Exactly soluble system of relativistic two-body interaction*

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An exactly solvable model for relativistic two-body interaction, consisting of two harmonic oscillators coupled via a massless scalar field, is studied. Elimination of the field from the dynamics of the sources by imposing boundary conditions leads to differential-difference equations for the particles. These equations can be solved to show that, in general, a solution is not uniquely determined by the initial data at some instant, but that the specification of the data on a finite time segment is necessary. In contrast to this, prescribing the field on some initial surface does lead to a normal Cauchy problem for particles and field. Further, we discuss radiation damping. For finite total energy (and physically reasonable values for the constants of the model) the system shows a dissipative behavior; any energy initially stored by the oscillators is eventually transferred to the field.

I. INTRODUCTION

The relativistic two-body problem has withstood the attack of many physicists and is one of the classical unsolved problems in physics. The main difficulty to this problem comes from the finite propagation velocity of the relativistic interaction, giving rise to a complicated system of difference-differential equations. Very little is known about the general character of the solutions of such a system. The only exact solution for the two-body problem in electrodynamics was given by Schild,[1] where two charges orbiting each other interact via half retarded plus half advanced potentials. Some authors[2-5] have argued that the relativistic two-body problem when formulated in terms of action at a distance, using either retarded, advanced, or some combinations of the field, does not lead to a normal Cauchy problem on the particle level. This argument is based on general considerations such as the finite interaction velocity and the absence of planes of absolute simultaneously. However, no proof of this argument was given. Driver[2] has studied a two-body system, proposed by Synge,[6] restricted to one-dimensional motion. His results seem to confirm the above statement. However, the model has the deficiency that the field is eliminated from the beginning, thereby omitting radiation reaction. This makes a comparison with the Cauchy problem of *particles* and *field* impossible.

This paper intends to give a modest contribution to the understanding of the Cauchy problem of relativistic two-body interaction. We study a model, which was first presented in Ref. 7, consisting of two nonrelativistic harmonic oscillators coupled via a relativistic massless scalar field in Minkowski space. The oscillators sit at fixed points in space and thus have no translational degrees of freedom. Physically one can think of two atoms in a solid influencing each other by radiation. We gain by this that the system of particles and field is exactly soluble, and therefore gives insight into the mathematical structure of the Cauchy problem. Because the exact solutions to the system do not only describe the interaction between the particles but also include radiation reaction, one can study the problem of radiation damping.

In Sec. II we give the Lagrangian for our model and derive from it the dynamical equations. The set of coupled equations can be solved by the method of Laplace transform to obtain the general solution in terms of the initial values of *particles* and *field*.

In Sec. III we take the "point limit" to our system, i.e., the limiting case where the interaction between particles and field is confined to points. It is in this limit that the characteristic features of the Cauchy problem (Sec. IV) are displayed most clearly; the dynamics of the particles is given by difference-differential equations, and our main results can be summarized as follows:

While it is possible to formulate the dynamics of the system as a Cauchy problem for *particles* and *field*, the elimination of the field by boundary conditions, e.g., by prescribing the incoming field, does *not* lead to a normal Cauchy problem on the particle level. Instead, the specification of the particle data on a finite time segment is necessary to determine the solution uniquely.

In Sec. V we discuss the long-time behavior of the solutions. Without imposing any time-asymmetric condition, e.g., no incoming radiation, the system shows a dissipative behavior: For *finite* *total energy* (and physically reasonable values of the constants in our model) any energy initially stored by the oscillators is eventually transferred to the field. There exists, however, a range in the values of the constants for which the system becomes unstable and exponentially growing solutions exist. This instability can be traced back to a non-positive-definite energy.

Section VI gives the explicit analytical form of the solutions for the difference-differential equations by integrating over intervals of the light distance between the particles. A very compact expression is obtained giving the solution up to the Nth interval.

Finally, in Sec. VII we give the computer plots of the time development for the oscillators for three special cases.

II. THE MODEL AND ITS SOLUTION

The model is defined by the following Lagrange function:

$$L=\tfrac{1}{2}[\dot{Q}_1^2(t)-\omega_0^2Q_1^2(t)+\dot{Q}_2^2(t)-\omega_0^2Q_2^2(t)] + \lambda\int d^3x[\rho(\vec{x}-\vec{a})Q_1(t)+\rho(\vec{x}+\vec{a})Q_2(t)]\Phi(\vec{x},t) + \tfrac{1}{2}\int d^3x\{\dot{\Phi}^2(\vec{x},t)-[\vec{\nabla}\Phi(\vec{x},t)]^2\}, \quad (2.1)$$

where $Q_1(t)$, $Q_2(t)$ are the oscillator variables, ω_0 is the bare spring constant, $\Phi(\vec{x},t)$ is a massless scalar field, $\rho(\vec{x})=\rho(|\vec{x}|)$ is a spherical symmetric "charge distribution" normalized to one,

$$\int d^3x\,\rho(\vec{x})=1,$$

and λ is the coupling constant between particle and field; the overdot means differentiation with respect to t.

The system is soluble because ρ does not depend on Q. This corresponds to the dipole approximation in electrodynamics. (For a single-source model in electrodynamics see Ref. 8; the scalar-field single-source model was first given by Schwabl and Thirring,[9] and a detailed discussion of radiation damping by Aichelburg and Beig, Refs. 10 and 11.)

From the Lagrangian (2.1) one derives a set of coupled equations

$$\ddot{Q}_1(t)+\omega_0^2Q_1(t)=\lambda\int d^3x\,\rho(\vec{x}-\vec{a})\Phi(\vec{x},t), \quad (2.2a)$$

$$\ddot{Q}_2(t)+\omega_0^2Q_2(t)=\lambda\int d^3x\,\rho(\vec{x}+\vec{a})\Phi(\vec{x},t), \quad (2.2b)$$

$$\Box\Phi(\vec{x},t)=\lambda\rho(\vec{x}-\vec{a})Q_1(t)+\lambda\rho(\vec{x}+\vec{a})Q_2(t), \quad (2.2c)$$

where $\Box=\partial^2/\partial t^2-\Delta$.

Our aim is to solve Eqs. (2.2) in terms of the values of Q_1, $\dot{Q}_1$, Q_2, $\dot{Q}_2$, Φ, and $\dot{\Phi}$ on an initial spacelike hypersurface, e.g., $t=0$, for $t>0$.

