INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control 2007; **17**:1734–1753
Published online 28 March 2007 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/rnc.1190

Exact state estimation for linear systems with unknown inputs based on hierarchical super-twisting algorithm

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SUMMARY

A robust hierarchical observer is designed for linear time invariant systems with unknown bounded inputs under conditions of strong observability, providing exact state estimation. The main condition for designing the state estimator is the, so-called, strong observability condition. The supertwisting (second-order sliding mode) algorithm is used in each step of the hierarchy; the continuity of the supertwisting output injection allows to reconstruct a vector formed by some full column rank matrix premultiplied by the state vector, and that vector is obtained *in a finite time and without any sort of filtration*. For the case when the unknown inputs are considered as constant uncertain parameters, the continuous version of the least-square method is developed. Two numerical examples illustrate the efficiency of the suggested technique. Copyright © 2007 John Wiley & Sons, Ltd.

Received 27 June 2005; Revised 24 October 2006; Accepted 14 February 2007

KEY WORDS: state observation; sliding mode control; linear systems

1. INTRODUCTION

1.1. Antecedents and motivation

The problem of state observation for systems with unknown inputs has been one of the most important in modern control theory during the last two decades [1, 2]. Usually, the design of

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Contract/grant sponsor: Mexican CONACyT (Consejo Nacional de Ciencia y Tecnologa); contract/grant number: 43807-Y
Contract/grant sponsor: Programa de Apoyo a Proyectos de Investigacin e Innovacin Tecnolgica (PAPIIT) UNAM; contract/grant number: 107006-2

observers for systems with unknown inputs requires the system to have relative degree one with respect to the unknown inputs; this restriction allows the decomposition of the state vector into two parts, the first part is not affected directly by the unknown inputs and needs to be observable, and the second part of the vector is completely known. The specific feature of a majority of the suggested observers is that they are asymptotically efficient when any uncertainties in the dynamic model description are absent, that is, *they provide an exponential convergence of the estimates to the corresponding state dynamics only asymptotically*, *or*, *in other words*, *in infinite time* (see [3], for example). Some observers, in the presence of any bounded unknown inputs, guarantee the error convergence to a zone proportional to the bound of these unknown inputs (see, e.g. [4]).

The paper [5] deals with an approach concerning a finite time exact state estimate in the absence of uncertainties, and that approach requires the simultaneous consideration of two asymptotic observers where the second one contains a *delay* in its dynamics.

On the other hand, the problem of state observation has been actively developed within *Variable Structure Theory* using the *Sliding Mode* approach. Sliding mode observers are widely used due to their attractive features: robustness with respect to some class of uncertainties and possibilities of current identification (estimation) of the uncertainties based on the equivalent output injection (see, for example, the corresponding chapters in the textbooks [6, 7] and the recent tutorials [8–10]). To ensure the finite convergence, one idea that has been suggested is to transform the system into a triangular form and use a step-by-step sliding mode observer based on first-order sliding modes, allowing the successive reconstruction of each component of the transformed state vector *via* the equivalent values (see, e.g. [11–14]). The methodology previously mentioned ensures finite time convergence theoretically since its realization requires a filtration of the equivalent control at each step.

In the last two decades the second-order sliding mode algorithms have been designed (see [15–19] and references therein). One of such algorithm is the *supertwisting algorithm* [20] keeping the advantages of sliding mode controllers. A *robust exact differentiator*, based on supertwisting algorithm, was designed in [21]. Such *differentiator* ensures a finite time convergence to the values of the corresponding derivatives and provides the best possible accuracy of the derivatives for the given value even considering deterministic noise, sampling step and in the case of discrete measurements. That is why the application of the supertwisting algorithm for observation and identification seems to be a reasonable choice.

1.2. Main contributions

1. A state hierarchical observer for linear time invariant systems with unknown bounded inputs under conditions of strong observability is proposed, providing the exact reconstruction of the state components.
2. We suggest to design, at each level of the hierarchy, sliding surfaces using the algorithm given in [22]. The continuity of the *supertwisting algorithm* [20] allows the recovery of an invertible matrix premultiplied by the state vector, so after finite time the *exact value* of the state vector is recovered *without any filtration*. *Moreover*, *the system does not need to be transformed to any canonical or triangular form*.
3. For the case when parameter uncertainty could be considered as unknown inputs which do not create invariant zeros in the system, the design of the hierarchical observer allows also

the estimation of the parametric uncertainties. No filters are needed in the parameter estimation process.

1.3. Structure of the paper

In Section 2 the observation process in the presence of unknown inputs, affecting the state and appearing explicitly in the output, is described; in Section 2.1 the model description and the problem formulation are presented. Section 3 deals with the case when the relative degree of the system is one and there exist parametric uncertainties; in this case we can also identify (estimate) the parametric uncertainties, which is carried out in Section 3.4. Section 4 deals with two numerical examples showing the effectiveness of the proposed method.

1.4. Main notation

Throughout this paper the following notations are used. By $F \in \mathbb{R}^{r\times q}$ we will denote any matrix of the corresponding size, its pseudoinverse is denoted by F^{+}. Specifically, if rank $F = r$, then $F^{+} = F^{\top}(FF^{\top})^{-1}$ and if rank $F = q$, then $F^{+} = (F^{\top}F)^{-1}F^{\top}$. In the case when rank $F = p$, we denote the matrix $F^{\perp} \in \mathbb{R}^{r-p\times r}$ as a matrix that is orthogonal to F, i.e. $F^{\perp}F = 0$ and rank $F^{\perp} = r - p$. It should be noted that the matrix $F^{\perp}$ is not unique. We also denote $F^{\perp\perp} \in \mathbb{R}^{p\times r}$ as a matrix such that

$$\det\begin{pmatrix} F^{\perp\perp} \\ F^{\perp} \end{pmatrix} \neq 0$$

2. EXACT STATE ESTIMATION

2.1. Plant's model and problem formulation

Let us consider a multi-state linear system given by the following ordinary differential equation (ODE):

$$\dot{x}(t) = Ax(t) + Bu(t) + Dw(x,t), \quad x(0) = x_0 \tag{1a}$$

$$y(t) = Cx(t) + Fw(x,t), \quad t \geqslant 0 \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is a control, $y(t) \in \mathbb{R}^p$ $(1 \leqslant p < n)$ is the output of the system, $w(x,t) \in \mathbb{R}^q$ is an unknown bounded input, that is, $\|w(x,t)\| \leqslant w^{+} < \infty$. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{n\times q}$, and $F \in \mathbb{R}^{p\times q}$ are known constant. The pair $\{u(t), y(t)\}$ is assumed to be measurable (available) at any time $t \geqslant 0$. The current states $x(t)$ as well as the initial state x_0 are not available.

Problem formulation: Estimate the state vector $x(t)$ for all $t \geqslant \varepsilon > 0$, based on the available information $\{u(\tau), y(\tau)\}_{\tau\in[0,t]}$.

Note that if for all $t \geqslant \varepsilon > 0$ the suggested estimate $\hat{x}(t)$ exactly coincides with $x(t)$, then we deal with an exact (non-asymptotic) state estimation process.

