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CRANFIELD

EXACT SUBHARMONIC OSCILLATIONS

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SUMMARY

It is demonstrated that a certain class of nonlinear differential equations of second-order, representative of various physical situations, possesses exact periodic solutions which are subharmonics of the forcing frequency. Conditions for the asymptotic stability and, thereby, the physical existence of these subharmonics are also obtained.

CONTENTS

1. Introduction

An asymptotically stable linear system forced by a sinusoidal input will, after the decay of the transient motion, possess a sinusoidal output whose amplitude and phase angle will, in general, differ from that of the input, but whose frequency will be the same. Under similar conditions the behaviour of nonlinear systems is usually very different in that the output may not even be periodic and when it is it may have a waveform containing the fundamental forcing frequency together with higher harmonics and, less often, subharmonics, i.e. oscillations having frequencies which are rational proper fractions of the fundamental.

Theoretical investigations of subharmonic resonance have been limited to a restricted class of nonlinear differential equations. Most of the known results are concerned with Duffing's equation, Van der Pol's equation and modest generalizations of these. The conditions for the existence of subharmonic solutions to Duffing's equation

$$\ddot{x} + b\dot{x} + c_1 x + c_3 x^3 = Q \sin \omega t, \; b \geqslant 0, \qquad \text{Eq. 1.1}$$

where $\ddot{x} = dx/dt$, have been discussed by Stoker in Ref. 1, Chapter 4, whilst the same problem for a slightly more general class of second-order equation has been very fully examined by Hayashi in Ref. 2, Chapters 7 and 9. An analysis of subharmonic resonance associated with the forced Van der Pol equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = Q \sin \omega t, \; \mu > 0, \qquad \text{Eq. 1.2}$$

often referred to as subharmonic frequency entrainment, is given by Minorsky in Ref. 4, Chapter 17 and also by Hayashi in Ref. 2, Chapter 12. Experimental verification of the existence of such subharmonic oscillations in physical systems, both mechanical and electrical, has been obtained by Ludeke, Ref. 3 and Hayashi, Ref. 2. In addition Hayashi presents extensive analogue computer solutions of Eq. 1.1 and Eq. 1.2 showing the nature of subharmonic resonance.

As far as the author is aware, in all previous theoretical studies the subharmonic solutions obtained refer to equations in which the nonlinear terms were implicitly small e.g. $c_3 \ll c_1$ in Eq. 1.1 and $\mu \ll 1$ in Eq. 1.2. Further, these solutions were not exact. That is to say the solution, which was approximate, consisted of the sum of subharmonic, fundamental and higher harmonic terms, implying that the waveform was not of pure subharmonic form. In the present study a more general nonlinear equation, 2.1, is considered and the magnitudes of the coefficients b_1, b_2, c_1 and c_3 are determined in such a manner that they produce a prescribed subharmonic oscillation of frequency one-third of that of the forcing frequency. The existence of such an oscillation in a physical system will, of course, depend on its asymptotic stability and appropriate stability criteria are derived for the case when the oscillation amplitude or frequency is large.

2. An exact subharmonic solution

Consider the equation

$$\ddot{x} + (b_1 + b_2 x^2)\dot{x} + c_1 x + c_3 x^3 = \tfrac{1}{4} Q \sin 3\omega t, \qquad \text{Eq. 2.1}$$

which when $b_1 > 0$, $b_2 = 0$ reduces to Duffing's form and when $b_1 < 0$, $b_2 > 0$, $c_1 > 0$, $c_3 = 0$ reduces to Van der Pol's form. The object of the following analysis is to demonstrate that with the correct choice of coefficients b_1, b_2, c_1 and c_3, Eq. 2.1 possesses a solution of the form

$$x = A \operatorname{Sin}\omega t + B \operatorname{Cos}\omega t \qquad \text{Eq. 2.2}$$

Such a solution is clearly a pure subharmonic of one-third of the forcing frequency 3ω.

