

# Exactly fourteen intrinsically knotted graphs have 21 edges

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Johnson, Kidwell, and Michael showed that intrinsically knotted graphs have at least 21 edges. Also it is known that  $K_7$  and the thirteen graphs obtained from  $K_7$  by  $\nabla Y$  moves are intrinsically knotted graphs with 21 edges. We prove that these 14 graphs are the only intrinsically knotted graphs with 21 edges.

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## 1 Introduction

Throughout the article we will take an embedded graph to mean a graph embedded in  $R^3$ . We call a graph  $G$  *intrinsically knotted* if every embedding of the graph contains a knotted cycle. Conway and Gordon [2] showed that  $K_7$ , the complete graph with seven vertices, is an intrinsically knotted graph. A graph  $H$  is *minor* of another graph  $G$  if it can be obtained from  $G$  by contracting or deleting some edges. An intrinsically knotted graph is *minor minimal intrinsically knotted* provided no proper minor is intrinsically knotted. Robertson and Seymour [9] proved that there are only finite minor minimal intrinsically knotted graphs, but finding the complete set of them is still an open problem. However, it is well known that  $K_7$  and the thirteen graphs obtained from this graph by  $\nabla Y$  moves are minor minimal intrinsically knotted; see Conway and Gordon [2], and Kohara and Suzuki [6].

A  $\nabla Y$  move is an exchanging operation that removes all edges of a triangle  $abc$  and inserts a new vertex  $v$  and three edges  $va, vb$  and  $vc$  as in Figure 1. Its reverse operation is called a  $Y\nabla$  move. Since  $\nabla Y$  moves preserve intrinsic knottedness (see Motwani, Raghunathan, and Saran [7]), we will only consider triangle-free graphs in the article.

From the work of Johnson, Kidwell, and Michael [5], it follows that any intrinsically knotted graph consists at least 21 edges. Here is the main theorem.

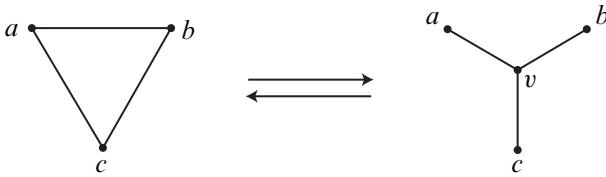


Figure 1:  $\nabla Y$  and  $Y\nabla$  moves

**Theorem 1** *The only triangle-free intrinsically knotted graphs with exactly 21 edges are  $H_{12}$  and  $C_{14}$ . ( $H_{12}$  and  $C_{14}$  were described by Kohara and Suzuki in [6].)*

Kohara and Suzuki [6] found fourteen intrinsically knotted graphs. Goldberg, Mattman, and Naimi [3] constructed twenty graphs derived from  $H_{12}$  and  $C_{14}$  by  $Y\nabla$  moves as in Figure 2, and they showed that these six graphs,  $N_9$ ,  $N_{10}$ ,  $N_{11}$ ,  $N'_{10}$ ,  $N'_{11}$ , and  $N'_{12}$ , are not intrinsically knotted. This fact was proved by Hanaki, Nikkuni, Taniyama, and Yamazaki [4] independently. Theorem 1 guarantees that all intrinsically knotted graphs with 21 edges can be obtained from  $H_{12}$  and  $C_{14}$  by  $Y\nabla$  moves. Thus, we have the following theorem.

**Theorem 2** *The only intrinsically knotted graphs with exactly 21 edges are  $K_7$  and the thirteen graphs obtained from  $K_7$  by  $\nabla Y$  moves.*

This theorem gives us the complete set of fourteen minor minimal intrinsically knotted graphs with 21 edges.

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## 2 Terminology

From now on let  $G = (V, E)$  denote a triangle-free graph with 21 edges. Here  $V$  and  $E$  denote the sets of all vertices and edges of  $G$ , respectively. For any two distinct vertices  $a$  and  $b$ , let  $\hat{G}_{a,b} = (\hat{V}_{a,b}, \hat{E}_{a,b})$  denote the graph obtained from  $G$  by deleting two vertices  $a$  and  $b$ , and then contracting an edge incident to a vertex of degree 1 or

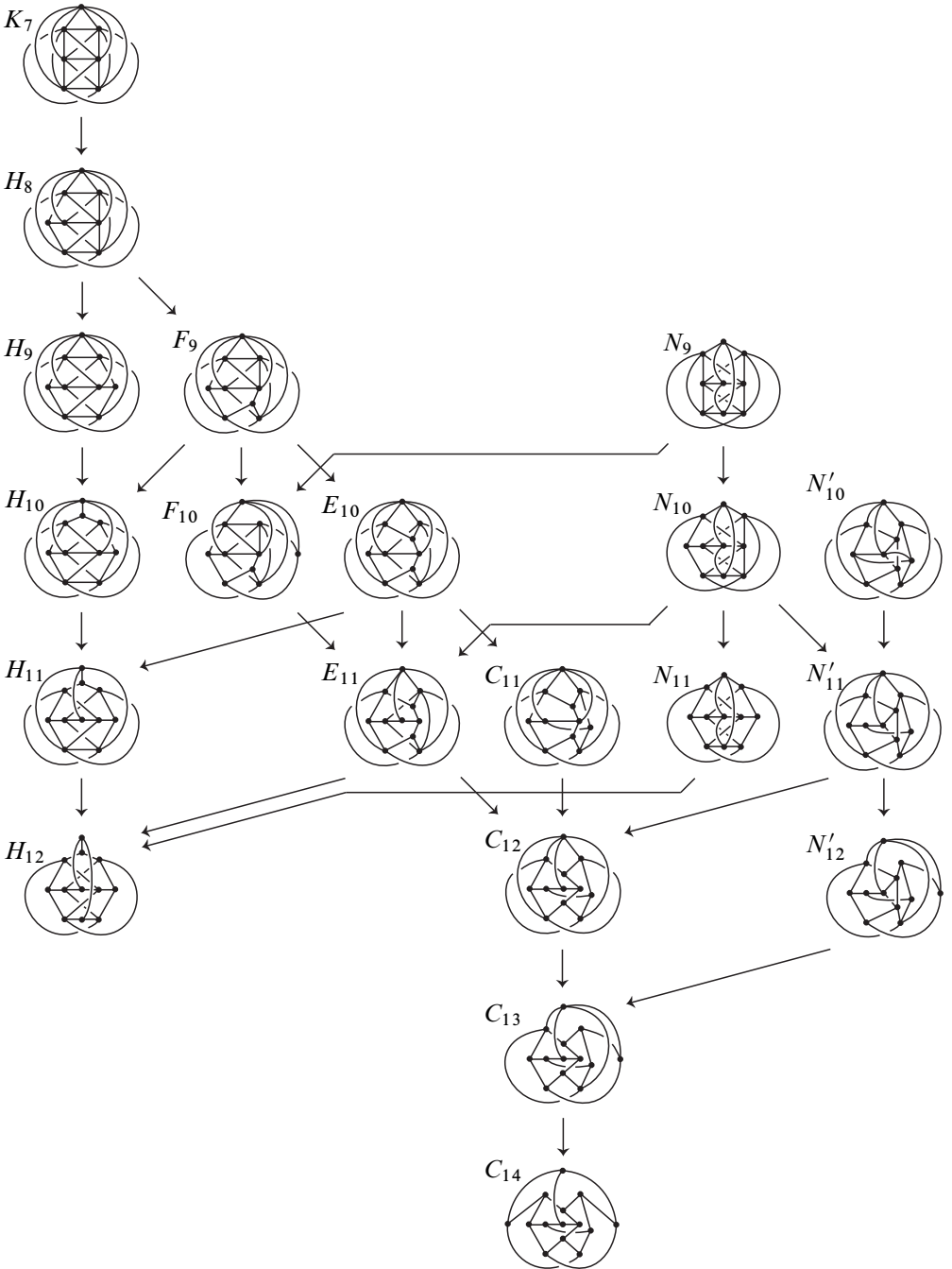


Figure 2: The graph  $K_7$  and 19 more related graphs, where each arrow represents a  $\nabla Y$  move

2 repeatedly until no vertices of degree 1 or 2 exist. Removing vertices means deleting interiors of all edges incident to these vertices as well as the resulting isolated vertices.