We use the method of Laplace transform and define

$$q_1(s)=\int_0^\infty dt\,e^{-st}Q_1(t) \quad (2.3)$$

similarly for $q_2(s)$, and

$$\phi(\vec{x},s)=\int_0^\infty dt\,e^{-st}\,\Phi(\vec{x},t). \quad (2.4)$$

Multiplying Eqs. (2.2a), (2.2b) with e^{-st} and integrating over $0<t<\infty$, we obtain after some partial integrations

$$(s^2+\omega_0^2)q_1(s)=sQ_1(0)+\dot{Q}_1(0) + \lambda\int d^3x\,\rho(\vec{x}-\vec{a})\phi(\vec{x},s), \quad (2.5a)$$

$$(s^2+\omega_0^2)q_2(s)=sQ_2(0)+\dot{Q}_2(0) + \lambda\int d^3x\,\rho(\vec{x}+\vec{a})\phi(\vec{x},s); \quad (2.5b)$$

the same procedure for Eq. (2.2c) gives

$$(s^2-\Delta)\phi(\vec{x},s)=s\,\Phi(\vec{x},0)+\dot{\Phi}(\vec{x},0)+\lambda\rho(\vec{x}-\vec{a})q_1(s) + \lambda\rho(\vec{x}+\vec{a})q_2(s). \quad (2.5c)$$

Solving for $\phi(\vec{x},s)$ from Eq. (2.5c) yields

$$\phi(\vec{x},s)=\int d^3x'\,\frac{e^{-s|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} \times[\lambda\rho(\vec{x}'-\vec{a})q_1(s)+\lambda\rho(\vec{x}'+\vec{a})q_2(s) + s\Phi(\vec{x}',0)+\dot{\Phi}(\vec{x}',0)] \quad (2.6)$$

Inserting for $\phi(\vec{x},s)$ into the right-hand side (RHS) of Eq. (2.5a) gives three contributions: the term containing $q_1(s)$ is the self-field leading to radiation reaction and to a renormalization of the spring constant ω_0; the term with $q_2(s)$ gives the interaction of q_1 with q_2, while the terms depending on the field show the influence of the initial field on q_1. From Eq. (2.5a) we thus get

$$C(s)q_1(s)=sQ_1(0)+\dot{Q}_1(0)+D(s)q_2(s) + \lambda\int d^3x\,d^3x'\rho(\vec{x}-\vec{a})\frac{e^{-s|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} \times[s\Phi(\vec{x}',0)+\dot{\Phi}(\vec{x}',0)], \quad (2.7)$$

where

$$C(s)=s^2+\omega_0^2-\frac{\lambda^2}{4\pi}\int d^3x\,d^3x'\,\frac{\rho(\vec{x}-\vec{a})\rho(\vec{x}'-\vec{a})}{|\vec{x}-\vec{x}'|}\,e^{-s|\vec{x}-\vec{x}'|} = s^2+\bar{\omega}^2-\frac{\lambda^2}{4\pi}\int d^3x\,d^3x'\,\frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}(e^{-s|\vec{x}-\vec{x}'|}-1), \quad (2.8)$$

where we have introduced the renormalized spring constant $\bar{\omega}$:

$$\bar{\omega}^2 = \omega_0{}^2 - \frac{\lambda^2}{4\pi}\int d^3x\, d^3x' \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} \tag{2.9}$$

and

$$D(s) = \frac{\lambda^2}{4\pi}\int d^3x\, d^3x' \frac{\rho(\vec{x}-\vec{a})\rho(\vec{x}'+\vec{a})}{|\vec{x}-\vec{x}'|}\, e^{-s|\vec{x}-\vec{x}'|}\,. \tag{2.10}$$

An equation similar to Eq. (2.7) is obtained for q_2 from Eq. (2.5b) and Eq. (2.6), with q_1, q_2 interchanged and the argument $\vec{a}$ replaced by $-\vec{a}$ in the last term. Note that $C(s)$ and $D(s)$ are unaffected by this.

The coupled equation for the oscillators can now be decoupled by introducing the variables

$$Q_\pm(t) = Q_1(t) \pm Q_2(t) \quad \text{and} \quad q_\pm(s) = q_1(s) \pm q_2(s)\,,$$

so that we finally have

$$[C(s) \pm D(s)]q_\pm(s) = sQ_\pm(0) + \dot{Q}_\pm(0) + (F_1 \pm F_2)\,, \tag{2.11}$$

where

$$F_{1(2)} = \lambda \int d^3x\, d^3x' \rho(\vec{x}\mp\vec{a})\frac{e^{-s|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} \times[s\Phi(\vec{x}',0)+\dot{\Phi}(\vec{x}',0)]\,.$$

From Eqs. (2.11) one obtains by the inversion formula for the Laplace transform the dynamics of the oscillators for arbitrary Cauchy data:

$$Q_\pm(t) = \frac{1}{2\pi i}\int_c ds\, e^{st} q_\pm(s)\,, \tag{2.12}$$

where c is an integration path in complex s plane parallel to the imaginary axis and to the right of all singularities of the integrand.

Similarly, the initial-value solution for the field is obtained by inserting for $q_1(s)$ and $q_2(s)$ from Eqs. (2.11) into Eq. (2.6) and applying the inverse Laplace transformation

$$\Phi(\vec{x},t) = \frac{1}{2\pi i}\int_c ds\, e^{st}\phi(\vec{x},s)\,. \tag{2.13}$$

We have thus succeeded in constructing the complete initial-value solution to our system for $t>0$. Whether or not the solution can be integrated explicitly depends on the structure of $\rho(\vec{x})$. In the next section we study the point limit, for which the explicit integration is possible.

III. THE POINT LIMIT

We now take the limit $\rho(\vec{x}) \to \delta^3(\vec{x})$; then the oscillators couple to the field only at $\vec{x}=\pm\vec{a}$. This simplifies the dynamics of the oscillators considerably, so that an explicit integration is possible. On the other hand, the point structure of the sources makes an infinite renormalization of the spring constant necessary and moreover causes a discontinuity in the field along the light cone for certain Cauchy data.