Differentiating Eq. 2.2 with respect to t gives

$$\dot{x} = \omega(A \operatorname{Cos}\omega t - B \operatorname{Sin}\omega t) \qquad \text{Eq. 2.3}$$

and

$$\ddot{x} = -\omega^2 (A \operatorname{Sin}\omega t + B \operatorname{Cos}\omega t) \qquad \text{Eq. 2.4}$$

Squaring and cubing Eq. 2.2 gives

$$x^2 = \tfrac{1}{2}(A^2 + B^2) + AB \operatorname{Sin} 2\omega t - \tfrac{1}{2}(A^2 - B^2) \operatorname{Cos} 2\omega t \qquad \text{Eq. 2.5}$$

and

$$x^3 = \tfrac{3}{4} A(A^2 + B^2) \operatorname{Sin}\omega t + \tfrac{3}{4} B(A^2 + B^2) \operatorname{Cos}\omega t$$

$$+ \tfrac{1}{4} A(3B^2 - A^2) \operatorname{Sin} 3\omega t - \tfrac{1}{4} B(3A^2 - B^2) \operatorname{Cos} 3\omega t \qquad \text{Eq. 2.6}$$

Multiplying Eq. 2.5 and Eq. 2.3 gives

$$x^2\dot{x} = -\tfrac{1}{4}\omega B(A^2 + B^2) \operatorname{Sin}\omega t + \tfrac{1}{4}\omega A(A^2 + B^2) \operatorname{Cos}\omega t$$

$$+ \tfrac{1}{4}\omega B(3A^2 - B^2) \operatorname{Sin} 3\omega t + \tfrac{1}{4}\omega A(3B^2 - A^2) \operatorname{Cos} 3\omega t \qquad \text{Eq. 2.7}$$

Substituting the proposed solution 2.2 into 2.1, utilizing Eqs. 2.3, 2.4, 2.6 and 2.7, and equating coefficients of the distinct terms, i.e. $\operatorname{Sin}\omega t$, $\operatorname{Cos}\omega t$, $\operatorname{Sin} 3\omega t$ and $\operatorname{Cos} 3\omega t$ respectively, gives rise to the following relations:

$$(c_1 - \omega^2) A - b_1\omega B - \tfrac{1}{4} b_2\omega B(A^2 + B^2) + \tfrac{3}{4} c_3 A(A^2 + B^2) = 0 \qquad \text{Eq. 2.8}$$

$$(c_1 - \omega^2) B + b_1\omega A + \tfrac{1}{4} b_2\omega A(A^2 + B^2) + \tfrac{3}{4} c_3 B(A^2 + B^2) = 0 \qquad \text{Eq. 2.9}$$

$$\tfrac{1}{4} b_2\omega B(3A^2 - B^2) + \tfrac{1}{4} c_3 A(3B^2 - A^2) = \tfrac{1}{4} Q \qquad \text{Eq. 2.10}$$

$$\tfrac{1}{4} b_2\omega A(3B^2 - A^2) - \tfrac{1}{4} c_3 B(3A^2 - B^2) = 0 \qquad \text{Eq. 2.11}$$

Eliminating c_3 between Eq. 2.10 and 2.11 gives

$$\tfrac{1}{4} b_2\omega \left[B^2(3A^2 - B^2)^2 + A^2(3B^2 - A^2)^2\right] = \tfrac{1}{4} Q\, B(3A^2 - B^2)$$

or

$$b_2 = \frac{Q}{\omega} \cdot \frac{B(3A^2 - B^2)}{(A^2 + B^2)^3} \qquad \text{Eq. 2.12}$$

and in a similar manner

$$c_3 = Q.\ \frac{A(3B^2 - A^2)}{(A^2 + B^2)^3} \qquad \text{Eq. 2.13}$$

Substituting for b_2 and c_3 in Eq. 2.8 and Eq. 2.9 then gives

$$(c_1 - \omega^2)A - b_1\omega B + \tfrac{1}{4}\,Q\,f_1(A, B) = 0 \qquad \text{Eq. 2.14}$$

