\rho_V}\right)\right]^2[\![\frac{1}{\rho}]\!] = 0\,. \tag{5.28}$$

Next we define the functions

$$h(p_V,p_L) = \left(\frac{m}{kT_0}\right)^{3/2}\frac{1}{\sqrt{2\pi}}\left[\frac{K_0}{\rho_0}\ln\frac{\rho_L}{\rho_0} - \frac{kT_0}{m}\ln\frac{p_V}{p_0} - \frac{1}{2}[\![p]\!]\left(\frac{1}{\rho_L}+\frac{1}{\rho_V}\right)\right] \tag{5.29}$$

and

$$f(p_V,p_L) = [\![p]\!] + h^2(p_V,p_L)p_V^2[\![\frac{1}{\rho}]\!]$$

for $p_V \ge 0$ and $p_L \ge p_V$. The roots of the latter function are the solutions of (5.28).

(1) Let us consider $p_V = p_0$, i.e. the saturation pressure. Then for $p_L = p_0$ we have $f(p_0,p_L) = 0$. So (p_0,p_0) is a solution of (5.28). It obviously satisfies (5.27) with $z = 0$.

(2) We note that

$$\frac{\partial f}{\partial p_V}(p_0,p_0) = -1 \qquad \text{and} \qquad \frac{\partial f}{\partial p_L}(p_0,p_0) = 1\,.$$

Accordingly in a neighborhood of $p_V = p_0$, relation (5.28) implicitly defines a function $p_L(p_V)$ with $p'_L(p_V) > 0$. This means, in a neighborhood of $p_V = p_0$, that relation (5.28) has a solution that satisfies the inequalities (5.27).

(3) By our assumption we consider a temperature regime, where in (3.3) we have

$$p_L = p_0 + K_0 \frac{\rho_L}{\rho_0} - K_0 < p_0 + K_0 \frac{\rho_L}{\rho_0} - (p_0 - p_V)\,.$$

Therefore

$$1 - \frac{p_L - p_V}{K_0} \frac{\rho_0}{\rho_L} > 0,$$

and we conclude that

$$\frac{\partial h}{\partial p_L}(p_V, p_L) = \left(\frac{m}{kT_0}\right)^{3/2} \frac{1}{\sqrt{2\pi}} \left[\frac{1}{2}\left(\frac{1}{\rho_L} - \frac{1}{\rho_V}\right) + \frac{1}{2} \frac{p_L - p_V}{K_0} \frac{\rho_0}{\rho_L} \frac{1}{\rho_L}\right] < 0\,.$$

For any fixed p_V the function $h(p_V, p_L)$ is strictly decreasing in p_L. Due to $p_L \geq p_V$ it attains its maximum at $p_L = p_V$.

(4) Next we calculate

$$\frac{\partial f}{\partial p_L}(p_V, p_L) = 1 - p_V^2 \cdot h^2(p_V, p_L) \frac{1}{\rho_L^2} \frac{\rho_0}{K_0} + p_V^2 h(p_V, p_L) \frac{\partial h}{\partial p_L}(p_V, p_L) [\![\frac{1}{\rho}]\!].$$

Let us consider any p_V^*, p_L^* such that $f(p_V^*, p_L^*) = 0$ and (5.27) is satisfied. Let us further consider that $z > 0$. Then we obtain

$$\frac{\partial f}{\partial p_L}(p_V^*, p_L^*) > 1 - \frac{K_0}{\rho_0} \frac{\rho_0}{K_0} = 0\,.$$

On the other hand, if $z < 0$, then

$$\frac{\partial f}{\partial p_L}(p_V^*, p_L^*) > 1 - \frac{\rho_V^{*2}}{\rho_L^{*2}} \frac{a_V^2}{a_L^2} - \left(\frac{m}{kT_0}\right)^{3/2} \frac{p_V^*}{\sqrt{2\pi}} \rho_V a_V \frac{1}{\rho_V^{*2}} > 0\,.$$

So p_L^* is a simple root of $f(p_V^*, \cdot)$ with $\frac{\partial f}{\partial p_L}(p_V^*, p_L^*) > 0$.

(5) Because of $f(p_V^*, \cdot) \to -\infty$ for $p_L \to \infty$ it is clear that f has a further root $p_L^{**} > p_L^*$ with $\frac{\partial f}{\partial p_L}(p_V^*, p_L^{**}) \leq 0$. Accordingly (p_V^*, p_L^{**}) does not satisfy the inequality (5.27); see step 4. Moreover, by monotonicity of h there is no further root $p_L > p_L^*$ that satisfies (5.27); see step 3.

By the same arguments as before there is no further root $p_L < p_L^*$.

(6) We have seen that in a neighborhood of $p_V = p_0$ for every fixed p_V^* there exists a unique p_L^* such that $f(p_V^*, p_L^*) = 0$ and (5.27) are satisfied. Next we want to show that this is true for every $0 \leq p_V < p_0$.

Assume that there exists a $p_V < p_0$ such that there is no solution p_L with $f(p_V, p_L) = 0$. Then by the previous results we conclude that there exist p_V^*, p_L^* with $p_V < p_V^* < p_0$ such that $f(p_V^*, p_L^*) = 0$ and $\frac{\partial f}{\partial p_L}(p_V^*, p_L^*) = 0$. Accordingly the solution (p_V^*, p_L^*) does not satisfy the right-hand side of inequalities (5.27). For (p_V^*, p_L^*) we estimate

$$z(p_V^*, p_L^*) < p_V^* h(p_V^*, p_V^*) < -\left(\frac{m}{kT_0}\right)^{(3/2)} \frac{p_V^*}{\sqrt{2\pi}} \frac{kT_0}{m} \ln \frac{p_V^*}{p_0} = -\frac{a_V}{\sqrt{2\pi}} \rho_V^* \ln \frac{p_V^*}{p_0}\,.$$

The expression $-\frac{a_V}{\sqrt{2\pi}}\rho_V^* \ln\frac{p_V^*}{p_0}$ attains its maximum at $\hat{p}_V = p_0\exp(-1)$. Accordingly we get

$$-\frac{a_V}{\sqrt{2\pi}}\rho_V^* \ln\frac{p_V^*}{p_0} \leq \frac{a_V}{\sqrt{2\pi}}\hat{\rho}_V < a_L\rho_L\,.$$

This contradicts the above statement that (p_V^*, p_L^*) does not satisfy the right-hand side of inequalities (5.27). We conclude, that for every fixed $0 < p_V^* < p_0$ there exists a unique p_L^*, such that $f(p_V^*, p_L^*) = 0$ and (5.27) are satisfied.

(7) Taking $p_V^* = \tilde{p}$ one can easily check that the root $(\tilde{p}, p_L(\tilde{p}))$ satisfies (5.27). Accordingly by an argumentation analogous to step 6 this is true for every p_V^* with $p_0 \leq p_V^* < \tilde{p}$. Now the first statement of Lemma 5.4 is proven for all $0 \leq p_V^* < \tilde{p}$.

Applying this solution to the kinetic relation (5.23) we obtain the mass flux z across the interface. □

Remark 5.5. For shorter and more clear notation we will often use instead of ρ_L the quantity $\left(\frac{p_L - p_0}{K_0} + 1\right)\rho_0$ given by the equation of state (3.3). This fact one should keep in mind when calculating partial derivatives $\partial/\partial p_L$.

Corollary 5.6. For every temperature 273.15K$\leq T_0 \leq$ 623.15K and given p_V^* the first root of $f(p_V^*, \cdot)$ satisfies (5.27).

Proof. It is obvious that for $p_V = p_L > p_0$ the function h is negative whereas for $p_V = p_L < p_0$ the function h is positive. Accordingly in the latter case we have

$$p_V \cdot h(p_V, p_L) < -\left(\frac{m}{kT_0}\right)^{3/2} \frac{p_V}{\sqrt{2\pi}}\frac{kT_0}{m}\ln\frac{p_V}{p_0} \leq \frac{p_0\exp(-1)}{a_V\sqrt{2\pi}} < a_L\rho_L\,. \tag{5.30}$$

This proves the statement that the right-hand side of (5.27) is always satisfied.

For the left-hand side of (5.27) this statement is clear by step 7 of the proof of Lemma 5.4. □

5.3.4. *Monotonicity of* $p_L^*(p_V^*)$.

Lemma 5.7. By (5.28) the implicitly defined function $p_L^*(p_V^*)$ is strictly increasing. Here p_L^* denotes the uniquely defined root of (5.28) for given p_V^*.