In a graph, the distance between two vertices  $a$  and  $b$  is the number of edges in the shortest path connecting them and is denoted by  $\text{dist}(a, b)$ . The degree of a vertex  $a$  is denoted by  $\text{deg}(a)$ . To count the number of edges of  $\widehat{G}_{a,b}$ , we introduce some notation.

- $E(a)$  is the set of edges which are incident to  $a$ .
- $V(a) = \{c \in V \mid \text{dist}(a, c) = 1\}$ .
- $V_n(a) = \{c \in V \mid \text{dist}(a, c) = 1, \text{deg}(c) = n\}$ .
- $V_n(a, b) = V_n(a) \cap V_n(b)$ .
- $V_Y(a, b) = \{c \in V \mid \exists d \in V_3(a, b) \text{ such that } c \in V_3(d) \setminus \{a, b\}\}$ .

First consider the graph  $G \setminus \{a, b\}$  for some distinct vertices  $a$  and  $b$ . In this graph each vertex of  $V_3(a, b)$  has degree 1, and each vertex of  $V_3(a), V_3(b)$  (not in  $V_3(a, b)$ ), and  $V_4(a, b)$  has degree 2. To derive  $\widehat{G}_{a,b}$ , we first delete all edges incident to  $a$  and  $b$  from  $G$ , and then we also delete the remaining edges incident to  $V_3(a, b)$ , and finally we contract one edge of the remaining pair of edges incident to each vertex of  $V_3(a), V_3(b)$  (not in  $V_3(a, b)$ ),  $V_4(a, b)$ , and  $V_Y(a, b)$  as dotted lines in Figure 3(a). Thus, we have the following equation counting the number of edges of  $\widehat{G}_{a,b}$  which is called a *count equation*:

$$|\widehat{E}_{a,b}| = 21 - |E(a) \cup E(b)| - (|V_3(a)| + |V_3(b)| - |V_3(a, b)| + |V_4(a, b)| + |V_Y(a, b)|).$$

For short,  $NE(a, b) = |E(a) \cup E(b)|$  and  $NV_3(a, b) = |V_3(a)| + |V_3(b)| - |V_3(a, b)|$ . If  $a$  and  $b$  are adjacent vertices (ie  $\text{dist}(a, b) = 1$ ), then all of  $V_3(a, b), V_4(a, b)$ , and  $V_Y(a, b)$  are empty because  $G$  is triangle-free. Note that this manner of deriving  $\widehat{G}_{a,b}$  must be handled in a slightly different way when there is a vertex  $c$  in  $V$  such that more than one vertex of  $V(c)$  are contained in  $V_3(a, b)$  as in Figure 3(b). In this case, we usually delete or contract more edges incident to  $c$ , even though  $c$  is not in  $V_Y(a, b)$ .

A graph is  $n$ -apex if one can remove  $n$  vertices from the graph to obtain a planar graph. The following lemma gives an important condition for a graph to be not intrinsically knotted.

**Lemma 3** [1; 8] *If  $G$  is 2-apex, then  $G$  is not intrinsically knotted.*

The following two lemmas play an important role for  $G$  to be 2-apex.

**Lemma 4** *If  $|\widehat{E}_{a,b}| \leq 8$ , then  $\widehat{G}_{a,b}$  is a planar graph. Thus,  $G$  is not intrinsically knotted.*

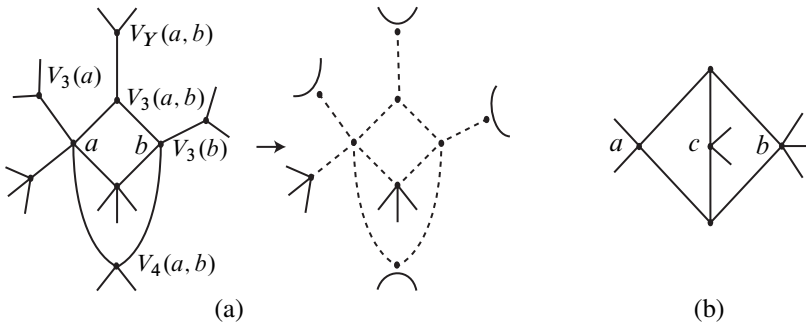


Figure 3: Deriving  $\hat{G}_{a,b}$

**Lemma 5** If  $|\hat{E}_{a,b}| = 9$ , then  $\hat{G}_{a,b}$  is either a planar graph or homeomorphic to  $K(3, 3)$ . Furthermore, if  $\hat{G}_{a,b}$  is not homeomorphic to  $K(3, 3)$ , then  $G$  is not intrinsically knotted.

The graph  $K(3, 3)$  is a bipartite graph where each part has three vertices and each vertex is adjacent to every vertex in the opposite part, and so it is a triangle-free graph and every vertex has degree 3.

To prove [Theorem 1](#), we will show that any triangle-free graph with 21 edges is eventually either a 2–apex or homeomorphic to one of  $H_{12}$  or  $C_{14}$ . Since intrinsically knotted graphs have at least 21 edges [\[5\]](#), it is sufficient to consider simple and connected graphs having no vertex of degree 1 or 2. Our process is constructing all possible such triangle-free graph  $G$  with 21 edges, deleting two suitable vertices  $a$  and  $b$  of  $G$ , and then counting the number of edges of  $\hat{G}_{a,b}$ . If  $\hat{G}_{a,b}$  has 9 edges or less, we can use [Lemma 4](#) or [Lemma 5](#) in order to show that  $G$  is not intrinsically knotted. In the event that  $\hat{G}_{a,b}$  is not planar, we will show that  $G$  is homeomorphic to  $H_{12}$  or  $C_{14}$ .

Before describing the proof of [Theorem 1](#), we introduce more notation. Since  $G$  is triangle-free, for any vertex  $a$  of  $G$ , no two vertices in  $V(a)$  are adjacent. This means that  $E(b)$  and  $E(c)$  do not contain an edge in common for any two distinct vertices  $b$  and  $c$  in  $V(a)$ . We set:

- $E^2(a) = \bigcup_{b \in V(a)} E(b)$ .
- $E \setminus E^2(a) = \{e_1(a), \dots, e_{21-n}(a)\}$  if  $|E^2(a)| = n < 21$ .

$e_i(a)$  is called an *extra edge*, and the two endpoints of the edge are denoted as  $x_i(a)$  and  $y_i(a)$ , where  $\deg(x_i(a)) \geq \deg(y_i(a))$ .

In order to visualize  $G$ , we perform the following steps. First choose a vertex  $a$  with the maximal degree among all vertices and draw  $E^2(a)$ . If  $|E^2(a)| < 21$ , draw  $E \setminus E^2(a)$  apart from  $E^2(a)$  as in Figure 4(a). Then all vertices of degree 1 of  $E^2(a)$  and  $E \setminus E^2(a)$  are merged into some vertices of degree at least 3 without adding new edges as in Figure 4(b). Let  $\bar{V}(a)$  denote the set of all such vertices, and let  $[\bar{V}(a)]$  denote a sequence of the degrees of vertices in  $\bar{V}(a)$  as follows:

- $\bar{V}(a) = V \setminus (V(a) \cup \{a\}) = \{\bar{v}_1(a), \dots, \bar{v}_m(a)\}$  with  $\deg(\bar{v}_i(a)) \geq \deg(\bar{v}_{i+1}(a))$ .
- $[\bar{V}(a)] = [\deg(\bar{v}_1(a)), \dots, \deg(\bar{v}_m(a))]$ .
- $||[\bar{V}(a)]|| = \deg(\bar{v}_1(a)) + \dots + \deg(\bar{v}_m(a))$ .

The graph in Figure 4(b) is an example satisfying  $\deg(a) = 5$ ,  $|V_3(a)| = 1$ ,  $|E^2(a)| = 19$ , and  $[\bar{V}(a)] = [4, 4, 4, 3, 3]$ .

