

Integrability for the full spectrum of planar AdS/CFT



Pedro Vieira

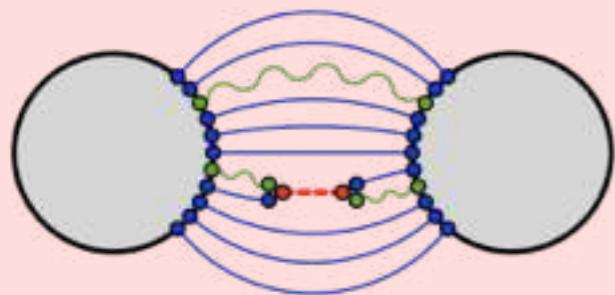
[with N.Gromov, V.Kazakov and A.Kozak]

Plan

- AdS/CFT, motivation
- 2D integrable models at finite volume
- Integrability in AdS/CFT
 - ◆ Asymptotic spectrum
 - ◆ Exact planar spectrum and the Konishi operator

CFT

N=4 SYM in 4d



$$\mathcal{L} = F^2 + (D\Phi)^2 + [\Phi, \Phi]^2 + \bar{\Psi}\mathcal{D}\Psi$$

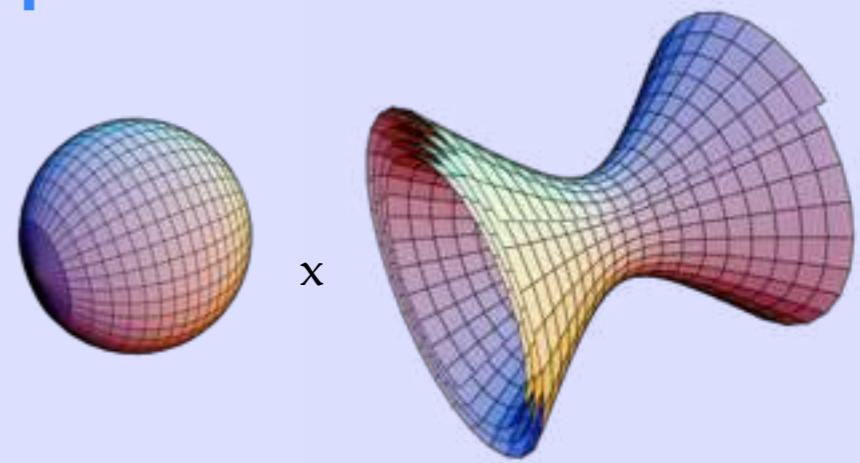
$$\lambda = g_{YM}^2 N = \frac{R^4}{\alpha'^2} = g_s N$$

Planar Limit $N \rightarrow \infty$

Planar CFT

AdS

type IIB in $\text{AdS}_5 \times S^5$

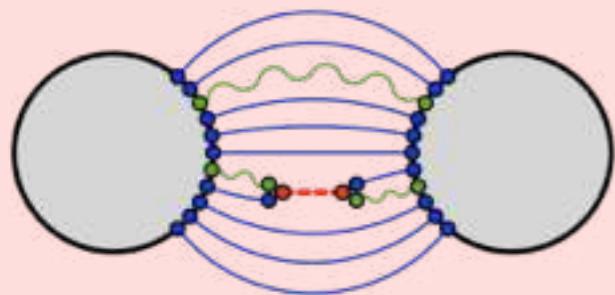


$$\mathcal{L} = (\partial x)^2 + \Lambda(x^2 - 1) + \dots$$

Free Strings

CFT

N=4 SYM in 4d



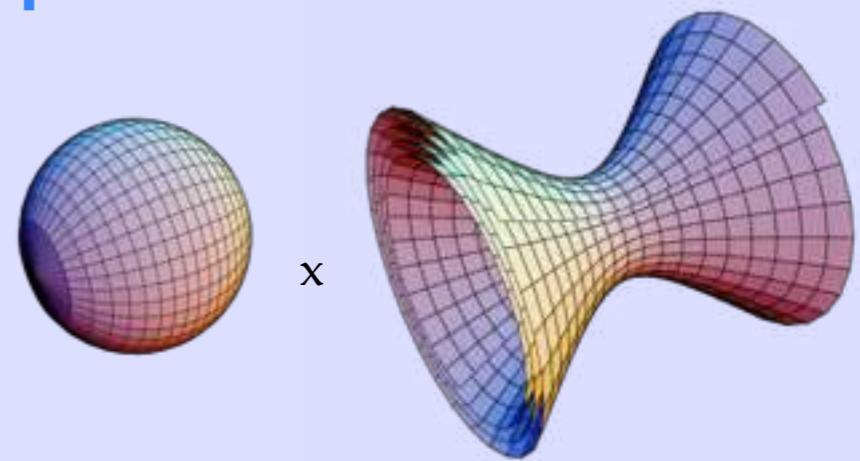
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Strongly coupled gauge theory

Perturbative gauge theory

AdS

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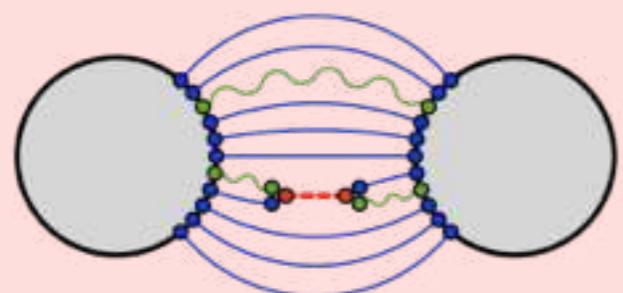
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Classical strings

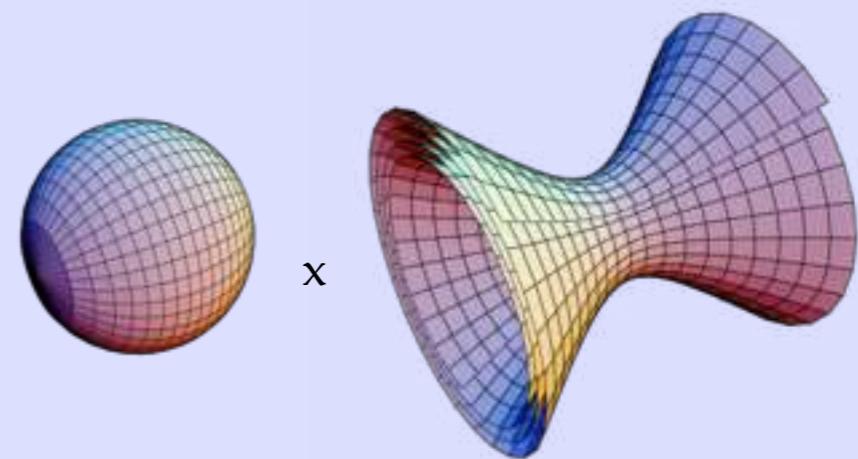
Highly quantum strings

Solve = ?

- Correlation functions:
 - 2 pt functions (**spectrum of anomalous dimensions**)
 - 3 pt functions (structure constants)
- S-matrix



anomalous dimensions



string energies

Integrability

Asymptotic Bethe equations and integrable 2D QFT

In 1+1D $Q_1 = \sum p_j , \quad Q_2 = \sum p_j^2 , \Rightarrow \{p_1, p_2\} = \{p'_1, p'_2\}$

Asymptotic Bethe equations and integrable 2D QFT

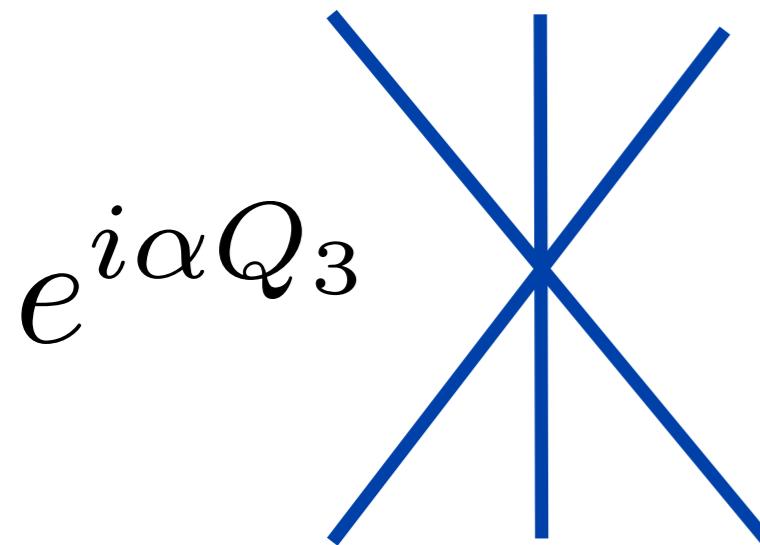
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If (integrability!) $\exists Q_3 = \sum p_j^3 \Rightarrow \{p_1, p_2, p_3\} = \{p'_1, p'_2, p'_3\}$

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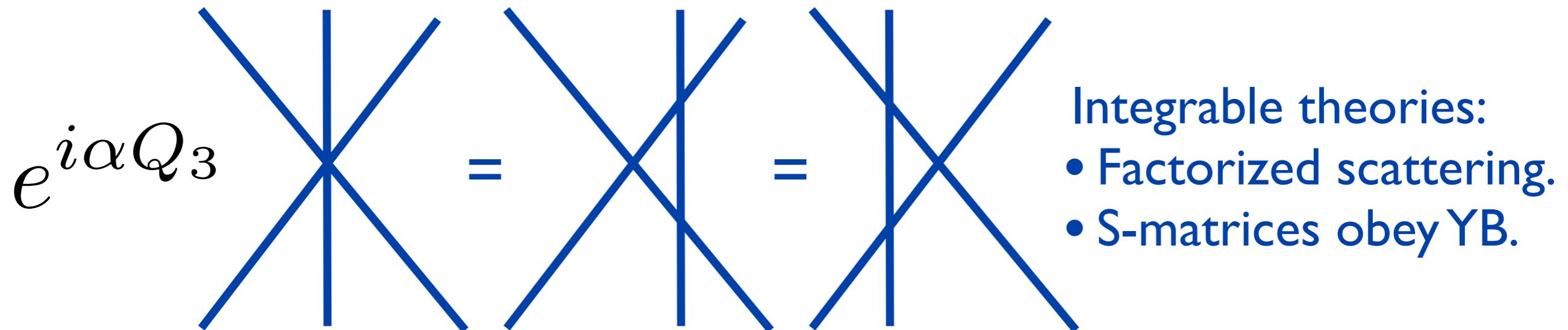
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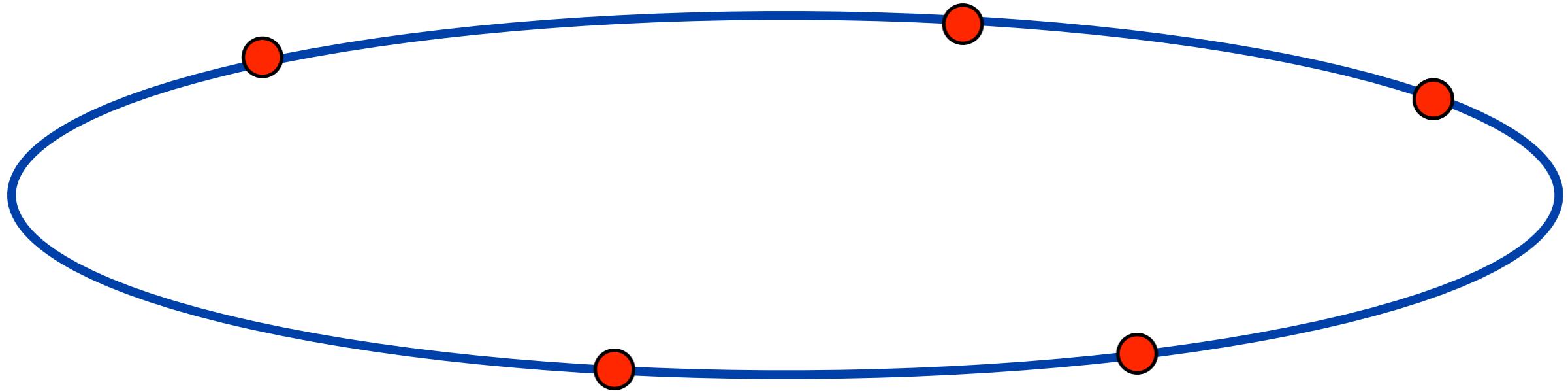
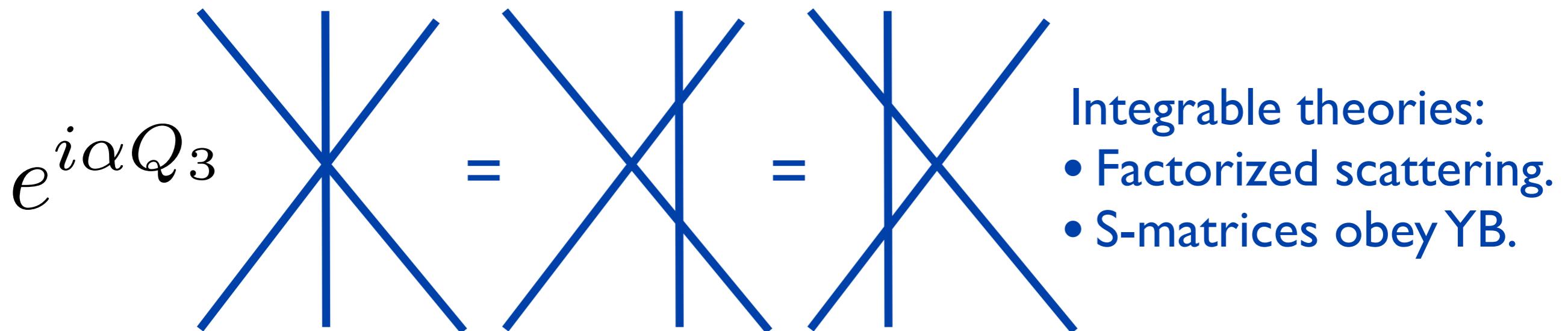
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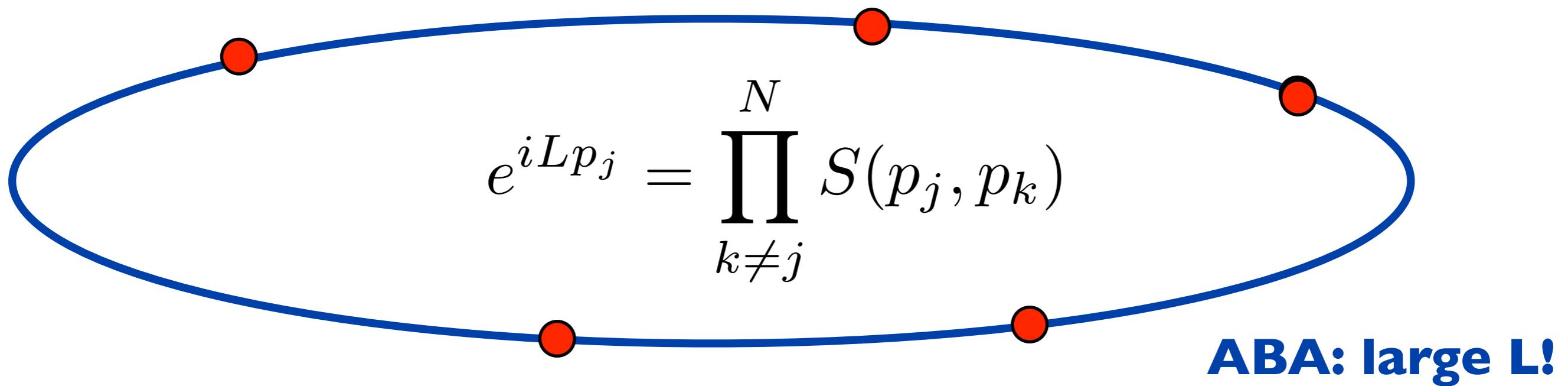
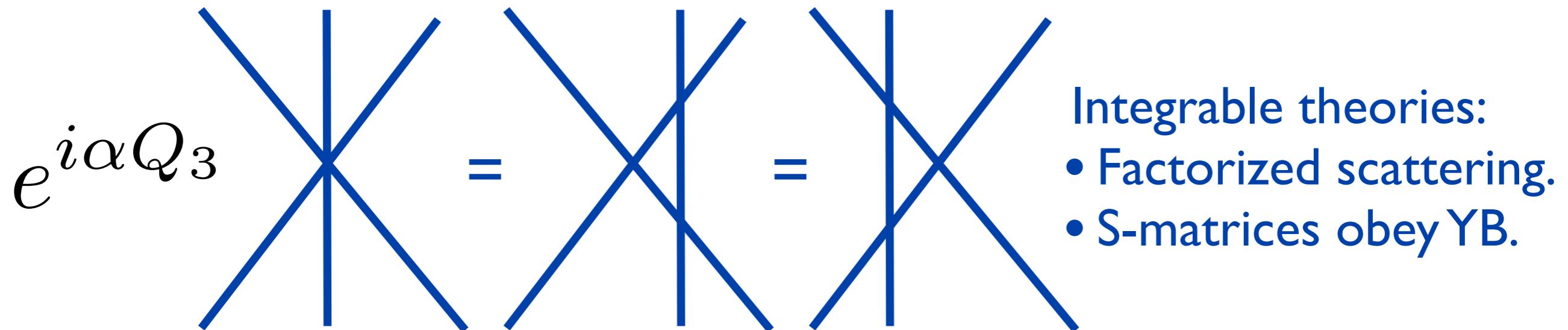
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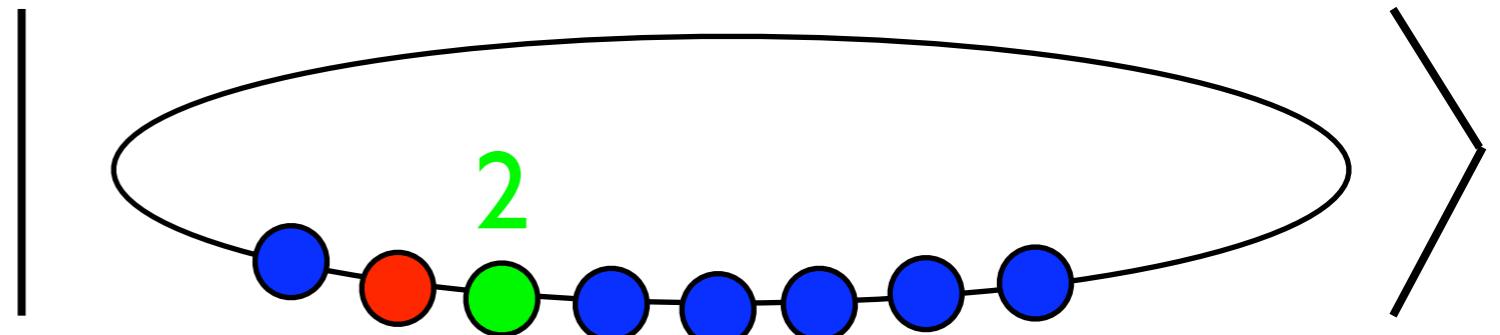