Proof. By the implicit function theorem we know that

$$p_L^{*\,\prime}(p_V^*) = -\frac{\partial f}{\partial p_V}(p_V^*, p_L^*) \Bigg/ \frac{\partial f}{\partial p_L}(p_V^*, p_L^*)\ .$$

From the last subsection we know that $\frac{\partial f}{\partial p_L}(p_V^*, p_L^*) > 0$. So we only have to show that

$$\frac{\partial f}{\partial p_V}(p_V^*, p_L^*) < 0\,.$$

We calculate

$$\frac{\partial f}{\partial p_V}(p_V^*, p_L^*) = -1 + p_V^* \cdot h^2(p_V^*, p_L^*)\left(\frac{2}{\rho_L^*} - \frac{1}{\rho_V^*}\right) + p_V^2 \cdot h(p_V^*, p_L^*)\frac{\partial h}{\partial p_V}(p_V^*, p_L^*)[\![\frac{1}{\rho^*}]\!]\,.$$

Let us assume that $z < 0$. Then

$$\begin{aligned}\frac{\partial h}{\partial p_V}(p_V^*, p_L^*) &= \left(\frac{m}{kT_0}\right)^{3/2} \frac{1}{\sqrt{8\pi}} \left([\![\frac{1}{\rho^*}]\!] + \frac{[\![p^*]\!]}{\rho_V^* p_V^*} \right) \\ &= \left(\frac{m}{kT_0}\right)^{3/2} \frac{1}{\sqrt{8\pi}} \left(1 - \frac{z^2}{p_V^* \rho_V^*}\right) [\![\frac{1}{\rho^*}]\!] < 0\end{aligned}$$

and consequently $\frac{\partial f}{\partial p_V}(p_V^*, p_L^*) < 0$. If $z > 0$ and $z \le \rho_V^* a_V$, then

$$\frac{\partial f}{\partial p_V}(p_V^*, p_L^*) < -1 + \left(\frac{m}{kT_0}\right)^{3/2} \frac{p_V^*}{\sqrt{2\pi}} \rho_V^* \left(\frac{kT_0}{m}\right)^{1/2} \frac{1}{\rho_V^{*2}} < 0\,.$$

Finally, for $z > 0$ and $z > \rho_V^* a_V$ the above statement is obvious. $\square$

COROLLARY 5.8. During a condensation process both pressures are larger than the saturation pressure

$$p_0 < p_V < p_L$$

whereas during an evaporation process we have

$$p_V < p_L < p_0\,.$$

This is a direct consequence of the last lemma and the fact that $p_L(p_0) = p_0$.

5.3.5. *Monotonicity of* $z[\![\frac{1}{\rho^*}]\!]$. Due to Lemma 5.4, for given p_V^* the mass flux z is uniquely defined, because $f(p_V^*, p_L) = 0$ has only a single admissible solution. Next we prove a further monotonicity relation.

LEMMA 5.9. For given temperature 273.15K$\le T_0 \le$ 623.15K the expression $z[\![\frac{1}{\rho^*}]\!]$ is strictly increasing in p_V^*, where z depends on the function $p_L^*(p_V^*)$ implicitly defined by (5.28).

Proof. We have

$$\frac{dz[\![\frac{1}{\rho}]\!]}{dp_V}(p_V^*, p_L^*) = \left(\frac{\partial z[\![\frac{1}{\rho}]\!]}{\partial p_V} + \frac{\partial z[\![\frac{1}{\rho}]\!]}{\partial p_L} p_L^{*\,\prime} \right)(p_V^*, p_L^*) = \left(\frac{\partial z[\![\frac{1}{\rho}]\!]}{\partial p_V} - \frac{\partial z[\![\frac{1}{\rho}]\!]}{\partial p_L} \frac{\frac{\partial f}{\partial p_V}}{\frac{\partial f}{\partial p_L}} \right)(p_V^*, p_L^*).$$

Using previous results we will show that

$$\left(\frac{\partial z[\![\frac{1}{\rho}]\!]}{\partial p_V} \frac{\partial f}{\partial p_L} - \frac{\partial z[\![\frac{1}{\rho}]\!]}{\partial p_L} \frac{\partial f}{\partial p_V} \right)(p_V^*, p_L^*) > 0\,. \tag{5.31}$$

Calculating all the derivatives we obtain

$$\begin{aligned}\left(\frac{\partial z[\![\frac{1}{\rho}]\!]}{\partial p_V} \frac{\partial f}{\partial p_L} - \frac{\partial z[\![\frac{1}{\rho}]\!]}{\partial p_L} \frac{\partial f}{\partial p_V} \right)(p_V^*, p_L^*) &= \left(\frac{m}{kT_0}\right)^{3/2} \frac{1}{\rho_V^* \sqrt{2\pi}} \left\{ [\![\frac{1}{\rho^*}]\!]^2 \left(\rho_V^* p_V^* - z^4 \frac{1}{\rho_L^{*2}} \frac{\rho_0}{K_0} \right) \right. \\ &\left. + \left[\frac{K_0}{\rho_0} \ln \frac{\rho_L^*}{\rho_0} - \frac{kT_0}{m} \ln \frac{p_V^*}{p_0} - \frac{1}{2} [\![p^*]\!] \left(\frac{1}{\rho_L^*} + \frac{1}{\rho_V^*} \right) \right] \frac{\rho_V^*}{\rho_L^*} \left(1 - \frac{\rho_0}{K_0} \frac{p_L^*}{\rho_L^*} \right) \right\}.\end{aligned}$$

Let us first consider that $z > 0$. Then for $z^2 \le \rho_V^* \rho_L^* a_V a_L$ the above statement is obvious.

Assume that z is positive with $z^2 > \rho_V^* \rho_L^* a_V a_L$. Then because $z > 0$ we conclude that

$$
\begin{aligned}
& -\frac{kT_0}{m} \ln \frac{p_V^*}{p_0} - \frac{1}{2}[\![p^*]\!] \left(\frac{1}{\rho_L^*} + \frac{1}{\rho_V^*} \right) > 0 \\
\implies \quad & -\frac{kT_0}{m} \ln \frac{p_V^*}{p_0} + \frac{1}{2}[\![\frac{1}{\rho^*}]\!] \left(\frac{1}{\rho_L^*} + \frac{1}{\rho_V^*} \right) \rho_V^* \rho_L^* a_V a_L > 0 \\
\implies \quad & -\frac{kT_0}{m} \ln \frac{p_V^*}{p_0} - \frac{\rho_L^*}{2\rho_V^*} a_V^2 > 0 \\
\iff \quad & \rho_L^* + 2\rho_V^* \ln \frac{p_V^*}{p_0} < 0 \,. \qquad (5.32)
\end{aligned}
$$

By some simple calculations we find that for fixed ρ_L^* the expression $\rho_L^* + 2\rho_V^* \ln \frac{p_V^*}{p_0}$ attains its minimum for $\hat{p}_V = p_0 \cdot \exp(-1)$. Accordingly we have

$$
\rho_L^* + 2\rho_V^* \ln \frac{p_V^*}{p_0} \geq \rho_L^* - 2\hat{\rho}_V > 0 \,.
$$

This is a contradiction to (5.32) and we conclude that $z^2 \leq \rho_V^* \rho_L^* a_V a_L$. This implies the above statement for positive z.

Now let us consider $z < 0$. We obtain

$$
\begin{aligned}
& [\![\frac{1}{\rho^*}]\!]^2 \left(\rho_V^* p_V^* - z^4 \frac{1}{\rho_L^{*2}} \frac{\rho_0}{K_0} \right) \\
& + \left[\frac{K_0}{\rho_0} \ln \frac{\rho_L^*}{\rho_0} - \frac{kT_0}{m} \ln \frac{p_V^*}{p_0} - \frac{1}{2}[\![p^*]\!] \left(\frac{1}{\rho_L^*} + \frac{1}{\rho_V^*} \right) \right] \frac{\rho_V^*}{\rho_L^*} \left(1 - \frac{\rho_0}{K_0} \frac{p_L^*}{\rho_L^*} \right) \\
> {} & [\![\frac{1}{\rho^*}]\!]^2 \left(\rho_V^{*2} a_V^2 - \frac{\rho_V^{*4} a_V^4}{\rho_L^{*2} a_L^2} \right) + \left[\frac{K_0}{\rho_0} \ln \frac{\rho_L^*}{\rho_0} - \frac{kT_0}{m} \ln \frac{p_V^*}{p_0} - \frac{1}{2}[\![p^*]\!] \left(\frac{1}{\rho_L^*} + \frac{1}{\rho_V^*} \right) \right] \frac{\rho_V^*}{\rho_L^*} \\
& \geq a_V^2 \left([\![\frac{1}{\rho^*}]\!]^2 \rho_V^{*2} \left(1 - \frac{\rho_V^{*2} a_V^2}{\rho_L^{*2} a_L^2} \right) - \sqrt{2\pi} \frac{\rho_V^*}{\rho_L^*} \right) .
\end{aligned}
$$

This expression is obviously positive, because $\rho_V / \rho_L < 1/4$; cf. (5.21). Accordingly the proof of Lemma 5.9 is complete. $\square$

REMARK 5.10. If we exclude phase transitions, this means that we use the trivial kinetic relation $z = 0$. Then Lemma 5.4 and Lemma 5.7 remain valid. It is quite evident that we have $p_L^* = p_V^*$. The expression $z[\![\frac{1}{\rho}]\!]$ of Lemma 5.9 becomes zero and is clearly nonstrictly increasing in p_V^*.