and

$$(c_1 - \omega^2)B + b_1\omega A + \tfrac{1}{4}\,Q\,f_2(A, B) = 0 \qquad \text{Eq. 2.15}$$

where

$$f_1(A, B) = (-3A^4 + 6A^2B^2 + B^4)/(A^2 + B^2)^2$$

and $\qquad$ Eq. 2.16

$$f_2(A, B) = 8AB^3/(A^2 + B^2)^2$$

Eliminating $(c_1 - \omega^2)$ and ωb_1 in turn between Eq. 2.14 and Eq. 2.15 then gives

$$b_1 = \frac{Q}{4\omega}\left\{ Bf_1(A, B) - Af_2(A, B) \right\} /(A^2 + B^2)$$

$$= \frac{Q}{4\omega}\ .\ B(-3A^4 - 2A^2B^2 + B^4)/(A^2 + B^2)^3$$

or $\quad b_1 = -\frac{Q}{4\omega}\ .\ \frac{B(3A^2 - B^2)}{(A^2 + B^2)^2} \qquad$ Eq. 2.17

and

$$c_1 - \omega^2 = -\frac{3Q}{4}\ .\ \frac{A(3B^2 - A^2)}{(A^2 + B^2)^2} \qquad \text{Eq. 2.18}$$

It follows that

$$b_2 = -4b_1/(A^2 + B^2) \qquad \text{Eq. 2.19}$$

and $\quad c_3 = \frac{-4}{3}\,(c_1 - \omega^2)/(A^2 + B^2) \qquad$ Eq. 2.20

Special Cases

Certain important special cases arise in the following ways.

Case 1. $b_1 = b_2 = 0$.

For this condition to be so, it follows from Eq. 2.17 and Eq. 2.19 that either $B = 0$ or $B^2 = 3A^2$. Consider then the two sub-cases.

Case 1(a). $B = 0$.

Then $c_1 - \omega^2 = 3Q/4A$ and $c_3 = -Q/A^3$. The original differential equation becomes

$$\ddot{x} + (\omega^2 + 3Q/4A)\,x - Q/A^3 . x^3 = \tfrac{1}{4}\,Q \text{ Sin } 3\omega t \qquad \text{Eq. 2.21}$$

with a solution

$$x = A \text{ Sin } \omega t \qquad \text{Eq. 2.22}$$

It will be observed that Eq. 2.21 is in the form of Duffing's equation without damping.

Case 1(b). $B^2 = 3A^2$.

Then $c_1 - \omega^2 = -3Q/8A$ and $c_3 = Q/8A^3$, and the equation becomes

$$\ddot{x} + (\omega^2 - 3Q/8A)\,x + Q/8A^3 . x^3 = \tfrac{1}{4}\,Q \text{ Sin } 3\omega t \qquad \text{Eq. 2.23}$$

with a solution

$$x = A(\text{Sin } \omega t + 3^{\frac{1}{2}} . \text{ Cos } \omega t) = B(3^{-\frac{1}{2}} \text{ Sin } \omega t + \text{Cos } \omega t) \qquad \text{Eq. 2.24}$$

This equation is again in Duffing's form.

Case 2. $c_3 = 0$, $c_1 = \omega^2$.

This condition implies that either $A = 0$, or $A^2 = 3B^2$. Consider these sub-cases.

Case 2(a). $A = 0$.

Then $b_1 = Q/4\omega B$ and $b_2 = -Q/\omega B^3$, and the equation becomes

$$\ddot{x} + Q/4\omega B. \;(1 - 4/B^2 x^2)\,\dot{x} + \omega^2 x = \tfrac{1}{4}\,Q \text{ Sin } 3\omega t \qquad \text{Eq. 2.25}$$

with a solution

$$x = B \text{ Cos } \omega t \qquad \text{Eq. 2.26}$$

Case 2(b). $A^2 = 3B^2$.