Spin chains in N=4

$$\mathrm{tr}\left(\Phi_1\Phi_2(D_3)^2(\Phi_1)^5\dots\right)$$

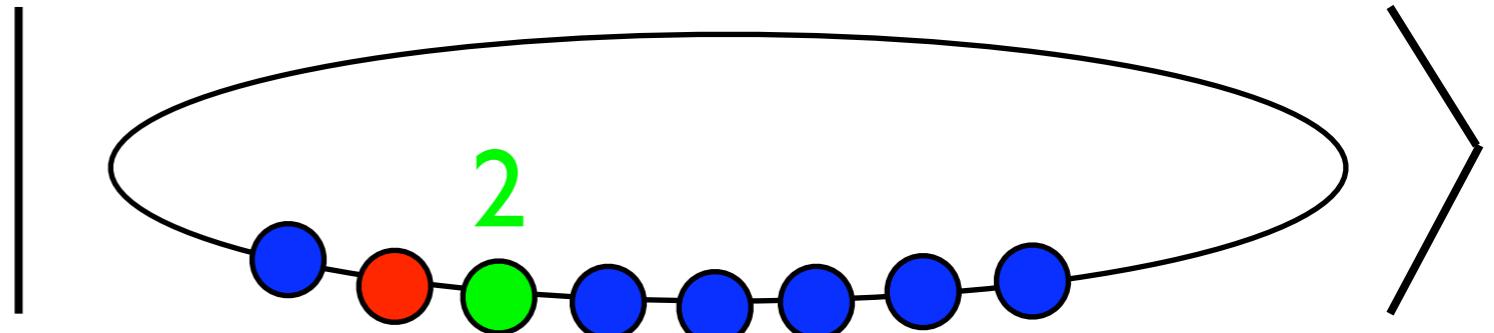
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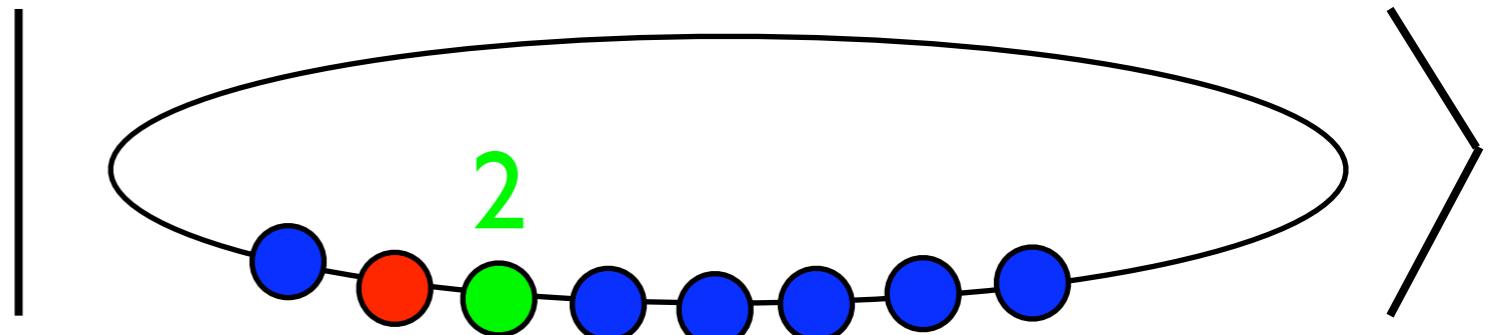
Mixing matrix or
Dilatation operator

$$\mathcal{O}_A^{ren}(x) = (e^{\hat{H} \log \Lambda})_{AB} \mathcal{O}_B(x)$$

$$\langle \mathcal{O}_A^{ren}(x) \mathcal{O}_B^{ren}(y) \rangle = \frac{\delta_{AB}}{|x - y|^{2\Delta}}$$

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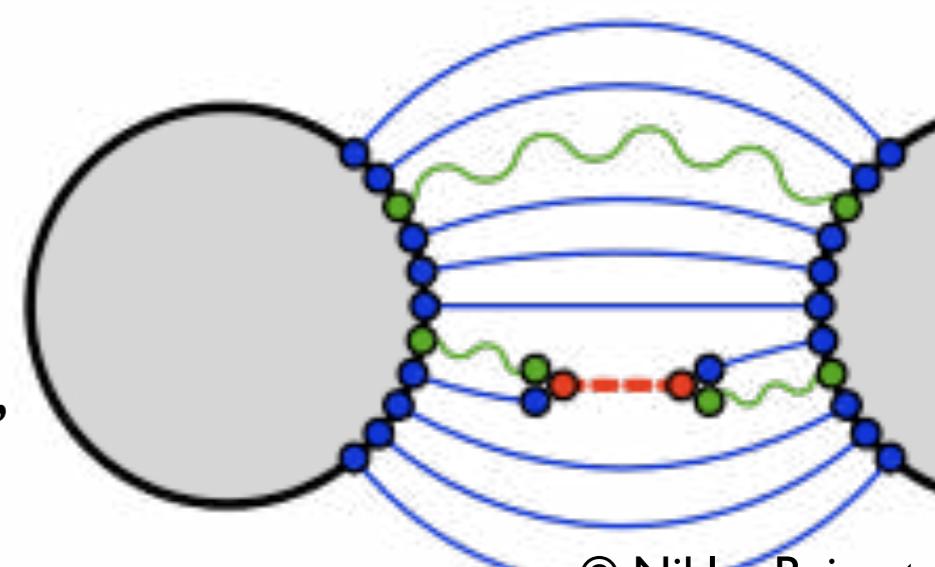


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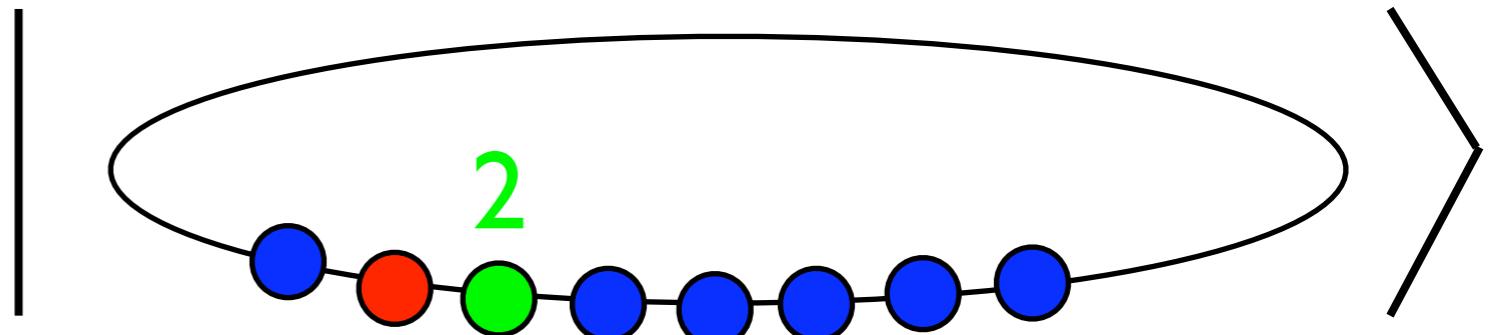
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H is nearest neighbors to leading order in perturbation theory,
next to nearest neighbors at next to leading order etc...



Spin chains in N=4

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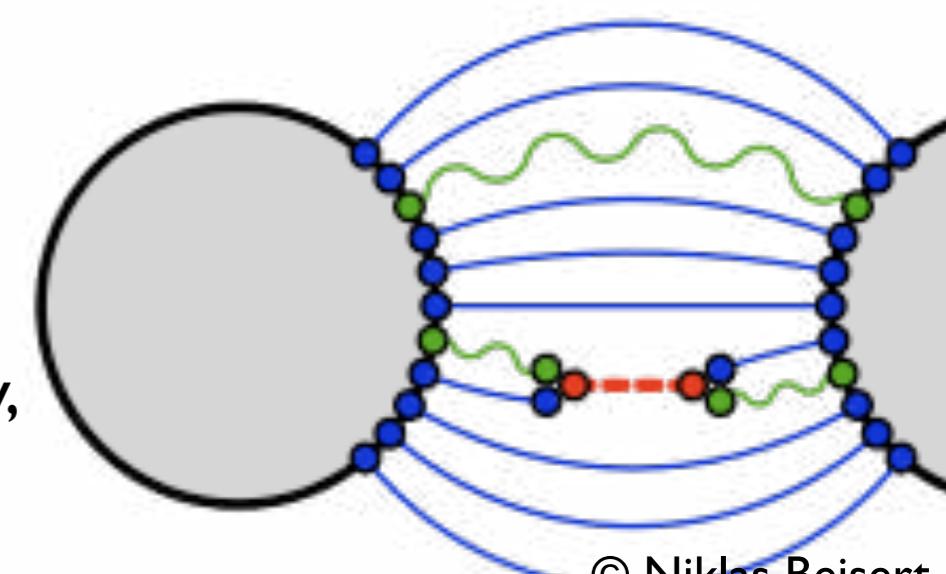
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Integrable spin chain
Hamiltonian