In the point limit Eqs. (2.11) simplify to

$$h_\pm(s)q_\pm(s) = sQ_\pm(0) + \dot{Q}_\pm(0) + \lambda[\phi_H(\vec{a},s) \pm \phi_H(-\vec{a},s)]\,, \tag{3.1}$$

where

$$h_\pm(s) = C(s) \pm D(s) = s^2 + 2\Gamma s + \bar{\omega}^2 \mp 2\Gamma\frac{e^{-2as}}{2a}\,, \qquad \Gamma = \frac{\lambda^2}{8\pi}\,, \tag{3.2}$$

and

$$\phi_H(\pm\vec{a},s) = \lambda\int d^3x \frac{e^{-s|\vec{x}\mp\vec{a}|}}{4\pi|\vec{x}\mp\vec{a}|}[s\Phi(\vec{x},0)+\dot{\Phi}(\vec{x},0)] \tag{3.3}$$

is the homogeneous part of the field at $\vec{x}=\pm\vec{a}$. Because of the structure of the charge distribution only the spherically symmetric part of the initial field with respect to $\vec{x}=\vec{a}$ and $\vec{x}=-\vec{a}$ couples to Q_1 and Q_2, respectively. We define

$$\Psi_{\pm a}(|\vec{x}\mp\vec{a}|) = \frac{|\vec{x}\mp\vec{a}|}{4\pi}\int d\Omega_{\pm a}\Phi(\vec{x},0)\,, \qquad \chi_{\pm a}(|\vec{x}\mp\vec{a}|) = \frac{|\vec{x}\mp\vec{a}|}{4\pi}\int d\Omega_{\pm a}\dot{\Phi}(\vec{x},0)\,, \tag{3.4}$$

where $d\Omega_{\pm a}$ means angular integration around the points $\vec{x}=\pm\vec{a}$.

With these definitions we write

$$\phi_H(\pm\vec{a},s) = \lambda\int_0^\infty d\xi\, e^{-s\xi}[s\Psi_{\pm a}(\xi)+\chi_{\pm a}(\xi)]\,.$$

For the total field we have in the point limit from Eq. (2.6)

$$\phi(\vec{x},s) = \lambda\frac{e^{-s|\vec{x}-\vec{a}|}}{4\pi|\vec{x}-\vec{a}|}q_1(s) + \lambda\frac{e^{-s|\vec{x}+\vec{a}|}}{4\pi|\vec{x}+\vec{a}|}q_2(s) + \phi_H(\vec{x},s)\,, \tag{3.5}$$

which after Laplace transforming gives, for $t>0$,

$$\Phi(\vec{x},t) = \frac{\lambda}{4\pi}\left[\frac{Q_1(t-|\vec{x}-\vec{a}|)}{|\vec{x}-\vec{a}|}\theta(t-|\vec{x}-\vec{a}|) + \frac{Q_2(t-|\vec{x}+\vec{a}|)}{|\vec{x}+\vec{a}|}\theta(t-|\vec{x}+\vec{a}|)\right] + \Phi_H(\vec{x},t)\,. \tag{3.6}$$

where θ is the step function. Note that the Laplace transform is especially suitable for obtaining the initial-value solution because the total field is automatically decomposed into $\Phi=\Phi_I+\Phi_H$, where Φ_I satisfies the inhomogeneous wave equation and has vanishing Cauchy data, while Φ_H is a free field with initial values $\Phi(\vec{x},0), \dot{\Phi}(\vec{x},0)$ [see Eq. (3.3)]. It is evident that the inhomogeneous part of the field is discontinuous along $t=\pm|\vec{x}+\vec{a}|$. The total field *is* continuous if the following conditions between initial field and the sources are satisfied:

$$\lim_{\vec{x}\to\pm\vec{a}} \Psi_{\pm a}(|\vec{x}\mp\vec{a}|)=\frac{\lambda}{4\pi}Q_{1(2)}(0),$$
$$\lim_{\vec{x}\to\pm\vec{a}} \chi_{\pm a}(|\vec{x}\mp\vec{a}|)=\frac{\lambda}{4\pi}\dot{Q}_{1(2)}(0). \tag{3.7}$$

Physically, these conditions mean that in the neighborhood of the point sources the field should display also for $t=0$ the $1/r$ Coulomb-type singularity in order that the spring constant be always the renormalized one. This is necessary because charge conservation is not implied for the scalar field (for further details see Ref. 10). Henceforth, we shall assume the continuity conditions (3.7) to be valid, which implies that we cannot dispose completely of Ψ and χ anymore. If we, nevertheless, in what follows write, e.g., "$\Psi_a=0$," this has to be understood outside an arbitrary small neighborhood of $\vec{x}=\vec{a}$.

Taking into account the continuity condition (3.7), we write the oscillator equations (3.1) as

$$h_\pm(s)q_\pm(s)=(s+2\Gamma)Q_\pm(0)+\dot{Q}_\pm(0) +\lambda\int_0^\infty d\xi\, e^{-s\xi}\{\Psi'_{+a}(\xi)+\chi_{+a}(\xi) \pm[\Psi'_{-a}(\xi)+\chi_{-a}(\xi)]\}, \tag{3.8}$$

where a prime denotes the derivative with respect to the argument. The additional 2Γ term comes from $\phi_H(\pm\vec{a},s)$ after a partial integration and using conditions (3.7).

In the following section we discuss the Cauchy problem for the point model in detail.

IV. THE CAUCHY PROBLEM

Suppose that the field at the initial surface vanishes, i.e., $\Psi_{\pm a}=0$ and $\chi_{\pm a}=0$, then Eq. (3.8) is simply

$$h_\pm(s)q_\pm(s)=(s+2\Gamma)Q_\pm(0)+\dot{Q}_\pm(0), \tag{4.1}$$

where $h_\pm(s)$ is given by Eq. (3.2).