Then $b_1 = -Q/8\omega B$ and $b_2 = Q/8\omega B^3$, and the equation becomes

$$\ddot{x} + Q/8\omega B\,(-1 + 1/B^2 . x^2)\,\dot{x} + \omega^2 x = \tfrac{1}{4}\,Q \text{ Sin } 3\omega t \qquad \text{Eq. 2.27}$$

with a solution

$$x = B(3^{\frac{1}{2}} \text{ Sin } \omega t + \text{Cos } \omega t) = A(\text{Sin } \omega t + 3^{-\frac{1}{2}} \text{ Cos } \omega t) \qquad \text{Eq. 2.28}$$

Eq. 2.27 with Q > 0 and B > 0 is in the form of a forced Van der Pol equation.

It is clear that the analysis above provides an interesting range of exact subharmonic solutions to Eq. 2.1 which may be utilized in various ways, however, the existence of such oscillations in a physical system will depend on their asymptotic stability and thought must now be given to the determination of stability criteria which will distinguish the physically observable (e.g. by using an analogue computer) oscillations from the others.

3. Stability criteria

One method of determining the stability of the solutions of Eq. 2.1 is by the use of the "variational equation" defined in Ref. 5, p. 322. Writing Eq. 2.1 in the vector form

$$\left.\begin{aligned} \dot{y}_1 &= y_2 = F_1(y_2) \\ \dot{y}_2 &= \tfrac{1}{4} Q \operatorname{Sin} \omega t - (b_1 + b_2 y_1^2) y_2 - c_1 y_1 - c_3 y_1^3 = F_2(y_1, y_2) \end{aligned}\right\} \quad \text{Eq. 3.1}$$

where $y_j = x_j + \xi_j$ and x_1 is given by Eq. 2.2, the variational equation is given by

$$\dot{\xi} = F_x(x).\xi \qquad \text{Eq. 3.2}$$

where F_x is the Jacobian matrix whose coefficients are

$$\partial F_1/\partial x_1 = 0, \quad \partial F_1/\partial x_2 = 1,$$

$$\partial F_2/\partial x_1 = -2b_2 x_1 x_2 - c_1 - 3c_3 x_1^2, \quad \partial F_2/\partial x_2 = -(b_1 + b_3 x_1^2)$$

The variational equation becomes

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(c_1 + 2b_3 x_1 x_2 + 3c_3 x_1^2) & -(b_1 + b_2 x_1^2) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \text{Eq. 3.3}$$

or in scalar form

$$\ddot{\xi} + (b_1 + b_2 x^2)\dot{\xi} + (c_1 + 3c_3 x^2 + 2b_2 x\dot{x})\xi = 0, \qquad \text{Eq. 3.4}$$

where $x = x_1$ is given by Eq. 2.2.

On substituting from Eq. 2.2, 2.3 and 2.5 into 3.4 this equation becomes

$$\begin{aligned} \ddot{\xi} &+ \left\{ \left[b_1 + \tfrac{1}{2} b_2 (A^2 + B^2) \right] + b_2 \left[AB \operatorname{Sin} 2\omega t - \tfrac{1}{2}(A^2 - B^2) \operatorname{Cos} 2\omega t \right] \right\} \dot{\xi} \\ &+ \left\{ \left[c_1 + \frac{3}{2} c_3 (A^2 + B^2) \right] + \left[\omega b_2 (A^2 - B^2) + 3c_3 AB \right] \operatorname{Sin} 2\omega t \right. \\ &\qquad \left. + \left[2\omega b_2 AB - \frac{3}{2} c_3 (A^2 - B^2) \right] \operatorname{Cos} 2\omega t \right\} \xi = 0, \end{aligned}$$

which upon writing $z = \omega t$ and substituting from Eqs. 2.19 and 2.20 becomes

$$\xi'' + \xi = \epsilon \left\{ \left[a_1 + a_2 \text{ Sin } 2z + a_3 \text{ Cos } 2z \right] \xi + \left[a_4 + a_5 \text{ Sin } 2z + a_6 \text{ Cos } 2z \right] \xi' \right\} \qquad \text{Eq. 3.5}$$

where

$$\epsilon = Q/\omega^2(A^2 + B^2)^3$$

$$a_1 = -\tfrac{3}{4} A(A^2 + B^2)(3B^2 - A^2)$$

$$a_2 = B^3(5A^2 + B^2)$$

$$a_3 = \tfrac{1}{2} A(3A^4 + 5B^4)$$

$$a_4 = -\tfrac{1}{4} B(A^2 + B^2)(3A^2 - B^2)$$

$$a_5 = -AB^2(3A^2 - B^2)$$

$$a_6 = \tfrac{1}{2} B(A^2 - B^2)(3A^2 - B^2)$$

and $\xi' = d\xi/dz$.