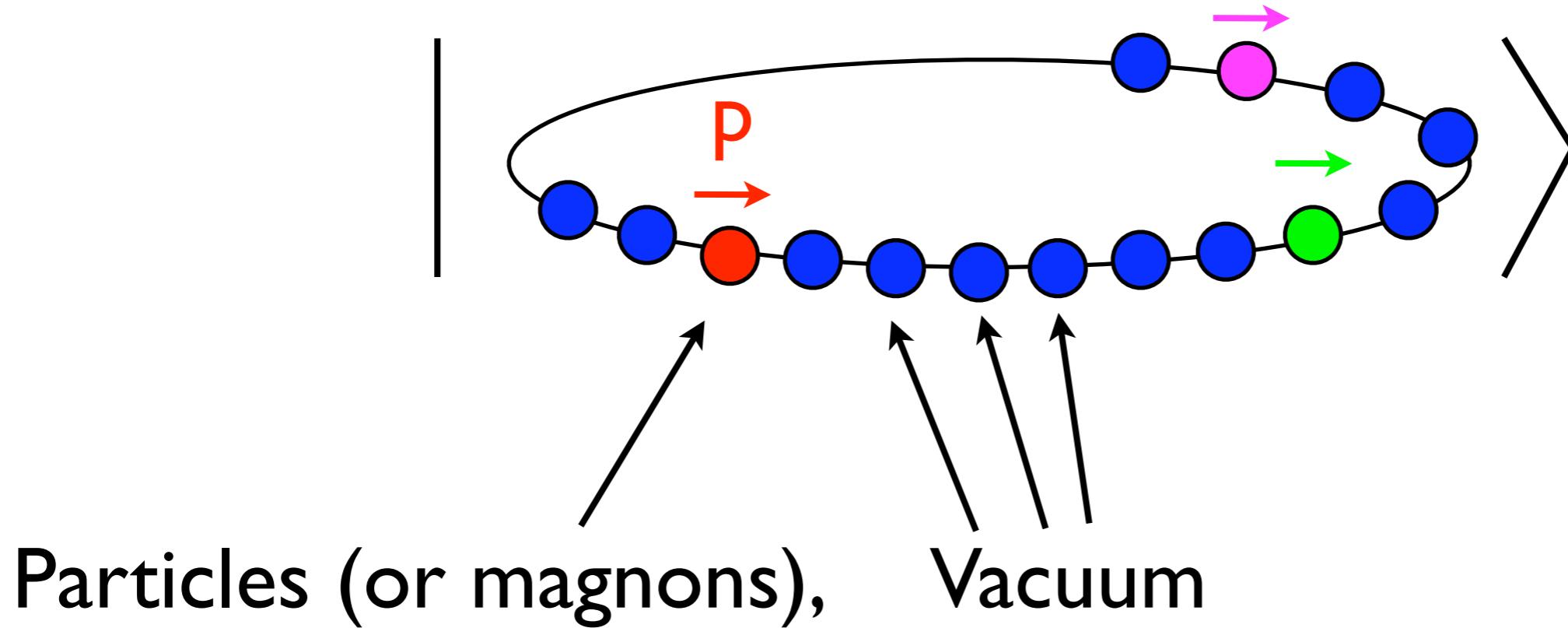
[Minahan, Zarembo; Beisert, Staudacher]

$$\langle \mathcal{O}_A^{ren}(x) \mathcal{O}_B^{ren}(y) \rangle = \frac{\delta_{AB}}{|x - y|^{2\Delta}}$$

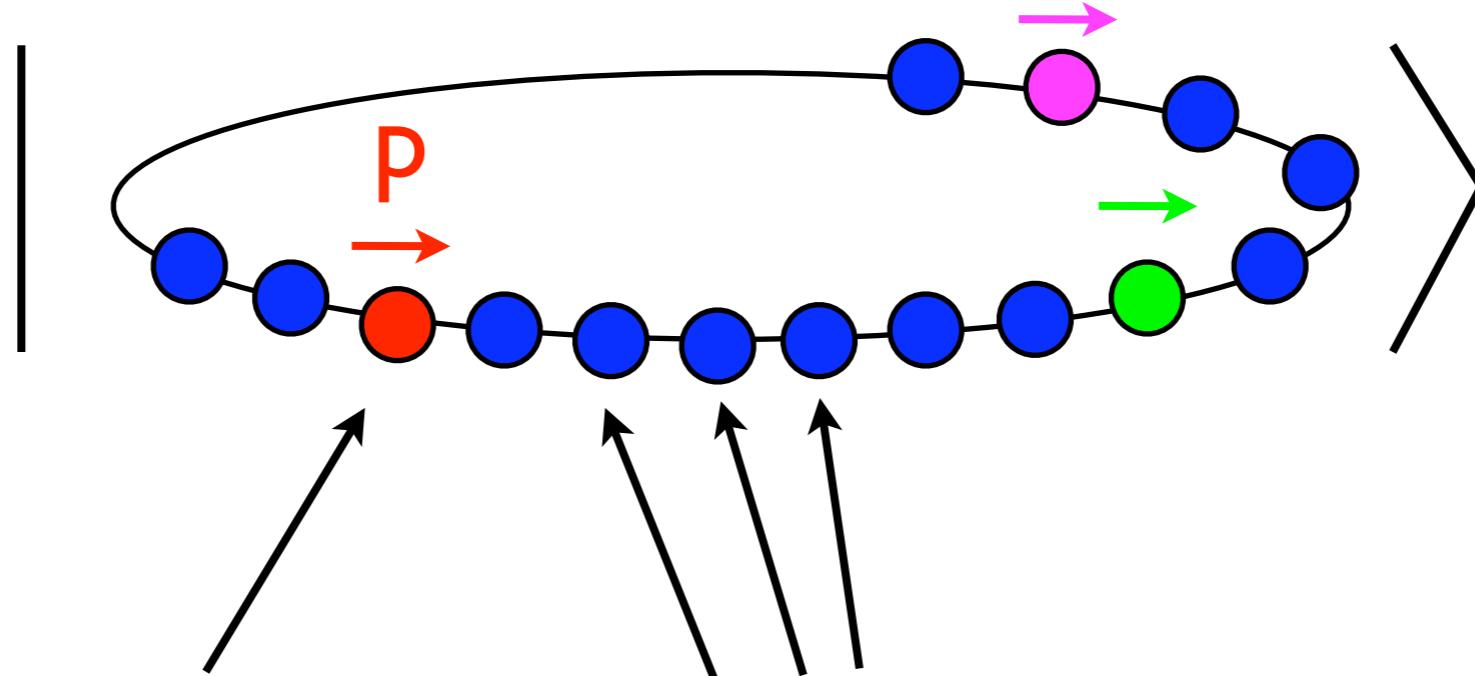
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2d S-matrix in N=4



2d S-matrix in N=4



Particles (or magnons), Vacuum

Particles can scatter, e.g.

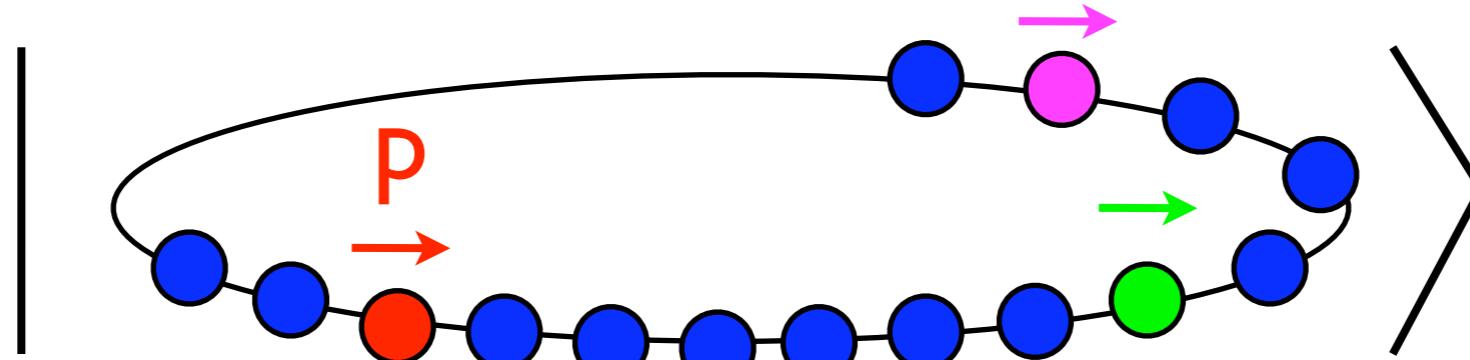
$$S(p, k)$$

[Staudacher]

Particles transform in $\text{PSU}(2|2)^2$ extended

[Beisert]

2d S-matrix in N=4



$$H \longrightarrow S(p,k)$$

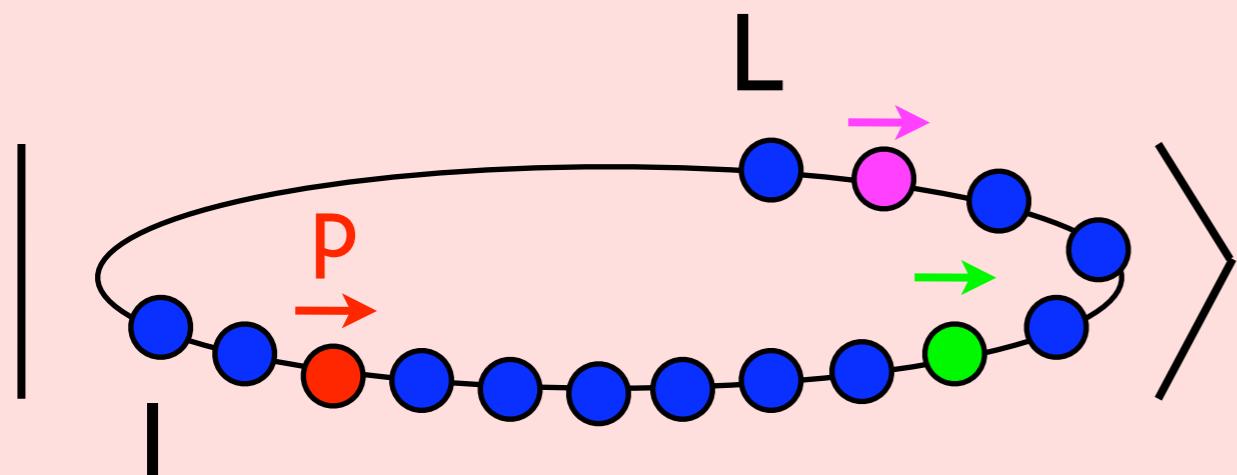
$$\text{PSU}(2,2|4) \longrightarrow \text{PSU}(2|2)^2 \text{ extended}$$

S-matrix (up to a scalar factor) and
magnon dispersion relation
almost fixed by symmetry

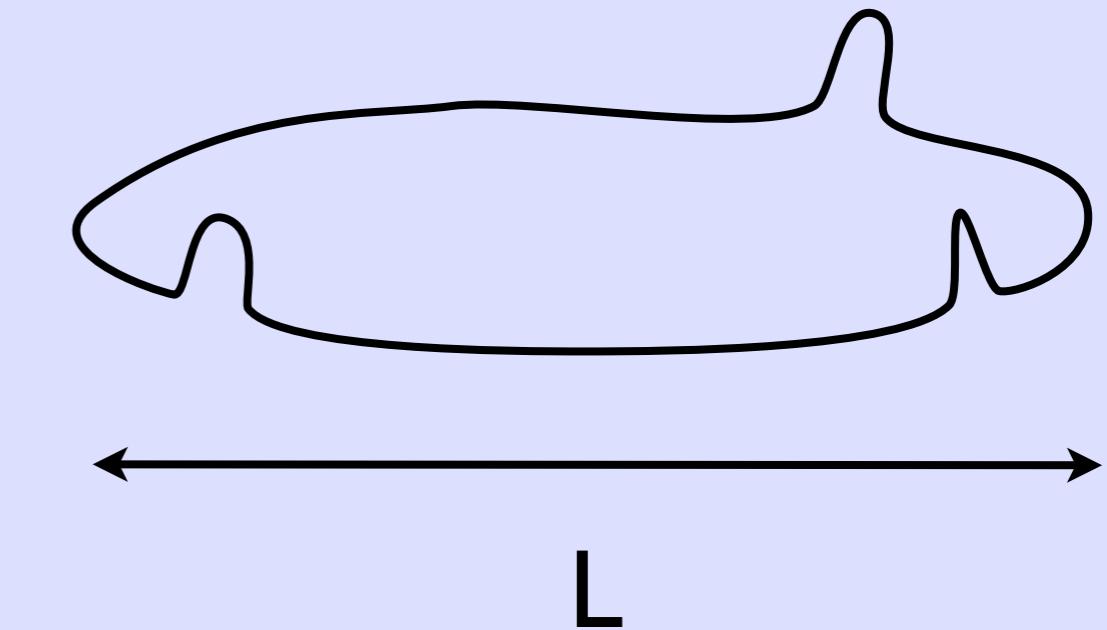
[Beisert, Aryutunov,
Frolov, Zamaklar]

2d S-matrix in AdS/CFT

Spin chain magnons in an operator with L fields



Previous arguments hold for both string and gauge theory



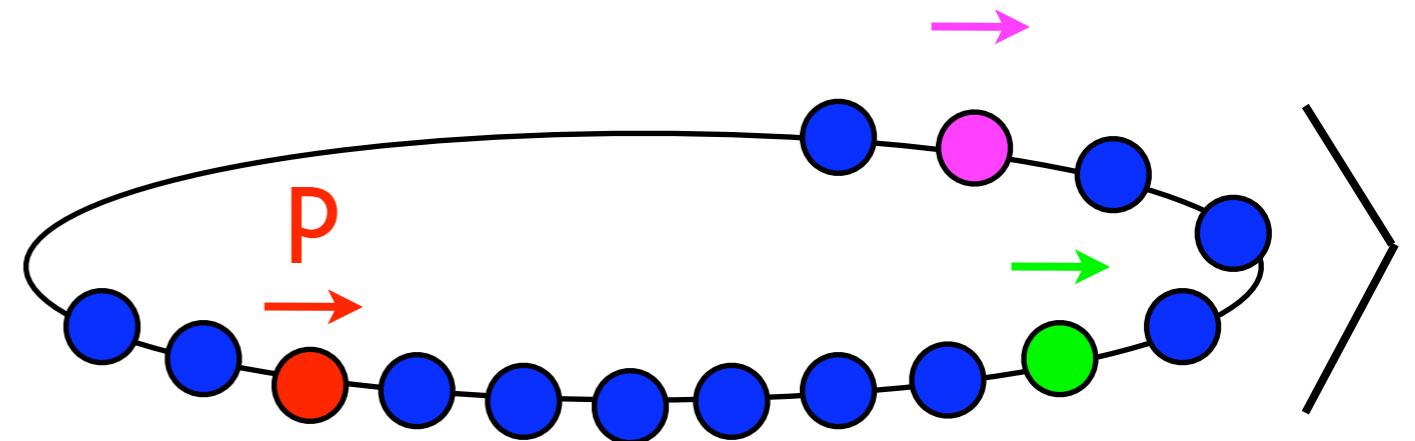
Integrability at strong coupling:
[Bena, Polchinski, Roiban],[KMMZ],...