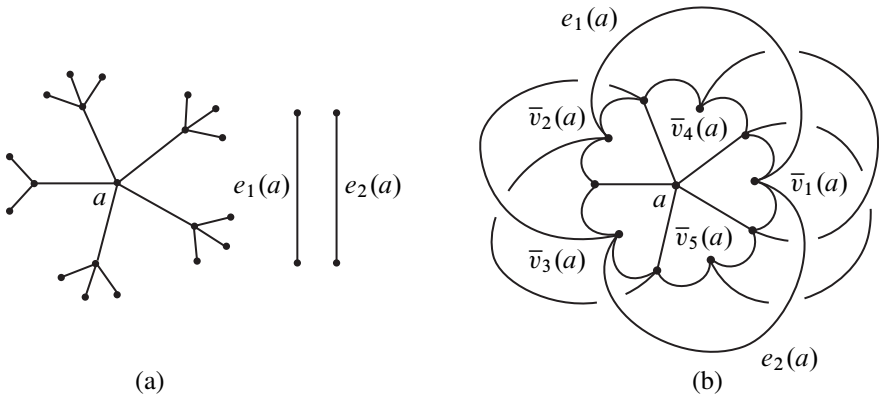


Figure 4: Visualization of  $G$

The remaining three sections of the article are devoted to the proof of Theorem 1. From now on,  $a$  denotes one of vertices with maximal degree in  $G$ . The proof is divided into three parts according to the degree of  $a$ . In Section 3 we show that any graph  $G$  with  $\deg(a) \geq 5$  cannot be intrinsically knotted. In Section 4 we show that an intrinsically knotted graph with  $\deg(a) = 4$  is exactly  $H_{12}$ . Finally, in Section 5 we show that any intrinsically knotted graph, all of whose vertices have degree 3, is always  $C_{14}$ .

### 3 $\deg(a) \geq 5$

In this section we will show that for some  $a', b' \in V$  either  $|\hat{E}_{a',b'}| \leq 8$  or  $|\hat{E}_{a',b'}| = 9$ , but that  $\hat{G}_{a',b'}$  is not homeomorphic to  $K(3, 3)$  by showing that it contains a vertex of degree more than 3 or a triangle (or sometimes a bigon). Then, as a conclusion,  $G$  is not intrinsically knotted by Lemmas 4 and 5. Recall that  $G$  has 21 edges, every vertex has degree at least 3, and  $a$  has the maximal degree among them.

### 3.1 Case $\deg(a) \geq 6$ or $\deg(a) = 5$ with $|V_3(a)| \geq 4$

If  $\deg(a) \geq 6$ , then  $|V_3(a)| \geq 3$ . Let  $c$  be any vertex in  $V_3(a)$ . Choose a vertex  $b$  which has the maximal degree among  $V(c) \setminus \{a\}$ . Then  $|E(b)| + |V_Y(a, b)| \geq 4$ , since  $|V_Y(a, b)| \geq 1$  when  $\deg(b) = 3$ . Note that  $|V_3(b)| \geq |V_3(a, b)|$ . By the count equation,  $|\hat{E}_{a,b}| \leq 8$  in  $\hat{G}_{a,b}$ .

Suppose that  $\deg(a) = 5$  and  $|V_3(a)| \geq 4$ . The proof is similar to the previous paragraph.

### 3.2 Case $\deg(a) = 5$ and $|V_3(a)| = 3$

Let  $b$  and  $c$  be two vertices of  $V(a) \setminus V_3(a)$ . First, suppose that both of them have degree 5. Then  $NE(a, b) = 9$  and  $|V_3(a)| = 3$ , so  $|\hat{E}_{a,b}| \leq 9$ . Furthermore, the vertex  $c$  has degree 4 in  $\hat{G}_{a,b}$ , so it follows that  $\hat{G}_{a,b}$  is not homeomorphic to  $K(3, 3)$ . Thus,  $G$  is not intrinsically knotted by Lemma 5.

Now assume that one of them, say  $b$ , has degree 4. If  $V(b) \setminus \{a\}$  consists of three vertices, all of which are of degree 3, then  $NE(a, b) = 8$  and  $NV_3(a, b) = 6$ , so  $|\hat{E}_{a,b}| \leq 7$ . If not, let  $d$  be a vertex of  $V(b)$  which has degree at least 4. Then  $NE(a, d) \geq 9$ ,  $|V_3(a)| = 3$ , and  $|V_4(a, d)| \geq 1$ , because  $V_4(a, d) \ni b$ . This implies that  $|\hat{E}_{a,d}| \leq 8$ .

### 3.3 Case $\deg(a) = 5$ and $|V_3(a)| = 0$

First, suppose that  $V(a)$  contains a vertex of degree 5, say  $c$ . Since  $G$  has 21 edges, the other four vertices of  $V(a)$  have degree 4. By the previous cases, it is sufficient to suppose that  $|V_3(c)| \leq 2$ . So  $V(c) \setminus \{a\}$  has at least two vertices, say  $b$  and  $d$ , of degree 4 or 5. Since  $|E^2(a)| = 21$  and  $G$  is triangle-free, all edges of  $E(b)$  must be incident to different vertices of  $V(a)$ , so  $|V_4(a, b)| \geq 3$ . This implies that  $|\hat{E}_{a,b}| \leq 9$ . Since  $\hat{G}_{a,b}$  has the vertex  $d$  of degree at least 4, it follows that  $\hat{G}_{a,b}$  is not homeomorphic to  $K(3, 3)$ .

Now, assume that all vertices of  $V(a)$  have degree 4, giving  $|E^2(a)| = 20$ . Let  $e_1(a)$  be the extra edge and recall that two endpoints of  $e_1(a)$  are  $x_1(a)$  and  $y_1(a)$  with  $\deg(x_1(a)) \geq \deg(y_1(a))$ . Since  $G$  is triangle-free, all edges of  $E(x_1(a)) \cup E(y_1(a))$  except  $e_1(a)$  must be incident to different vertices of  $V(a)$ . Thus the degrees of  $x_1(a)$  and  $y_1(a)$  must be either 4 and 3, or 3 and 3, respectively. If  $\deg(x_1(a)) = 4$ , then  $|V_4(a, x_1(a))| = 3$  and  $|V_3(x_1(a))| = 1$ , so  $|\hat{E}_{a,x_1(a)}| = 8$ . If not,  $[\bar{V}(a)]$  is either  $[5, 3, 3, 3, 3]$  or  $[4, 4, 3, 3, 3]$ , because  $||\bar{V}(a)|| = 17$ . Thus  $\bar{v}_1(a)$  has degree 5 or 4 and differs from  $x_1(a)$  and  $y_1(a)$ , so  $|V_4(a, \bar{v}_1(a))| \geq 4$ . Therefore,  $|\hat{E}_{a,\bar{v}_1(a)}| \leq 8$ .

### 3.4 Case $\text{deg}(a) = 5$ and $|V_3(a)| = 1$

In this case,  $V(a)$  contains four vertices of degree 4 or 5. Let  $n$  be the number of such vertices of degree 4, and so we have  $4 - n$  vertices of degree 5, where  $n = 2, 3, 4$ . This implies that  $|E^2(a)| = 21 + (2 - n)$ , and  $n - 2$  extra edges exist. If  $\bar{V}(a)$  contains a vertex  $\bar{v}_1(a)$  of degree 5, then five edges of  $E(\bar{v}_1(a))$  are extra edges or incident to different vertices in  $V(a)$ . For any of the above  $n$ , at least two among these edges are incident to vertices of degree 4 in  $V(a)$ . Then  $NE(a, \bar{v}_1(a)) = 10$ ,  $|V_3(a)| = 1$ , and  $|V_4(a, \bar{v}_1(a))| \geq 2$ , implying  $|\hat{E}_{a, \bar{v}_1(a)}| \leq 8$ .

Now, suppose that  $\bar{V}(a)$  contains vertices of degree 3 or 4 only. If  $n = 2$ ,  $||\bar{V}(a)|| = 16$ , and so  $[\bar{V}(a)]$  is either  $[4, 4, 4, 4]$  or  $[4, 3, 3, 3, 3]$ . For any vertex  $b$  in  $V_5(a)$ , four edges of  $E(b)$  must be incident to different vertices of  $\bar{V}(a)$ . Indeed, these four edges are incident to four vertices of degree 4, or at least three edges among them are incident to vertices of degree 3 in  $\bar{V}(a)$ . This means that the vertex  $b$  has degree 5 with either  $V_3(b) = 0$  or  $V_3(b) \geq 3$ . Both cases are dealt with in previous cases 3.3, 3.1, and 3.2.