It is interesting to note that the inverse Laplace transform of Eqs. (4.1) for $t>0$ is given by

$$\ddot{Q}_\pm(t)+2\Gamma\dot{Q}_\pm(t)+\bar{\omega}^2Q_\pm(t)=\pm\frac{\Gamma}{a}Q_\pm(t-2a)\theta(t-2a). \tag{4.2}$$

Naturally we considered only solutions for which a Laplace transform exists, in other words which grow at most as e^{ct}; however, this is always the case for Eqs. (4.2) (see Appendix B). Equations (4.2) are difference-differential equations of retarded type, because the retarded time $t-2a$ appears as argument in the oscillator variables. It is known (see Ref. 12) that, in general, solutions to difference-differential equations are not uniquely determined by the set of initial data, but that the variables have to be given over an interval equal to the retardation time. However, for the Eqs. (4.2) this is not the case because of the θ function on the RHS, which ensures that in the first interval (i.e., for $0\le t<2a$) these equations are ordinary differential equations and thus lead to a normal Cauchy problem for the particles. Physically this is evident, since the field at the initial surface vanishes; the oscillators do not influence each other in the first interval and perform free damped oscillations during this period.

If the field *does not* vanish for $t=0$, the time development of the system is determined by the set of Cauchy data for *oscillators* and *field*.

Suppose, now, that instead of prescribing the (total) field at some initial surface, we would like to give the *incoming* field. As usual, we decompose the total field as

$$\Phi(\vec{x},t)=\Phi_{\rm ret}(\vec{x},t)+\Phi_{\rm in}(\vec{x},t), \tag{4.3}$$

where $\Phi_{\rm in}$ satisfies the free-field equation and

$$\Phi_{\rm ret}(\vec{x},t)=\frac{\lambda}{4\pi}\left[\frac{Q_1(t-|\vec{x}-\vec{a}|)}{|\vec{x}-\vec{a}|}+\frac{Q_2(t-|\vec{x}+\vec{a}|)}{|\vec{x}+\vec{a}|}\right], \tag{4.4}$$

We can introduce this decomposition into Eqs. (3.8). Note that $\Phi_{\rm ret}$ fulfills our continuity conditions Eqs. (3.7). For simplicity, first let $\Phi_{\rm in}=0$; then e.g.,

$$\Psi_a(\xi)=\frac{\lambda}{4\pi}Q_1(-\xi)+\frac{\lambda}{4\pi}\xi\int d\Omega_a\frac{Q_2(-|\vec{\xi}+2\vec{a}|)}{4\pi|\vec{\xi}+2\vec{a}|}, \tag{4.5}$$

with $\vec{\xi}=\vec{x}+\vec{a}$, $\xi=|\vec{\xi}|$, and it can be shown that

$$\frac{\lambda}{4\pi}\int_0^\infty d\xi\, e^{-s\xi}[\Psi'_a(\xi)+\chi_a(\xi)] =\frac{\Gamma}{a}e^{-2as}\int_{-2a}^0 dt\, e^{-st}Q_2(t). \tag{4.6}$$

A completely analogous expression with Q_1 on the RHS is obtained from the Ψ_{-a}, χ_{-a} terms of Eqs. (3.8). Thus, if the field is *purely retarded*, the dynamical equations for the oscillators read

$$h_\pm(s)q_\pm(s)=(s+2\Gamma)Q_\pm(0)+\dot{Q}_\pm(0) \pm\frac{\Gamma}{a}e^{-2as}\int_{-2a}^0 dt\, e^{-st}Q_\pm(t). \tag{4.7}$$

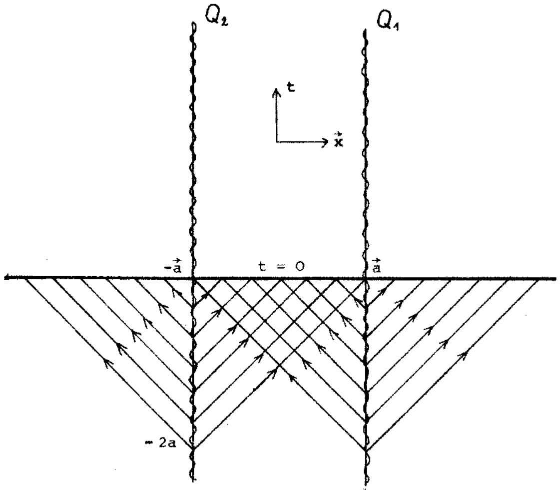

FIG. 1. The two oscillators Q_1 and Q_2 interact along the light cone; energy can be carried away by the field. If Φ_{in} is given, then the data of Q_1 and Q_2 for $-2a < t \leq 0$ are necessary to determine the solution uniquely for $t > 0$.

In this case $q_\pm$ does not only depend on the initial data $Q_\pm(0)$ and $\dot{Q}_\pm(0)$ but also on $Q_\pm(t)$ for $-2a<t<0$. This can also be inferred by looking at the inverse Laplace transform of Eqs. (4.7), which is

$$\ddot{Q}_\pm(t) + 2\Gamma\dot{Q}_\pm(t) + \omega^2 Q_\pm(t) = \pm\frac{\Gamma}{a} Q_\pm(t-2a) . \qquad (4.8)$$

There is an important difference between Eqs. (4.2) and Eqs. (4.8): The absence of the θ function in the latter, which implies that in this case one has difference-differential equations for all t.

If the incoming field does not vanish, then this field has to be added as an external force on the RHS of Eqs. (4.7), so that we may write

$$\begin{aligned} h_\pm(s)q_\pm(s) &= (s+2\Gamma)Q_\pm(0) + \dot{Q}_\pm(0) \\ &\pm \frac{\Gamma}{a} e^{-2as}\int_{-2a}^{0} dt\, e^{-st} Q_\pm(t) \\ &+ \lambda[\phi_{in}(\vec{a}, s) \pm \phi_{in}(-\vec{a}, s)] , \end{aligned} \qquad (4.9)$$

where $\phi_{in}(\pm\vec{a}, s)$ is the Laplace transform of $\phi_{in}(\pm\vec{a}, t)$.

Let us summarize: Eqs. (3.8) and Eqs. (4.9) are equivalent forms of describing the dynamics of the oscillators. The difference lies in the decomposition of the total field. In Eqs. (3.8) the total field is decomposed into $\Phi = \Phi_I + \Phi_H$, where the inhomogeneous solution is nonzero only in the forward light cone [see Eq. (3.6)]. The time development is determined by giving the field on $t=0$ and the Cauchy data for the particles. To obtain Eqs. (4.9) we used the splitting of the field into *retarded* and *incoming part*. We see that prescribing the incoming field does not in general lead to a Cauchy problem for the particles (see Fig. 1). We say "in general" because, if it happens that the incoming field is exactly compensated by the retarded field at $t=0$, i.e., the total field vanishes at the initial surface, then, of course, the time development for $t>0$ follows from Eqs. (4.1) or, equivalently, Eqs. (4.2).

V. ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS

We would like to know the behavior of our system for $t\to\infty$. For this it is convenient to start with the Laplace-transformed equations given in Sec. IV. First we discuss the homogeneous equations, i.e., no field at the initial-surface Eqs. (4.1), or no incoming field, Eqs. (4.7). For $t>0$ we have from Eqs. (4.1)

$$Q_\pm(t) = \frac{1}{2\pi i}\int_c ds \frac{e^{st}}{h_\pm(s)}[(s+2\Gamma)Q_\pm(0) + \dot{Q}_\pm(0)] \quad (5.1a)$$

and from Eqs. (4.7)

$$Q_\pm(t) = \frac{1}{2\pi i}\int_c ds\, e^{st}\frac{f_\pm(s)}{h_\pm(s)} , \qquad (5.1b)$$

where $f_\pm(s)$ is the left-hand side (LHS) of Eqs. (4.7),

$$\begin{aligned} f_\pm(s) &= (s+2\Gamma)Q_\pm(0) + \dot{Q}_\pm(0) \\ &\pm\frac{\Gamma}{a} e^{-2as}\int_{-2a}^{0} dt\, e^{-st} Q_\pm(t) . \end{aligned}$$

In order to integrate Eqs. (5.1a) and (5.1b) we have to know the distribution of the (infinitely many) zeros of the characteristic functions $h_\pm(s)$. In Appendix A we prove that the integrals can be expressed as sums of the residues of their integrands.

Thus,

$$Q_\pm(t) = \sum_{i=1}^{\infty} e^{s_i t} p_\pm^i , \quad t>0 , \qquad (5.2)$$

where the s_i satisfy $h_\pm(s_i)=0$, and $e^{s_i t}p_\pm^i$ are the residues of the integrands either in Eqs. (5.1a) or Eqs. (5.1b). (Since $h_\pm$ has in general no multiple zeros, the p^i do not depend on t.) Once we know Eqs. (5.2), the asymptotic behavior is determined by the zero with the largest real part. It is clear that if $h_\pm(s)$ has only zeros for which $\mathrm{Re}\, s_i<0$, then

$$\lim_{t\to\infty} Q_\pm(t) = 0 . \qquad (5.3)$$

Note that this asymptotic behavior then follows for the solutions to both equations (4.1) and (4.8), because they have the same $h_\pm(s)$.

In Appendix B we sketch a proof that there are only zeros of $h_\pm(s)$ with negative real part if and only if $\bar{\omega}^2 > \Gamma/a$. Thus, provided that this condition is fulfilled, Eqs. (5.3) follow. Under the same conditions one shows that also

$$\lim_{t\to\infty}\dot{Q}_\pm(t)=0\,, \tag{5.4}$$

and therefore the system has a dissipative behavior in the sense that the energy initially stored in the oscillators is eventually completely transferred to the field.

In what follows we discuss the asymptotic behavior for the case that the field at the initial surface does *not vanish*. The complete initial-value solution is contained in Eqs. (3.8), and because we are now interested in the time development due to some initial field, let $Q_\pm(0)=\dot{Q}_\pm(0)=0$. Then we have from Eqs.(3.8), by inverse Laplace transformation,

$$Q_\pm(t)=\int_c ds\, e^{st}h_\pm^{-1}(s)\times\int_0^\infty d\xi\, e^{-s\xi}\{\Psi'_a(\xi)+\chi_a(\xi)\pm[\Psi'_{-a}(\xi)+\chi_{-a}(\xi)]\}. \tag{5.5}$$

Expressing the s integration as the sum over the residues (assuming only simple roots) gives

$$Q_\pm(t)=\sum_{i=0}^\infty\int_0^t d\xi\,\frac{e^{s_i(t-\xi)}}{h'_\pm(s_i)}\{\Psi'_a(\xi)+\chi_a(\xi)\pm[\Psi'_{-a}(\xi)+\chi_{-a}(\xi)]\}\,. \tag{5.6}$$

Now, if

$$\lim_{\xi\to\infty}\Psi'_{\pm a}(\xi)=\lim_{\xi\to\infty}\chi_{\pm a}(\xi)=0 \tag{5.7}$$

and $\mathrm{Re}(s_i)<0$, each integral of the sum vanishes for $t\to\infty$. (See Ref. 12, Appendix A).

If we confine ourselves to fields which fall off with $r\to\infty$ fast enough to ensure that the total field energy is finite, which is guaranteed if

$$\begin{aligned}&|\vec{\nabla}\Phi(\vec{x},0)|=O(r^{-3/2-\epsilon}),\quad \epsilon>0\\ &\dot{\Phi}(\vec{x},0)=O(r^{-3/2-\epsilon}),\end{aligned} \tag{5.8}$$

it follows that

$$\lim_{t\to\infty}Q_\pm(t)=\lim_{t\to\infty}\dot{Q}_\pm(t)=0\,. \tag{5.9}$$

We want to remark that, owing to the point character of the sources and the continuity conditions (3.7), the total field energy is always divergent. What we mean here is the field energy, say, outside a sphere of finite radius containing the sources.

It is worth mentioning that, without any time-asymmetric condition on the field, the asymptotic behavior (5.9) follows.

The above argument was given for the initial-value solution. However, an analogous discussion can be given for Eqs. (3.9), i.e., the case where the *incoming field* is prescribed. If one expresses the incoming field in terms of its data on some hypersurface, e.g., $t=0$, and requires a spatial decrease as demanded in Eqs. (5.8), then again Eqs. (5.9) follow.

Now we turn to the case where $\bar{\omega}^2<\Gamma/a$. We know from Appendix B that $h_+(s)$ has then a positive real zero $h_+(\gamma)=0$, $\gamma>0$, and $Q_+(t)=e^{\gamma t}$ is a solution to Eqs. (4.2) and (4.8).

In order to investigate the cause of this instability of our system we look at the total energy.