Worldsheet excitations in light-cone gauged string theory. 2D QFT in a circle of size L

see also
[Aryutunov, Frolov, Plefka, Zamaklar]

Bethe Equations

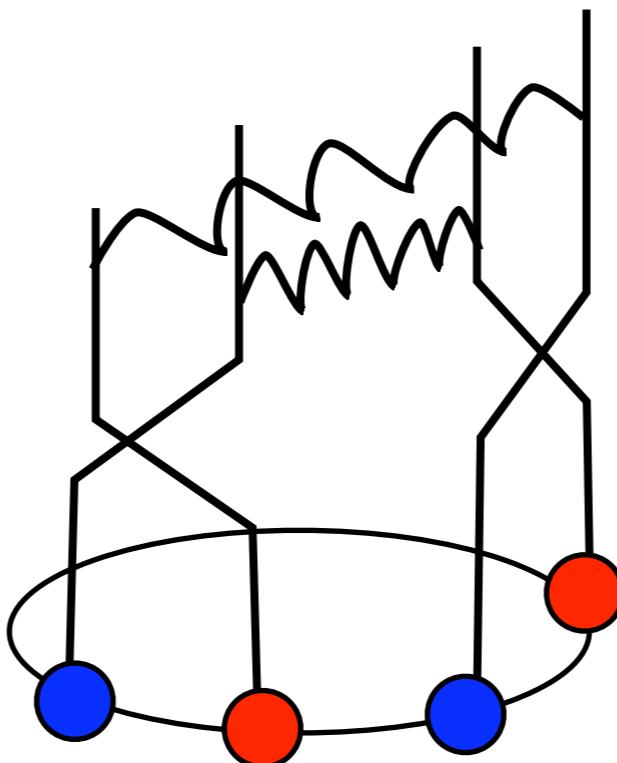
$$0 = \left(e^{iLp_j} \prod_{k \neq j}^M \hat{S}(p_j, p_k) - 1 \right)$$



$$\Delta = J + \sum_{j=1}^M \sqrt{1 + \lambda \sin^2 \frac{p_j}{2}} + \dots$$

[Staudacher; Beisert, Staudacher; Janik; Beisert, Eden, Staudacher;
Beisert, Hernandez, Lopez; Arutyunov, Frolov, Zamaklar]

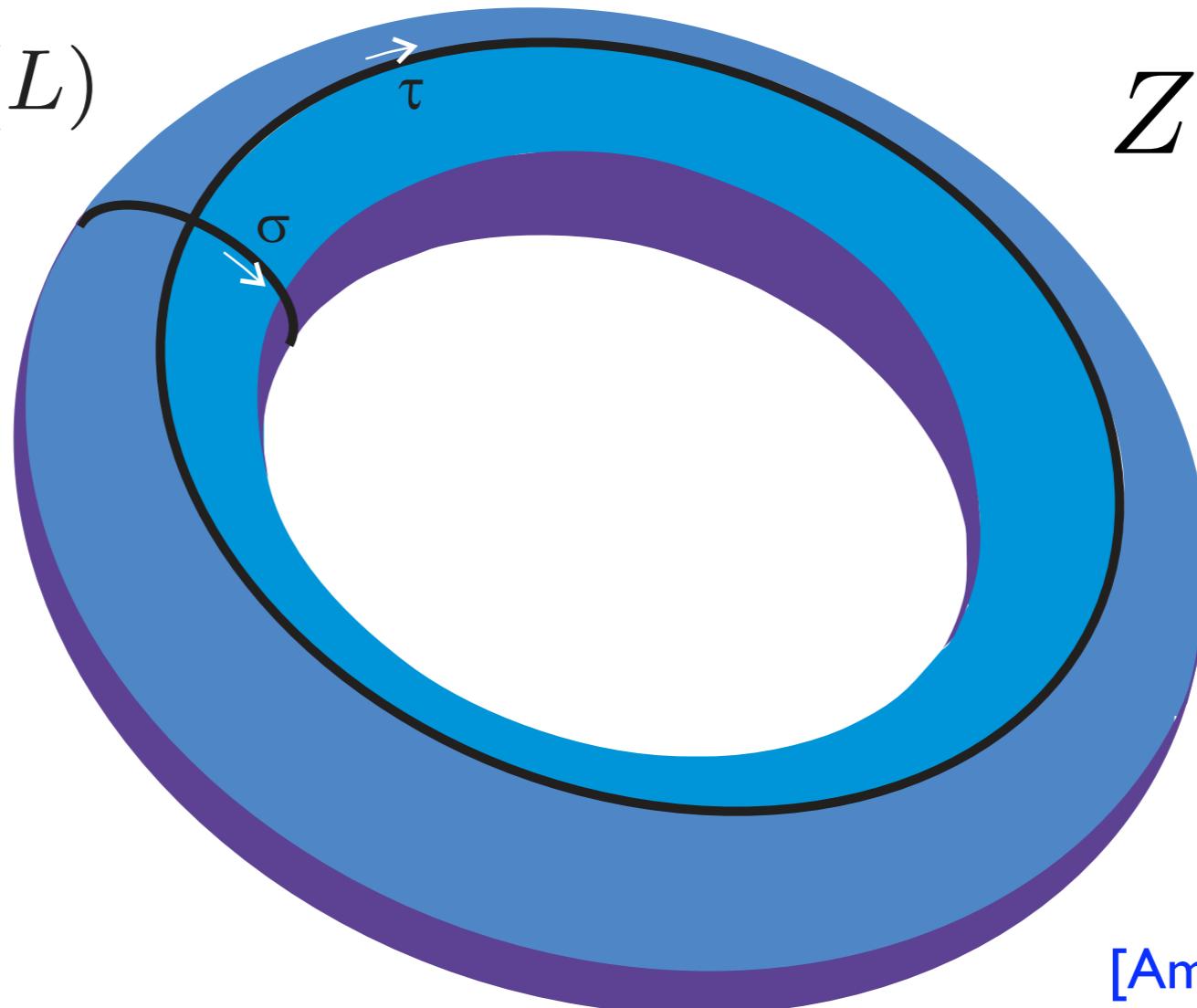
What about small operators?



TBA Wick rotation and exact ground state energy

$$Z = e^{-RE_0(L)}$$

Length: L
 Temperature: 0
 Asymptotic Bethe Ansatz



$$Z = e^{-Rf(L)}$$

Length: infinity
 Temperature: 1/L
 Exact Bethe Ansatz

On the mirror theory in AdS/CFT:

[Ambjorn, Janik, Kristjansen]
 [Arutyunov, Frolov]

Ground state energy at size L = free energy per unit length at temperature 1/L

Recipe

- Compute the particle dispersion relation
- Compute the two body S-matrix
- Write down the mirror BAE in the continuum limit (introducing one density for each kind of particle and bound state)
- Compute the entropy and energy and free energy by a saddle point computation
 - ◆ Saddle point equations: Y-system equations (in or functional form)
 - ◆ Free energy at saddle point: Exact vacuum energy
- Pick extra singularities in the free energy and in the integral equations to get the energy of excited states [Dorey and Tateo]

The Dorey-Tateo trick

Vacuum equations

$$E_0 = \int \frac{du}{2\pi i} \partial_u \epsilon(u) \log(1 + Y(u))$$

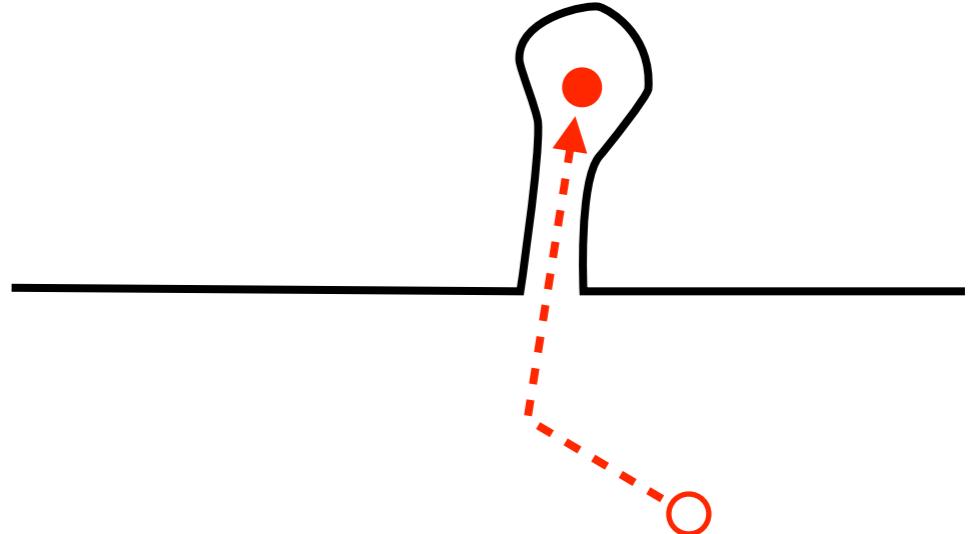
$$\log Y(u) = i L p(u) + \int \frac{dv}{2\pi i} \partial_v \log S(u, v) \log(1 + Y(v))$$

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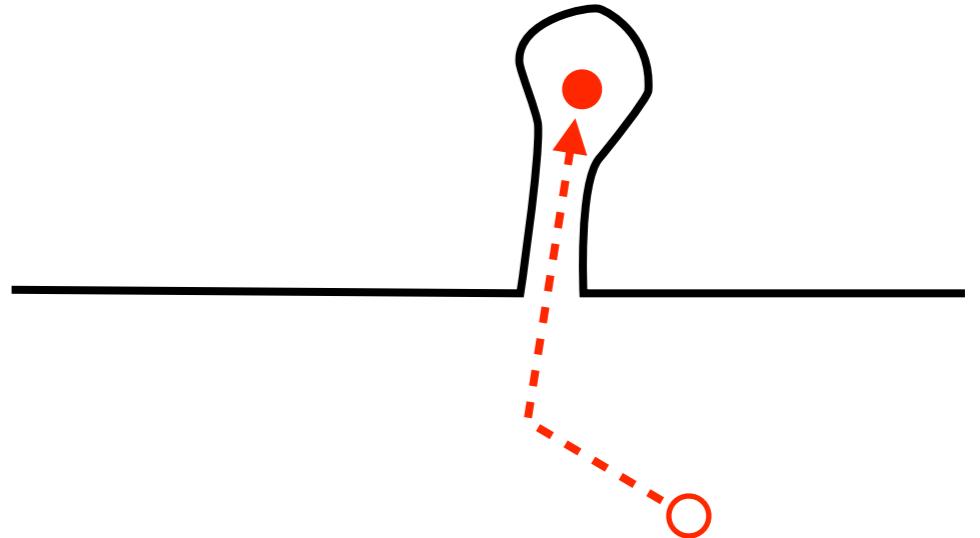
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Vacuum equations **become**

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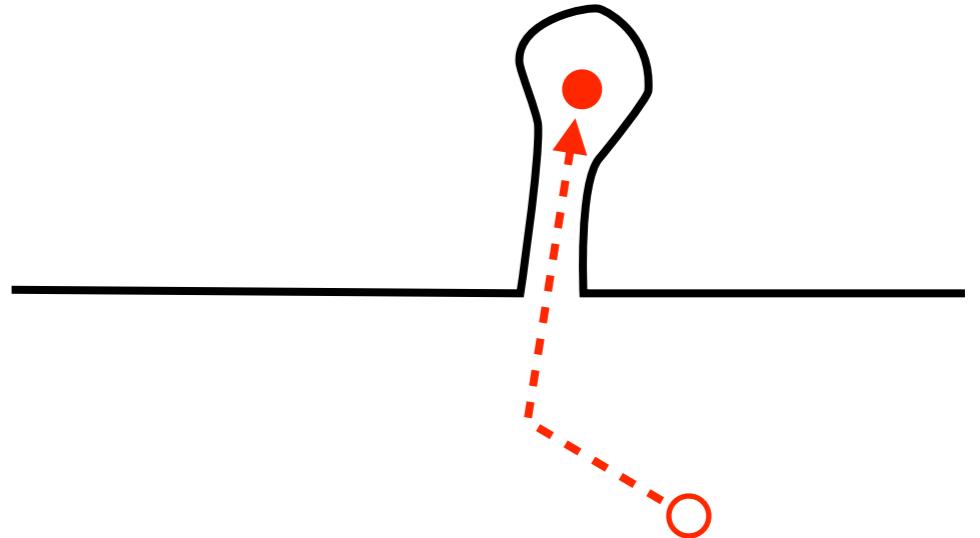
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Exact Bethe Equations: $Y(u_j) = -1$

Large L

$$Y(u) \simeq e^{iLp(u)} \prod_{j=1}^N S(u, u_j)$$

$$E(L) \simeq \sum_{j=1}^N \epsilon(u_j) + \int \frac{du}{2\pi i} \partial_u \epsilon(u) e^{ip(u)L} \prod_{j=1}^N S(u, u_j)$$

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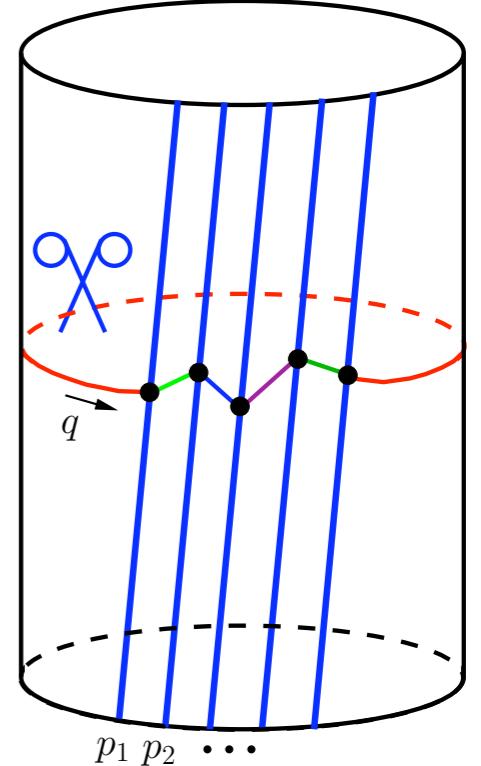
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Asymptotic Bethe Ansatz + Luscher corrections come out of large L