If  $n = 3$ ,  $||\bar{V}(a)|| = 17$ , and so  $[\bar{V}(a)] = [4, 4, 3, 3, 3]$ . Let  $V_5(a) = \{b\}$ . To avoid the case 3.2, four edges of  $E(b)$  must be incident to two vertices of degree 4 and two vertices of degree 3 in  $\bar{V}(a)$ , which are  $\bar{v}_1(a)$ ,  $\bar{v}_2(a)$ ,  $\bar{v}_3(a)$ , and  $\bar{v}_4(a)$ . Then there is a vertex  $c$  of  $V_4(a)$  such that at most one edge of  $E(c)$  is incident to  $\bar{v}_3(a)$  and  $\bar{v}_4(a)$ , ie two edges of  $E(c)$  are incident to  $\bar{v}_1(a)$ ,  $\bar{v}_2(a)$ , or  $\bar{v}_5(a)$ . This implies that  $NE(b, c) = 9$  and  $NV_3(b, c) + |V_4(b, c)| \geq 4$ , implying  $|\hat{E}_{b, c}| \leq 8$ .

Finally, if  $n = 4$ ,  $||\bar{V}(a)|| = 18$ , and so  $[\bar{V}(a)]$  is either  $[4, 4, 4, 3, 3]$  or  $[3, 3, 3, 3, 3, 3]$ . Recall that two extra edges exist. In the former case let  $\{\bar{v}_1(a), \bar{v}_2(a), \bar{v}_3(a)\}$  be the three vertices of degree 4 in  $\bar{V}(a)$ . For each  $i = 1, 2, 3$ , if more than two edges of  $E(\bar{v}_i(a))$  are incident to  $V_4(a)$ , then  $NE(a, \bar{v}_i(a)) = 9$ ,  $|V_3(a)| = 1$ , and  $|V_4(a, \bar{v}_i(a))| \geq 3$ , implying  $|\hat{E}_{a, \bar{v}_i(a)}| \leq 8$ . So, each of at least two edges of  $E(\bar{v}_i(a))$  must be either incident to the unique vertex of  $V_3(a)$  or an extra edge. Since  $G$  is triangle-free, one of three vertices, say  $\bar{v}_1(a)$ , has the property that  $E(\bar{v}_1(a))$  contains both extra edges, and  $V(\bar{v}_1(a))$  and  $V(\bar{v}_i(a))$  for each  $i = 2, 3$  cannot share a vertex in  $V(a)$ . This implies that  $V(\bar{v}_2(a))$  and  $V(\bar{v}_3(a))$  coincide as in Figure 5(a). Then  $NE(\bar{v}_2(a), \bar{v}_3(a)) = 8$ , and either  $|V_4(\bar{v}_2(a), \bar{v}_3(a))| = 4$  or  $|V_4(\bar{v}_2(a), \bar{v}_3(a))| = 3$  and  $|V_3(\bar{v}_2(a))| = 1$ . Thus,  $|\hat{E}_{\bar{v}_2(a), \bar{v}_3(a)}| \leq 9$ . In  $\hat{G}_{\bar{v}_2(a), \bar{v}_3(a)}$  the vertex  $a$  still has degree 4 or 5 so that  $\hat{G}_{\bar{v}_2(a), \bar{v}_3(a)}$  is not homeomorphic to  $K(3, 3)$ .

In the latter case, let  $V_4(a) = \{b_1, b_2, b_3, b_4\}$ . We claim that for some  $i, j = 1, 2, 3, 4$ ,  $|V_3(b_i, b_j)| \leq 1$ . Suppose not; that is,  $|V_3(b_i, b_j)| \geq 2$  for all combinations of  $i$  and  $j$ . By some combinatorics we can derive that all 12 edges of  $E(b_1) \cup E(b_2) \cup E(b_3) \cup E(b_4) \setminus E(a)$  are incident to only four vertices of  $\bar{V}(a)$  as in Figure 5(b).



This means that two extra edges must be incident to the remaining two vertices of  $\bar{V}(a)$  at both endpoints. But a bigon is not allowed. Therefore, without loss of generality,  $|V_3(b_1, b_2)| \leq 1$ . Then  $NE(b_1, b_2) = 8$  and  $NV_3(b_1, b_2) \geq 5$ , implying  $|\hat{E}_{b_1, b_2}| \leq 8$ .

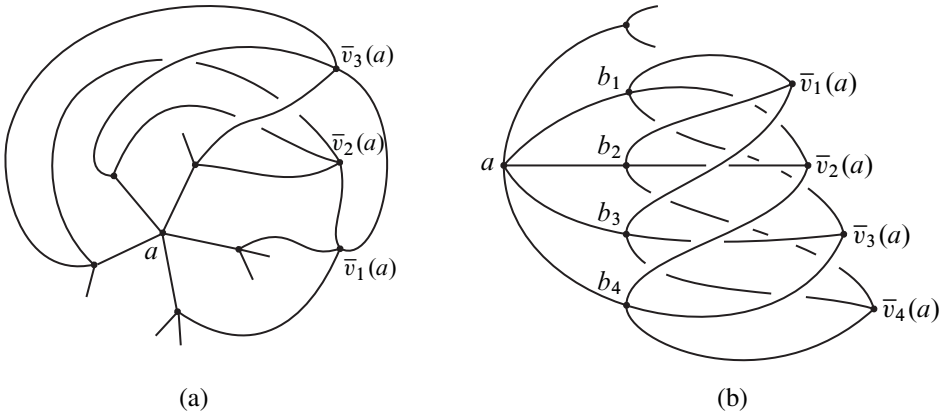


Figure 5:  $[4, 4, 4, 3, 3]$  and  $[3, 3, 3, 3, 3, 3]$  cases

### 3.5 Case $\text{deg}(a) = 5$ and $|V_3(a)| = 2$

If  $V(a)$  contains a vertex of degree 5, say  $b$ , then the previous four cases guarantee that we only consider that  $|V_3(b)| = 2$ , so  $NV_3(a, b) = 4$ , which implies  $|\hat{E}_{a, b}| = 8$ . Therefore we assume that  $V(a)$  contains three vertices of degree 4. In this case three extra edges exist. Since  $|\bar{V}(a)| = 19$ ,  $[\bar{V}(a)]$  is one of  $[5, 5, 5, 4]$ ,  $[5, 5, 3, 3, 3]$ ,  $[5, 4, 4, 3, 3]$ ,  $[4, 4, 4, 4, 3]$ , or  $[4, 3, 3, 3, 3, 3]$ .

If, for some vertex  $\bar{v}_i(a)$  with degree 5, one edge of  $E(\bar{v}_i(a))$  is incident to  $V_4(a)$ , then  $NE(a, \bar{v}_i(a)) = 10$ ,  $|V_3(a)| = 2$ , and  $|V_4(a, \bar{v}_i(a))| \geq 1$ , implying  $|\hat{E}_{a, \bar{v}_i(a)}| \leq 8$ . Thus, three edges of  $E(\bar{v}_i(a))$  are extra edges and the remaining two edges are incident to  $V_3(a)$ . In the first two cases,  $[5, 5, 5, 4]$  and  $[5, 5, 3, 3, 3]$ , both  $E(\bar{v}_1(a))$  and  $E(\bar{v}_2(a))$  share three extra edges, but  $G$  does not have a bigon. In the third case,  $[5, 4, 4, 3, 3]$ ,  $E(\bar{v}_1(a))$  contains three extra edges and one of these extra edges must be incident to  $\bar{v}_4(a)$  or  $\bar{v}_5(a)$ , both of which have degree 3. Then  $NE(a, \bar{v}_1(a)) = 10$  and  $NV_3(a, \bar{v}_1(a)) \geq 3$ , implying  $|\hat{E}_{a, \bar{v}_1(a)}| \leq 8$ .

If, for some vertex  $\bar{v}_i(a)$  with degree 4, two edges of  $E(\bar{v}_i(a))$  are incident to  $V_4(a)$ , then  $NE(a, \bar{v}_i(a)) = 9$ ,  $|V_3(a)| = 2$ , and  $|V_4(a, \bar{v}_i(a))| \geq 2$ , implying  $|\hat{E}_{a, \bar{v}_i(a)}| \leq 8$ . Thus, at most one edge of  $E(\bar{v}_i(a))$  is incident to  $V_4(a)$ . In the fourth case,  $[4, 4, 4, 4, 3]$ , at least twelve among sixteen edges incident to four vertices of degree 4 in  $\bar{V}(a)$  are not incident to  $V_4(a)$ . This is impossible because there are only two vertices in  $V_3(a)$  and three extra edges. In the last case,  $[4, 3, 3, 3, 3, 3]$ , since only one

edge of  $E(\bar{v}_1(a))$  is possibly incident to  $V_4(a)$ , there is a vertex  $b$  in  $V_4(a)$  such that three edges of  $E(b)$  are incident to vertices of degree 3 in  $\bar{V}(a)$ . Then  $NE(a, b) = 8$  and  $NV_3(a, b) \geq 5$ , implying  $|\hat{E}_{a,b}| \leq 8$ .