From Ref. 5, Chapter 13, Theorem 2.1 it will follow that the solution Eq. 2.2 of Eq. 2.1, with b_1, b_2, c_1 and c_3 given by Eqs. 2.17, 2.19, 2.18 and 2.20 respectively, will be asymptotically stable provided the variational Eq. 3.5 is asymptotically stable. The form of the solution of Eq. 3.5 is known from Floquet's theorem, Ref. 5, Chapter 3, Theorem 5.1, and as shown in Refs. 6 and 7 the stability is determined by the signs of the 'characteristic exponents'. In the case when the periodic terms in the coefficients are small compared with some of the constant terms the approximate evaluation of the characteristic exponents is possible by a method given in Ref. 6, Chapter 8 and used in Ref. 7. It will be observed in Eq. 3.5 that for general values of A, B and ω the coefficients ϵa_2, ϵa_3, ..., ϵa_5, of the periodic terms are not small compared with unity, the coefficients of the ξ and ξ'' terms on the left hand side of the equation. Thus the characteristic exponents may not be evaluated by this method for general values of A, B and ω. However, when $A^2 + B^2$ or ω^2 is sufficiently large, all the coefficients ϵa_1, ..., ϵa_5 will be small compared to unity and the approximate values of the characteristic exponents may be evaluated by the method of Ref. 6, Chapter 8. In this case ϵ may be looked upon as the appropriate 'small parameter'.

Eq. 3.5 may be written in system form as

$$\xi' = C\xi + \epsilon \Phi(z)\xi, \qquad \text{Eq. 3.6}$$

where

ξ = col (ξ_1, ξ_2), the column vector,

$$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\Phi(z) = \begin{bmatrix} 0 & 0 \\ f_1 & f_2 \end{bmatrix},$$

$$f_1 = -(a_1 + a_2 \text{ Sin } 2z + a_3 \text{ Cos } 2z)$$

and

$$f_2 = -(a_4 + a_5 \text{ Sin } 2z + a_6 \text{ Cos } 2z)$$

The characteristic equation of C is

$$\det(C - \lambda E) = 0, \qquad \text{Eq. 3.7}$$

which has the roots $\lambda_{1,2} = \pm i$. Reduction of C to diagonal form is then achieved by means of the similarity transformation

$$T C T^{-1} = \text{diag}(+i, -i) = D, \qquad \text{Eq. 3.8}$$

where

$$T = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \qquad \text{Eq. 3.9}$$

and

$$T^{-1} = \tfrac{1}{2} i \begin{bmatrix} -i & -i \\ 1 & -1 \end{bmatrix} \qquad \text{Eq. 3.10}$$

Transforming Eq. 3.6 by means of this transformation gives

$$y' = Dy + \epsilon \Psi(z) y, \qquad \text{Eq. 3.11}$$

where

$$\Psi(z) = T \Phi(z) T^{-1} \qquad \text{Eq. 3.12}$$

From Eqs. 3.9, 3.10 and 3.12

$$\Psi(z) = \tfrac{1}{2} i \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} 0 & 0 \\ f_1 & f_2 \end{bmatrix} \begin{bmatrix} -i & -i \\ 1 & -1 \end{bmatrix}$$

or

$$\Psi(z) = \tfrac{1}{2} i \begin{bmatrix} -f_1 - if_2 & -f_1 + if_2 \\ f_1 + if_2 & f_1 - if_2 \end{bmatrix} \qquad \text{Eq. 3.13}$$