REMARK 5.11. During the proof of Lemma 5.9 we observe that the smallest $p_L \geq 0$ with $f(p_V^*, p_L) = 0$ identically satisfies the inequalities

$$
-a_V \rho_V \leq z \leq \sqrt{a_V a_L} \sqrt{\rho_V \rho_L} < a_L \rho_L \,, \qquad (5.33)
$$

which is a sharper result than the inequality (5.27).

6. Explicit solutions of the Riemann problem for isothermal Euler equations for two phases with different equations of state. Now let us consider two phase flows, where from now on for all examples the left phase (initially $x < 0$) is assumed to be water vapor, whereas the right phase (initially $x > 0$) is assumed to be liquid water. The different phases are characterized by different equations of state, given in (3.2) and (3.3). We consider the Riemann problem

$$\rho(x,0)=\begin{cases}\rho_- = \rho_V & \text{for} \quad x<0\\ \rho_+ = \rho_L & \text{for} \quad x>0\end{cases} \qquad \text{and} \qquad v(x,0)=\begin{cases}v_- = v_V & \text{for} \quad x<0\\ v_+ = v_L & \text{for} \quad x>0\,.\end{cases} \tag{6.1}$$

The solution consists of 4 constant states that are separated by 2 classical waves and the phase boundary. Accordingly we have three possible wave patterns; see Figure 4.

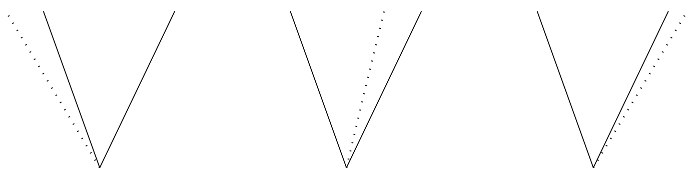

FIG. 4. Wave patterns. Solid line: classical waves. Dashed line: vapor-liquid interface

6.1. *Case 1: Two-phase flow without phase transition.* Let us first consider the case where the phase transition is excluded, i.e. $z=0$. In this case we have

LEMMA 6.1. There exists no solution of wave pattern types a) and c), which include the cases of the coincidence of the classical waves with the phase boundary.

The lemma will be proven at the end of this section.

Now we consider Case b). For solutions of that type we use the following notation for the 4 constant states:

$$\mathbf{W}_V = \begin{bmatrix}\rho_V\\ v_V\end{bmatrix}, \quad \mathbf{W}_V^* = \begin{bmatrix}\rho_V^*\\ v_V^*\end{bmatrix}, \quad \mathbf{W}_L^* = \begin{bmatrix}\rho_L^*\\ v_L^*\end{bmatrix}, \quad \mathbf{W}_L = \begin{bmatrix}\rho_L\\ v_L\end{bmatrix}. \tag{6.2}$$

To find the exact solution we extend the procedure that is described for single gas flows by Toro in [20]. We aim to derive a function

$$f(p,\mathbf{W}_V,\mathbf{W}_L) = f_V(p,\mathbf{W}_V) + f_L(p,\mathbf{W}_L) + (v_L - v_V)\,, \tag{6.3}$$

such that the only root $p=p^*$ is the solution for the pressure p_V^* of the Riemann problem (2.1)-(2.2), (6.1). The functions f_V and f_L are the increments that relate the initial velocities v_V, v_L to v_V^* and v_L^*, resp., only in terms of the initial data and the unknown solution p^*. This means that

$$v_V^* = v_V - f_V(p_*,\mathbf{W}_V) \qquad \text{and} \qquad v_L^* = v_L + f_L(p_*,\mathbf{W}_L)\,. \tag{6.4}$$

This procedure makes use of the constancy of pressure and velocity across the phase boundary, $v_V^* = v_L^*$ and $p_V^* = p_L^*$, which is due to $z=0$.

Because p is constant in the star region, we choose p^* to be the unknown and eliminate ρ_V^*, ρ_L^*. However, for shorter notation we keep the initial data ρ_V, ρ_L.

We use the results in (5.2), (5.4) and (5.16). For a left wave we replace $'$ and $''$ by $_V$ and $^*_V$, resp. On the other hand, for a right wave, $'$ and $''$ are replaced by $^*_L$ and $_L$. We end up with the following.

THEOREM 6.2 (**Solution of isothermal two-phase Euler equations without phase transition**). Let $f(p, \mathbf{W}_V, \mathbf{W}_L)$ be given as

$$f(p, \mathbf{W}_V, \mathbf{W}_L) = f_V(p, \mathbf{W}_V) + f_L(p, \mathbf{W}_L) + \Delta v\ ,\ \Delta v = v_L - v_V,$$

where the functions f_V and f_L are given by

$$f_V(p, \mathbf{W}_V) = \begin{cases} \frac{p-p_V}{\sqrt{\rho_V p}} & \text{if } p > p_V \text{ (shock)}, \\ -a_V \ln p_V + a_V \ln p & \text{if } p \le p_V \text{ (rarefaction)}, \end{cases}$$

$$f_L(p, \mathbf{W}_L) = \begin{cases} \frac{p-p_L}{\sqrt{K_0 \rho_L \left(\frac{p-p_0}{K_0}+1\right)}} & \text{if } p > p_L \text{ (shock)}, \\ -a_L \ln \frac{\rho_L}{\rho_0} + a_L \ln\left(\frac{p-p_0}{K_0} + 1\right) & \text{if } p \le p_L \text{ (rarefaction)}. \end{cases}$$

If the function $f(p, \mathbf{W}_V, \mathbf{W}_L)$ has a root p^* with $0 < p^* \le \tilde{p}$ and with $\tilde{p}$ as in Section 3, then this root is unique and is the unique solution for pressure p^*_V of the Riemann problem (2.1)-(2.2), (6.1). The velocity v^*_V can be calculated as follows:

$$v^*_V = \frac{1}{2}(v_V + v_L) + \frac{1}{2}\left(f_L(p_*, \mathbf{W}_L) - f_V(p_*, \mathbf{W}_V)\right)\ .$$

Proof. The function f is strictly monotone increasing in p with $f(p, \mathbf{W}_V, \mathbf{W}_L) \to -\infty$ for $p \to 0$. Therefore f has at most one unique root, which is by construction the solution for the pressure p^*_V of the Riemann problem considered. The second part of the theorem is an immediate consequence of (6.4). □

For given initial data one can define the sets of states that can be connected to the initial states by a single shock or rarefaction wave. These sets define curves in the p-v-phase plane, where the intersection point (p^*, v^*) is the solution due to Theorem 6.2; see Figure 5. In Figure 5 the curve C_V belongs to the vapor phase, whereas the curve C_L

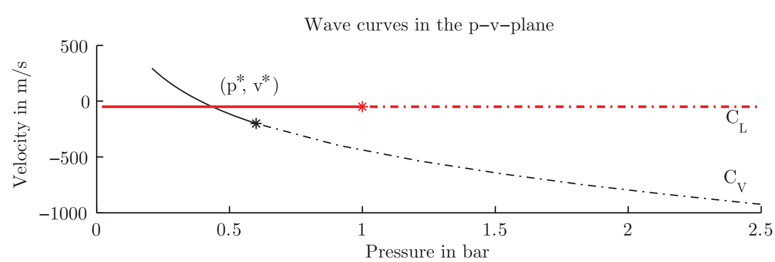

FIG. 5. Wave curves in the p-v-phase plane

belongs to the liquid phase. The solid lines denote those states that can be connected to the initial states, indicated by a star, by a rarefaction wave. Along the dash-dotted lines we have states that may be connected to the initial states by a shock wave. The wave curves in Figure 5 belong to the data of the second example in Section 8.

Theorem 6.3 (**Sufficient condition for solvability**). Let us consider the Riemann problem (2.1)-(2.2), (6.1). We have two cases.

- For $p_L < p_V(\tilde{\rho}) = \tilde{p}$ the above Riemann problem is solvable if and only if

$$f(\tilde{p}, \mathbf{W}_V, \mathbf{W}_L) = \frac{\tilde{p} - p_V}{\sqrt{\rho_V \tilde{p}}} + \frac{\tilde{p} - p_L}{\sqrt{\rho_L}(\tilde{p} - p_0 + K_0)} + \Delta v \geq 0\,.$$

- For $p_L \geq \tilde{p}$ the above Riemann problem is solvable if and only if

$$f(\tilde{p}, \mathbf{W}_V, \mathbf{W}_L) = \frac{\tilde{p} - p_V}{\sqrt{\rho_V \tilde{p}}} + a_L \ln\left(\frac{p_L - p_0 + K_0}{\tilde{p} - p_0 + K_0}\right) + \Delta v \geq 0\,.$$

Proof. As seen before, f is strictly monotone increasing in p with $f(p, \mathbf{W}_V, \mathbf{W}_L) \to -\infty$ for $p \to 0$. Accordingly f has a unique root if and only if $f(p, \mathbf{W}_V, \mathbf{W}_L) \geq 0$ for $p \to \tilde{p}$. □

Remark 6.4 (**Complete solution**). Theorems 6.2 and 6.3 allow us to calculate the pressure and the velocity in the star region as well as the interface velocity. From the equations of state (3.2) and (3.3) we find the densities ρ_V^*, ρ_L^* of the star region respectively. In the case of shock waves the relation (5.16) gives the shock speeds.