### 4 deg(a) = 4

Since  $|V| = |V_4| + |V_3|$  and  $4|V_4| + 3|V_3| = 2|E|$ , the pair  $(|V_4|, |V_3|)$  has three choices: (3, 10), (6, 6), and (9, 2). Here,  $V_n$  denotes the set of vertices of degree  $n$ . As in the preceding section, we will show that for some  $a', b' \in V$  either  $|\hat{E}_{a',b'}| \leq 8$  or  $|\hat{E}_{a',b'}| = 9$ , but  $\hat{G}_{a',b'}$  is not homeomorphic to  $K(3, 3)$ , implying that  $G$  is not intrinsically knotted. But one exception occurs so that  $G$  can possibly be  $H_{12}$  when  $(|V_4|, |V_3|) = (6, 6)$ .

#### 4.1 Case $(|V_4|, |V_3|) = (3, 10)$

First suppose that  $V_4$  has a vertex  $a$  such that all four vertices of  $V(a)$  have degree 3. Let  $b_1$  and  $b_2$  be the other vertices of  $V_4$ . For each  $i = 1, 2$ ,  $NE(a, b_i) = 8$ . If there is a vertex of  $V_3(b_i)$  which is not contained in  $V(a)$ , then  $NV_3(a, b_i) \geq 5$ , implying  $|\hat{E}_{a,b_i}| \leq 8$ . Thus each vertex of  $V(b_1)$  is the vertex  $b_2$  or contained in  $V(a)$ , and similarly for  $b_2$ . This implies that the number of vertices of  $V_3$  which have distance 1 or 2 from the vertex  $a$  is at most 6. Take a vertex  $c$  of  $V_3$  with distance at least 3 from  $a$ . Since each vertex of  $V(c)$  is neither  $b_1$  nor  $b_2$ , it has degree 3. Thus  $NE(a, c) = 7$  and  $NV_3(a, c) \geq 7$ , implying  $|\hat{E}_{a,c}| \leq 7$ .

Now, we only need to consider the case that each vertex of  $V_4$  is adjacent to at least one vertex of degree 4. Then, without loss of generality, we have vertices  $a, b$  and  $c$  of  $V_4$  such that  $V(b)$  contains  $a$  and  $c$ . If  $V_3(a)$  and  $V_3(c)$  do not coincide, then  $|V_4(a, c)| = 1$  and  $NV_3(a, c) \geq 4$ , implying  $|\hat{E}_{a,c}| \leq 8$ . If  $V_3(a)$  and  $V_3(c)$  coincide and  $|V_Y(a, c)| \geq 2$ , then  $|V_4(a, c)| = 1$  and  $NV_3(a, c) = 3$ , implying  $|\hat{E}_{a,c}| \leq 7$ . If not, for the unique vertex  $d$  of  $V_Y(a, c)$ ,  $V_3(a) = V_3(c) = V(d)$ . Then, for a vertex  $b'$  of  $V_3(b)$ ,  $V_3(b')$  is disjoint from  $V_3(a)$ . Thus  $NE(a, b') = 7$ ,  $NV_3(a, b') = 5$ , and  $|V_4(a, b')| = 1$ , implying  $|\hat{E}_{a,b'}| \leq 8$ .

#### 4.2 Case $(|V_4|, |V_3|) = (6, 6)$

Consider the subgraph  $H$  of  $G$  consisting of all edges whose both end vertices have degree 4. Since  $G$  has six vertices of degree 3 and the same number of vertices of degree 4,  $H$  is not empty set.

**Claim 1** *If  $H$  has a vertex of degree 1, then  $G$  is not intrinsically knotted.*

**Proof** Suppose that  $H$  has a vertex  $a$  of degree 1. Let  $b$  be the unique vertex of degree 4 in  $V(a)$ . If  $|V_3(b)| = 3$ , then  $NE(a, b) = 7$  and  $NV_3(a, b) = 6$ , implying  $|\hat{E}_{a,b}| \leq 8$ . Thus, there is another vertex  $c$  of  $V_4(b)$ , and so we let  $V(c) = \{b, d_1, d_2, d_3\}$ .

First, assume that  $|V_3(c)| = 0$ . So the two vertices of  $V(b) \setminus \{a, c\}$  must have degree 3, because the six vertices  $a, b, c, d_1, d_2$ , and  $d_3$  in  $V_4$  are all different. Thus  $NE(a, b) = 7$  and  $NV_3(a, b) = 5$ , so  $|\hat{E}_{a,b}| \leq 9$ . Since  $\hat{G}_{a,b}$  has another vertex  $d_1$  of degree 4, it follows that  $\hat{G}_{a,b}$  is not homeomorphic to  $K(3, 3)$ .

Second, assume that  $|V_3(c)| = 1$ , say  $d_1 \in V_3(c)$ . If  $d_1$  is not one of the vertices in  $V(a)$ , then  $NE(a, c) = 8$  and  $NV_3(a, c) + |V_4(a, c)| = 5$ , implying  $|\hat{E}_{a,c}| \leq 8$ . So we may assume that  $d_1$  is in  $V(a)$  and let  $V(d_1) = \{a, c, v_1\}$ . If  $v_1$  has degree 3, then  $NV_3(a, c) + |V_4(a, c)| = 4$  and  $V_Y(a, c) = \{v_1\}$ , implying  $|\hat{E}_{a,c}| \leq 8$ . Otherwise  $v_1$  has degree 4 and it is different from  $d_2$  and  $d_3$ . For any  $i = 2, 3$ , each vertex of  $V(d_i) \setminus \{c\}$  either has degree 3 or is  $v_1$ . Thus  $NE(d_2, d_3) = 8$  and  $NV_3(d_2, d_3) + |V_4(d_2, d_3)| \geq 4$ , implying  $|\hat{E}_{d_2, d_3}| \leq 9$ . But  $\hat{G}_{d_2, d_3}$  has a triangle containing vertices  $a, b$  and  $d_1$ . See Figure 6(a).

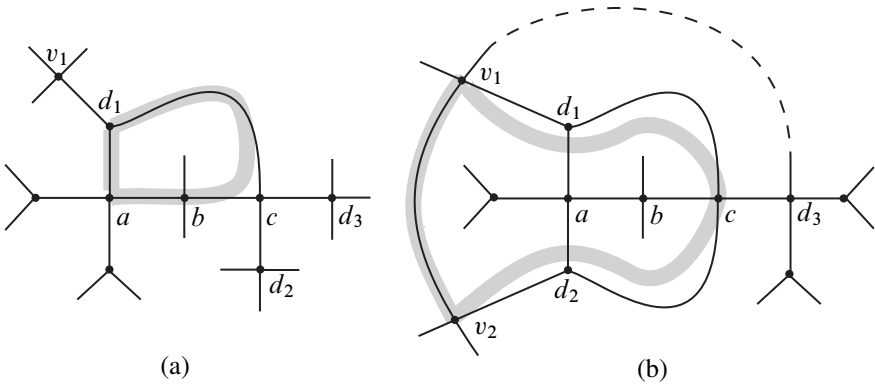


Figure 6: Some nonintrinsically knotted cases

Last, assume that  $|V_3(c)| \geq 2$  and let  $d_1$  and  $d_2$  be two such vertices. As in the previous case, we may say that  $d_1$  and  $d_2$  are in  $V(a)$ , and  $V(d_i) = \{a, c, v_i\}$  for  $i = 1, 2$  where  $v_i$  has degree 4. When  $v_1 = v_2$ ,  $|V_3(a)| = 3$ ,  $|V_4(a, c)| = 1$ , and  $v_1$  has degree 2 when we construct  $\hat{G}_{a,c}$ , implying  $|\hat{E}_{a,c}| \leq 8$ . When  $\text{dist}(v_1, v_2) \geq 2$ , three cases occur as follows:  $|V_3(v_1)| \geq 3$ ,  $|V_3(v_2)| \geq 3$ , or for both  $i = 1, 2$   $|V_3(v_i)| = 2$  and  $V_4(v_i) = V_4 \setminus \{a, c, v_1, v_2\}$ . All three cases satisfy that  $NV_3(v_1, v_2) + |V_4(v_1, v_2)| \geq 4$ , implying  $|\hat{E}_{v_1, v_2}| \leq 9$ . But  $\hat{G}_{v_1, v_2}$  has a bigon containing vertices  $a$  and  $c$ . Finally, when  $\text{dist}(v_1, v_2) = 1$ , two cases occur as follows. If  $d_3$  has degree 3, then by the same reason as before we may say that  $d_3$  is also in  $V(a)$ , and  $V(d_3) = \{a, c, v_3\}$  where  $v_3$  has degree 4. By the previous argument any pair of  $v_1, v_2$  and  $v_3$  has distance 1. This

implies that  $G$  contains a triangle. If  $d_3$  has degree 4, then  $|V_3(d_3)| \geq 2$ , because at most one vertex of  $V(d_3)$  can be  $v_1$  or  $v_2$ . Thus,  $NV_3(a, d_3) \geq 4$ , implying  $|\widehat{E}_{a,d_3}| \leq 9$ . But  $\widehat{G}_{a,d_3}$  has a triangle containing vertices  $c, v_1$  and  $v_2$ . See Figure 6(b).  $\square$