From the Lagrangian (2.1) one derives the total *conserved* energy of the system

$$E=E_Q+E_\Phi+E_{Q\Phi}\,, \tag{5.10}$$

where

$$\begin{aligned}E_Q&=\tfrac{1}{2}(\dot{Q}_1^{\,2}+\dot{Q}_2^{\,2})+\frac{\omega_0^{\,2}}{2}(Q_1^{\,2}+Q_2^{\,2})\,,\\ E_\Phi&=\tfrac{1}{2}\int d^3x\,[\dot{\Phi}^2+(\vec{\nabla}\Phi)^2]\,,\\ E_{Q\Phi}&=-\lambda\int d^3x\,[\rho(\vec{x}-\vec{a})Q_1(t)+\rho(\vec{x}+\vec{a})Q_2(t)]\Phi(\vec{x},t)\,.\end{aligned} \tag{5.11}$$

In the point limit the energy diverges, but if we split off the singular part of the field by introducing the auxiliary field $\bar{\Phi}$,

$$\Phi(\vec{x},t)=\bar{\Phi}(\vec{x},t)+\frac{\lambda}{4\pi}\frac{Q_1(t)}{|\vec{x}-\vec{a}|}+\frac{\lambda}{4\pi}\frac{Q_2(t)}{|\vec{x}+\vec{a}|}\,, \tag{5.12}$$

and use the divergent terms to renormalize $\omega_0^{\,2}$ to $\bar{\omega}^2$, as it was done in the equations of motion, we finally arrive at

$$E=\tfrac{1}{4}(\dot{Q}_+^{\,2}+\dot{Q}_-^{\,2})+\frac{1}{4}\left(\bar{\omega}^2-\frac{\Gamma}{a}\right)Q_+^{\,2}+\frac{1}{4}\left(\bar{\omega}^2+\frac{\Gamma}{a}\right)Q_-^{\,2}+\int d^3x\,[\dot{\bar{\Phi}}^2+(\vec{\nabla}\bar{\Phi})^2]\,. \tag{5.13}$$

Evidently, if $\bar{\omega}^2<\Gamma/a$, the second term in Eq. (5.13) becomes negative and the total energy cannot be written as a sum of quadratic terms. Thus, despite the fact that the total energy is conserved and the system contains radiation damping, the negative interaction energy may cause an instability of the system. (A similar instability was found in classical meson pair theory.[13,14]) Is this instability physically relevant? If we think of two atoms with electromagnetic interaction, one has for optical transitions $\bar{\omega}\simeq10^{15}$ sec^{-1}, $\Gamma\simeq10^{8}$ sec^{-1}, and a delay time $a\sim10^{-18}$ sec, so that $\Gamma/a\simeq10^{26}$ while $\bar{\omega}^2\simeq10^{30}$, and the condition $\Gamma/a>\bar{\omega}^2$ cannot be satisfied.

VI. EXPLICIT SOLUTION

In order to find the explicit dynamics for the oscillators the equations (5.2) are not convenient because one would need to know the location of the infinitely many zeros of $h(s)$.

Here we will give two alternative methods, with the help of which one can solve Eqs. (4.2) and (4.8) explicitly. Our first way, suitable especially for Eqs. (4.2), consists in starting with the interval $0 \le t \le 2a$, and then going from one interval to the next one, imposing continuity conditions for $Q(t)$ and $\dot{Q}(t)$. The solution for the Nth interval $2a(N-1) \le t < 2aN$ is then given by

$$Q_\pm(t) = e^{-\Gamma t} \sum_{j=1}^{N} \epsilon_\pm^{j-1} Q_\pm^{(j)}(t - 2a(j-1)), \tag{6.1}$$

where

$$\epsilon_\pm = \pm \frac{\Gamma}{a\omega^2} e^{2\Gamma a}, \quad N = \left[\frac{t}{2a}\right] + 1,$$

and $e^{-\Gamma t} Q_\pm^{(1)}(t)$ is the solution of the damped oscillator equation

$$Q_\pm^{(1)}(t) = [\dot{Q}_\pm(0) + \Gamma Q_\pm(0)] \frac{\sin\omega t}{\omega} + Q_\pm(0)\cos\omega t \tag{6.2}$$

with $\omega^2 = \overline{\omega}^2 - \Gamma^2$, while $Q_\pm^{(j)}(t)$ results by folding $(j-1)$ times with the oscillator Green's function, which is represented by the kernel

$$Q_\pm^{(j)}(t) = \omega \int_0^t dt' \sin\omega(t - t') Q_\pm^{(j-1)}(t') . \tag{6.3}$$

The proof can be given by induction: Eqs. (6.1) are true for $N = 1$; assume that $Q_\pm(t)$ is represented as in (6.1) for N replaced by $N-1$, then the solution for the Nth interval is given by

$$Q_\pm(t) = Q_\pm^H(t) + e^{-\Gamma t} \sum_{j=1}^{N-1} \epsilon_\pm^j \int_{2(N-1)a}^{t} dt' \omega \sin\omega(t - t') Q_\pm^{(j)}(t' - 2a(j-1)), \tag{6.4}$$

where $Q_\pm^H(t)$ denote solutions of the "homogeneous equation" (that equation for which the delay term is set equal to zero) in order to make $Q_\pm(t)$ and $\dot{Q}_\pm(t)$ continuous.

$$Q_\pm^H(t) = e^{-\Gamma t} Q_\pm^{(1)}(t) + e^{-\Gamma t} \sum_{j=2}^{N-1} \epsilon_\pm^{j-1} \int_{(j-1)2a}^{(N-1)2a} dt' \omega \sin\omega(t - t') Q_\pm^{(j-1)}(t' - 2a(j-1)); \tag{6.5}$$

adding up gives again Eq. (6.1).