For a left (right) rarefaction wave the head and tail speeds can be obtained from (5.2) or (5.4) respectively. The solution inside the fans is given by (5.3), respectively (5.5).

Finally we give the proof of Lemma 6.1.

Proof. We denote the states between the classical waves with two stars. The states between the right wave and the phase boundary have one star; also see Figure 6. Assume

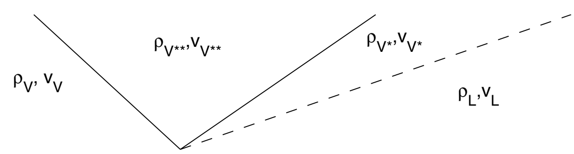

Fig. 6. Notation, wave pattern type c)

that the solution is of wave pattern type c). Then the interface is moving with speed $w = v_L = v_{V*}$. Let us further assume that the right wave is a shock wave moving with speed S_2. Obviously the condition $w \geq S_2$ must hold. To find S_2 we use $(5.16)_1$ and $(5.16)_2$. We replace $'$ and $''$ by $_{V**}$ and $_{V*}$, resp. We obtain

$$S_2 = w + \frac{a_V \rho_{V*}}{\sqrt{\rho_{V*}\rho_{V**}}}\,,$$

which contradicts the condition $w \geq S_2$.

On the other hand if the right wave is a rarefaction wave, then the head speed is given by $a_V + v_{V*}$; see Subsection 5.1. This is likewise a contradiction to the condition $w = v_L = v_{V*} \geq a_V + v_{V*}$. If the phase boundary lies within the rarefaction wave or at its tail we obtain the analogous contradiction in the wave speeds.

Accordingly there is no solution of type c). In an analogous manner we may discuss the case of wave pattern type a). □

6.2. *Case 2: Two-phase flow with phase transition.* The lemma corresponding to Lemma 6.1 is much more complicated in this case. For this reason we must discuss all three cases from Figure 4 and we start with Case b).

6.2.1. *Solutions of type b).* To find the solution for the Riemann problem (2.1)-(2.2), (6.1) with phase transition we use the same strategy as before. Due to phase transition we have $v_L^* \neq v_V^*$ at the interface, which gives us a further term in the resulting algebraic equation. Moreover, a further challenge results from the inequality of the pressures $p_L^* \neq p_V^*$. Nevertheless we are able to construct a function

$$f_z(p, \mathbf{W}_V, \mathbf{W}_L) = f_V(p, \mathbf{W}_V) + f_L(p_L^*(p), \mathbf{W}_L) + z[\![\frac{1}{\rho}]\!] + (v_L - v_V) \tag{6.5}$$

such that the only root $p = p^*$ is the solution for the pressure p_V^* of the Riemann problem (2.1)-(2.2), (6.1) with phase transition. The functions f_V and f_L relate the initial velocities v_V, v_L to v_V^* and v_L^*, respectively, only in terms of the initial data and the unknown solution p^* as well as the implicitly defined function $p_L^*(p^*)$.

As before we use the results in (5.2), (5.4) and (5.16). For a left wave we repalce $'$ and $''$ by ${}_V$ and ${}^*_V$, respectively. On the other hand, for a right wave, $'$ and $''$ are replaced by ${}^*_L$ and ${}_L$. We end up with the following.

Theorem 6.5 (**Solution of isothermal two-phase Euler equations with phase transition**). Let $f_z(p, \mathbf{W}_V, \mathbf{W}_L)$ be given as

$$f_z(p, \mathbf{W}_V, \mathbf{W}_L) = f_V(p, \mathbf{W}_V) + f_L(p_L^*(p), \mathbf{W}_L) + z[\![\frac{1}{\rho}]\!] + \Delta v \ , \ \Delta v = v_L - v_V, \tag{6.6}$$

where the functions f_V and f_L are given by

$$\begin{aligned} f_V(p, \mathbf{W}_V) &= \begin{cases} \frac{p - p_V}{\sqrt{\rho_V p}} & \text{if } p > p_V \text{ (shock)}, \\ -a_V \ln p_V + a_V \ln p & \text{if } p \leq p_V \text{ (rarefaction)}, \end{cases} \\ f_L(p, \mathbf{W}_L) &= \begin{cases} \frac{p_L^*(p) - p_L}{\sqrt{K_0 \rho_L \left(\frac{p_L^*(p) - p_0}{K_0} + 1\right)}} & \text{if } p_L^*(p) > p_L \text{ (shock)}, \\ -a_L \ln \frac{\rho_L}{\rho_0} + a_L \ln \left(\frac{p_L^*(p) - p_0}{K_0} + 1\right) & \text{if } p_L^*(p) \leq p_L \text{ (rarefaction)}. \end{cases} \end{aligned}$$

The function $p_L^*(p)$ is implicitly defined by (5.28) and z is given by (5.23).
If the function $f_z(p, \mathbf{W}_V, \mathbf{W}_L)$ has a root p^* with $0 < p^* \leq \tilde{p}$, see Section 3, this root is unique.

If further

$$p^* > p_V, \qquad \text{we must have} \qquad z > -a_V \sqrt{\rho_V \rho_V^*}\,. \tag{6.7}$$

In this case the root p^* is the unique solution for the pressure p_V^* for a b)-type solution of the Riemann problem (2.1)-(2.2), (6.1) with phase transition and the complete solution is uniquely determined.

If there is no root or condition (6.7) is not satisfied, then the Riemann problem has no solution.

Proof. The function f_z is strictly increasing in p. This follows from Lemma 5.7 and Lemma 5.9. Further we have $f_z \to -\infty$ for $p \to 0$. Therefore f_z has at most one unique root, which is by construction the solution for p_V^* of the considered Riemann problem.

Then by Lemma 5.4 the pressure $p_L^*(p)$ and the mass flux z are uniquely defined. The corresponding densities can be obtained from the equations of state (3.2), (3.3). To find the velocities in the star regions one can use (5.2), (5.4) for rarefactions or (5.16) for shocks. The interface velocity can be obtained from (2.5).

The further calculations are the same as in the case of isothermal Euler equations without phase transition; see the proof for Theorem 6.2 and the remarks following. $\square$

REMARK 6.6. The additional condition (6.7) in Theorem 6.5 is necessary to guarantee that $S_1 \leq w$ in the case of a 1-shock propagating through the gas. If this condition is not satisfied, the root p^* of (6.6) is meaningless.

As in the case of no phase transition in the previous section, one can construct the solution in the $p-v-$phase plane. We define the same sets of states as before. Moreover, for every state that can be connected to (p_V, v_V) by a single wave, there exists a uniquely defined state (p_L^*, v_L^*) that can be connected to (p_V, v_V) by a phase boundary due to the kinetic relation (5.23). These states define a further wave curve $C_{L'}$; see Figure 7. The

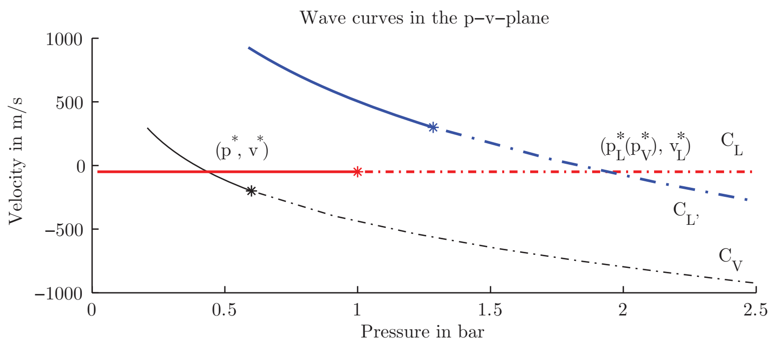

FIG. 7. Wave curves in the $p - v-$phase plane

curves C_L and C_V are identical to the case before. The curve $C_{L'}$ is newly defined, where the solid part of $C_{L'}$ corresponds to the solid part of C_V. The intersection point of the curves $C_{L'}$ and C_L is the solution for $(p_L^*(p_V^*), v_L^*)$ due to Theorem 6.5. As before the wave curves in Figure 7 belong to the data of the second example in Section 8.

THEOREM 6.7 (**Sufficient condition for solvability I**). Let us consider the Riemann problem (2.1)-(2.2), (6.1). If the Riemann problem considered for Case 1 is solvable, then the same Riemann problem is also solvable taking into account phase transition due to the kinetic relation (5.23).

The proof is obvious by the monotonicity properties of f_z. For details, see the following corollary and its proof.

COROLLARY 6.8. Let p^* be the solution of the pressure in the star region of the Riemann problem (2.1)-(2.2), (6.1) for Case 1. Then for the solutions p_V^* and $p_L^*(p_V^*)$ of the same Riemann problem for Case 2 we have:

(1) $p^* = p_0$ implies that $p_V^* = p_L^*(p_V^*) = p_0$.
(2) $p^* < p_0$ implies that $p^* < p_L^*(p_V^*) < p_0$.