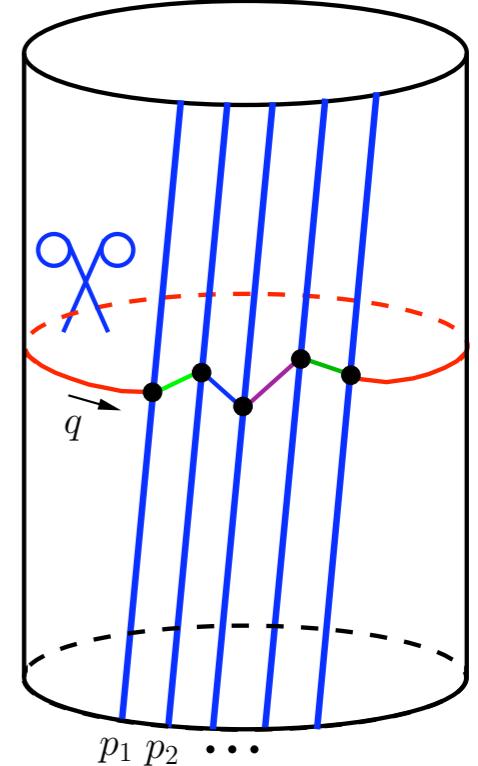
[Luscher; Janik, Lukowsky]

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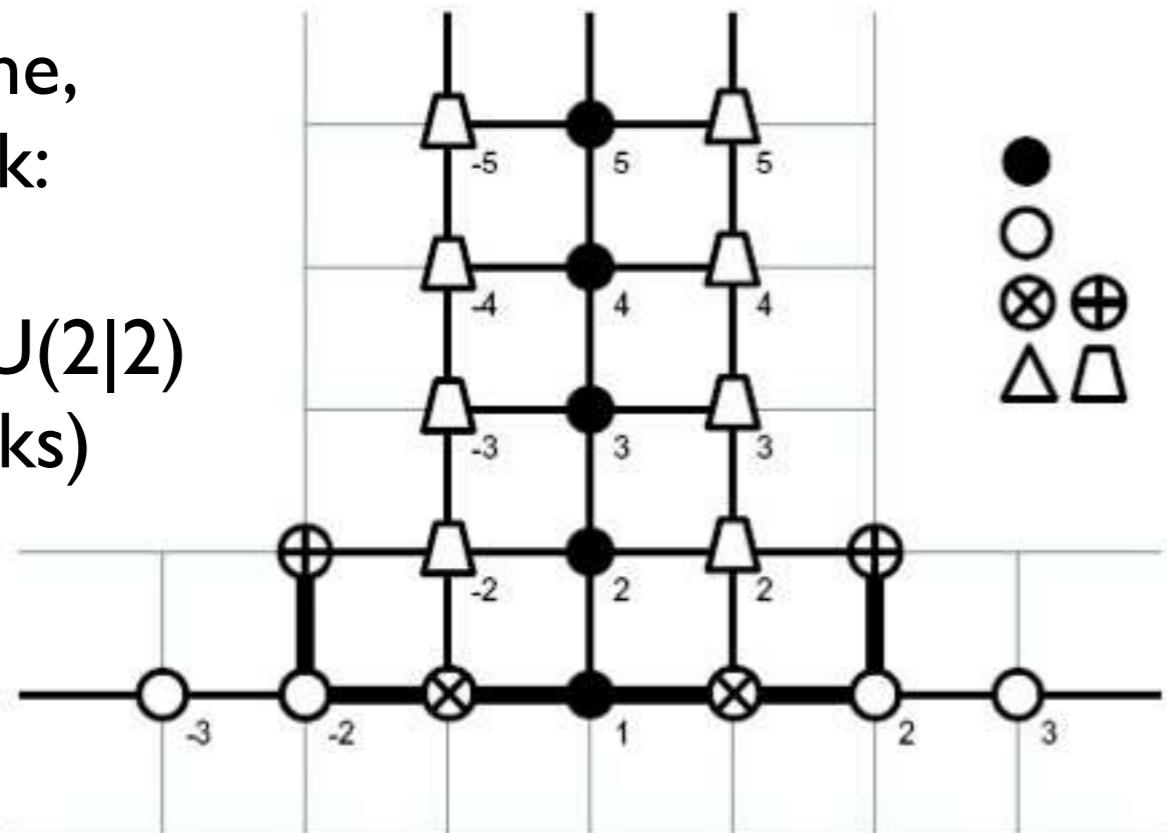
At finite L we simply iterate

$$\log Y(u) = iLp(u) + \sum_{j=1}^N \log S(u, u_j) + \int \frac{dv}{2\pi i} \partial_v \log S(u, v) \log(1 + Y(v))$$

The AdS/CFT Y-system

a,s plane,
T-hook:

(glues 2 SU(2|2)
fat hooks)



[Gromov,Kazakov,PV]

[Gromov,Kazakov,Kozak, PV]

middle node roots/strings

boson roots/strings

v/w fermion roots

pyramids

AdS/CFT vacuum TBA equations:

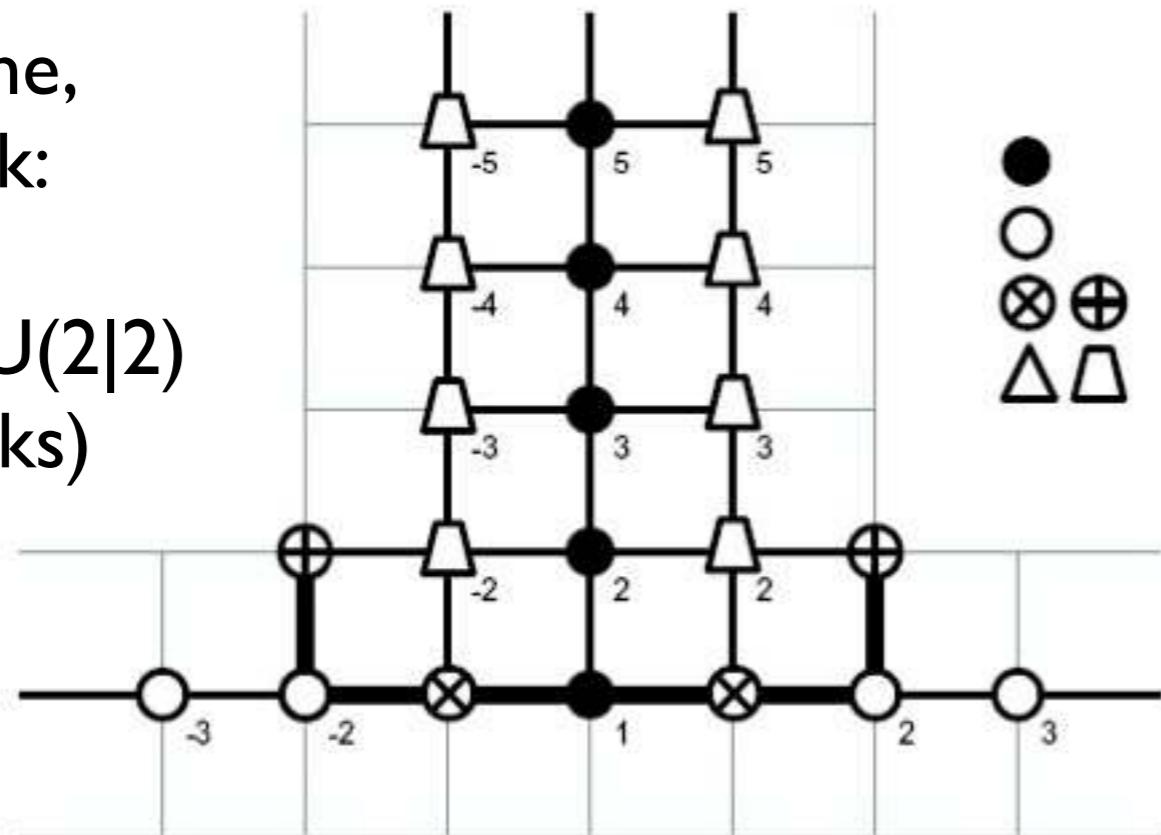
[Bombardelli, Fioravanti, Tateo],

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$$f^\pm = f(u \pm i/2)$$

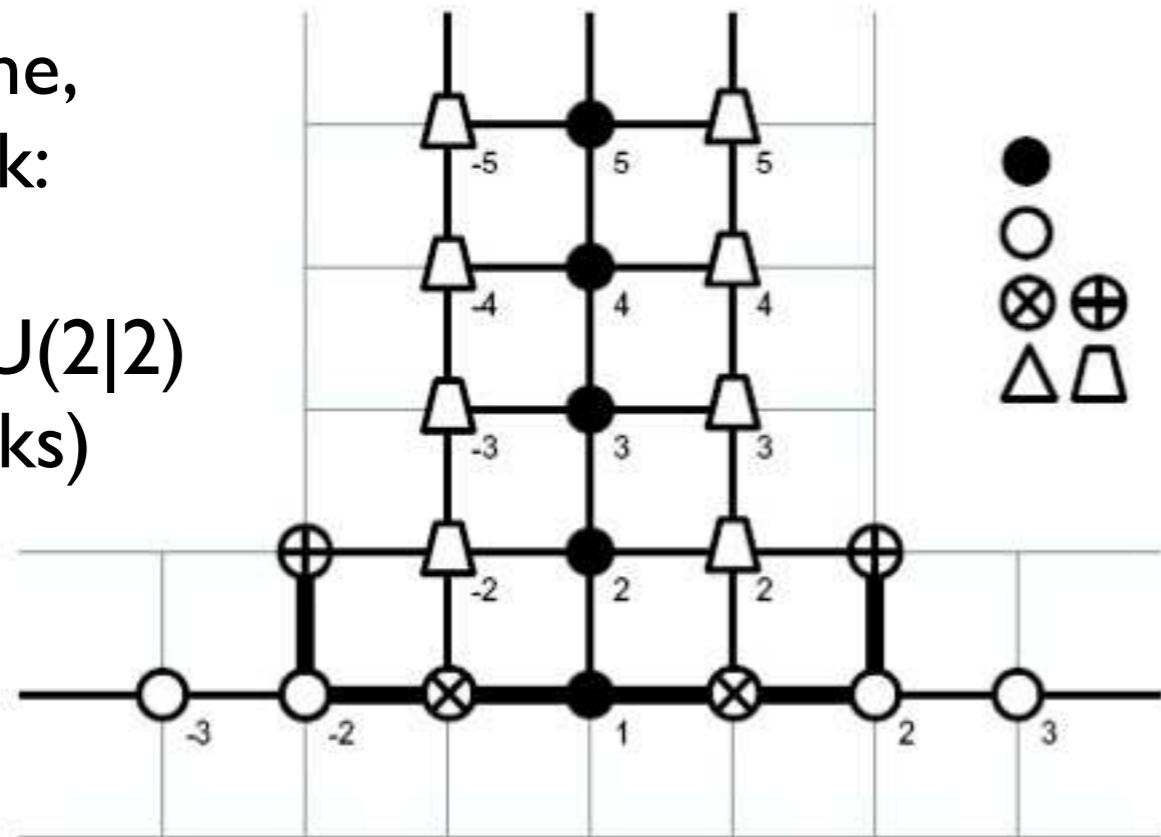
+ boundary conditions

+ analyticity

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The energy depends on the magnon dispersion relation
and on the Y-functions:

$$E = \sum_j \epsilon_1(u_{4,j}) + \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi i} \frac{\partial \epsilon_a^*}{\partial u} \log (1 + Y_{a,0}^*(u))$$

[Gromov,Kazakov,PV]

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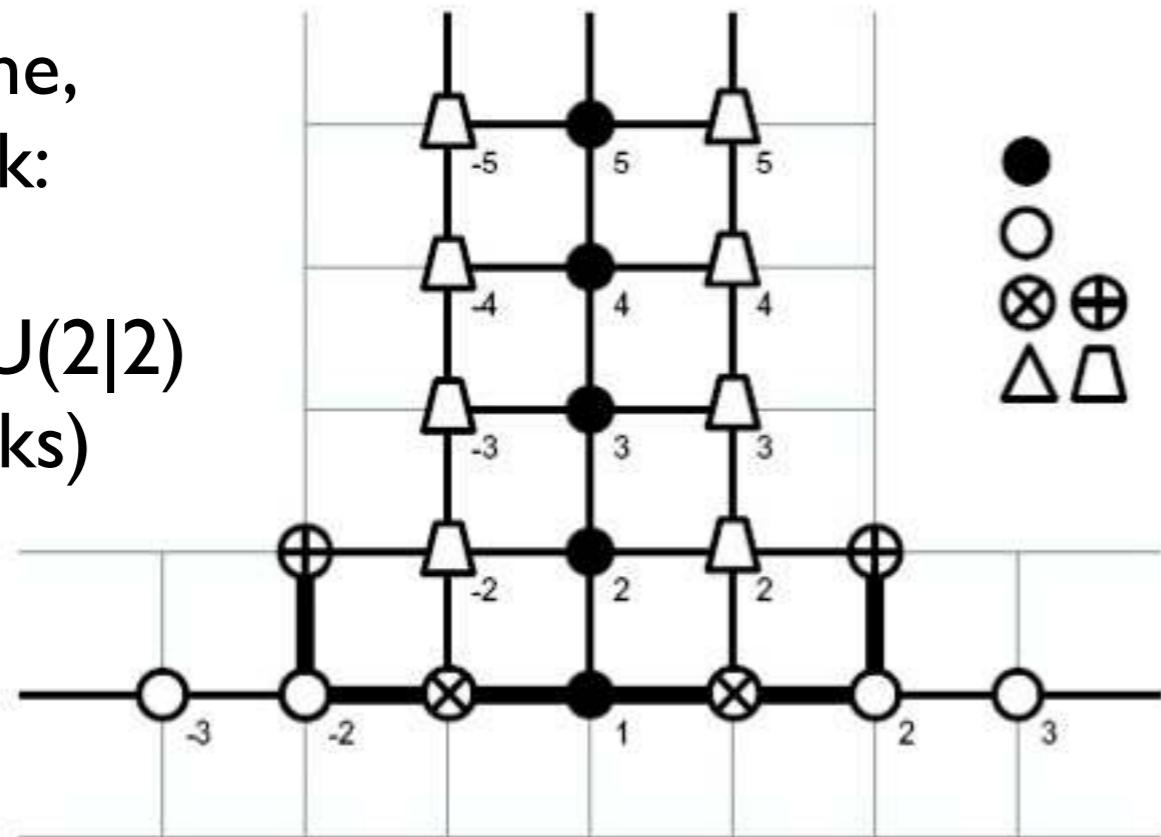
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$$Y_{1,0}(u_{4,j}) = -1$$