A straightforward manipulation gives the iterates of $\sin\omega t$ ($= S^{(j)}$) and $\cos\omega t$ ($= C^{(j)}$):

$$S^{(j)}(t) = \sum_{k=1}^{j} d_{jk} (t\omega)^{k-1} \sin\omega\left(t - \frac{\pi}{2}(k-1)\right), \tag{6.6a}$$

and clearly

$$C^{(j)}(t) = \frac{1}{\omega} \frac{d}{dt} S^{(j)}(t), \tag{6.6b}$$

$$d_{jk} = \frac{(2j-k-1)!}{(j-k)!\,(j-1)!\,(k-1)!\,2^{2j-k-1}} . \tag{6.6c}$$

Next we turn to the second method, which is more elegant and easier to handle. We start from the Laplace-transformed solution to Eqs. (4.7) i.e., Eqs. (5.1b). We write $h_\pm(s) \equiv g(s) \pm (\Gamma/a) e^{-2as}$ and expand $h_\pm^{-1}(s)$ as a geometrical series

$$h_\pm^{-1}(s) = \sum_{n=0}^{\infty} \left(\pm \frac{\Gamma}{a}\right)^n \frac{e^{-2ans}}{g^{n+1}(s)} . \tag{6.7}$$

The path of integration in Eqs. (5.1b) can be chosen always such that

$$|g(s)| > \left|\frac{\Gamma}{a} e^{-2as}\right| , \tag{6.8}$$

and the series is absolutely convergent. We observe that $g^{-(n+1)}(s)$ corresponds to the n-fold application of the operator

$$\left(\frac{d^2}{dt^2} + 2\Gamma \frac{d}{dt} + \overline{\omega}^2\right)^{-1} \equiv G . \tag{6.9}$$

Let us for the moment introduce the inverse Laplace transform of $g^{-(n+1)}(s) f_\pm(s)$: $(G^n F)(t)$; by a simple change of variables we get from Eqs. (5.1b) and (6.7)

$$\begin{aligned} q_\pm(s) &= \int_0^\infty dt\, e^{-st} Q_\pm(t) \\ &= \sum_{n=0}^{\infty} \left(\pm \frac{\Gamma}{a}\right)^n e^{-2ans} \\ &\quad \times \int_0^\infty dt\, e^{-st} \theta(t - 2an)(G^n F)(t - 2an) . \end{aligned} \tag{6.10}$$

Now taking the Laplace transform yields

$$Q_\pm(t) = \sum_{n=0}^{\infty} \left(\pm \frac{\Gamma}{a}\right)^n \theta(t - 2an) Q_\pm^{(n)}(t - 2an), \tag{6.11}$$

where

$$Q_\pm^{(n)}(t) = \frac{1}{2\pi i} \int_c ds \frac{e^{st} f_\pm(s)}{g^{n+1}(s)} . \tag{6.12}$$

In this way we arranged the terms to get a finite number for finite t.

In Sec. VII we will use Eqs. (6.11) to get explicit

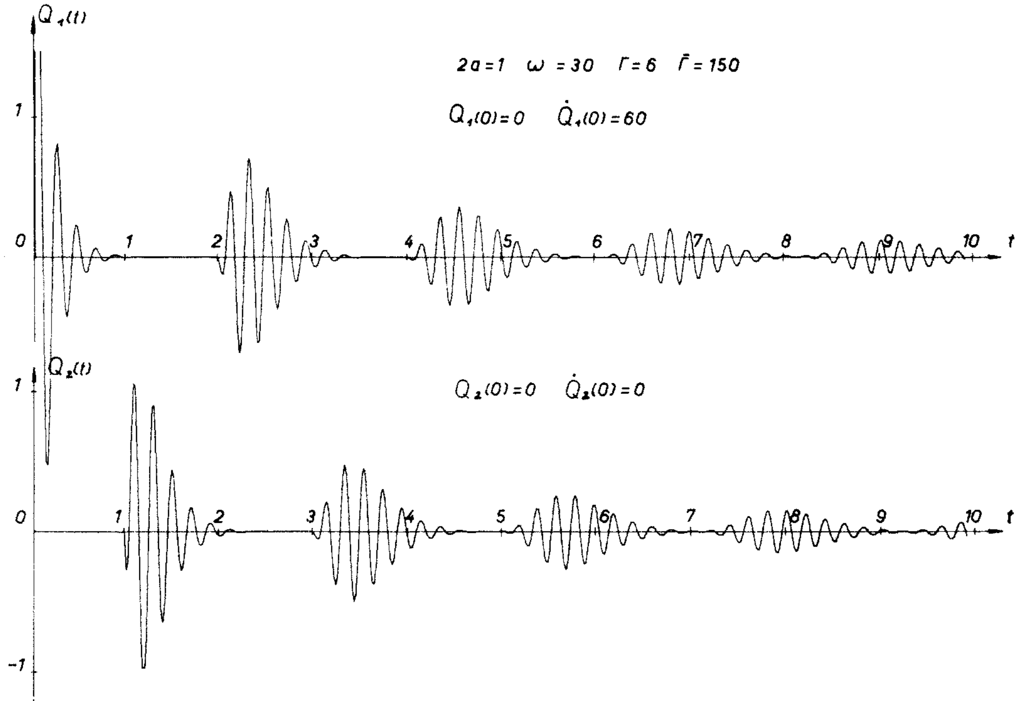

FIG. 2. This figure shows the time development of the oscillators as given by Eqs. (4.2) (no field at the initial surface). The amplitudes are multiplied by a factor $\bar{\Gamma}/\Gamma=25$ after each interval to obtain a better resolution. Note the dissipative behavior due to radiation damping.