(3) $p^* > p_0$ implies that $p_0 < p_V^* < p^*$.

Proof. The first statement is obvious. Now let us consider $p^* < p_0$. Consider that $p_L^*(p_V^*) = p_0$. Then we have an equilibrium and therefore $p_V^* = p_0$ and further $f_z(p_0, \mathbf{W}_V, \mathbf{W}_L) = f(p_0, \mathbf{W}_V, \mathbf{W}_L) > 0$. Now $f_z(p_0, \mathbf{W}_V, \mathbf{W}_L) > 0$ and we obtain due to the monotonicity of f_z that $p_L^*(p_V^*) < p_0$. On the other hand, if $p_L^*(p_V^*) = p_*$, then $f_z(p_V^*, \mathbf{W}_V, \mathbf{W}_L) < f(p^*, \mathbf{W}_V, \mathbf{W}_L) = 0$ and we conclude the other inequality $p^* < p_L^*(p_V^*)$. The argumentation for the third statement is analogous. □

THEOREM 6.9 (**Sufficient condition for solvability II)**. Let us consider the Riemann problem (2.1)-(2.2), (6.1) with phase transition. This Riemann problem is solvable by a b)-type solution if and only if

$$f_z(\tilde{p}, \mathbf{W}_V, \mathbf{W}_L) \geq 0$$

and (6.7) is satisfied.

Proof. The statement is obvious, because the above requirement guarantees that the function f_z has a root. □

6.2.2. *Further solutions.* As in Section 6.1 we want to discuss the existence of further solutions for the Riemann problem (2.1)-(2.2), (6.1) with phase transition. We obtain

LEMMA 6.10. There is no solution of type a).

Proof. Assume there is a solution of type a). Then analogously to solutions of type c) in Section 6.1 we denote the constant states by (ρ_V, v_v), (ρ_{L*}, v_{L*}), (ρ_{L**}, v_{L**}), (ρ_L, v_L); see Figure 6. Obviously in that case we have a condensation process and therefore $z < 0$. Assume the left wave is a rarefaction wave. Then the head speed is given by $v_{L*} - a_L$ and

$$w = \frac{z}{\rho_{L*}} + v_{L*} \leq v_{L*} - a_L \tag{6.8}$$

must hold. We obtain $z \leq -a_L \rho_{L*}$. This contradicts (5.27), and therefore there is no solution of type a) with a left rarefaction.

Similarly, for a left shock wave,

$$w = \frac{z}{\rho_{L*}} + v_{L*} < v_{L*} - a_L \sqrt{\frac{\rho_{L**}}{\rho_{L*}}}$$

must hold. This is a stronger inequality than (6.8), and therefore it cannot be satisfied. This proves the above statement. □

LEMMA 6.11. Consider the Riemann problem (2.1)-(2.2), (6.1) with phase transition. If $p_L \geq p_0$ there is no solution of type c).

Proof. A solution of type c) implies an evaporation process. This requires that $p_L < p_0$. □

LEMMA 6.12. Consider the Riemann problem (2.1)-(2.2), (6.1) with phase transition. For sufficiently large p_L with $p_L \leq p_0$ there is no solution of type c).

Proof. Assume there is a solution of type c). Then analogously to the previous case of an a)-type solution for a right rarefaction,

$$w = \frac{z}{\rho_{V*}} + v_{V*} \geq v_{V*} + a_V$$

must hold. On the other hand, for a right shock wave we have

$$w = \frac{z}{\rho_{V*}} + v_{V*} > v_{V*} + a_V \sqrt{\frac{\rho_{V**}}{\rho_{V*}}}\,.$$

Accordingly

$$\frac{z}{\rho_{V*}} + v_{V*} < v_{V*} + a_V \qquad \Longleftrightarrow \qquad \frac{z}{\rho_{V*} a_V} < 1$$

is sufficient to guarantee that there is no solution of type c). Due to $z > 0$ we obtain from (5.23) by a simple estimate

$$\frac{z}{\rho_{V*} a_V} < -\frac{1}{\sqrt{2\pi}} \ln \frac{p_{V*}}{p_0}\,.$$

Therefore, if

$$-\frac{1}{\sqrt{2\pi}} \ln \frac{p_{V*}}{p_0} \leq 1 \qquad \Longleftrightarrow \qquad p_{V*} \geq p_0 \exp(-\sqrt{2\pi})$$

there is no solution of type c). By the strict monotonicity of $p_L(p_{V*})$ the proof is complete; see Lemma 5.7. $\square$

REMARK 6.13. Note that the inequality $p_L \geq p_L(p_0 \exp(-\sqrt{2\pi}))$ is sufficient, but not necessary, for the statement of the above lemma.

7. 3-Phase flow.

7.1. *Condensation by compression.* Now let us consider the Riemann problem for the isothermal Euler equations with the following initial data for $\rho_{V\pm} \in [0, \tilde{p}]$:

$$\rho(x,0) = \begin{cases} \rho_- = \rho_{V-} & \text{for} \quad x < 0 \\ \rho_+ = \rho_{V+} & \text{for} \quad x > 0 \end{cases} \qquad \text{and} \qquad v(x,0) = \begin{cases} v_- = v_{V-} & \text{for} \quad x < 0 \\ v_+ = v_{V+} & \text{for} \quad x > 0\,. \end{cases} \tag{7.1}$$

This means that we have a Riemann problem for a vapor phase only. Using the results of Section 5 we easily obtain

THEOREM 7.1 (**Solution of classical isothermal Euler equations**). Let the function f_{VV} be given as

$$f_{VV}(p, \mathbf{W}_{V-}, \mathbf{W}_{V+}) = f_{V-}(p, \mathbf{W}_{V-}) + f_{V+}(p, \mathbf{W}_{V+}) + \Delta v\ ,\ \Delta v = v_{V+} - v_{V-},$$

where the functions f_{V-} and f_{V+} are given by

$$f_{V-}(p, \mathbf{W}_{V-}) \quad = \quad \begin{cases} \frac{p - p_{V-}}{\sqrt{\rho_{V-} p}} & \text{if } p > p_{V-} \text{ (shock)}, \\ -a_V \ln p_{V-} + a_V \ln p & \text{if } p \leq p_{V-} \text{ (rarefaction)}, \end{cases}$$

$$f_{V+}(p, \mathbf{W}_{V+}) \quad = \quad \begin{cases} \frac{p - p_{V+}}{\sqrt{\rho_{V+} p}} & \text{if } p > p_{V+} \text{ (shock)}, \\ -a_V \ln p_{V+} + a_V \ln p & \text{if } p \leq p_{V+} \text{ (rarefaction)}. \end{cases}$$

If the function $f_{VV}(p, \mathbf{W}_{V-}, \mathbf{W}_{V+})$ has a root p^* with $0 < p^* \le \tilde{p}$, this root is unique and is the unique solution for pressure p_V^* of the Riemann problem (2.1)-(2.2), (7.1). The velocity v_V^* is given by

$$v_V^* = \frac{1}{2}(v_{V-} + v_{V+}) + \frac{1}{2}\left(f_{V+}(p_*, \mathbf{W}_{V+}) - f_{V-}(p_*, \mathbf{W}_{V-})\right).$$

In principle this result is known with some small modifications; see for instance the book of Toro [20]. In the literature one usually looks for a pressure p^* that is a root of the above algebraic equation. Due to $f_{VV} \to -\infty$ for $p \to 0$ and $f_{VV} \to +\infty$ for $p \to +\infty$ there is always a solution. The latter case is physically not meaningful because a sufficiently high pressure in a gas will lead to a phase transition to a liquid or solid phase. In contrast we only consider solutions that satisfy the inequality $0 < p^* \le \tilde{p}$, where $\tilde{p}$ denotes the maximal possible gas pressure. As a consequence one can find Riemann initial data without a solution. If this happens we follow the following strategy.

Definition 7.2 (**Nucleation criterion**). If there is no solution to the Riemann problem (2.1)-(2.2), (7.1) according to Theorem 7.1, then nucleation occurs.

If this criterion is fulfilled, we look for a solution with two transition fronts (phase boundaries) and two classical waves. Next we discuss the possible wave patterns for condensation.

Lemma 7.3. If there is a solution of the Riemann problem (2.1)-(2.2), (7.1) consisting of two classical waves and two phase boundaries, then no wave is propagating through the liquid. Waves may only occur in the gas.

Proof. Assume there is a solution with a classical wave propagating through the liquid phase. W.l.o.g. this wave is a left going wave. We denote the states to the left and right of this wave by $_{L*}$ and $_{L**}$, respectively. Furthermore, on the left-hand side of this wave there is a phase boundary propagating with speed w_1. The state left to the phase boundary is denoted by $_{V*}$.

Obviously we have a condensation process. Accordingly $p_* > p_0$ and $p_{L*} > p_0$. This configuration is impossible due to Lemma 6.10. Analogously we discuss the case of a right going wave. □

We conclude that both waves propagate through the vapor phase. The possible wave patterns are given in Figure 8.