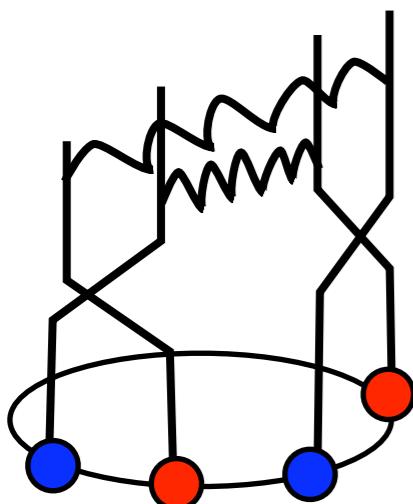
Konishi operator at weak coupling

Konishi multiplet:: [Bianchi,Kovacs,Rossi,Stanev],[Eden,Jarczak,Sokatchev,Stanev]

$$Konishi = \text{tr}[Z, X]^2 = \left| \begin{array}{c} \text{Diagram: 4 vertices in a circle, 2 blue, 2 red, with a wavy line above} \end{array} \right\rangle - \left| \begin{array}{c} \text{Diagram: 4 vertices in a circle, 2 blue, 2 red, with a wavy line below} \end{array} \right\rangle$$

$$\Delta_{Konishi} = 12g^2 - 48g^4 + 336g^6 + (-2584 + 384\zeta_3 - 1440\zeta_5)g^8$$

Reproduces the YM 4 loop computation involving more than 130000 Feynman diagrams! (in components)

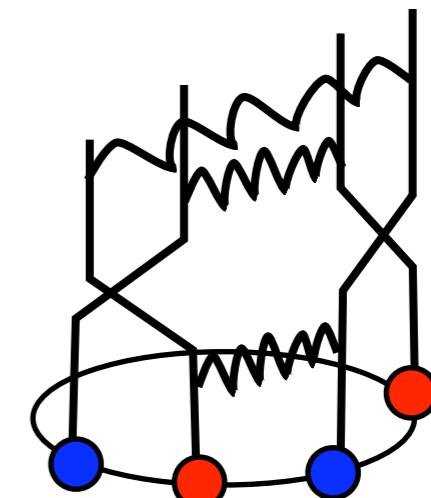


+

[Bajnok,Janik],
[Gromov,Kazakov,PV]

[Fiamberti, Stambrogio,
Sieg, Zanon][Velizhanin]

5 loops last Tuesday: [Bajnok,Hegedus,Janik,Lukowsky]



Konishi at any coupling

[Gromov,Kazakov,Kozak, PV]

(generalizing the Dorey-Tateo trick and using the Aryutunov-Frolov choice of branches)

- We start with large L solution predicted from the Y-system in functional form and then we “simply” iterate the integral equations:

$$\log Y_{\otimes} = K_{m-1} * \log(1 + 1/Y_{O_m}) / (1 + Y_{\Delta_m})$$

$$+ \mathcal{R}^{(0m)} * \log(1 + Y_{\bullet_m}) + \log \frac{-R^{(+)}}{R^{(-)}}$$

$$\log Y_{\Delta_n} = \mathcal{M}_{nm} * \log(1 + Y_{\bullet_m}) - K_{n-1} \circledast \log(1 + Y_{\otimes})$$

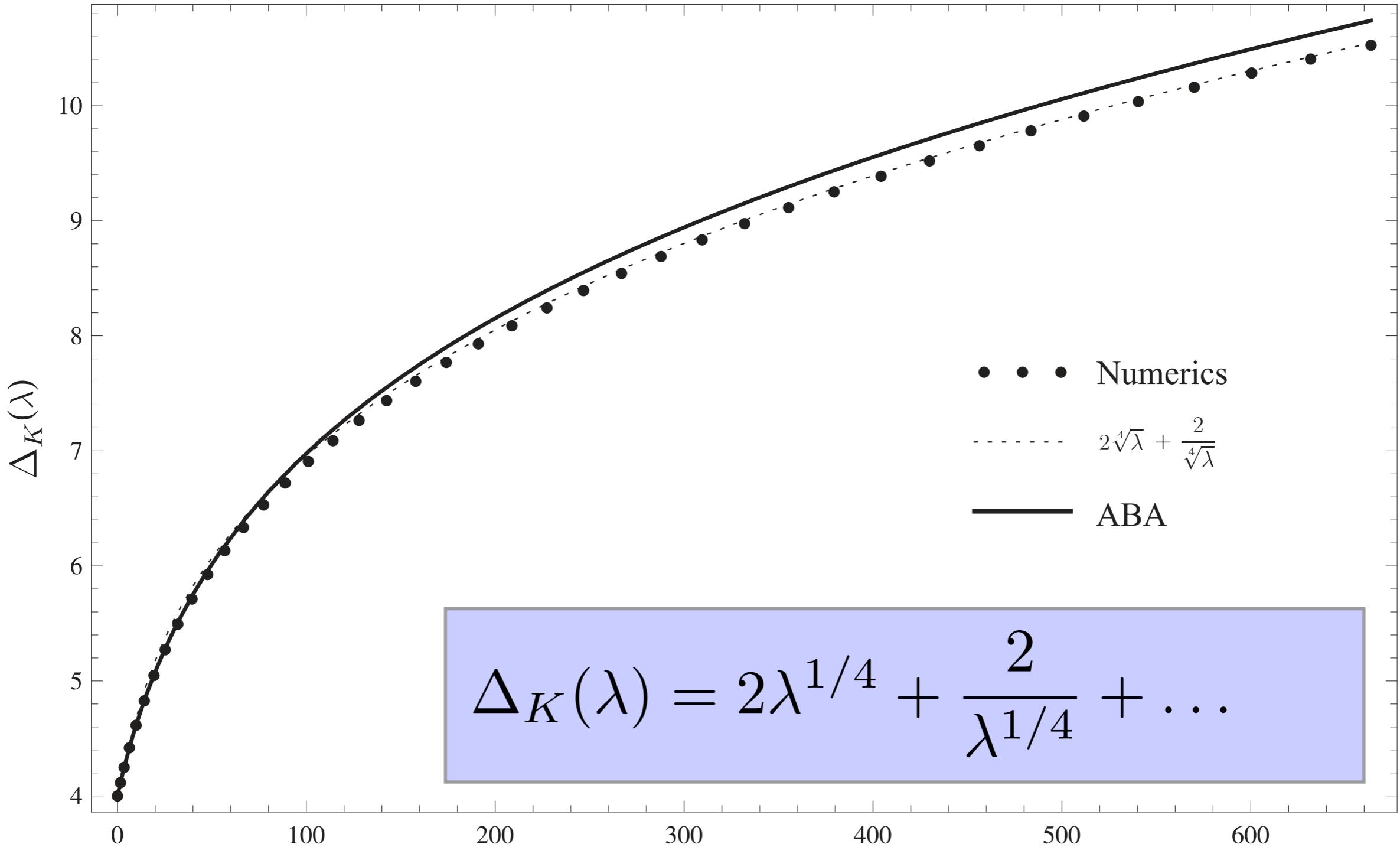
$$- K_{n-1,m-1} * \log(1 + Y_{\Delta_m}) + \log \frac{R_n^{(+)} B_{n-2}^{(+)}}{R_n^{(-)} B_{n-2}^{(-)}}$$

$$\log Y_{O_n} = K_{n-1,m-1} * \log(1 + 1/Y_{O_m}) + K_{n-1} \circledast \log(1 + Y_{\otimes})$$

$$\begin{aligned} \log Y_{\bullet_n} = & \mathcal{T}_{nm} * \log(1 + Y_{\bullet_m}) + 2\mathcal{R}^{(n0)} \circledast \log(1 + Y_{\otimes}) \\ & + \mathcal{N}_{nm} * \log(1 + Y_{\Delta_m}) + i\Phi_n . \end{aligned}$$

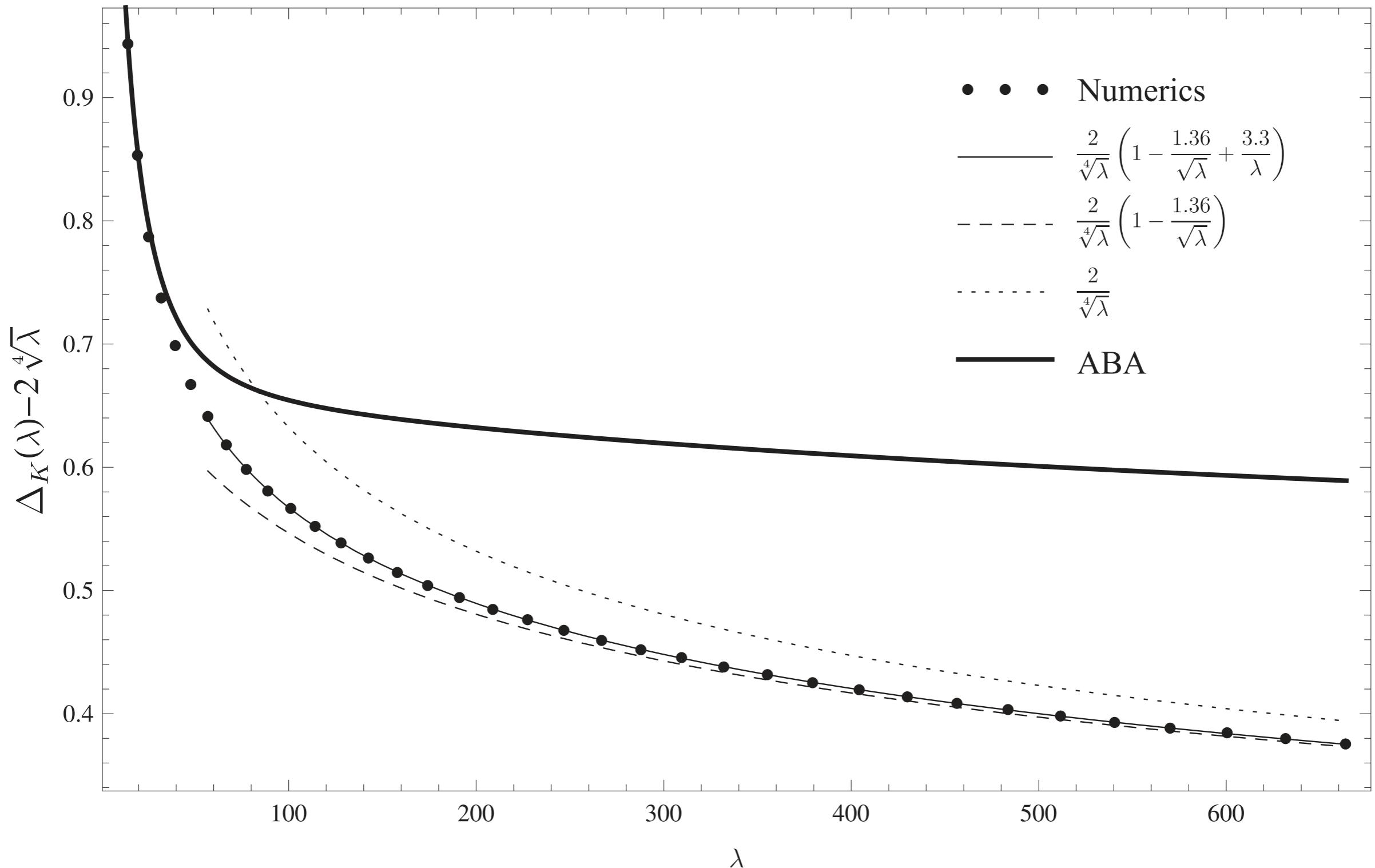
Konishi at *any* coupling

Konishi state



Konishi at *any* coupling

Konishi state

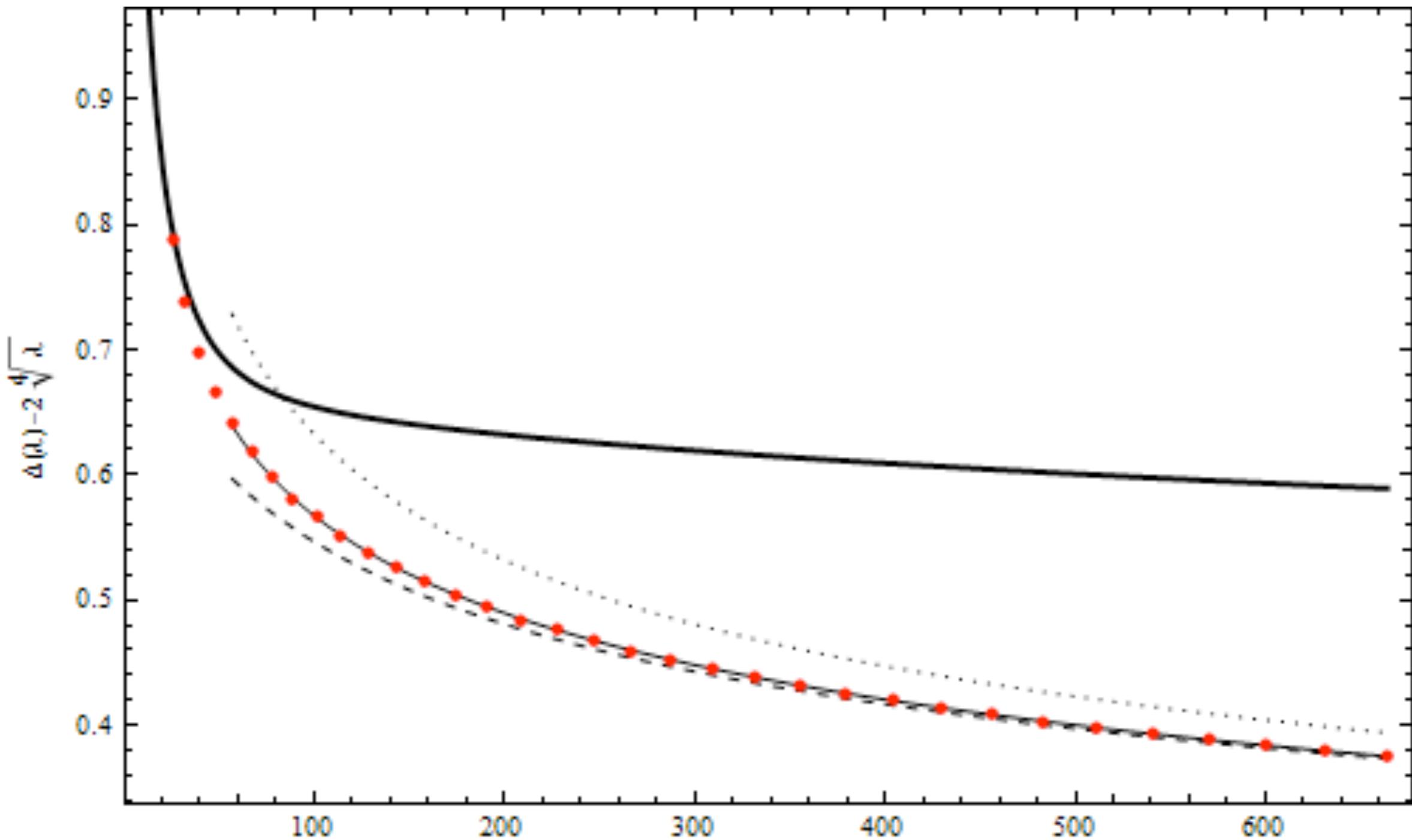


Konishi at *any* coupling

- “Simply” iterate = more than 1000h of computer time
- Integers are absolutely non-trivial!