Before designing a state estimator, we recall some definitions and properties which justify the procedure that we will suggest to design the state estimator. Since $u(t)$ is known and its effect can

DOI: 10.1002/rnc

be compensated for any observer, without loss of generality, one can assume that $u(t) = 0$ for all $t \geqslant 0$. That is, consider the following state equations:

$$\dot{x}(t) = Ax(t) + Dw(x, t), \quad x(0) = x_0 \tag{2a}$$

$$y(t) = Cx(t) + Fw(x, t), \quad t \geqslant 0 \tag{2b}$$

We recall some definitions corresponding to properties of (2) (see [1, 22–24]).

Definition 1
$\mathscr{V}$ is a null-output (A, D) invariant subspace if for every $x \in \mathscr{V}$ there exist some w such that $(Ax + Dw) \in \mathscr{V}$ and $(Cx + Fw) = 0$. $\mathscr{V}^*$ is the maximal null-output (A, D) invariant subspace, i.e. $\mathscr{V} \subseteq \mathscr{V}^*$ for each subspace $\mathscr{V}$.

Subspace $\mathscr{V}^*$ is called the weakly unobservable subspace of (2).

Definition 2
We say that system (2) has invariant zeros if

$$\left\{ s \in \mathbb{C} : \text{rank}\, P(s) < n + \text{rank} \begin{pmatrix} D \\ F \end{pmatrix} \right\} \neq \emptyset \quad \text{where } P(s) = \begin{bmatrix} sI - A & -D \\ C & F \end{bmatrix} \tag{3}$$

$P(s)$ is known as the Rosenbrock matrix for system (2).

Definition 3
System (2) is called strongly observable if, and only if for any initial condition x_0 and for any unknown input $w(t)$, the condition $y(t) = 0$ for all $t \geqslant 0$ implies that $x(t) = 0$ for all $t \geqslant 0$.

The following statements are equivalent (see, e.g. [1, 22–24])

(i) System (2) is strongly observable.
(ii) System (2) has no invariant zeros.
(iii) $\mathscr{V}^* = 0$.

It means that if system (2) has invariant zeros, then there exists an initial condition $x_0 = \xi$ and an unknown input $w(t)$ such that $y(t) = 0$ for all $t \geqslant 0$ and $x(t)$ being not equal to zero for all $t \geqslant 0$. So, in that case it would be impossible to make an estimation, independently of $w(t)$, of the state $x(t)$.

Therefore, throughout the paper it will be assumed that

A1. System (2) has no invariant zeros.

We will suggest an state estimator that has a hierarchical design and uses the supertwisting algorithm for its construction. We use the supertwisting as a substitute for a differentiator, so we will try to construct a vector formed by an analogous of the observability matrix multiplied by the state vector, all taking into account the presence of the unknown inputs. For such a goal we will use the following algorithm [22]. This design allows the reconstruction of the state vector independently of the unknown inputs.

DOI: 10.1002/rnc

Algorithm 1 (Molinari [22])

Step 0: Set $k=0$, and set $M_0=0$

Step k: Evaluate

$$\Gamma_k = \begin{bmatrix} M_kD & M_kA \\ F & C \end{bmatrix} \tag{4}$$

and let T_k any non-singular matrix reducing Γ_k to

$$T_k\Gamma_k = \begin{bmatrix} J_{k+1} & L_{k+1} \\ 0 & M_{k+1} \end{bmatrix} \tag{5}$$

where J_{k+1} has full row rank. J_{k+1} has q columns and M_{k+1} has n columns.

Step k + 1: Set $k=k+1$ and go to Step k.

The matrix M_{k+1} could be reduced to the form

$$\begin{bmatrix} M_{k+1} \\ 0 \end{bmatrix}$$

In [22] was proven that

$$\mathscr{V}^* = \ker M_n \tag{6}$$

By assumption A1 and the equivalences (ii) and (iii), it follows that $\mathscr{V}^*=0$. Hence, (6) implies rank $M_n=n$, that is, M_n has full column rank.

Let l be the least positive integer such that rank $M_l=n$. This means that $\mathscr{V}^*=\ker M_l=0$. Thus, we may select T_k as

$$T_k := \begin{bmatrix} T_{k,\mathrm{a}} \\ T_{k,\mathrm{b}} \end{bmatrix} \quad \text{where } T_{k,\mathrm{a}} = \begin{pmatrix} M_kD \\ F \end{pmatrix}^{\perp\perp} \quad \text{and} \quad T_{k,\mathrm{b}} = \begin{pmatrix} M_kD \\ F \end{pmatrix}^{\perp} \quad \text{for } k=0,1,\ldots,l-1 \tag{7}$$

Thus, T_k selected as in (7) satisfies (5) and M_{k+1} takes the form

$$M_1 = F^{\perp}C, \quad M_{k+1} = \begin{pmatrix} M_kD \\ F \end{pmatrix}^{\perp} \begin{pmatrix} M_kA \\ C \end{pmatrix} = T_{k,\mathrm{b}} \begin{pmatrix} M_kA \\ C \end{pmatrix} \quad \text{for } k=1,\ldots,l-1 \tag{8}$$

Below we will show that we can recover each vector $M_{k+1}x(t)$ until we obtain $M_lx(t)$ and, consequently, by the pseudoinversion of M_l, we can recover $x(t)$.

2.2. Auxiliary dynamic systems

The use of the supertwisting algorithm requires the knowledge of some bounds. Let us show that the consideration of the following dynamic system (linear observer) ensures the knowledge of these bounds. First, design the following dynamic system:

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) + K(y(t) - C\tilde{x}(t)) \tag{9}$$

where K must be designed such that

A2. The eigenvalues of $\tilde{A} := (A-KC)$ have negative real part.

DOI: 10.1002/rnc

Letting $\tilde{e}(t) = x(t) - \tilde{x}(t)$, from (1) and (9), the dynamic equations governing $\tilde{e}(t)$ can be represented as

$$\dot{\tilde{e}}(t) = [A - KC]\tilde{e}(t) + [D + KF]w(t) = \tilde{A}\tilde{e}(t) + [D + KF]w(x, t)$$

Since $||w(x, t)|| \leqslant w^+$, it is well known that $\tilde{e}(t)$ is of bounded norm, i.e. there exist some constants $\gamma, \eta, \mu > 0$ such that

$$||\tilde{e}(t)|| \leqslant \gamma \exp(-\mu t)||\tilde{e}(0)|| + \eta w^+ \tag{10}$$

Thus, the inequality in (10) yields the following statement:

$$\tilde{e}^+ > \eta w^+ \text{ implies } ||\tilde{e}(t)|| < \tilde{e}^+ \quad \text{for all } t > -\frac{1}{\mu}\ln\frac{\tilde{e}^+ - \eta w^+}{\gamma||\tilde{e}(0)||} \tag{11}$$

This means that if $\tilde{e}^+ > \eta w^+$, in finite time, $\tilde{e}(t)$ will be bounded by $\tilde{e}^+$. Below we will need a bound for $||\tilde{e}(t)||$, and (11) ensures that we can always satisfy such a requirement.