**Claim 2** *If  $H$  is not a cycle with 6 edges, then  $G$  is not intrinsically knotted.*

**Proof** By Claim 1, if  $H$  is not a cycle with 6 edges, then  $H$  contains a cycle with 4 or 5 edges. First assume that  $H$  contains a cycle with 5 edges. Let  $\{a_1, \dots, a_5\}$  be the set of five vertices of the cycle appearing in clockwise order. If the remaining vertex  $b$  of  $V_4$  is contained in some  $V(a_i)$ , say  $i = 1$ , then  $b$  must have distance 1 from one of  $a_3$  and  $a_4$ , say  $a_3$ , by Claim 1. See Figure 7. If  $V_3(a_2) \neq V_3(b)$ ,  $NV_3(a_2, b) + |V_4(a_2, b)| \geq 5$ , implying  $|\widehat{E}_{a_2,b}| \leq 8$ . Otherwise,  $V_3(a_2) = V_3(b)$ . Let  $c_1$  and  $c_3$  be the vertices of  $V_3(a_1)$  and  $V_3(a_3)$ , respectively. If  $c_1 = c_3$ , we still have  $|\widehat{E}_{a_2,b}| \leq 9$  and  $\widehat{G}_{a_2,b}$  has a triangle containing vertices  $a_5, a_4$  and  $c_1 = c_3$ . If  $c_1 \neq c_3$ , then  $|\widehat{E}_{a_1,a_3}| \leq 9$  and  $\widehat{G}_{a_1,a_3}$  has a bigon as in the figure.

If  $b$  is not contained in  $V(a_i)$  for any  $i = 1, \dots, 5$ , then  $|V_3(a_i)| = 2$ . If there is a pair of vertices  $a_i$  and  $a_{i+2}$  (or  $a_{i-3}$  if  $i = 4, 5$ ) such that  $V_3(a_i)$  and  $V_3(a_{i+2})$  are disjoint, then  $NV_3(a_i, a_{i+2}) + |V_4(a_i, a_{i+2})| = 5$ , implying  $|\widehat{E}_{a_i,a_{i+2}}| \leq 8$ . Otherwise, for any pair of vertices  $a_i$  and  $a_{i+2}$  (or  $a_{i-3}$  if  $i = 4, 5$ ),  $V_3(a_i)$  and  $V_3(a_{i+2})$  share vertices. Then they must share only one vertex as in Figure 7(b). Since there is only one extra vertex  $b$  of degree 4, for some pair of vertices  $a_i$  and  $a_{i+2}$ ,  $NV_3(a_i, a_{i+2}) + |V_4(a_i, a_{i+2})| = 4$  and  $V_Y(a_i, a_{i+2}) \geq 1$ , implying  $|\widehat{E}_{a_i,a_{i+2}}| \leq 8$ .

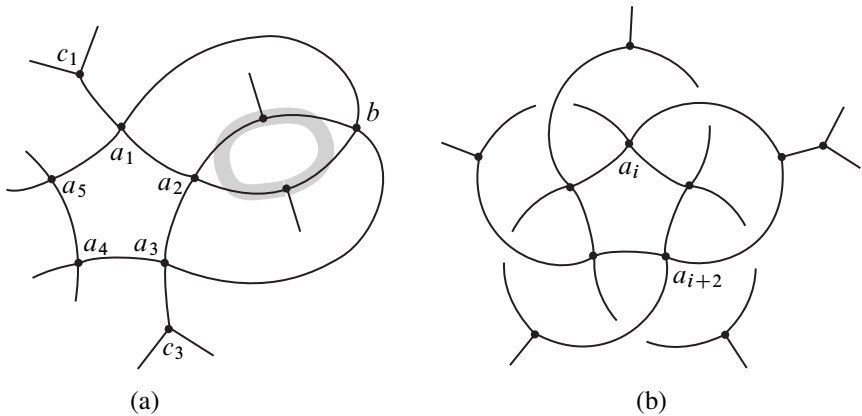


Figure 7: Cycle with 5 edges

Now, assume that  $H$  contains a cycle with 4 edges. Let  $\{a_1, \dots, a_4\}$  be the set of four vertices of the cycle appearing in clockwise order. If  $V(a_1)$  and  $V(a_3)$  (or similarly for  $V(a_2)$  and  $V(a_4)$ ) share only two vertices,  $a_2$  and  $a_4$ , then the remaining two

vertices of  $V_4$  must be contained in  $V(a_1) \cup V(a_3)$ . Otherwise, since  $V(a_1) \cup V(a_3)$  has four more vertices other than  $a_2$  and  $a_4$ ,  $NV_3(a_1, a_3) \geq 3$  and  $|V_4(a_1, a_3)| = 2$ , implying  $|\hat{E}_{a_1, a_3}| \leq 8$ . By **Claim 1**, the two vertices have distance 1, so  $H$  contains a cycle with 5 edges which was dealt in the previous case. If  $V(a_1)$  and  $V(a_3)$  (or similarly for  $V(a_2)$  and  $V(a_4)$ ) share exactly three vertices,  $a_2$ ,  $a_4$  and  $b$ , then let  $c_1$  and  $c_3$  be the remaining vertices of  $V(a_1)$  and  $V(a_3)$ , respectively. If both  $c_1$  and  $c_3$  have degree 3, then  $NV_3(a_1, a_3) + |V_4(a_1, a_3)| \geq 5$ . If both have degree 4, then  $H$  contains a cycle with 5 edges as in the previous case. Finally, if only  $c_1$  (or similarly  $c_3$ ) has degree 4, then, by **Claim 1**,  $V(c_1)$  contains another vertex, say  $d$ , of  $V_4$ , and also  $d$  must have distance 1 from one of  $a_2$  and  $a_4$ , say  $a_4$ , as in **Figure 8(a)**. So  $NV_3(a_4, c_1) + |V_4(a_4, c_1)| \geq 4$ , implying  $|\hat{E}_{a_4, c_1}| \leq 9$ , and  $\hat{G}_{a_4, c_1}$  has a triangle containing vertices  $a_2$ ,  $a_3$ , and  $b$ . Now we may assume that  $V(a_1) = V(a_3)$  and  $V(a_2) = V(a_4)$ . Then  $NV_3(a_1, a_3) + |V_4(a_1, a_3)| = 4$ , implying  $|\hat{E}_{a_1, a_3}| \leq 9$ , and so  $\hat{G}_{a_1, a_3}$  has a bigon as in **Figure 8(b)**.  $\square$