Konishi at *any* coupling

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Conclusions

- We proposed a set of Y-system (integral and functional) equations describing the exact planar spectrum of AdS/CFT.
- We did not find any singularity from the perturbative deep into the strong coupling regime. On the contrary we found many nice cancelations ensuring nice properties such as reality of the Y-system.
Integers in predictions below are absolutely non-trivial.

- New predictions:

5 loops:

[Bajnok,Hegedus,Janik,Lukowsky]

Strong coupling: [Roiban,Tseytlin]

[Carlini Vallilo, Mazzucato], using
pure spinor methods, in progress

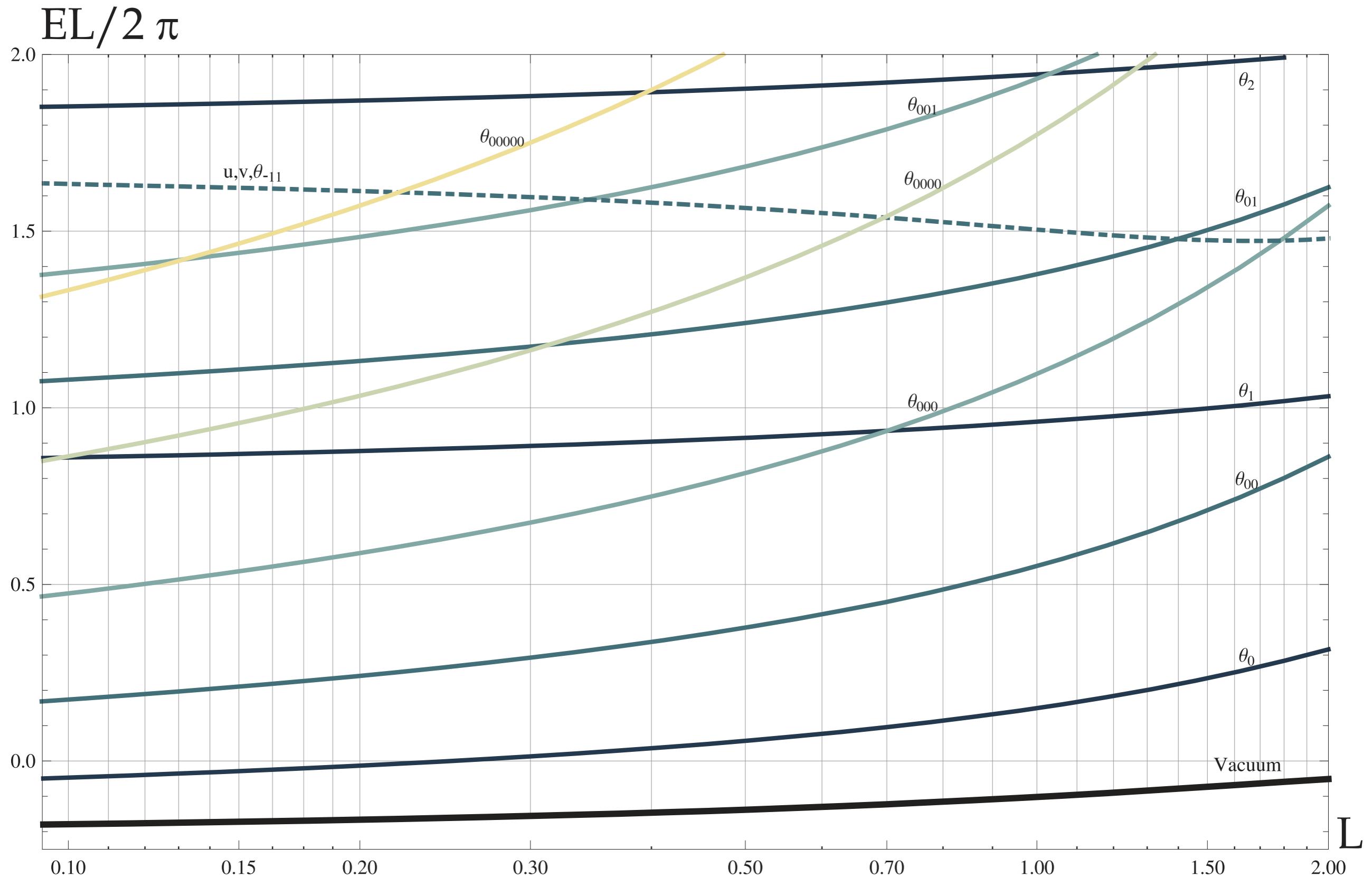
$$\Delta_K(\lambda) = 2\lambda^{1/4} + \frac{2}{\lambda^{1/4}} + \dots$$

$$\dots \simeq -\frac{2.66}{\lambda^{3/4}} + \frac{6.4}{\lambda^{5/4}} + \mathcal{O}(\lambda^{-7/4})$$

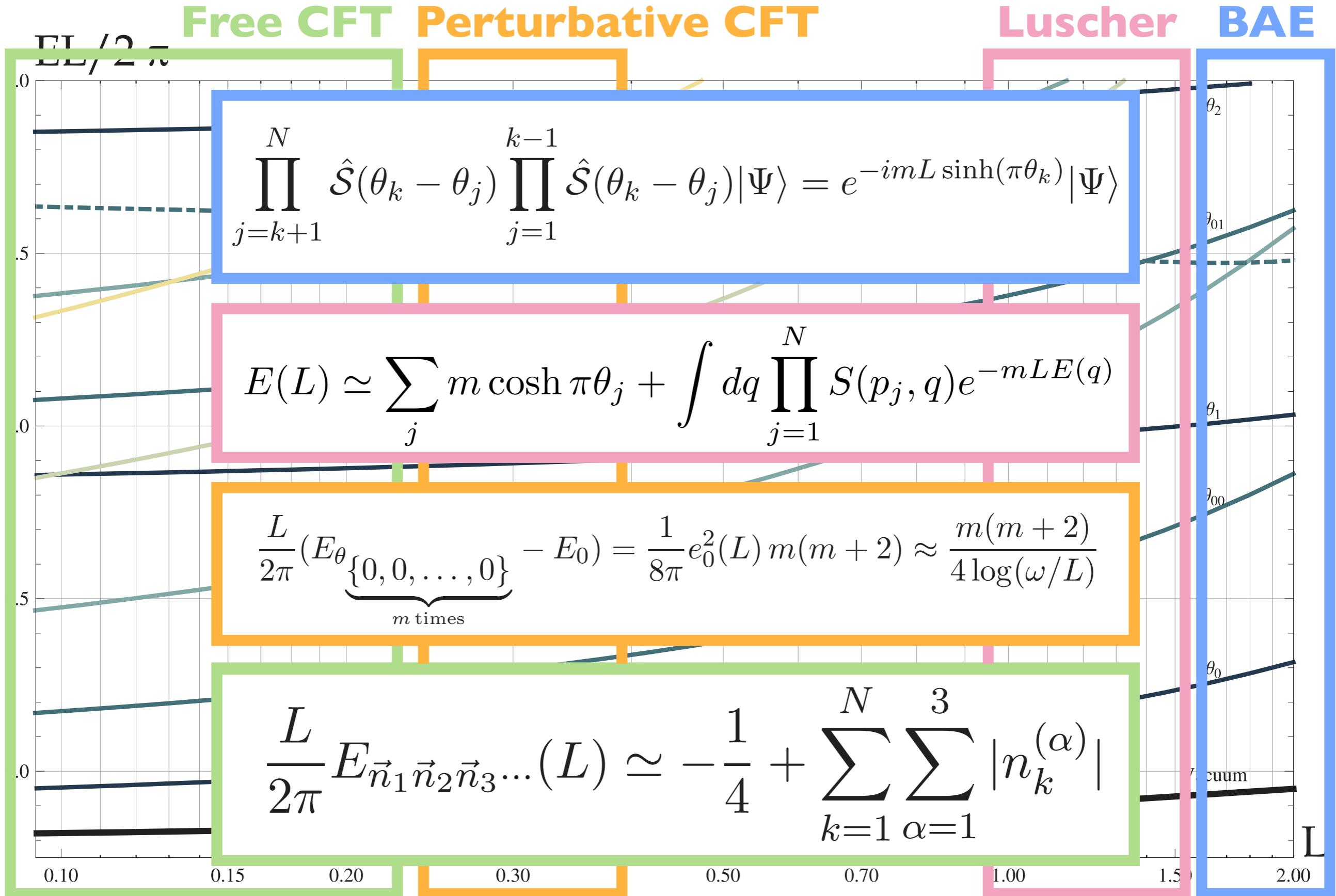
- Future: One single DdV integral equation (as for the SU(2) PCF)... ?
Generalize to all states.

Extra slides

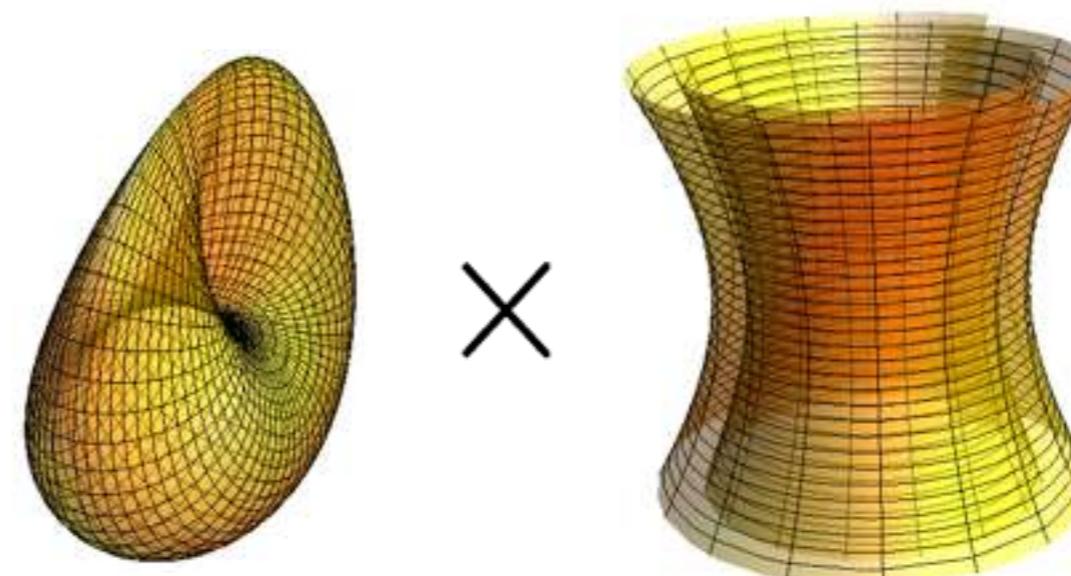
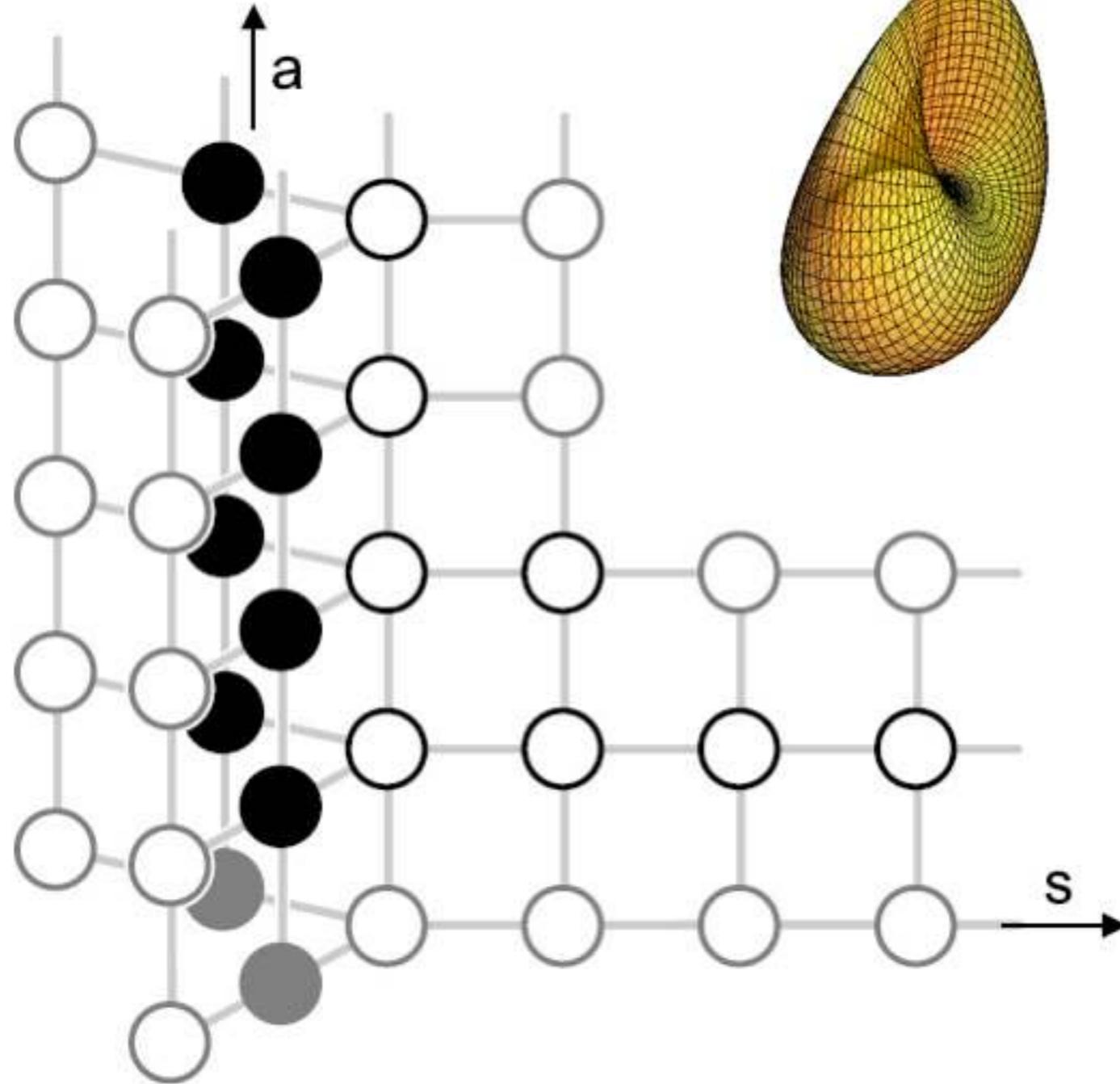
The exact spectrum of the SU(2) PCF