In (8) we can see that before recovering $M_{k+1}x(t)$, one needs to recover $M_k x(t)$ for $k = 1, \ldots, l-1$; therefore, this subsection is devoted to the recovery of the vectors $M_k x(t)$ $(k = 1, \ldots, l)$. The first vector $M_1 x(t)$ is already available, since $M_1 x(t) = F^{\perp}Cx(t) = F^{\perp}y(t)$.

Now, to recover the vector $M_2 x(t)$ an *auxiliary vector* is designed in the following form:

$$\dot{x}_{\mathrm{a}} = A\tilde{x} + Bu \tag{12}$$

We design an *output injection* $v^{(1)}$ using the '*supertwisting*' algorithm (see [20]) and its components are defined as follows:

$$v_i^{(1)} = z_i^{(1)} + \lambda_1 |s_i^{(1)}|^{1/2} \operatorname{sign}(s_i^{(1)}), \quad \dot{z}_i^{(1)} = \alpha_1 \operatorname{sign}(s_i^{(1)}) \tag{13}$$

All the solutions of the dynamic systems are defined in Filippov's sense [25]. Since $M_1 x = F^{\perp}y$, the variable $s^{(1)}$ is given by the formula:

$$s^{(1)}(y(t), x_{\mathrm{a}}(t)) = T_{1,\mathrm{b}}\left[\begin{pmatrix} F^{\perp}y(t) \\ \int_{\tau=0}^{t} y(\tau)\,\mathrm{d}\tau \end{pmatrix} - \begin{pmatrix} F^{\perp}Cx_{\mathrm{a}}(t) \\ C\int_{\tau=0}^{t} \tilde{x}(\tau)\,\mathrm{d}\tau \end{pmatrix}\right] - \int_{\tau=0}^{t} v^{(1)}(\tau)\,\mathrm{d}\tau \tag{14}$$

The dimension of the vector $v^{(1)}$ is the same as the dimension of $s^{(1)}$ and it is equal to the number of rows of $T_{1,\mathrm{b}}$ that depends on the specific values that the matrices of the system take. Thus, in view of (1), (12), (7), and (8), the time derivative of $s^{(1)}$ is

$$\dot{s}^{(1)}(t) = T_{1,\mathrm{b}}\begin{pmatrix} F^{\perp}CA \\ C \end{pmatrix}[x(t) - \tilde{x}(t)] - v(t) = M_2[x(t) - \tilde{x}(t)] - v(t) \tag{15}$$

Now, choose the scalar gains λ_1, α_1 so that the following conditions are fulfilled:

$$\alpha_1 > \beta_1 \geqslant ||M_2||(||\tilde{A}||\tilde{e}^+ + ||D + KF||w^+)$$

$$\lambda_1 > \frac{(\alpha_1 + \beta_1)(1 + \theta)}{(1 - \theta)}\sqrt{\frac{2}{\alpha_1 - \beta_1}}, \quad 0 < \theta < 1 \tag{16}$$

DOI: 10.1002/rnc

where $\tilde{e}^+$ satisfies (11). In view of (11), one can always satisfy (16) in finite time. It was shown in [20] that if the bounds in (16) are fulfilled, then there exists a finite time t_1 such that the following equalities

$$s^{(1)}(t) = \dot{s}^{(1)}(t) = 0, \quad t \geqslant t_1 \tag{17}$$

hold, where t_1 is the reaching time. From (13), it is clear that if $s^{(1)} = 0$, then $v^{(1)} \equiv z^{(1)}$, so the vector $M_2 x(t)$ can be recovered from (15) in the following manner:

$$M_2 x(t) = M_2 \tilde{x}(t) + z^{(1)}(t) \quad \text{for } t \geqslant t_1 \tag{18}$$

Recursively, we can follow the same procedure to obtain $M_3 x(t)$, $M_4 x(t)$, and so on. Below we give the general design of the auxiliary system and the sliding surfaces with their corresponding output injection.

(a) Design the dynamics of the auxiliary system in the form:

$$\dot{x}_a(t) = A\tilde{x}(t) + Bu \tag{19}$$

The output injection $v^{(k)}$ at the kth level is designed as a 'super-twisting' controller (see [20]):

$$v_i^{(k)} = z_i^{(k)} + \lambda_k |s_i^{(k)}|^{1/2} \operatorname{sign}(s_i^{(k)}), \quad \dot{z}_i^{(k)} = \alpha_k \operatorname{sign}(s_i^{(k)}) \tag{20}$$

where λ_k and α_k are constants satisfying

$$\left.\begin{aligned} &\alpha_k > \beta_k \geqslant \|M_{k+1}\|(\|\tilde{A}\|\tilde{e}^+ + \|D + KF\|w^+) \\ &\lambda_k > \frac{(\alpha_k + \beta_k)(1+\theta)}{(1-\theta)}\sqrt{\frac{2}{\alpha_k - \beta_k}}, \quad 0<\theta<1 \end{aligned}\right\} 1 \leqslant k < l-1$$

$$\left.\begin{aligned} &\alpha_{l-1} > \beta_{l-1} \geqslant \|\tilde{A}\|\tilde{e}^+ + \|D + KF\|w^+ \\ &\lambda_{l-1} > \frac{(\alpha_{l-1} + \beta_{l-1})(1+\theta)}{(1-\theta)}\sqrt{\frac{2}{\alpha_{l-1} - \beta_{l-1}}}, \quad 0<\theta<1 \end{aligned}\right\}, \quad k = l-1 \tag{21}$$

where $\tilde{e}^+$ should satisfy (11).

(b) Since $M_1 x(t) = F^{\perp} y(t)$ and $M_l^+ M_l = I$ (M_l^+ defined as in 1.4) the variables $s^{(k)}$ and $z^{(k)}$ are related as

$$s^{(k)}(z^{(k-1)}, x_a) = \begin{cases} T_{1,b}\left[\begin{pmatrix} F^{\perp} y(t) \\ \int_{\tau=0}^{t} y(\tau)\,d\tau \end{pmatrix} - \begin{pmatrix} M_1 x_a(t) \\ \int_{\tau=0}^{t} C\tilde{x}(\tau)\,d\tau \end{pmatrix}\right] - \int_{\tau=0}^{t} v^{(1)}(\tau)\,d\tau, & k=1 \\ T_{k,b}\left[\begin{pmatrix} M_k \tilde{x} + z^{(k-1)} \\ \int_{\tau=0}^{t} y(\tau)\,d\tau \end{pmatrix} - \begin{pmatrix} M_k x_a \\ \int_{\tau=0}^{t} C\tilde{x}(\tau)\,d\tau \end{pmatrix}\right] - \int_{\tau=0}^{t} v^{(k)}(\tau)\,d\tau, & 1<k<l-1 \\ M_l^+ T_{l-1,b}\left[\begin{pmatrix} M_{l-1} \tilde{x} + z^{(l-2)} \\ \int_{\tau=0}^{t} y(\tau)\,d\tau \end{pmatrix} - \begin{pmatrix} M_{l-1} x_a \\ \int_{\tau=0}^{t} C\tilde{x}(\tau)\,d\tau \end{pmatrix}\right] - \int_{\tau=0}^{t} v^{(l-1)}(\tau)\,d\tau, & k=l-1 \end{cases} \tag{22}$$

The following lemma establishes how the vectors $M_k x(t)$ can be recovered by the second-order sliding motions ($s^{(k)} = \dot{s}^{(k)} = 0$).