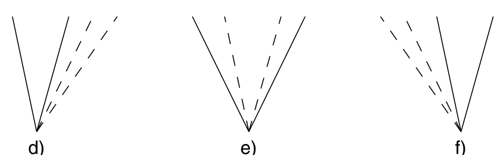

Fig. 8. Wave patterns. Solid line: classical waves. Dashed line: vapor-liquid interface

Lemma 7.4. There are no solutions of wave pattern types *d*) and *f*).

Proof. Let us assume that the solution is of wave pattern type d). This corresponds to solutions of wave pattern type c) in Section 6.2.2; see Figure 4. We have seen that such solutions can only occur for very low pressures, which implies evaporation; see Lemma 6.11 and Lemma 6.12. Here we have a condensation process, so wave pattern type d) is impossible. Analogously we can exclude solutions of wave pattern type f). $\square$

Accordingly the only possible wave configuration is of type e). We use the notation as given in Figure 9 and obtain

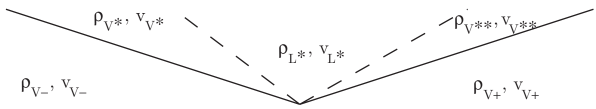

FIG. 9. Notation, wave pattern type e).

LEMMA 7.5. Assume there is a solution of wave pattern type e). Then $p_{V*} = p_{V**}$.

Proof. For given p_{V*} the pressure p_{L*} is uniquely defined; cf. Lemma 5.4. The function $p_{L*}(p_{V*})$ is strictly monotone; see Lemma 5.7. For the second phase boundary we have to use the modified kinetic relation (5.24). We obtain the same pressure function $p_{L*}(p_{V**}) = p_{L*}(p_{V*})$ with the same monotonicity properties as in Section 5.3.4. $\square$

Using the results of the previous sections and taking into account that there are two phase boundaries we can formulate the following.

THEOREM 7.6 (**Solution of isothermal Euler equations for two gases with phase transition**). Consider the Riemann problem (2.1)-(2.2), (7.1) and assume that the nucleation criterion is satisfied. Let $f_{VVz}(p, \mathbf{W}_{V-}, \mathbf{W}_{V+})$ be given as

$$f_{VVz}(p, \mathbf{W}_{V-}, \mathbf{W}_{V+}) = f_{V-}(p, \mathbf{W}_{V-}) + f_{V+}(p, \mathbf{W}_{V+}) + 2z[\![\frac{1}{\rho}]\!] + v_{V+} - v_{V-} \ ,$$

where the functions f_{V-} and f_{V+} are given by

$$\begin{aligned} f_{V-}(p, \mathbf{W}_{V-}) &= \begin{cases} \frac{p-p_{V-}}{\sqrt{\rho_{V-}p}} & \text{if } p > p_{V-} \text{ (shock)}, \\ -a_V \ln p_{V-} + a_V \ln p & \text{if } p \le p_{V-} \text{ (rarefaction)}, \end{cases} \\ f_{V+}(p, \mathbf{W}_{V+}) &= \begin{cases} \frac{p-p_{V+}}{\sqrt{\rho_{V+}p}} & \text{if } p > p_{V+} \text{ (shock)}, \\ -a_V \ln p_{V+} + a_V \ln p & \text{if } p \le p_{V+} \text{ (rarefaction)}. \end{cases} \end{aligned}$$

Here z is given by (5.23) and $[\![\frac{1}{\rho}]\!] = \frac{1}{\rho_{L*}} - \frac{1}{\rho_{V*}}$. The function $p_L^*(p)$ is implicitly defined by (5.28).

If the function f_{VVz} has a root with $p_0 < p \le \tilde{p}$, then this root is the only one. Furthermore, this root is the unique solution for the pressure $p_{V*} = p_{V**}$ of the Riemann problem (2.1)-(2.2), (7.1) for the vapor pressure in the star regions. The liquid velocity v_{L*} can be calculated by

$$v_{L*} = \frac{1}{2}(v_{V-} + v_{V+}) + \frac{1}{2}\left(f_{V+}(p_*) - f_{V-}(p_*)\right) \ .$$

By previous results it is obvious that the function f_{VVz} has at most one root. By construction this root is the solution for the pressure of the vapor phase in the two star regions in Figure 9.

The further calculations to find the complete solution are analogous to previous calculations.

REMARK 7.7. Note that $v_{V*} \neq v_{V**}$ with $v_{V*} + v_{V**} = 2v_{L*}$.

THEOREM 7.8 (**Sufficient condition for solvability I**). Consider the Riemann problem (2.1)-(2.2), (7.1). This problem is solvable without phase transition if and only if

$$f_{VV}(\tilde{p}, \mathbf{W}_{V-}, \mathbf{W}_{V+}) \geq 0\,.$$

Proof. This statement is obvious by the monotonicity of f_{VV}. □

THEOREM 7.9 (**Sufficient condition for solvability II**). Consider the Riemann problem (2.1)-(2.2), (7.1) and assume that the nucleation criterion due to Definition 7.2 is satisfied. Taking phase transition into account this problem is solvable if and only if

$$f_{VVz}(\tilde{p}, \mathbf{W}_{V-}, \mathbf{W}_{V+}) \geq 0\,.$$

Proof. This statement is obvious due to the monotonicity of f_{VVz}. □

7.2. *Evaporation by expansion.* In the following we consider the Riemann problem for the isothermal Euler equations with initial data $\rho_{L\pm} \geq \rho_{\min}$:

$$\rho(x,0) = \begin{cases} \rho_- = \rho_{L-} & \text{for} \quad x < 0 \\ \rho_+ = \rho_{L+} & \text{for} \quad x > 0 \end{cases} \quad \text{and} \quad v(x,0) = \begin{cases} v_- = v_{L-} & \text{for} \quad x < 0 \\ v_+ = v_{L+} & \text{for} \quad x > 0; \end{cases} \tag{7.2}$$

i.e., the initial data only contain two states in a liquid phase.

We have seen that at a planar phase boundary the liquid pressure is always positive. It is known from applications that negative liquid pressures are possible. They give rise to cavitation in the liquid; see Doering [6]. Recall that in the liquid-vapor case a negative liquid pressure is forbidden; see (5.18). Now, in the liquid-liquid case we may meet negative pressures. The smallest pressure in the liquid is $p_{\min}$.

Using that definition we obtain

THEOREM 7.10 (**Solution of isothermal Euler equations for two states of a liquid without phase transition**). Let $f_{LL}(p, \mathbf{W}_{L-}, \mathbf{W}_{L+})$ be given as

$$f_{LL}(p, \mathbf{W}_{L-}, \mathbf{W}_{L+}) = f_{L-}(p, \mathbf{W}_{L-}) + f_{L+}(p, \mathbf{W}_{L+}) + \Delta v\ ,\ \Delta v = v_{L+} - v_{L-},$$

where the functions f_{L-} and f_{L+} are given by

$$f_{L-}(p, \mathbf{W}_{L-}) \quad = \quad \begin{cases} \frac{p - p_{L-}}{\sqrt{\rho_{L-}(p - p_0 + K_0)}} & \text{if } p > p_{L-} \text{ (shock),} \\ -a_L \ln \frac{\rho_{L-}}{\rho_0} + a_L \ln \left(\frac{p - p_0}{K_0} + 1 \right) & \text{if } p \leq p_{L-} \text{ (rarefaction),} \end{cases}$$

$$f_{L+}(p, \mathbf{W}_{L+}) \quad = \quad \begin{cases} \frac{p - p_{L+}}{\sqrt{\rho_{L+}(p - p_0 + K_0)}} & \text{if } p > p_{L+} \text{ (shock),} \\ -a_L \ln \frac{\rho_{L+}}{\rho_0} + a_L \ln \left(\frac{p - p_0}{K_0} + 1 \right) & \text{if } p \leq p_{L+} \text{ (rarefaction).} \end{cases}$$

If the function $f_{LL}(p, \mathbf{W}_{L-}, \mathbf{W}_{L+})$ has a root p^* with $p_{\min} \le p^*$, this root is unique and is the unique solution for the pressure p_L^* of the Riemann problem (2.1)-(2.2), (7.2). The velocity v_L^* is calculated from

$$v_L^* = \frac{1}{2}(v_{L-} + v_{L+}) + \frac{1}{2}\left(f_{L+}(p_*) - f_{L-}(p_*)\right) .$$

REMARK 7.11. For simplicity in our calculations we choose $p_{\min} = 0$, but also lower values are possible. Our theoretical results are general and do not depend on the special value of $p_{\min}$.

Analogous to the above nucleation criterion we give the

DEFINITION 7.12 (**Cavitation criterion**). If there is no solution of the Riemann problem (2.1)-(2.2), (7.2) according to Theorem 7.10, then we may encounter cavitation.

If this criterion is fulfilled, we look for a solution involving a vapor phase with two transition fronts (phase boundaries) and two classical waves. As before we discuss the possible wave patterns.