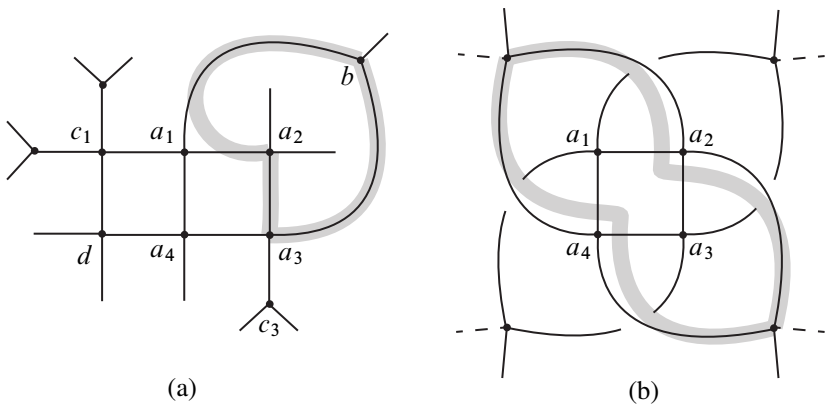


Figure 8: Cycle with 4 edges

By **Claim 2**,  $H$  is exactly a cycle with 6 edges. Let  $\{a_1, \dots, a_6\}$  be the set of six vertices of the cycle with  $a_i$  adjacent to  $a_{i+1}$  for  $i = 1, \dots, 5$ , and  $a_6$  adjacent to  $a_1$ . First, suppose that there is not a vertex  $b$  in  $V_3$  such that  $V(b) = \{a_1, a_3, a_5\}$ . If  $V_3(a_1)$  and  $V_3(a_3)$  are disjoint, then  $NV_3(a_1, a_3) + |V_4(a_1, a_3)| = 5$ . If  $V_3(a_1)$  and  $V_3(a_3)$  share exactly one vertex  $c$ , then the vertex of  $V(c) \setminus \{a_1, a_3\}$  is not  $a_5$ , so it should be one of  $V_Y(a_1, a_3)$ . Thus  $NV_3(a_1, a_3) + |V_4(a_1, a_3)| + |V_Y(a_1, a_3)| = 5$ . If  $V_3(a_1)$  and  $V_3(a_3)$  are same, then  $NV_3(a_1, a_5) + |V_4(a_1, a_5)| = 5$ , because  $V_3(a_1)$  and  $V_3(a_5)$  are disjoint. All three cases guarantee that  $G$  is not intrinsically knotted. Therefore we may assume that there are two vertices  $b_1$  and  $b_2$  so that  $V(b_1) = \{a_1, a_3, a_5\}$  and  $V(b_2) = \{a_2, a_4, a_6\}$ . See **Figure 9(a)**.

Suppose that there is a vertex  $c$ , with  $c \neq b_1$ , so that  $V(c)$  contains  $a_1$  and  $a_3$ . Let  $d_2$  and  $d_5$  be the vertices of  $V_3(a_2)$  and  $V_3(a_5)$ , other than  $b_1$  and  $b_2$ , respectively. If  $d_2 \neq d_5$ , then  $NV_3(a_2, a_5) = 4$ . If  $d_2 = d_5$ , then  $NV_3(a_2, a_5) = 3$  and  $V_Y(a_2, a_5)$  is not empty. Both cases provide  $|\widehat{E}_{a_2, a_5}| \leq 9$ , and  $\widehat{G}_{a_2, a_5}$  has a triangle containing vertices  $a_1, a_3$ , and  $c$ . Therefore we may assume in general that for any vertex  $c$ , except  $b_1$  and  $b_2$ ,  $V(c)$  does not contain both  $a_i$  and  $a_{i+2}$  for any  $i = 1, 2, 3, 4$ , and both  $a_i$  and  $a_{i-4}$  for any  $i = 5, 6$ .

Now we conclude  $E \setminus \{E^2(b_1) \cup E^2(b_2)\}$  consists of three extra edges. Note that each vertex of these edges has degree 3, and there are four more vertices of degree 3 besides  $b_1$  and  $b_2$ . These two facts guarantee that these extra edges must be connected as a tree. This tree can be of two types; either all three edges are incident to one vertex  $d$ , or two edges are incident to different endpoints of the other edge  $e$ , respectively. In both cases, any two edges adjoined to the tree at the same vertex at the end must be also incident to  $a_i$  and  $a_{i+3}$ , respectively, for some  $i = 1, 2, 3$ . Therefore,  $G$  is one of three graphs as in Figure 9(b)–(c), depending on the type of the tree. The graph  $G$  in Figure 9(b) is  $H_{12}$ , which is intrinsically knotted. But the two graphs in Figure 9(c) are not intrinsically knotted because, for some  $i$ ,  $|\widehat{E}_{a_i, a_{i+2}}| \leq 9$ , and  $\widehat{G}_{a_i, a_{i+2}}$  has a triangle.

### 4.3 Case $(|V_4|, |V_3|) = (9, 2)$

Let  $b_1$  and  $b_2$  be the vertices of  $V_3$ . Since  $|V_3| = 2$ , there are at least three vertices,  $a_1, a_2$ , and  $a_3$ , in  $V_4$  such that all vertices of each  $V(a_i)$  have degree 4. If  $\text{dist}(a_1, a_2) = 1$ , then  $V(a_1) \cup V(a_2)$  consists of 8 vertices of  $V_4$ , and so let  $c$  be the ninth vertex. Let  $d$  be any vertex among  $V(a_1) \cup V(a_2) \setminus \{a_1, a_2\}$  which is not contained in  $V(c)$ . We assume that  $d$  is in  $V(a_1)$ . Then  $V(d)$  should be contained in  $V(a_2) \cup \{b_1, b_2\}$ . This implies that  $NE(a_2, d) = 8$  and  $|V_3(d)| + |V_4(a_2, d)| \geq 4$ , implying  $|\widehat{E}_{a_2, d}| \leq 9$ . Since  $c$  has degree 4 in  $\widehat{G}_{a_2, d}$ , it follows that  $\widehat{G}_{a_2, d}$  is not homeomorphic to  $K(3, 3)$ . We have the same result for any choices of pairs among  $a_1, a_2$ , and  $a_3$ .

Now assume that the distance between any pair among  $a_1, a_2$ , and  $a_3$  is at least 2. We separate into several cases according to the number  $|V_4(a_1, a_2)|$ . If  $V_4(a_1, a_2) = \emptyset$  (ie  $\text{dist}(a_1, a_2) > 2$ ), then  $|V_4| \geq 10$ , a contradiction. If  $V_4(a_1, a_2) = \{d\}$ , then  $V_4 = V(a_1) \cup V(a_2) \cup \{a_1, a_2\}$ . This implies that  $a_3 \in V(a_1) \cup V(a_2)$ , so  $\text{dist}(a_1, a_3) = 1$  or  $\text{dist}(a_2, a_3) = 1$ , both of which were dealt with in the previous case. If  $V_4(a_1, a_2) = \{d_1, d_2\}$ , then  $V(d_1) \cup V(d_2) \setminus \{a_1, a_2\}$  is contained in  $\{a_3, b_1, b_2\}$ . This implies that each  $V(d_i) \setminus \{a_1, a_2\}$  is a set of two vertices among  $\{a_3, b_1, b_2\}$ , so that  $|V_3(d_1, d_2)| + |V_4(d_1, d_2)| \geq 4$ , implying  $|\widehat{E}_{d_1, d_2}| \leq 9$ . Since at least two of four vertices in  $V(a_1) \cup V(a_2) \setminus \{d_1, d_2\}$  still have degree 4 in  $\widehat{G}_{d_1, d_2}$ , it follows