Lemma 1

Under the assumptions A1–A2, if the auxiliary state vector $x_a^{(k)}$ and the variable $s^{(k)}$, for all $k = 1, \dots, l-1$, are designed as in (19)–(22), then, from some finite time t_k, one has

$$M_{k+1}x(t) = M_{k+1}\tilde{x}(t) + z^{(k)}(t) \quad \text{for } k = 1, \dots, l-2 \tag{23a}$$

$$x(t) = \tilde{x}(t) + z^{(l-1)}(t) \tag{23b}$$

Proof

Let us prove Lemma 1 using induction. For $k = 1$, as it was shown above, there exist a finite time t_1 such that $M_2 x(t)$ is recovered by the equation

$$M_2 x(t) = M_2\tilde{x}(t) + z^{(1)}(t), \quad t \geqslant t_1$$

Now, suppose that there exist a finite time t_{j-1} such that (23) is true for some intermediate $k = j-1$. Thus, according to (22), $s^{(j)}$ is in the form

$$s^{(j)}(z^{(j-1)}(t), x_a(t)) = T_{j,b}\left[\begin{pmatrix} M_j\tilde{x}(t) + z^{(j-1)}(t) \\ \int_{\tau=0}^{t} y(\tau)\,d\tau \end{pmatrix} - \begin{pmatrix} M_j x_a(t) \\ \int_{\tau=0}^{t} C\tilde{x}(\tau)\,d\tau \end{pmatrix}\right] - \int_{\tau=0}^{t} v^{(j)}(\tau)\,d\tau \tag{24}$$

Substitution of $z^{(j-1)}(t)$, from (23), into (24) yields

$$s^{(j)}(t) = T_{j,b}\begin{pmatrix} M_j[x(t) - x_a(t)] \\ \int_{\tau=0}^{t} [y(\tau) - C\tilde{x}(\tau)]\,d\tau \end{pmatrix} - \int_{\tau=0}^{t} v^{(j)}(\tau)\,d\tau$$

for $t \geqslant t_{j-1}$. Thus, from (1), (19), and (8), the derivative of $s^{(j)}$ is obtained by the equation

$$\dot{s}^{(j)}(t) = M_{j+1}[x(t) - \tilde{x}(t)] - v^{(j)}(t) \tag{25}$$

Again, as it was shown in [20], if condition (21) is satisfied, then *a second-order sliding mode* is obtained, i.e.

$$s^{(j)}(t) = \dot{s}^{(j)}(t) = 0, \quad t \geqslant t_j \tag{26}$$

where t_j is the reaching time. Thus, from the structure of $v^{(j)}$, $s^{(j)} = 0$ implies $v^{(j)} \equiv z^{(j)}$. Then in view of (26), the equality (23) for $k = j$ is deduced from (25). Therefore (23) is true for $k = 1, \dots, l-2$.

In particular, (23) is true for $k = l-2$; therefore, since $M_l^+ M_l = I$, if $s^{(l)}$ and $v^{(l)}$ are designed as in (20)–(22) one has

$$\dot{s}^{(l-1)}(t) = M_l^+ M_l[x(t) - \tilde{x}(t)] - v^{(j)}(t)$$

and, according with [20], by the condition (21) we have

$$s^{(l-1)}(t) = \dot{s}^{(l-1)}(t) = 0, \quad t \geqslant t_{l-1}$$

and so $x(t) = \tilde{x}(t) + z^{(l-1)}(t)$. □

DOI: 10.1002/rnc

2.3. *Design of the state estimator*

From (23) we have

$$x(t) = \tilde{x}(t) + z^{(l-1)}(t) \tag{27}$$

So, the state estimator is defined as

$$\hat{x}(t) = \tilde{x}(t) + z^{(l-1)}(t) \tag{28}$$

Now, we can resume the previous result in the following theorem.

Theorem 1
Under assumptions A1–A2,

$$\hat{x}(t) = x(t) \quad \text{for all } t \geqslant t_{l-1} \tag{29}$$

Proof
It follows immediately from (27) and (28). □

We conclude this section with the algorithm for the state estimator given in (28).

Algorithm 2 (Design of the state estimator)
Step A: Find the matrices M_k according the (8) with $T_{k,\mathrm{b}}$ as in (7). Determine the value of the least positive integer l so that rank $M_l = n$.
Step B: Design $\tilde{x}$ according to (9) with the gain K satisfying A2 and design the auxiliary system x_a as in (19).
Step C: Design $l-1$ sliding surface $s^{(k)}$ according to (22) and design the output injections $v^{(k)}$ following (20), each one fulfilling (21).
Step D: Design the state estimator using (28).

3. A PARTICULAR CASE: THE SYSTEM WITH PARAMETRIC UNCERTAINTIES

In this section we will assume that the unknown inputs represent some parameter uncertainties, that is, $w(x,t) = \Delta A(t)Gx(t)$ where $\Delta A(t) \in \mathbb{R}^{q\times r}$, $1 \leqslant r \leqslant n$ and $||\Delta A(t)|| \leqslant 1$, represents parametric uncertainty (maybe, time varying). $G \in \mathbb{R}^{r\times n}$ is a known constant matrix. As before, it is assumed that $||w(x,t)|| \leqslant w^+$. Here is assumed that there is no direct influence of the unknown inputs to the output, i.e. $F = 0$. Hence, system (1) takes the form

$$\dot{x}(t) = [A + D\Delta A(t)G]x(t) + Bu(t), \quad x(0) = x_0 \tag{30a}$$

$$y(t) = Cx(t), \quad t \geqslant 0 \tag{30b}$$

Additional assumptions are imposed to the system (30)

A3. Rank of C is p and rank of D is $q < p$, that is, the number of unknown inputs is less than the number of measurable outputs:

$$\operatorname{rank} D = q < \operatorname{rank} C = p$$

A4. $\operatorname{Rank}(CD) = q$.