LEMMA 7.13. *Assume there is a solution of the Riemann problem (2.1)-(2.2), (7.2) consisting of two classical waves and two phase boundaries. If further p_{L-}, p_{L+} are sufficiently large, then no wave is propagating through the vapor.*

The proof is analogous to the proof of Lemma 6.12. A sufficient lower bound for p_{L-}, p_{L+} is given in Remark 6.13.

LEMMA 7.14. *There is no solution of types d) and f); see Figure 8.*

The proof is analogous to the proof of Lemma 6.10.

Accordingly we construct solutions of type e); the notation is analogous to the notation in Figure 9. We obtain

LEMMA 7.15. *Assume there is a solution of wave pattern type e). Then $p_{L*} = p_{L**}$.*

The proof is analogous to the proof of Lemma 7.5.

The next theorem addresses wave pattern type e).

THEOREM 7.16 (**Solution for isothermal Euler equations for two liquids with phase transition**). *Consider the Riemann problem (2.1)-(2.2), (7.2) and assume the cavitation criterion is satisfied. Let $f_{LLz}(p, \mathbf{W}_{L-}, \mathbf{W}_{L+})$ be given as*

$$f_{LLz}(p, \mathbf{W}_{L-}, \mathbf{W}_{L+}) = f_{L-}(p_L(p), \mathbf{W}_{L-}) + f_{L+}(p_L(p), \mathbf{W}_{L+}) + 2z[\![\frac{1}{\rho}]\!] + v_{L+} - v_{L-} ,$$

with f_{L-} and f_{L+} according to

$$f_{L-}(p_L^*(p), \mathbf{W}_{L-}) = \begin{cases} \frac{p_L^*(p) - p_{L-}}{\sqrt{\rho_{L-}(p_L^*(p) - p_0 + K_0)}} & \text{if } p_L^*(p) > p_{L-} \text{ (shock)}, \\ -a_L \ln \frac{\rho_{L-}}{\rho_0} + a_L \ln\left(\frac{p_L^*(p) - p_0}{K_0} + 1\right) & \text{if } p_L^*(p) \le p_{L-} \text{ (rf.)}, \end{cases}$$

$$f_{L+}(p_L^*(p), \mathbf{W}_{L+}) = \begin{cases} \frac{p_L^*(p) - p_{L+}}{\sqrt{\rho_{L+}(p_L^*(p) - p_0 + K_0)}} & \text{if } p_L^*(p) > p_{L+} \text{ (shock)}, \\ -a_L \ln \frac{\rho_{L+}}{\rho_0} + a_L \ln\left(\frac{p_L^*(p) - p_0}{K_0} + 1\right) & \text{if } p_L^*(p) \le p_{L+} \text{ (rf.)} . \end{cases}$$

Here z is calculated from (5.23) and $[\![\frac{1}{\rho}]\!] = \frac{1}{\rho_{L*}} - \frac{1}{\rho_{V*}}$. The function $p_L^*(p)$ is implicitly defined by (5.28).

If the function f_{LLz} has a root with $p_{\min} \leq p$, then this root is unique. Further, this root uniquely determines the pressure p_V^* of the Riemann problem (2.1)-(2.2), (7.2) for the vapor pressure in the star region. Further, the vapor velocity v_{V*} is given by

$$v_{V*} = \frac{1}{2}(v_{L-} + v_{L+}) + \frac{1}{2}\left(f_{L+}(p_L^*(p_*)) - f_{L-}(p_L^*(p_*))\right).$$

Proof. Due to our previous results, it is obvious that the function f_{LLz} has at most one root. By construction this root is the solution for the pressure of the vapor phase in the star region.

The further calculations leading to the complete solution are analogous to previous calculations. □

THEOREM 7.17 (**Sufficient condition for solvability I**). Consider the Riemann problem (2.1)-(2.2), (7.2). This problem is solvable without phase transition if and only if

$$f_{LL}(p_{\min}, \mathbf{W}_{V-}, \mathbf{W}_{V+}) \leq 0.$$

Proof. This statement is obvious due to monotonicity of f_{LL}. □

THEOREM 7.18 (**Sufficient condition for solvability II**). Consider the Riemann problem (2.1)-(2.2), (7.2) and assume the cavitation criterion is satisfied. If we admit phase transition, this problem is always solvable.

Proof. This statement is obvious due to the fact that $z[\![\frac{1}{\rho}]\!] \to -\infty$ for $p_V^* \to 0$. □

8. Numerical results.

In the following section we discuss some numerical examples. The calculations need the Boltzmann constant k and the mass of a single water molecule m_W:

$$k = 1.380658 \cdot 10^{-23}\text{J/K} \qquad \text{and} \qquad m_W = \frac{2 \cdot 1.0079 + 15.9994}{6.02205 \cdot 10^{26}}\text{kg}.$$

The reference values used are found in [22].

8.1. *Example 1: 2-phase flow, wave structure independent of phase transition.* We consider an example in which the wave structure does not depend on whether a phase transition is modeled or not. The initial data and reference values for the first example are given by

$v_V = -100\text{m/s}$	$v_L = 100\text{m/s}$	$T_0 = 293.15\text{K}$	$K_0 = 10^9/0.45836\text{Pa}$
$p_V = 2300\text{Pa}$	$p_L = 1000\text{Pa}$	$\rho_0 = 1000/1.00184\text{kg/m}^3$	$p_0 = 2339\text{Pa}.$

Figure 10 shows for $z = 0$ the solution for velocity, pressure and density as well as the wave pattern. The phase boundary is indicated by the dotted line. Figure 11 gives the solution for the same problem with $z \neq 0$, i.e. with phase transition. Both solutions have similar wave patterns.

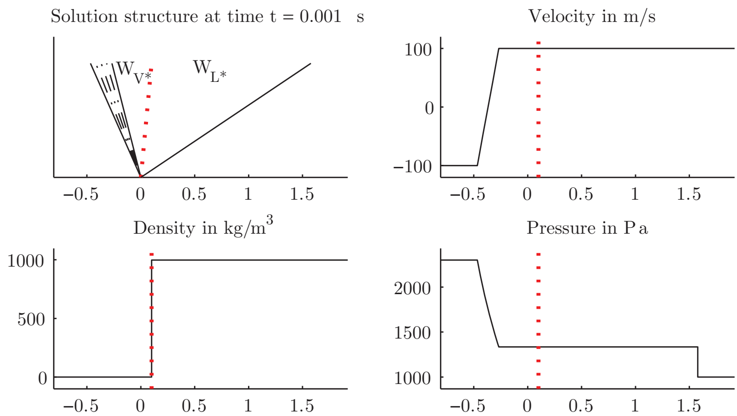

FIG. 10. Example 1, without phase transition

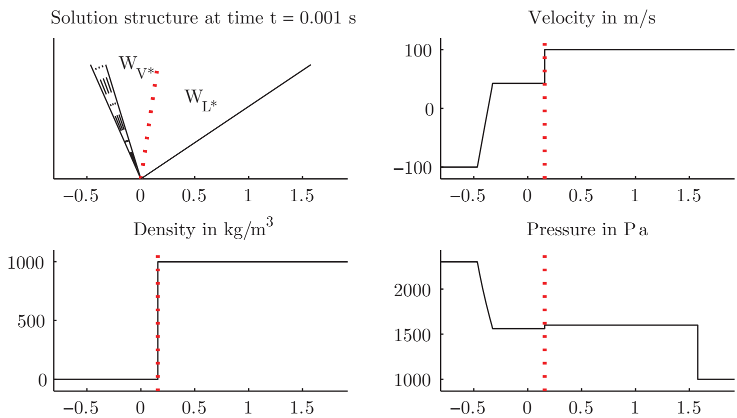

FIG. 11. Example 1, with phase transition

Note that in the plots for density and velocity the jump across the shock wave is so small that it is not visible in the chosen scale. This is generally true for classical waves inside the liquid phase. The difference is only visible in a local zoom.