As in Ref. 7, if γ be any finite complex number then the reduction to standard form is achieved by means of the transformation

$$y = \exp(\lambda_1 + \epsilon\gamma)z.\ \exp(D - \lambda_1 E)z.\,w, \qquad \text{Eq. 3.14}$$

where w is a two vector. Applying this transformation to Eq. 3.11 yields

$$w' = -\epsilon\gamma w + \epsilon \exp\left[-\operatorname{diag}(0,\ -2i)\,z\right].\,\Psi(z).$$
$$\exp \operatorname{diag}(0,\ -2i)\,z.\,w \qquad \text{Eq. 3.15}$$

which upon substitution from Eq. 3.13 gives

$$w' = -\epsilon\gamma w + \tfrac{1}{2}\epsilon i \operatorname{diag}(1,\ e^{2iz}) \begin{bmatrix} -f_1 - if_2 & -f_1 + if_2 \\ f_1 + if_2 & f_1 + if_2 \end{bmatrix} \operatorname{diag}(1,\ e^{-2iz}).w,$$

which, upon multiplying the matrices, becomes

$$w' = -\epsilon\gamma w + \epsilon\Gamma(z).\,w, \qquad \text{Eq. 3.16}$$

where

$$\Gamma(z) = \tfrac{1}{2} i \begin{bmatrix} -f_1 - if_2 & e^{-2iz}(-f_1 + if_2) \\ e^{2iz}(f_1 + if_2) & f_1 - if_2 \end{bmatrix} \qquad \text{Eq. 3.17}$$

Eq. 3.16 is now in the required standard form appropriate to the "totally degenerate" case where all the characteristic roots of C are imaginary.

If γ can be determined in such a way that Eq. 3.16 has a periodic solution of period $T = \pi$, the period of $\Phi(z)$, then this solution has, from Eq. 3.14, the form

$$y = \exp(\lambda_1 + \epsilon\gamma)z.\ p(z), \qquad \text{Eq. 3.18}$$

where p(z) is of period π. This implies that $\lambda_1 + \epsilon\gamma$ is a characteristic exponent of Eq. 3.11. The characteristic exponent is unchanged under a similarity transformation such as Eq. 3.8 so that $\lambda_1 + \epsilon\gamma$ is also a characteristic exponent of Eq. 3.6 and hence of Eq. 3.5.

The solution for γ in the first approximation may be obtained in a closely similar manner to that used in Ref. 7, pp. 17-18. For this purpose define the matrix

$$G(\gamma,\ \epsilon = 0) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \frac{1}{T}\int_0^T \Gamma(z)\,dz, \qquad \text{Eq. 3.19}$$

where T is the period, then γ may be evaluated from the determinantal equation

$$\det\left[G(\gamma,\ \epsilon = 0) - \gamma E\right] = 0 \qquad \text{Eq. 3.20}$$

From Eqs. 3.17 and 3.19 the coefficients of G are as follows:

$$G_{11} = \frac{i}{2\pi}\int_0^{\pi} (-f_1 - if_2)dz$$

$$= -\frac{1}{2\pi}\int_0^{\pi} (a_4 + a_5 \text{ Sin } 2z + a_6 \text{ Cos } 2z) - i(a_1 + a_2 \text{ Sin } 2z$$

$$a_3 \text{ Cos } 2z) \Big\} dz$$

$$= -\frac{1}{2\pi}\int_0^{\pi} (a_4 - ia_1)dz = -\tfrac{1}{2}(a_4 - ia_1),$$

$$G_{22} = -\tfrac{1}{2}(a_4 + ia_1),$$

$$G_{12} = \frac{i}{2\pi}\int_0^{\pi} e^{-2iz}(-f_1 + if_2)dz$$

$$= \frac{i}{2\pi}\int_0^{\pi} (\text{Cos } 2z - i \text{ Sin } 2z)(-f_1 + if_2)dz$$