3.1. Decomposition of the system

Define an $n \times n$ non-singular matrix P by

$$P := \begin{bmatrix} D^{\perp} \\ (CD)^{+}C \end{bmatrix}, \quad P^{-1} = [(I - D(CD)^{+}C)D^{\perp +} \;\; D] \tag{31}$$

where $D^{\perp} \in \mathbb{R}^{n-q \times n}$, $(CD)^{+} \in \mathbb{R}^{q \times p}$, and $D^{\perp +} \in \mathbb{R}^{n \times n-q}$ (see Section 1.4). And consider $r = [r_1^{\mathrm{T}} \;\; r_2^{\mathrm{T}}]^{\mathrm{T}} := Px$, $r_1 \in \mathbb{R}^{n-q}$. Thus, in the new variables, the following motion equations are obtained:

$$\begin{bmatrix} \dot{r}_1(t) \\ \dot{r}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} 0 \\ w(x,t) \end{bmatrix} \tag{32a}$$

$$y = [[I - CD(CD)^{+}]CD^{\perp +} \;\; CD] \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}, \quad w(x,t) = \Delta A(t)Gx(t) \tag{32b}$$

where $A_{11} \in \mathbb{R}^{n-q \times n-q}$, $B_1 \in \mathbb{R}^{n-q \times m}$, and the remaining partitions have suitable dimension. Now, define

$$\begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} := \begin{bmatrix} (CD)^{\perp} \\ (CD)^{+} \end{bmatrix} y = \begin{bmatrix} (CD)^{\perp}CD^{\perp +} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \quad \bar{y}_1 \in \mathbb{R}^{p-q}$$

that is

$$\bar{y}_1 = (CD)^{\perp} y = \bar{C} r_1 \quad \text{where } \bar{C} := (CD)^{\perp}CD^{\perp +} \tag{33a}$$

$$\bar{y}_2 = (CD)^{+} y(t) = r_2(t) \tag{33b}$$

Then, from (33), we can see that r_2 is measurable. Therefore, only the estimation of r_1 is required to complete the state estimation; further, the dynamic equations of r_1 does not include any unknown inputs.

Thus, in view of (32) and (33), the dynamics equations for r_1 are as follows:

$$\dot{r}_1(t) = A_{11}r_1(t) + A_{12}r_2(t) + B_1u(t)$$

$$\bar{y}_1(t) = \bar{C}r_1(t) \tag{34}$$

This means that to achieve an exact estimation of the state vector r_1, the pair $(A_{11}, \bar{C})$ should be observable. The following lemma states the conditions of the observability of the pair $(A_{11}, \bar{C})$ in terms of A, D, and C.

Lemma 2

The pair $(A_{11}, \bar{C})$ is observable if and only if (2) has no invariant zeros.

The proof is given in Appendix.

Hence, the assumption A1 yields that $(A_{11}, \bar{C})$ is observable.

DOI: 10.1002/rnc

3.2. Auxiliary dynamic systems and controls

From the previous definitions, we have that l is the least positive integer such that rank $M_l = n - q$. Following the procedure, used in 2, we have that M_k equal to

$$M_k = \begin{bmatrix} \bar{C} \\ \bar{C}A_{11} \\ \vdots \\ \bar{C}A_{11}^{k-1} \end{bmatrix} \tag{35}$$

Thus, we have that M_l is the observability matrix of $(A_{11}, \bar{C})$. So, we are able to estimate $r_1(t)$ in a finite time following the procedure suggested in the Section 2. But, in 2, due to the unknown inputs, we need to use $M_k x(t)$ on-line to recover $M_{k+1}x(t)$. Thus, instead of recovering $M_{k+1}r_1(t)$ in each step, we suggest to recover only $\bar{C}A_{11}^k r_1(t)$ in each step, for $k = 1, \dots, l-1$, and reconstruct $M_l r_1(t)$ with each one of the vectors $\bar{C}A_{11}^k r_1(t)$.

The method we suggest to recover $\bar{C}A_{11}^k r_1(t)$ is very similar to that used in 2 to recover $M_k x(t)$. The main difference is the design of s^k. Therefore, we will not repeat all details of the procedure, presenting below only the following algorithm.

Algorithm 3

Step A: Determine the value of the least positive integer l so that rank $M_l = n$, where M_l is defined as in (35).

Step B: Design $\tilde{r}$ as

$$\dot{\tilde{r}}(t) = A_{11}\tilde{r}(t) + A_{12}r_2(t) + B_1 u(t) + K(\bar{y}_1(t) - \bar{C}\tilde{r}(t))$$

Step C: Design the auxiliary system r_a as

$$\dot{r}_a(t) = A_{11}\tilde{r}(t) + A_{12}r_2(t) + B_1 u(t)$$

and the output injection $v^{(k)} \in \mathbb{R}^{p-q}$ is designed as

$$v_i^{(k)} = z_i^{(k)} + \lambda_k |s_i^{(k)}|^{1/2} \operatorname{sign}(s_i^{(k)}), \quad \dot{z}_i^{(k)} = \alpha_k \operatorname{sign}(s_i^{(k)}) \tag{36}$$

where λ_k and α_k are constants satisfying

$$\alpha_k > \beta_k \geqslant (\|\bar{C}A_{11}^{k+1}\|) r^+$$

$$\lambda_k > \frac{(\alpha_k + \beta_k)(1+\theta)}{(1-\theta)} \sqrt{\frac{2}{\alpha_k - \beta_k}}, \quad 0 < \theta < 1$$

with the sliding surface $s^{(k)} \in \mathbb{R}^{p-q}$ as

$$s^{(k)}(z^{(k-1)}, r_a) = \begin{cases} \bar{y}_1(t) - \bar{C}r_a(t) - \int_{\tau=0}^{t} v^{(1)}(\tau)\,d\tau & \text{for } k = 1 \\ z^{(k-1)}(t) + \bar{C}A_{11}^{k-1}\tilde{r}(t) - \bar{C}A_{11}^{k-1}r_a(t) - \int_{\tau=0}^{t} v^{(k)}(\tau)\,d\tau & \text{for } k > 1 \end{cases} \tag{37}$$

So, using Algorithm 3 and following the same procedure as in the proof of Lemma 1, we can deduce that for a finite time t_k the following identities hold:

$$s^{(k)}(t) = \dot{s}^{(k)}(t) = 0 \quad \text{for} = 1, \dots, l-1, \quad t \geqslant t_k \tag{38}$$

and so

$$\bar{C}A_{11}^{k}r_1(t) = \bar{C}A_{11}^{k}\tilde{r}(t) + z^{(k)} \quad \text{for } k = 1, \ldots, l-1, \quad t \geqslant t_k \tag{39}$$

3.3. State estimation

Now, using (39), we can construct $M_l r_1$ where M_l is defined as in (35):

$$M_l r_1(t) = M_l \tilde{r}(t) + z \quad \forall t \geqslant t_{l-1} \tag{40}$$

where

$$z^{\mathrm{T}} = [(\bar{y}_1 - \bar{C}\tilde{r})^{\mathrm{T}} \ \ (z^{(1)})^{\mathrm{T}} \ \ \ldots \ \ (z^{(l-1)})^{\mathrm{T}}]$$

The premultiplication of (40) by M_l^{+} leads to the equation

$$r_1(t) \equiv \tilde{r}(t) + M_l^{+} z(t) \quad \forall t \geqslant t_{l-1} \tag{41}$$

Again , a state estimator for $r_1(t)$ is proposed in the following form:

$$\hat{r}_1(t) = \tilde{r}(t) + M_l^{+} z(t) \tag{42}$$

From (41) and (42), it is clear that

$$\hat{r}_1(t) \equiv r_1(t) \quad \forall t \geqslant t_{l-1} \tag{43}$$

and, as a consequence, it provides an exact state estimator for the original system (30) *via* the following algebraic mapping:

$$\hat{x}(t) = P^{-1} \begin{bmatrix} \hat{r}_1(t) \\ (CD)^{+} y(t) \end{bmatrix}, \quad \hat{x}(t) \equiv x(t) \quad \forall t \geqslant t_{l-1} \tag{44}$$