The solutions to Example 1 for the intermediate states $v_{V*}, p_{V*}, v_{L*}, p_{L*}$ for both cases are summarized in the following table:

$v_{V*} = 100.0002\text{m/s}$	$v_{L*} = 100.0002\text{m/s}$	$v_{V*} = 42.5\text{m/s}$	$v_{L*} = 100.0004\text{m/s}$
$p_{V*} = 1335.3\text{Pa}$	$p_{L*} = 1335.3\text{Pa}$	$p_{V*} = 1561\text{Pa}$	$p_{L*} = 1699.5\text{Pa}\,.$

8.2. *Example 2: 2-phase flow, wave structure depending on phase transition.* We now consider an example in which the wave type changes when a phase transition is introduced. The second example relies on

$v_V = -200\text{m/s}$	$v_L = -50\text{m/s}$	$T_0 = 473.15\text{K}$	$K_0 = 10^9/0.88383\text{Pa}$
$p_V = 60000\text{Pa}$	$p_L = 100000\text{Pa}$	$\rho_0 = 1000/1.15651\text{kg/m}^3$	$p_0 = 1554670\text{Pa}\,.$

In the case without phase transition the solution is composed of two rarefaction waves, see Figure 12, whereas the solution with phase transition possesses two shock waves, see Figure 13. The corresponding wave curves are given in Figure 5 of Subsection 6.1 and Figure 7 of Subsection 6.2. The solutions to Example 2 for the intermediate states

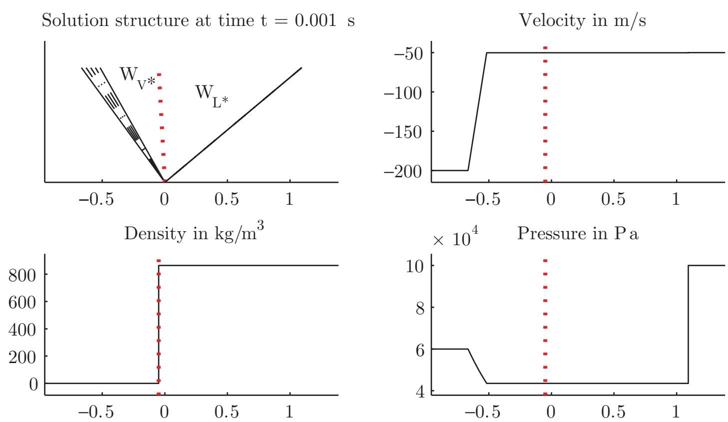

FIG. 12. Example 2, without phase transition

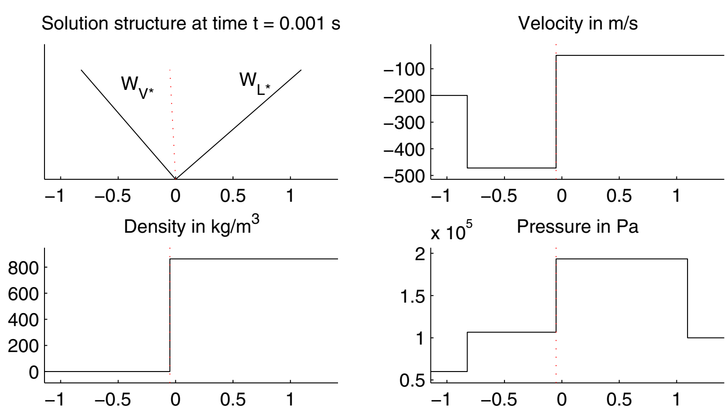

FIG. 13. Example 2, with phase transition

$v_{V*}, p_{V*}, v_{L*}, p_{L*}$ for both cases are summarized in the following table:

$v_{V*} = -50.057\text{m/s}$	$v_{L*} = -50.057\text{m/s}$	$v_{V*} = -472\text{m/s}$	$v_{L*} = -49.905\text{m/s}$
$p_{V*} = 43531\text{Pa}$	$p_{L*} = 43531\text{Pa}$	$p_{V*} = 106525\text{Pa}$	$p_{L*} = 193464\text{Pa}\,.$

8.3. *Example 3: Condensation by compression.* In the third example the data are

$v_{V-} = 2.7\text{m/s}$	$v_{V+} = -2.7\text{m/s}$	$T_0 = 363.15\text{K}$	$K_0 = 10^9/0.47316\text{Pa}$
$p_{V-} = 70000\text{Pa}$	$p_{V+} = 70000\text{Pa}$	$\rho_0 = 1000/1.03594\text{kg/m}^3$	$p_0 = 70182.4\text{Pa}.$

The solution at time $t = 0.001s$ is illustrated in Figure 14, including a zoom plot to show the details. Further, the solutions to Example 3 for the intermediate states $v_{V*}, p_{V*}, v_{L*}, p_{L*}, v_{V**}, p_{V**}$ are summarized in the following table:

$v_{V*} = 0.465\text{m/s}$	$v_{L*} = 0$	$v_{V**} = -0.465\text{m/s}$
$p_{V*} = 70383.04\text{Pa}$	$p_{L*} = 70383.13\text{Pa}$	$p_{V**} = 70383.04\text{Pa}\,.$

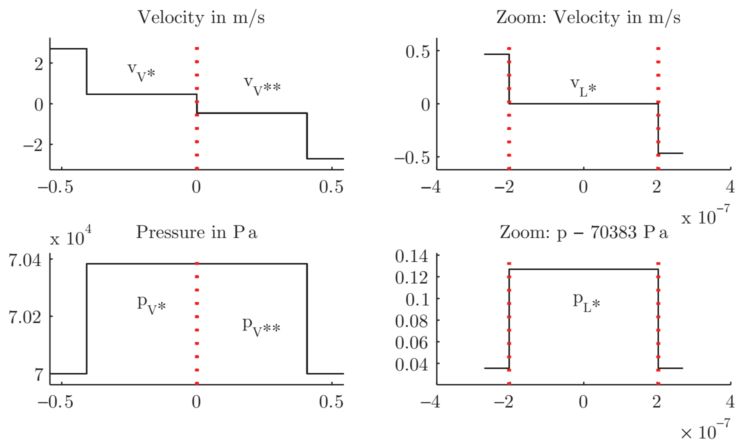

FIG. 14. Example 3, condensation by compression

8.4. *Examples 4 and 5: Evaporation by expansion.* At first we start from the data

$v_{L-} = -40\text{m/s}$	$v_{L+} = 40\text{m/s}$	$T_0 = 363.15\text{K}$	$K_0 = 10^9/0.47316\text{Pa}$
$p_{L-} = 60000\text{Pa}$	$p_{L+} = 60000\text{Pa}$	$\rho_0 = 1000/1.03594\text{kg/m}^3$	$p_0 = 70182.4\text{Pa}$

and show the result at time $t = 0.001s$ in Figure 15. The same phenomenon is produced now by different data, namely

$v_{L-} = -20\text{m/s}$	$v_{L+} = 30\text{m/s}$	$T_0 = 363.15\text{K}$	$K_0 = 10^9/0.47316\text{Pa}$
$p_{L-} = 30000\text{Pa}$	$p_{L+} = 40000\text{Pa}$	$\rho_0 = 1000/1.03594\text{kg/m}^3$	$p_0 = 70182.4\text{Pa}.$

Example 4 consists of two rarefaction waves and two phase transitions, whereas Example 5 exhibits two shock waves and two phase transitions; see Figure 16. The data for the intermediate states $v_{L*}, p_{L*}, v_{V*}, p_{V*}, v_{L**}, p_{**}$ for both examples are given in

$v_{L*} = -39.996\text{m/s}$	$v_{V*} = 0$	$v_{L**} = 39.996\text{m/s}$
$p_{L*} = 55188\text{Pa}$	$p_{V*} = 54665\text{Pa}$	$p_{L**} = 55188\text{Pa}$

and

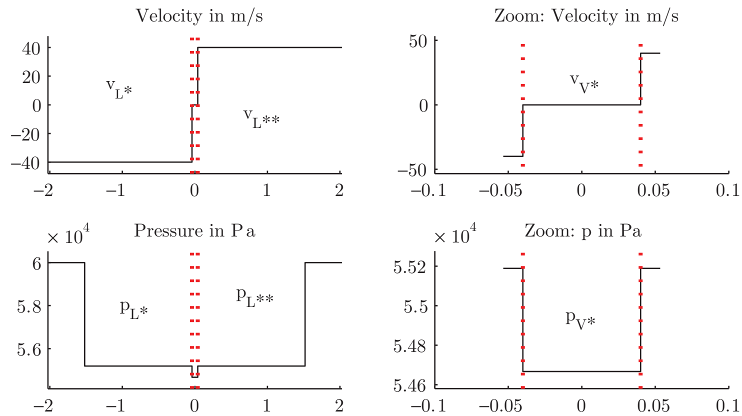

FIG. 15. Example 4, evaporation by expansion

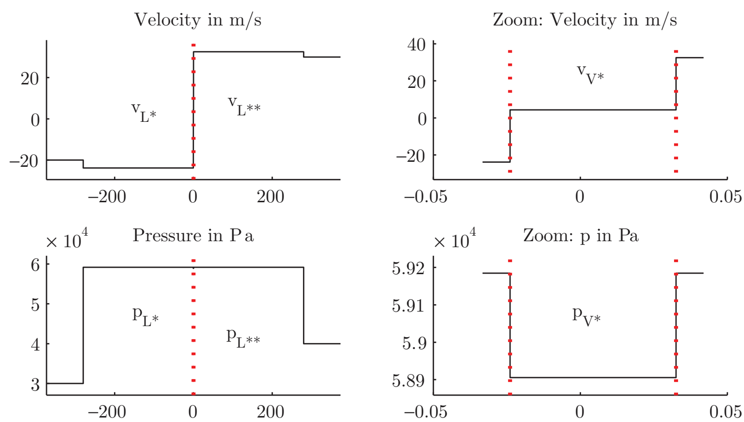

FIG. 16. Example 5, evaporation by expansion

$v_{L*} = -23.9\text{m/s}$	$v_{V*} = 4.3\text{m/s}$	$v_{L**} = 32.5\text{m/s}$
$p_{L*} = 59185\text{Pa}$	$p_{V*} = 58905\text{Pa}$	$p_{L**} = 59185\text{Pa}$.

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