$$= \frac{1}{2\pi}\int_0^{\pi} \Big\{ -(f_1 \text{ Sin } 2z + f_2 \text{ Cos } 2z) + i(f_2 \text{ Sin } 2z - f_1 \text{ Cos } 2z) \Big\} dz$$

$$= \frac{1}{2\pi}\int_0^{\pi} \Big\{ (a_2 - ia_5) \text{ Sin}^2 2z + (a_6 + ia_3) \text{ Cos}^2 2z \Big\} dz$$

$$= \frac{1}{4\pi}\int_0^{\pi} \Big\{ (a_2 - ia_5) + (a_6 + ia_3) \Big\} dz$$

$$= \frac{1}{4}\Big[(a_2 + a_6) + i(a_3 - a_5) \Big],$$

$$G_{21} = \frac{1}{4}\Big[(a_2 + a_6) - i(a_3 - a_5) \Big]$$

Substituting into Eq. 3.20 gives

$$y^2 - (G_{11} + G_{22})y + (G_{11} G_{22} - G_{12} G_{21}) = 0, \qquad \text{Eq. 3.21}$$

where

$$G_{11} + G_{22} = -a_4 \qquad \text{Eq. 3.22}$$

and

$$G_{11} G_{22} - G_{12} G_{21} = \frac{1}{4}(a_1^2 + a_4^2) - \frac{1}{16}\Big[(a_2 + a_6)^2 + (a_3 - a_5)^2 \Big] \qquad \text{Eq. 3.23}$$

Eq. 3.21 has the roots

$$\gamma_{1,2} = \tfrac{1}{2}\left\{ (G_{11} + G_{22}) \pm \left[(G_{11} + G_{22})^2 - 4(G_{11}G_{22} - G_{12}G_{21}) \right]^{\frac{1}{2}} \right\}$$
$$= \tfrac{1}{2}\left\{ (G_{11} + G_{22}) \pm \left[(G_{11} - G_{22})^2 + 4G_{12}G_{21} \right]^{\frac{1}{2}} \right\}$$

or

$$\gamma_{1,2} = \tfrac{1}{2}\left\{ -a_4 \pm \left[-a_1^2 + \tfrac{1}{4}\left((a_2 + a_6)^2 + (a_3 - a_5)^2 \right) \right]^{\frac{1}{2}} \right\} \quad \text{Eq. 3.24}$$

One characteristic exponent of Eq. 3.5 is, therefore, in the first approximation

$$\lambda_1 + \epsilon\gamma = i + \tfrac{1}{2}\epsilon\left\{ -a_4 \pm \left[-a_1^2 + \tfrac{1}{4}\left((a_2 + a_6)^2 + (a_3 - a_5)^2 \right) \right]^{\frac{1}{2}} \right\} \quad \text{Eq. 3.25}$$

A second characteristic exponent is the complex conjugate of the above expression. The asymptotic stability of the variational equation is determined entirely by the sign of the real part of the characteristic exponent, i.e. in the present case the real part of γ. Now

$$a_2 + a_6 = B^3(5A^2 + B^2) + \tfrac{1}{2}B(A^2 - B^2)(3A^2 - B^2)$$
$$= \frac{3}{2} B(A^2 + B^2)^2$$

and

$$a_3 - a_5 = \tfrac{1}{2}A(3A^4 + 5B^4) + AB^2(3A^2 - B^2)$$
$$= \frac{3}{2} A(A^2 + B^2)^2,$$

giving

$$\frac{1}{4}\left[(a_2 + a_6)^2 + (a_3 - a_5)^2 \right] = \frac{9}{16}(A^2 + B^2)^5.$$