3.4. Parameters estimation

Let us construct a state estimator for the variable r_2 in the following form:

$$\dot{\hat{r}}_2(t) = A_{21}\hat{r}_1(t) + A_{22}r_2(t) + B_2(t)u(t) + \bar{u}(t), \quad \hat{r}_2(0) = r_2(0)$$

Then, define the sliding surface to be designed as the difference between r_2 and $\hat{r}_2$, that is, $\sigma(t) = r_2(t) - \hat{r}_2(t)$. So, in view of (32) and (43), we derive

$$\dot{\sigma}(t) = A_{21}[r_1(t) - \hat{r}_1(t)] + \Delta A(t)Gx(t) - \bar{u}(t) \tag{45}$$

Designing $\bar{u}(t)$ as a super twisting control

$$\bar{u}_i(t) = \bar{z}_i(t) + \lambda|\sigma_i(t)|^{1/2}\,\mathrm{sign}(\sigma_i(t)), \quad \bar{z}_i(t) = \alpha \int_{\tau=0}^{t} \mathrm{sign}(\sigma_i(\tau))\,\mathrm{d}\tau$$

$$\alpha > \beta > \|G\| \, \|\dot{x}(t)\|, \quad \lambda > \frac{(\alpha+\beta)(1+\theta)}{(1-\theta)}\sqrt{\frac{2}{\alpha-\beta}}, \quad 0 < \theta < 1$$

we obtain

$$\sigma(t) = \dot{\sigma}(t) = 0 \quad \text{for all } t \geqslant 0 \tag{46}$$

Remark 1
Since $w(x,t) = \Delta A(t)Gx(t)$ is supposed to be bounded w^+, the constant β can be estimated from (30). Other alternative is to estimate $||G||\ ||\dot{x}(t)||$ on-line (due to (44)) and, based on such estimation, chose β sufficiently large.

Recalling that $\hat{r}_1(t) \equiv r_1(t)$ for all $t \geqslant t_{l-1}$, from (45) and(46), one gets

$$\Delta A(t)Gx(t) = \bar{z}(t) \quad \forall t \geqslant t_{l-1} \tag{47}$$

Since $\hat{x} = x(t)$ for all $t \geqslant t_{l-1}$, the estimation of the uncertain matrix $\Delta A(t)$ may be carried out by means of the identity in (47). Then any appropriate method, designed for parameter estimation, may be applied (see, e.g. [26]). Supposing that $\Delta A(t)$ is a constant matrix ($\Delta A(t) = \Delta A$), a continuous form of the least square (LS) procedure (used, for instance, in [27]) may be easily applied to generate the on-line estimates $(\overline{\Delta A}(t)G)$ of (ΔAG). Postmultiplying (47) by $x(t)$ and integrating one gets

$$(\Delta AG)\left[\int_{\theta=t_{l-1}}^{t} x(\theta)x^{\top}(\theta)\,\mathrm{d}\theta\right] = \int_{\theta=t_{l-1}}^{t} \bar{z}(\theta)x^{\top}(\theta)\,\mathrm{d}\theta$$

or

$$\Delta AG = \left[\int_{\theta=t_{l-1}}^{t} \bar{z}(\theta)x^{\top}(\theta)\,\mathrm{d}\theta\right]\left[\int_{\theta=t_{l-1}}^{t} x(\theta)x^{\top}(\theta)\,\mathrm{d}\theta\right]^{-1} \tag{48}$$

Define $\Gamma^{-1}(t) := \int_{\theta=t_{l-1}}^{t} x(\theta)x^{\top}(\theta)\,\mathrm{d}\theta$ and suppose that $\det(\Gamma(t)) \neq 0$ starting from some instant t_0. Then, the estimates $\overline{\Delta A}(t)G$ of ΔAG at time t can be defined as follows:

$$\overline{\Delta A}(t)G = \left[\int_{\theta=t_{l-1}}^{t} \bar{z}(\theta)x^{\top}(\theta)\,\mathrm{d}\theta\right]\Gamma(t), \quad t \geqslant t_0 \tag{49}$$

Thus, comparing (48) and (49), it is clear that

$$\overline{\Delta A}(t)G = \Delta AG \quad \text{for all } t > t_0 \tag{50}$$

The form given in (49) for estimating ΔAG can be rewritten in a differential form. Indeed, taking into account that $\dot{\Gamma}(t)\Gamma^{-1}(t) + \Gamma(t)\dot{\Gamma}^{-1}(t) = 0$ one has

$$\dot{\Gamma}(t) = -\Gamma(t)x(t)x^{\top}(t)\Gamma(t), \quad \Gamma(t_0) = \left[\int_{\theta=t_{l-1}}^{t_0} x(\theta)x^{\top}(\theta)\,\mathrm{d}\theta\right]^{-1}$$

Thus, estimate (49) can be rewritten in a differential form as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\overline{\Delta A}(t)G) = [\bar{z}(\theta) - (\overline{\Delta A}(t)G)x(t)]x^{\top}(t)\Gamma(t) \tag{51}$$

Identity (50) is fulfilled under the supposition that in the estimation process is made without, even small, disturbances, and there is no numerical errors. However, if this non-idealities appear (as we know, they always appear), the parameters estimated can still converge to the original ones. The sufficient conditions so that the identification error tends to zero are given in the following lines.

In the case when non-idealities appear, we can write $\Delta AGx(t) \equiv \bar{z}(t) + \varepsilon$ where ε is the error between $\Delta AGx(t)$ and $\bar{z}(t)$. Hence, using (48) and (51), the identification error can be expressed

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in the following manner:

$$||\Delta AG - \overline{\Delta A}(t)G|| = \left\|\left[\int_{\theta=t_{l-1}}^{t} [\Delta A(t)Gx(t) - \bar{z}(t)]x^{\top}(\theta)\,\mathrm{d}\theta\right]\Gamma(t)\right\|$$

$$= \left\|\left[\int_{\theta=t_{l-1}}^{t} \varepsilon(\theta)x^{\top}(\theta)\,\mathrm{d}\theta\right]\Gamma(t)\right\| = \left\|\left(\frac{1}{t}\int_{\theta=t_{l-1}}^{t} \varepsilon(\theta)x^{\top}(\theta)\,\mathrm{d}\theta\right)[t\Gamma(t)]\right\|$$

which tends to zero if

$$(1)\ \limsup_{t\to\infty} ||t\Gamma(t)|| < \infty, \quad (2)\ \frac{1}{t}\int_{\theta=t_{l-1}}^{t} \varepsilon(\theta)x^{\top}(\theta)\,\mathrm{d}\theta \underset{t\to\infty}{\rightarrow} 0$$

The *first condition* is fulfilled if the, so-called, *persistence excitation condition* holds

$$\liminf_{t\to\infty} \lambda_{\min}\left(\frac{1}{t}\int_{\theta=t_{l-1}}^{t} x(\theta)x^{\top}(\theta)\,\mathrm{d}\theta\right) > 0$$

since

$$\limsup_{t\to\infty} ||t\Gamma(t)|| = \left[\liminf_{t\to\infty} \lambda_{\min}(t\Gamma(t))^{-1}\right]^{-1} = \left[\liminf_{t\to\infty} \lambda_{\min}\left(\frac{1}{t}\int_{\theta=t_{l-1}}^{t} x(\theta)x^{\top}(\theta)\,\mathrm{d}\theta\right)\right]^{-1}$$

The *second condition* is usually fulfilled when the error (noise) $\varepsilon(t)$ is '*non-correlated*' with the state of the system $x(t)$. Thus, by means of the time variable term $\overline{\Delta A}(t)G$ we can estimate the constant matrix ΔAG.