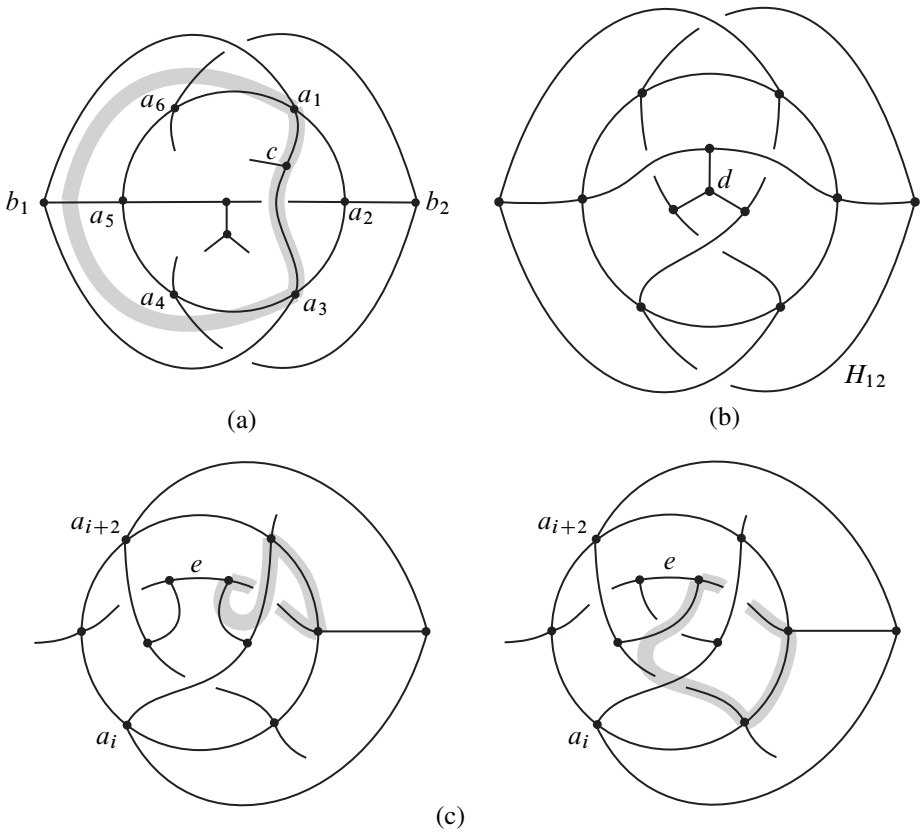


Figure 9: Constructing  $H_{12}$

that  $\widehat{G}_{a_1, a_2}$  is not homeomorphic to  $K(3, 3)$ . If  $V_4(a_1, a_2) = \{d_1, d_2, d_3\}$ , then  $V(d_1) \cup V(d_2) \cup V(d_3) \setminus \{a_1, a_2\}$  is contained in  $\{a_3, a_4, b_1, b_2\}$ , where  $a_3$  and  $a_4$  are the remaining two vertices of degree 4 other than  $V(a_1) \cup V(a_2) \cup \{a_1, a_2\}$ . Thus each  $V(d_i) \setminus \{a_1, a_2\}$  is the set of two vertices among  $\{a_3, a_4, b_1, b_2\}$ . This implies that  $|V_3(d_i, d_j)| + |V_4(d_i, d_j)| \geq 4$  for some  $i, j = 1, 2, 3$ , implying  $|\widehat{E}_{d_i, d_j}| \leq 9$ . Since at least one of three vertices  $V(a_1) \cup V(a_2) \setminus \{d_i, d_j\}$  still has degree 4 in  $\widehat{G}_{d_i, d_j}$ , it follows that  $\widehat{G}_{d_i, d_j}$  is not homeomorphic to  $K(3, 3)$ . Finally, if  $|V_4(a_1, a_2)| = 4$ , then  $|\widehat{E}_{a_1, a_2}| \leq 9$ . Since  $\widehat{G}_{a_1, a_2}$  still has the remaining three vertices of degree 4, it follows that  $\widehat{G}_{a_1, a_2}$  is not homeomorphic to  $K(3, 3)$ .

### 5 $\deg(a) = 3$

Since we are working on the graph with 21 edges and every vertex has degree 3, there are exactly 14 vertices. First, suppose that there exists a pair of vertices  $a$  and  $b$  with

$\text{dist}(a, b) \geq 4$ . Then  $E^2(a)$  and  $E^2(b)$  can share vertices, but they do not share edges in common. Since  $|E^2(a) \cup E^2(b)| = 18$  and  $|V(a) \cup V(b) \cup \{a, b\}| = 8$ , the 18 endpoints of  $E^2(a)$ ,  $E^2(b)$ , and three extra edges which are  $E \setminus \{E^2(a) \cup E^2(b)\}$ , meet at six vertices. If any two edges of  $E^2(a) \setminus E(a)$  (and similarly for  $b$ ) are incident to one vertex  $c$  of these six vertices, take the unique vertex  $d$  of  $V(a)$  which is not an endpoint of these two edges. Then  $NE(b, d) = 6$  and  $NV_3(b, d) = 6$ , implying  $|\hat{E}_{b,d}| = 9$ . But  $\hat{G}_{b,d}$  has a triangle containing  $c$  and the two vertices of  $V(a) \setminus \{d\}$ , so it follows that  $\hat{G}_{b,d}$  is not homeomorphic to  $K(3, 3)$ . If not, each of these six vertices is a common endpoint of one edge of  $E^2(a)$ , one edge of  $E^2(b)$ , and one extra edge. Now, take an extra edge  $e$  and let  $b_1$  and  $b_2$  be the two vertices of  $V(b)$  which have distance 1 from the endpoints of  $e$ . Let  $b_3$  be the remaining vertex of  $V(b)$ . Then  $NE(b_1, b_2) = 6$ ,  $NV_3(b_1, b_2) = 5$ , and  $V_Y(b_1, b_2) = \{b_3\}$ , implying  $|\hat{E}_{b_1,b_2}| = 9$ . But  $\hat{G}_{b_1,b_2}$  has a triangle containing  $a$  and two vertices of  $V(a)$ , so it follows that  $\hat{G}_{b_1,b_2}$  is not homeomorphic to  $K(3, 3)$ . See Figure 10(a).

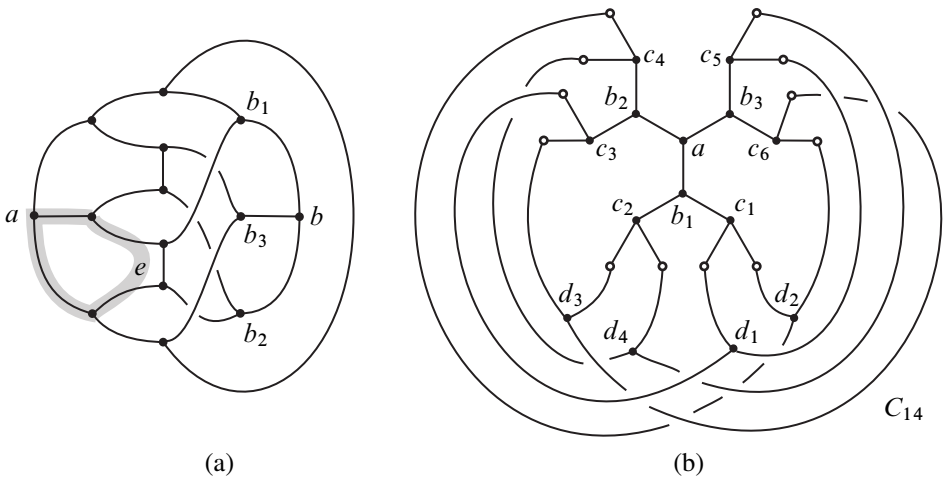


Figure 10: Constructing  $C_{14}$

Therefore, we assume that the distance between any pair of vertices cannot exceed 3. Now we construct the intrinsically knotted graph  $G$  satisfying these conditions. Take a vertex  $a$  and let  $V(a) = \{b_1, b_2, b_3\}$  and  $V(b_i) = \{a, c_{2i-1}, c_{2i}\}$  for  $i = 1, 2, 3$ . As in Figure 10(b), the graph  $E(a) \cup E(c_1) \cup \dots \cup E(c_6)$  consists of 21 edges and 22 vertices. We show this is the only way to draw the graph with 21 edges such that all vertices have distance at most 3 from  $a$  and 10 vertices  $a, b_1, b_2, b_3, c_1, \dots, c_5$ , and  $c_6$  have degree 3. Now we join 12 white dots in Figure 10(b) into 4 groups indicating the remaining 4 vertices by  $d_1, d_2, d_3$  and  $d_4$ . Thus each  $V(d_j)$ ,  $j = 1, 2, 3, 4$ , has three vertices among  $c_1, \dots, c_6$ . Since the distance between any  $c_i$  and  $c_{i'}$  cannot



exceed 3, the following two properties must be satisfied. The first property is that  $V(d_j)$  contains exactly one vertex from each group  $\{c_{2i-1}, c_{2i}\}$  for  $i = 1, 2, 3$ . For example, if  $V(d_1) = \{c_1, c_2, c_3\}$  (ie two vertices from the group  $\{c_1, c_2\}$ ), then we can connect  $c_1$  to at most two vertices among  $\{c_4, c_5, c_6\}$  through some  $E(d_j)$ . This means that the distance between  $c_1$  and one among  $\{c_4, c_5, c_6\}$  exceeds 3. The second property is that different  $V(d_j)$  and  $V(d_{j'})$  share at most one vertex. For example, if they share two vertices  $c_1$  and  $c_3$ , then  $\text{dist}(c_1, c_4) = 4$ . From these two properties, without loss of generality, we may say that

$$\begin{aligned} V(d_1) &= \{c_1, c_3, c_5\}, & V(d_2) &= \{c_1, c_4, c_6\}, \\ V(d_3) &= \{c_2, c_3, c_6\}, & V(d_4) &= \{c_2, c_4, c_5\} \end{aligned}$$

as drawn in [Figure 10\(b\)](#). This graph is exactly  $C_{14}$ .

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