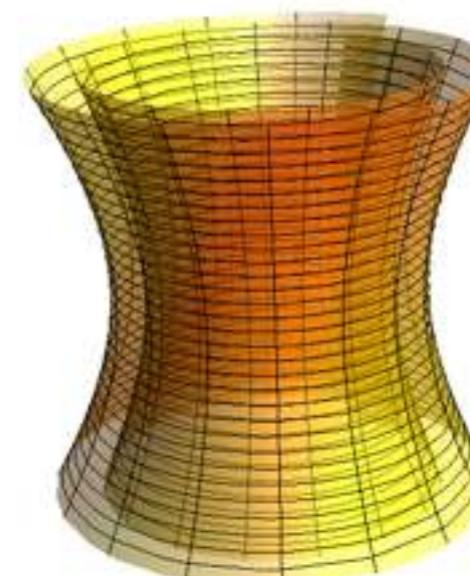
The exact spectrum of the SU(2) PCF



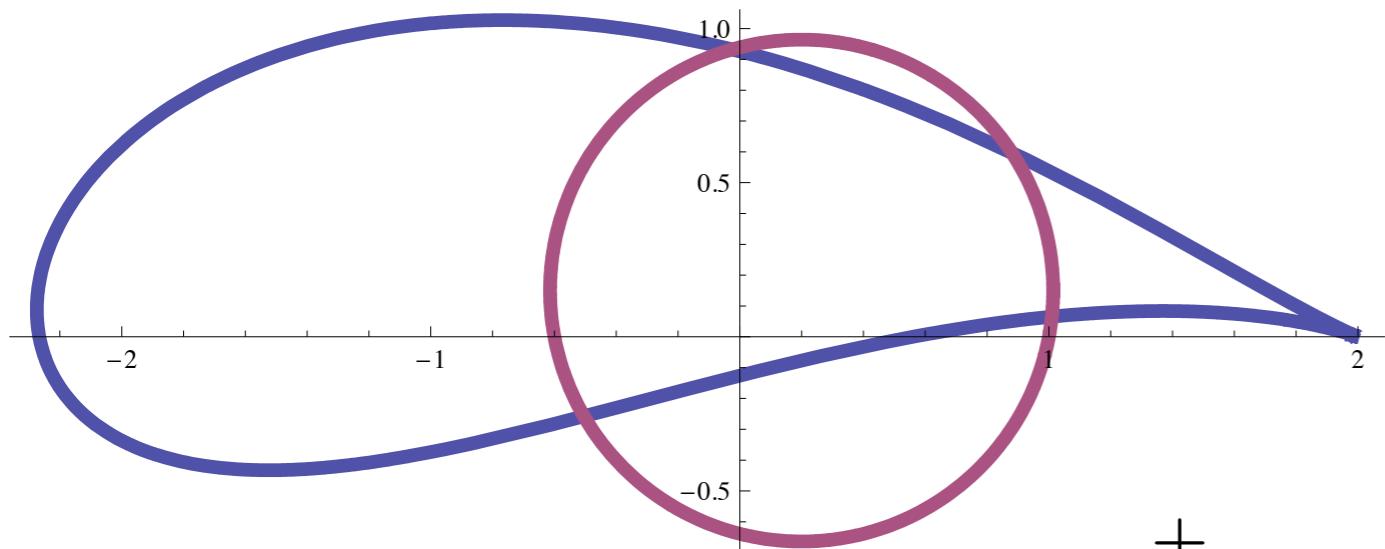
Full spectrum of planar ABJM



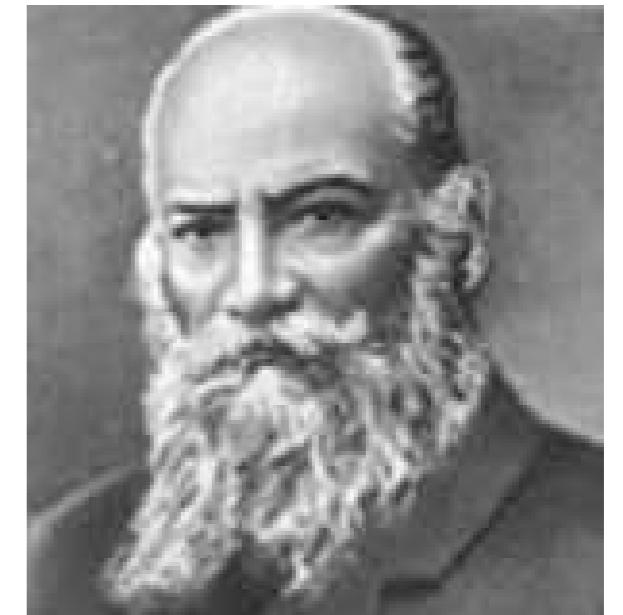
\times



Explicitly



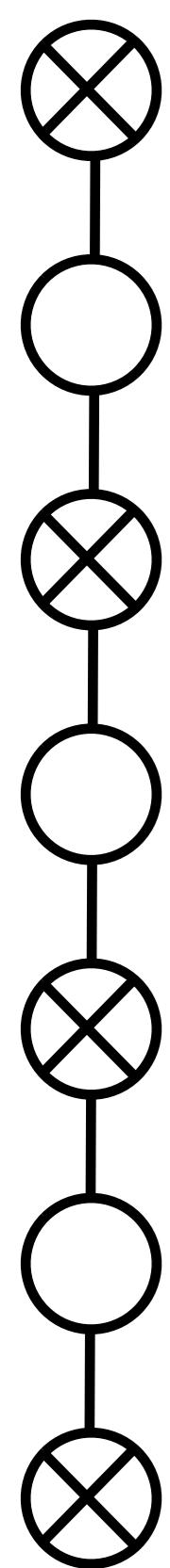
$$\frac{x^+}{x^-} = e^{ip}$$



$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}$$

$$x^+ + \frac{1}{x^+} + x^- + \frac{1}{x^-} = \frac{2u}{g} = 2 \left(x + \frac{1}{x} \right)$$

[Beisert, Staudacher; Beisert, Eden, Staudacher;
Beisert, Hernandez, Lopez, Aryutunov-Frolov]



$$1 = \frac{\mathbf{u}_1 - u_2 + \frac{i}{2}}{\mathbf{u}_1 - u_2 - \frac{i}{2}} \frac{\mathbf{x}_1 - \frac{1}{x_4^+}}{\mathbf{x}_1 - \frac{1}{x_4^-}},$$

$$1 = \frac{\mathbf{u}_2 - u_2 - i}{\mathbf{u}_2 - u_2 + i} \frac{\mathbf{u}_2 - u_1 + \frac{i}{2}}{\mathbf{u}_2 - u_1 - \frac{i}{2}} \frac{\mathbf{u}_2 - u_3 + \frac{i}{2}}{\mathbf{u}_2 - u_3 - \frac{i}{2}},$$

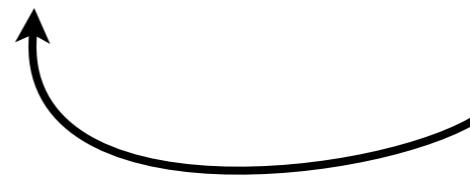
$$1 = \frac{\mathbf{u}_3 - u_2 + \frac{i}{2}}{\mathbf{u}_3 - u_2 - \frac{i}{2}} \frac{\mathbf{x}_3 - x_4^+}{\mathbf{x}_3 - x_4^-}$$

$$1 = \left(\frac{\mathbf{x}_4^-}{\mathbf{x}_4^+} \right)^L \frac{\mathbf{u}_4 - u_4 + i}{\mathbf{u}_4 - u_4 - i} \frac{x_1 - \frac{1}{\mathbf{x}_4^-}}{x_1 - \frac{1}{\mathbf{x}_4^+}} \frac{\mathbf{x}_4^- - x_3}{\mathbf{x}_4^+ - x_3} \frac{x_7 - \frac{1}{\mathbf{x}_4^-}}{x_7 - \frac{1}{\mathbf{x}_4^+}} \frac{\mathbf{x}_4^- - x_5}{\mathbf{x}_4^+ - x_5} \sigma_{\text{BES}}(\mathbf{u}_4, u_4),$$

$$1 = \frac{\mathbf{u}_5 - u_6 + \frac{i}{2}}{\mathbf{u}_5 - u_6 - \frac{i}{2}} \frac{\mathbf{x}_5 - x_4^+}{\mathbf{x}_5 - x_4^-}$$

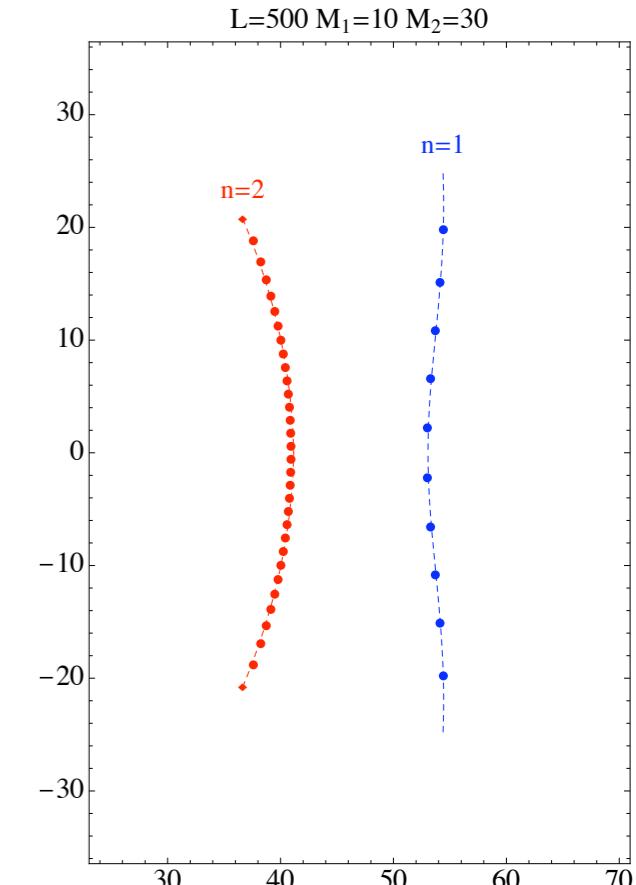
$$1 = \frac{\mathbf{u}_6 - u_6 - i}{\mathbf{u}_6 - u_6 + i} \frac{\mathbf{u}_6 - u_7 + \frac{i}{2}}{\mathbf{u}_6 - u_7 - \frac{i}{2}} \frac{\mathbf{u}_6 - u_5 + \frac{i}{2}}{\mathbf{u}_6 - u_5 - \frac{i}{2}},$$

$$1 = \frac{\mathbf{u}_7 - u_6 + \frac{i}{2}}{\mathbf{u}_7 - u_6 - \frac{i}{2}} \frac{\mathbf{x}_7 - \frac{1}{x_4^+}}{\mathbf{x}_7 - \frac{1}{x_4^-}},$$



$$\prod_{k=1}^{K_5} \frac{u_{6,j} - u_{5,k} + \frac{i}{2}}{u_{6,j} - u_{5,k} - \frac{i}{2}}$$

+ some square roots
(for mirror theory)



Dressing Phase

$$\sigma \equiv \exp [i\chi^{++} + i\chi^{--} - i\chi^{+-} - i\chi^{-+}] , \quad \chi^{\pm\pm} = \chi(u \pm i/2, v \pm i/2)$$

In physical kinematics:

[Beisert, Eden, Staudacher] (Dressing phase proposal)
 [Dorey,Hofman,Maldacena] (Integral representation)

$$\chi(u, v) = \frac{1}{i} \oint_{|z_1|=1} \frac{dz_1}{2\pi} \oint_{|z_2|=1} \frac{dz_2}{2\pi} \frac{1}{z_1 - x^{\text{ph}}(u)} \frac{1}{z_2 - x^{\text{ph}}(v)} \log \frac{\Gamma(iw_1 - iw_2 + 1)}{\Gamma(iu_2 - iu_1 + 1)}$$

In the mirror kinematics:

[Aryutunov, Frolov:Volin; Gromov,Kazakov,PV]

$$\begin{aligned} \hat{\chi}(u, v) &\equiv \frac{1}{i} \int_{|z_1|>1} \frac{dz_1}{2\pi} \int_{|z_2|>1} \frac{dz_2}{2\pi} \left[\frac{1}{(z_1 - x^{\text{mir}}(u))} - \frac{1}{(z_1 - \overline{x^{\text{mir}}}(u))} \right] \times \\ &\times \left[\frac{1}{(z_2 - x^{\text{mir}}(v))} - \frac{1}{(z_2 - \overline{x^{\text{mir}}}(v))} \right] \log \frac{\Gamma(iu_1 - iu_2 + 1)}{\Gamma(iu_2 - iu_1 + 1)} \end{aligned}$$