4. EXAMPLES

Here we consider two different examples: the first one includes the general case of Section 2, the second one deals with the case considered in Section 3.

Example 1
Consider system (1) with the following matrices

$$A = \begin{bmatrix} -0.86 & 0.68 & 0.12 & 0.66 & -1.03 \\ 0.23 & -1.56 & -2.58 & 1.23 & -0.21 \\ -1.2 & 1.83 & -0.56 & 0.51 & -1.83 \\ -1.77 & 1.08 & -1.2 & -1.74 & -1.87 \\ -0.52 & 1.2 & -1.48 & 1.06 & -2.07 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.5 \\ 0.7 \\ 1.2 \\ 1.4 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.5 & 0 \\ 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0.5 \\ 0 & 1 \\ 0 & 1.5 \end{bmatrix}, \quad u = 1.5\sin(2t), \quad w = \begin{bmatrix} 0.5\cos(2t) + 0.43 \\ 0.2\sin(t) + 0.23 \end{bmatrix}$$

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Following Algorithm 1, we obtain that the matrices M_i $(i = 1, 2, 3)$ are

$$M_1 = \begin{bmatrix} 0.96 & -0.14 & -0.22 & 0 & 0 \\ 0 & -0.83 & 0.55 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -0.59 & 0.47 & 0.62 & 0.34 & -0.55 \\ 1.15 & -0.76 & -0.70 & 0.19 & 0.22 \\ 0.31 & -0.52 & -1.41 & 0.39 & 0.44 \\ 0.47 & -2.28 & -1.11 & 0.59 & 0.67 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 0.446 & -1.727 & -1.451 & -0.357 & 0.698 \\ -0.875 & 1.578 & 1.133 & -1.871 & -0.469 \\ 1.129 & -0.302 & -0.397 & 0.035 & 0.154 \\ 0.258 & 0.395 & -0.794 & 0.071 & 0.308 \\ 0.388 & -0.906 & -0.191 & 0.107 & 0.462 \end{bmatrix}$$

and one can verify that $\det M_3 \neq 0$ so in this case $l = 3$. So, we design $\tilde{x}$ as in (9) with the gain K satisfying A2, x_a was designed in the form given by (19). The variables $s^{(k)}$, $v^{(k)}$ $(k = 1, 2)$ were designed according to (22) and (20), respectively. We used $\alpha_1 = 10$, $\lambda_1 = 15$, $\alpha_2 = 10$, and $\lambda_2 = 20$. Figures 1 and 2 show the trajectories of the state x and the state estimator $\hat{x}$ ($\hat{x}$ is represented by xe in the figures) where $\hat{x}$ was designed according to (28).

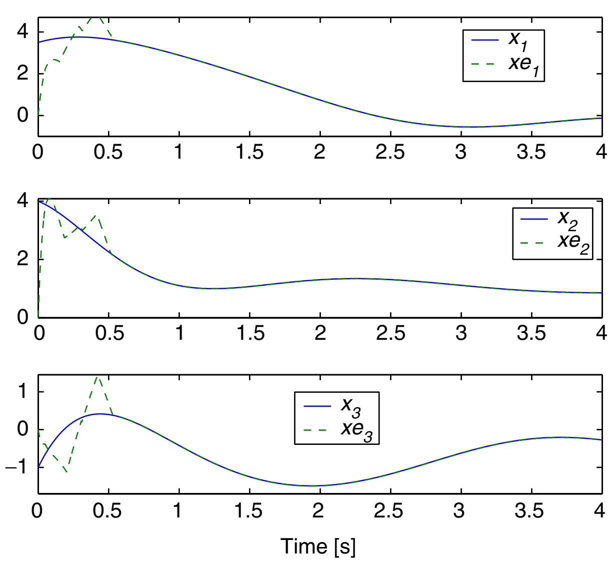

Figure 1. Trajectories of the first three components of x and $xe \equiv \hat{x}$ for Example 1.

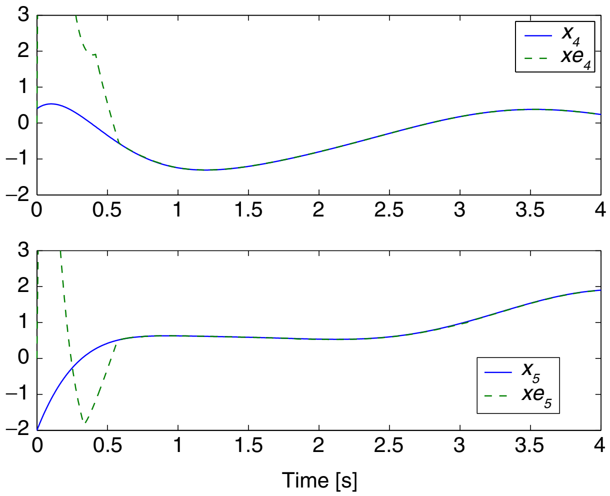

Figure 2. Trajectories of the last two components of x and $xe \equiv \hat{x}$ for Example 1.

Example 2
Consider system (30) with the parameters given by

$$A = \begin{bmatrix} 0 & -4 & 5.5 \\ 1 & -5 & 4.5 \\ 1 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$u = 4\sin(t), \quad \Delta A = [0.1 \ \ -0.15 \ \ 0.2], \quad G = I$$

First, the non-singular matrix P takes the form

$$P = \begin{bmatrix} D^{\perp} \\ (CD)^{+}C \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -0.7071 & 0 & 0.7071 \\ 0.6154 & 0.5385 & 0.3846 \end{bmatrix}$$

So the matrices of the reduced equation (34) are

$$A_{11} = \begin{bmatrix} -7.961 & 3.372 \\ 4.215 & -4.384 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 5.5 \\ -3.889 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \bar{C} = [-1.109 \ \ -0.392]$$

Thus, in this case we need to design only one sliding surface $s^{(1)}$ with the corresponding $v^{(1)}$, both are designed in the form (37) and (36), respectively, with the gains $\alpha_1 = 20$ and $\lambda_1 = 30$. Hence, the state estimator is represented as (44)

$$\hat{x}(t) = P^{-1} \begin{bmatrix} \hat{r}_1(t) \\ (CD)^{+}y(t) \end{bmatrix} = \begin{bmatrix} -0.538 & -0.543 & 1 \\ 1 & 0 & 0 \\ -0.538 & 0.870 & 1 \end{bmatrix} \begin{bmatrix} \hat{r}_1(t) \\ (CD)^{+}y(t) \end{bmatrix}$$

