Exact Vibration Solutions for Nonuniform Timoshenko Beams with Attachments

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The exact solution for the free vibration of a symmetric nonuniform Timoshenko beam with tip mass at one end and elastically restrained at the other end of the beam is derived. The two coupled governing characteristic differential equations are reduced into one complete fourth-order ordinary differential equation with variable coefficients in the angle of rotation due to bending. The frequency equation is derived in terms of the four normalized fundamental solutions of the differential equation. It can be shown that, if the coefficients of the reduced differential equation can be expressed in polynomial form, the exact fundamental solutions can be found by the method of Frobenius. Finally, several limiting cases are studied and the results are compared with those in the existing literature.

Nomenclature

$A(x)$ = cross-sectional area of the beam
$E(x)$ = Young's modulus of beam material
$G(x)$ = shear modulus of beam material
$I(x)$ = area moment inertia of the beam
$J(x)$ = mass moment of inertia of the beam per unit length
J_m = rotatory inertia attached at the right end of the beam
K_T, K_θ = translational and rotational spring constants at the left end of the beam, respectively
L = length of the beam
M = concentrated mass attached at the right end of the beam
M_b = total mass of the beam
$m(x)$ = mass of the beam per unit length
$Q(x)$ = beam shear rigidity, $\kappa G(x)A(x)$
$q(\xi)$ = dimensionless shear rigidity, $Q(\xi)/Q(0)$
$R(x)$ = beam bending rigidity, $E(x)I(x)$
$r(\xi)$ = dimensionless bending rigidity, $R(\xi)/R(0)$
$s(\xi)$ = dimensionless mass, $m(\xi)/m(0)$
$v(\xi)$ = dimensionless mass moment inertia, $J(\xi)/J(0)$
x = length variable of the beam
Y = beam lateral displacement
y = dimensionless lateral displacement, Y/L
α = dimensionless rotatory inertia of the attached mass, $J_m/[m(0)L^3]$
β_T, β_θ = dimensionless translational and rotational spring constants, respectively, $K_T L^3/R(0)$, $K_\theta L/R(0)$
γ = dimensionless concentrated mass, $M/[m(0)L]$
δ = dimensionless ratio of bending rigidity to shear rigidity at $x = 0$, $R(0)/[Q(0)L^2]$
η = dimensionless ratio of mass moment inertia to mass at $x = 0$, $J(0)/[m(0)L^2]$
κ = shear correction factor of the beam
λ = taper ratio of the beam
μ = dimensionless ratio of attached mass to total mass of the beam, M/M_b
ξ = dimensionless distance to the left end of the beam, x/L

ω = angular frequency of beam vibration
Ψ = angle of rotation due to bending
Ω^2 = dimensionless frequency, $m(0)\omega^2 L^4/R(0)$

Introduction

NONUNIFORM beams are widely used in many structural applications in order to optimize the distribution of weight and strength and sometimes to satisfy special architectural and functional requirements. Therefore, the analysis of nonuniform beams is of interest to many mechanical, aeronautical, and civil engineers.

It is a standard engineering practice to analyze beams of uniform or variable properties on the basis of Bernoulli-Euler beam theory. However, if the effect of shear distortion and rotatory inertia is considered, then a higher-order beam theory (Timoshenko beam theory) is required. Based on Bernoulli-Euler beam theory, the analysis of nonuniform beams has been studied by many authors via many different methods. A brief review of the work can be found in the work recently done by Lee and Kuo.[1,2] They made the static and dynamic analysis of a general elastically restrained nonuniform Bernoulli-Euler beam. The exact solution for the problem governed by a general self-adjoint fourth-order ordinary differential equation with arbitrarily polynomial varying coefficients were derived in Green's function form and concisely expressed in terms of the four normalized fundamental solutions of the system. Exact stiffness matrices for the analysis of nonuniform Bernoulli-Euler beams were developed by Karabalis and Beskos.[3]

For Timoshenko beams the governing characteristic differential equations are two coupled differential equations expressed in terms of two dependent variables: the flexural displacement and the angle of rotation due to bending. It is well known that, if a beam is uniform, then the two coupled differential equations can be uncoupled into two complete fourth-order ordinary differential equations in the flexural displacement and the angle of rotation due to bending.[4,5] However, this is not the case for nonuniform beams. Consequently, exact solutions for the problems were never given, and the problems were mainly studied by approximate methods such as the finite element method,[6] the optimized Rayleigh-Ritz method,[7] and the transfer matrix method.[8]

In this paper the exact solution for the free vibration of a symmetric nonuniform Timoshenko beam with tip mass at

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