Also

$$a_1^2 = \frac{9}{16} A^2(A^2 + B^2)^2 (3B^2 - A^2)^2$$
$$= \frac{9}{16}(A^2 + B^2)^2 (9A^2B^4 - 6A^4B^2 + A^6)$$

and, thereby,

$$\frac{1}{4}\left[(a_2 + a_6)^2 + (a_3 - a_5)^2 \right] - a_1^2 = \frac{9}{16}(A^2 + B^2)^2 \left[(A^2 + B^2)^3 - 9A^2B^4 + 6A^4B^2 - A^6 \right]$$

$$= \frac{9}{16}(A^2 + B^2)^2 B^2 (3A^2 - B^2)^2 \qquad \text{Eq. 3.26}$$

This expression is real and positive and, therefore, γ must be real. Since ϵ is also real and positive then the condition for asymptotic stability becomes $\gamma < 0$ or

$$-a_4 \pm \left\{ -a_1^2 + \frac{1}{4}\left[(a_2 + a_6)^2 + (a_3 - a_5)^2 \right] \right\}^{\frac{1}{2}} < 0$$

or

$$|a_4| > \left| \left\{ -a_1^2 + \frac{1}{4}\left[(a_2 + a_6)^2 + (a_3 - a_5)^2 \right] \right\}^{\frac{1}{2}} \right|$$

or

$$a_1^2 + a_4^2 - \frac{1}{4}\left[(a_2 + a_6)^2 + (a_3 - a_5)^2 \right] > 0,$$

which upon substitution from Eq. 3.26 and for a_4 becomes

$$\frac{1}{16} B^2(A^2 + B^2)^2 (3A^2 - B^2)^2 - \frac{9}{16} B^2(A^2 + B^2)^2 (3A^2 - B^2)^2 > 0$$

or

$$-\tfrac{1}{2} B^2(A^2 + B^2)^2 (3A^2 - B^2)^2 > 0 \qquad \text{Eq. 3.27}$$

The expression on the left-hand side of Eq. 3.27 is clearly negative and the inequality cannot be satisfied. It follows that for the conditions postulated, i.e. $A^2 + B^2$ or ω sufficiently large, the solution Eq. 2.2 is asymptotically unstable and, therefore, not physically observable as a steady oscillation.

4. Discussion

The analysis of the previous sections is unsatisfactory in that only a limited discussion of the stability has been possible and it is not known whether any of the subharmonic oscillations for moderate $A^2 + B^2$ and ω, such as to make $\epsilon > 1$, are asymptotically stable. Some confirmation of such stable solutions can be obtained from Ref. 2, p. 304, Figure 12.12, which shows the regions in which physically observable subharmonic solutions to the forced Van der Pol equation were obtained on an analogue computer. Taking $\omega = 1$, $B = 1$ and $Q = 32$ in Eq. 2.27 corresponds to a point, on the above figure, in a region where physically observable subharmonics of the appropriate frequency occur. The value of ϵ in this case is 0.5 and is clearly not in a region for which the stability criteria of Section 3 are valid.

Another comparison which may be made is with the result given by Stoker in Ref. 1, p. 106, Eq. 7.11, from which it follows that no subharmonic of order $\frac{1}{3}$ can exist with a frequency equal to that of the free oscillation of the linear system

($\beta = 0$ in Stoker's notation). From Eqs. 2.21 and 2.23 the corresponding frequency of the linear system would be

$$(\omega^2 \pm 3Q/4A)^{\frac{1}{2}},$$

whereas the frequency of the forced subharmonic is ω. For finite A the two frequencies cannot coincide unless Q = 0. The present analysis, therefore, agrees with Stoker on this point.

A technique which is often used to obtain periodic solutions of equations such as 2.1 is that due originally to Galerkin, in which the solution is taken in the form of a Fourier series whose greatest period is equal to that of the forcing frequency. Substitution into the equation and comparison of the coefficients then allows a determination of the coefficients of the Fourier series. The validity of this technique as a means of generating exact solutions has been demonstrated in particular cases by Cesari in Ref. 8. It is clear, however, that such a process is not valid, at least not without modification, for values of b_1, b_2, c_1 and c_3 for which subharmonic solutions exist. The present results should help in defining the values for which the Galerkin process has to be modified.

Finally, the fact that the present solutions have a pure subharmonic wave-form should allow a simpler physical (e.g. analogue computer with cathode ray oscilloscope display) demonstration of the existence of subharmonics.

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