DOI: 10.1002/rnc

where $\hat{r}_1(t)$ is designed according to (42). The trajectories of x and $\hat{x}$ (named in the figure xe) are depicted in Figure 3, and the observation error $x(t) - \hat{x}(t)$ is depicted in Figure 4. The estimation of ΔA was carried out according to (51). Let us denote $\overline{\Delta A}(t) = [ae_1(t)\ ae_2(t)\ ae_3(t)]$. The estimation $\overline{\Delta A}(t)$ is depicted in Figure 5. In the figure we can note that the convergence to the parameters is not so fast compared with the convergence of the observation error. This is because of the numerical and observation errors affect the parameter estimation algorithm; however, the parameters estimated stay in a zone of the original parameters.

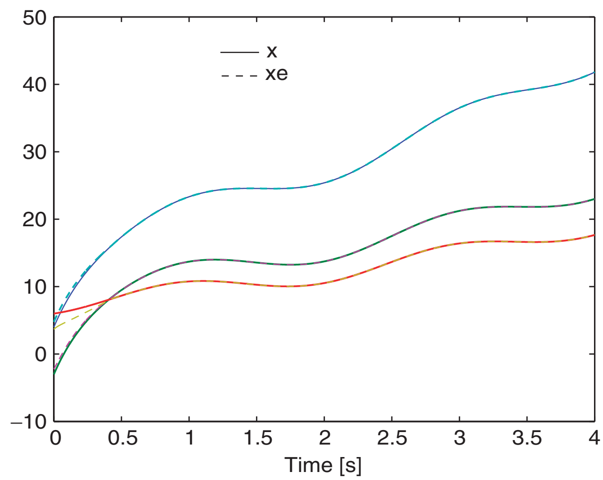

Figure 3. Trajectories of the vector state (x) and the estimator (xe) for Example 2.

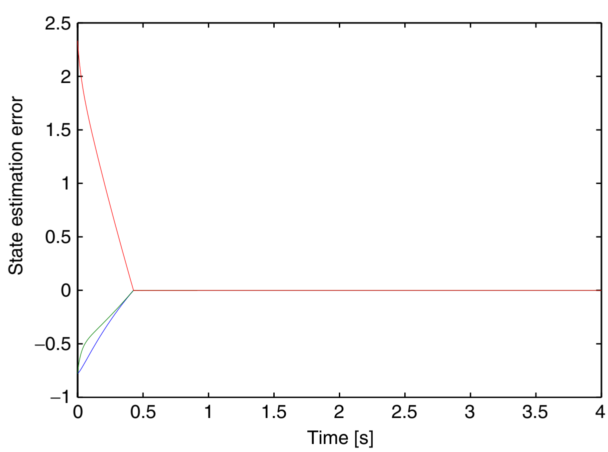

Figure 4. Estimation error $x(t) - \hat{x}(t)$ for Example 2.

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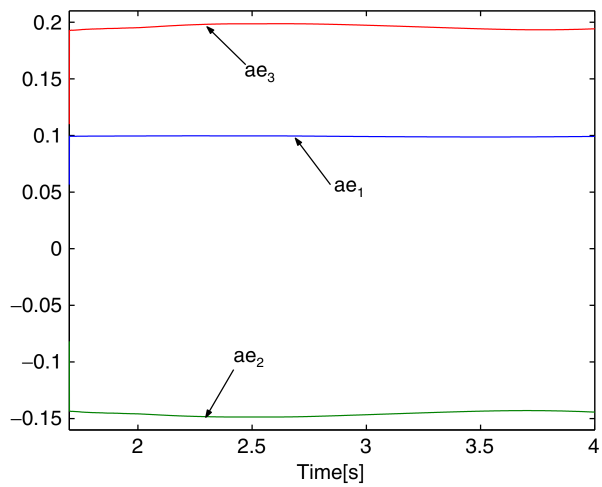

Figure 5. Estimation $\overline{\Delta A}(t)$ of the uncertainty parameter matrix ΔA.

5. CONCLUSIONS

We suggested a state estimator for linear time invariant systems in the presence of unknown inputs, which provides the *exact* values of the state vector in finite time. The conditions for the realization of the state estimator suggested are equivalent to the strong observability of the system and the knowledge of an upper bound of the unknown inputs. The suggested state estimator is presented in an algebraic (non-differential) form. The observation scheme is based on the concept of the, so-called, *hierarchical supertwisting observation strategy*, allowing reconstruction of the state vector *via* an algebraic equation. Specifically, the state vector appears as the sum of a linear observer and an equivalent output injection. For the case when the parameter uncertainty could be considered as unknown inputs which are not creating invariant zeros we use a continuous version of the LS method for parameter identification.

APPENDIX

Proof of Lemma 2

By the assertion of Lemma 2, we need to prove that

$$\operatorname{rank}\begin{bmatrix} sI - A & -D \\ C & 0 \end{bmatrix} = n + q \quad \Leftrightarrow \quad \operatorname{rank}\begin{bmatrix} sI - A_{11} \\ \bar{C} \end{bmatrix} = n - q$$

for all $s \in \mathbb{C}$.

Let us define the non-singular matrix

$$U := \begin{bmatrix} (CD)^{\perp} \\ (CD)^{+} \end{bmatrix}$$

Thus, in view of (32) and (33)

$$\operatorname{rank}\begin{bmatrix} sI-A & -D \\ C & 0 \end{bmatrix} = n+q$$

$$\Leftrightarrow \operatorname{rank}\left\{\begin{bmatrix} I_n & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & I_p \end{bmatrix}\begin{bmatrix} sI-A & -D \\ C & 0 \end{bmatrix}\begin{bmatrix} P^{-1} & 0 \\ 0 & I_p \end{bmatrix}\right\} = n+q$$

$$\Leftrightarrow \operatorname{rank}\begin{bmatrix} sI-PAP^{-1} & -PD \\ UCP^{-1} & 0 \end{bmatrix} = n+q$$

$$\Leftrightarrow \operatorname{rank}\begin{bmatrix} sI-A_{11} & -A_{12} & 0 \\ -A_{21} & sI-A_{22} & -I \\ \bar{C} & 0 & 0 \\ 0 & I & 0 \end{bmatrix} = n+q \quad \Leftrightarrow \quad \operatorname{rank}\begin{bmatrix} sI-A_{11} \\ \bar{C} \end{bmatrix} = n-q \qquad \square$$

ACKNOWLEDGEMENTS

L. Fridman gratefully acknowledges the financial support of this work by the Mexican CONACyT (Consejo Nacional de Ciencia y Tecnología), grant no. 43807-Y, and the Programa de Apoyo a Proyectos de Investigación e Innovación Tecnológica (PAPIIT) UNAM, grant no. 107006-2.

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