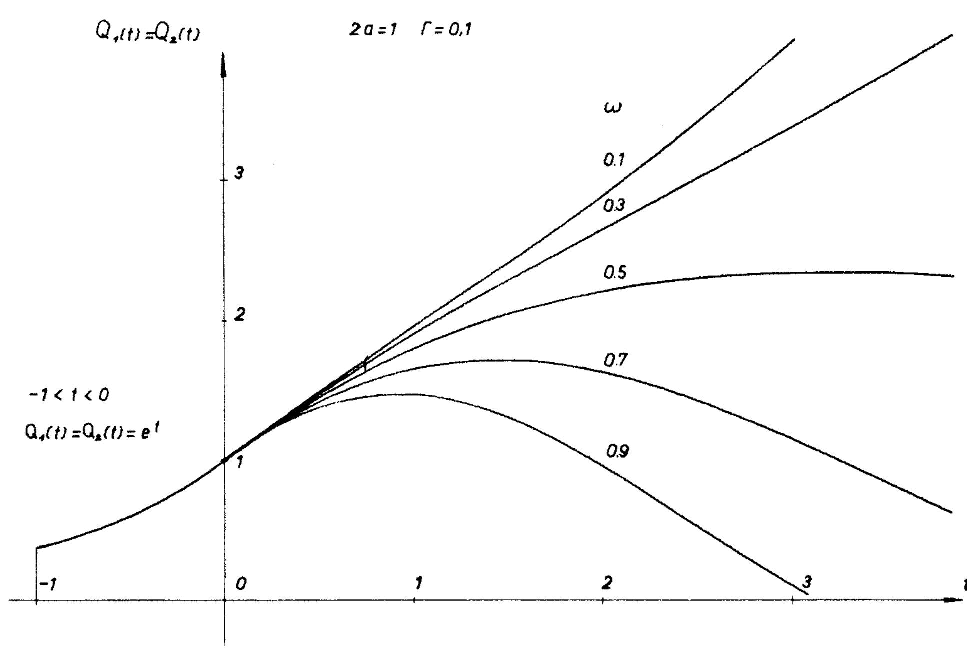

FIG. 3. This figure shows the time development of the oscillators for a purely retarded field, Eqs. (4.8), near the instability limit $\bar{\omega}^2=\omega^2+\Gamma^2=\Gamma/a$. For the chosen values this is at $\omega^2=(0.19)^{1/2}\simeq 0.436$. One sees that for $\omega<0.436$ the solutions tend to rise indefinitely.

solutions. There we will find that in the special case where $Q_\pm(t)=0$ for $-2a\leq t<0$, Eqs. (6.12) give the same answer as (6.6a), (6.6b), (6.6c).

VII. COMPUTER CALCULATIONS

To illustrate our calculations of the previous section, we have worked out three examples which show typical behavior of the oscillators. First we discuss a solution to Eqs. (4.2) so that both oscillators have a free damped time evolution along the first interval, while influencing each other afterwards. The explicit formula has already been given in Eqs. (6.1) and (6.6a), (6.6b), (6.6c). Figure 2 illustrates the time development of Eq. (4.2) for the one oscillator excited while the other is quiet. One sees clearly how the excitation is transferred with a time delay of $2a$ between the oscillators, as well as the dissipative behavior of the system due to radiation damping. To obtain a better graphical resolution we have introduced an enhancement factor for the amplitudes after each interval or, in other words we have changed the coupling between the oscillators $(\Gamma-\overline{\Gamma})$ relative to the self-coupling strength Γ.

Next we turn to solutions of Eq. (4.8) as given by Eq. (6.11). Let us choose first as initial condition an exponential function $Q_+(t)=e^{\gamma t}$, $Q_-(t)=0$ for $-2a\leq t<0$. $f_\pm(s)$ in Eqs. (5.1b) then become

$$f_\pm(s)=(s+2\Gamma)Q_\pm(0)+\dot{Q}_\pm(0)\pm\frac{\Gamma}{a}\,\frac{e^{-2as}-e^{-2a\gamma}}{\gamma-s}. \tag{7.1}$$

So we are left with the problem of integrating the expressions in Eq. (6.11), which is straightforward. The integration of the first two terms from Eq. (7.1) need not be reproduced here, since we naturally get the same answer as before, Eqs. (6.6a), (6.6b), (6.6c). Let us quote the results for the last two integrations:

$$\overline{Q}^{(n)}(t)=\frac{\Gamma}{2\pi i a}\int_c ds\,\frac{e^{st}}{(s-s_0)^n(s-s_0^*)^n}\,\frac{e^{-2as}-e^{-2a\gamma}}{\gamma-s}$$
$$\equiv\frac{\Gamma}{a}[I_1(t)+I_2(t)], \tag{7.2}$$
$$s_0=-\Gamma+i\omega.$$

Care must be taken in closing the path by an appropriate semicircle in the complex s plane:

$$I_2(t)=\theta(t)\left\{\frac{e^{\gamma(t-2a)}}{[(\gamma+\Gamma)^2+\omega^2]^n}-e^{-\Gamma t-2a\gamma}\sum_{k,j}C_{nkj}(t\omega)^{k-j}f_{kj}(t)\right\},$$
$$f_{kj}(t)=\frac{(j-1)!}{[(\gamma+\Gamma)^2+\omega^2]^j}\sum_{\sigma=1}^{j+1}\binom{j}{\sigma-1}(\gamma+\Gamma)^{j-\sigma+1}\omega^{\sigma-1}\sin\left(\omega t-\frac{\pi}{2}k+\frac{\pi}{2}\sigma\right), \tag{7.3}$$
$$C_{nkj}=\frac{(2n-k-1)!}{(n-k)!(n-1)!(j-1)!(k-j)!(2\omega)^{2n-k-1}},$$

and $I_1(t)$ is obtained from above by a simple change of variables $t\to t-2a$ and multiplication by $-e^{2a\gamma}$. With the help of these expressions it is now easy, without any numerical integration, to discuss the behavior of Q_1 and Q_2.

Figure 3 illustrates the instability phenomenon for $\overline{\omega}^2<\Gamma/a$ as discussed in Sec. V. We have chosen the data for both oscillators to be identical within the first interval, which naturally implies equality for all t. For $\omega<(0.19)^{1/2}$ the undamped rising is shown.

Our last example (Fig. 4) shows the propagation of a pulse of length α. The relevant expressions are obtained from the previous one by setting $\gamma=0$ and changing the arguments in the θ functions in Eq. (7.3). It is seen how this pulse is smoothed out; again a scale factor $\overline{\Gamma}/\Gamma$ has been introduced, so as to be able to plot the behavior.

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APPENDIX A

We will show that the integrals in Eqs. (5.1a), (5.1b) can be expressed as an infinite sum, over the residues of the integrands. We show this for Eq. (5.1b). To do so, we transform (4.8) into a system of first-order differential-difference equations by a change of variables (for simplicity we consider only $Q_+\equiv Q$):

$$y_1(t)=Q(t),\quad y_2(t)=\dot{Q}(t),$$
$$A_0\dot{y}(t)+B_0y(t)+B_1y(t-2a)=0,$$
$$A_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad B_0=\begin{pmatrix}0&-1\\\overline{\omega}^2&2\Gamma\end{pmatrix}, \tag{A1}$$
$$B_1=\begin{pmatrix}0&0\\-\Gamma/a&0\end{pmatrix},\quad y=\begin{pmatrix}y_1\\y_2\end{pmatrix}.$$

By Laplace transformation we obtain the solution

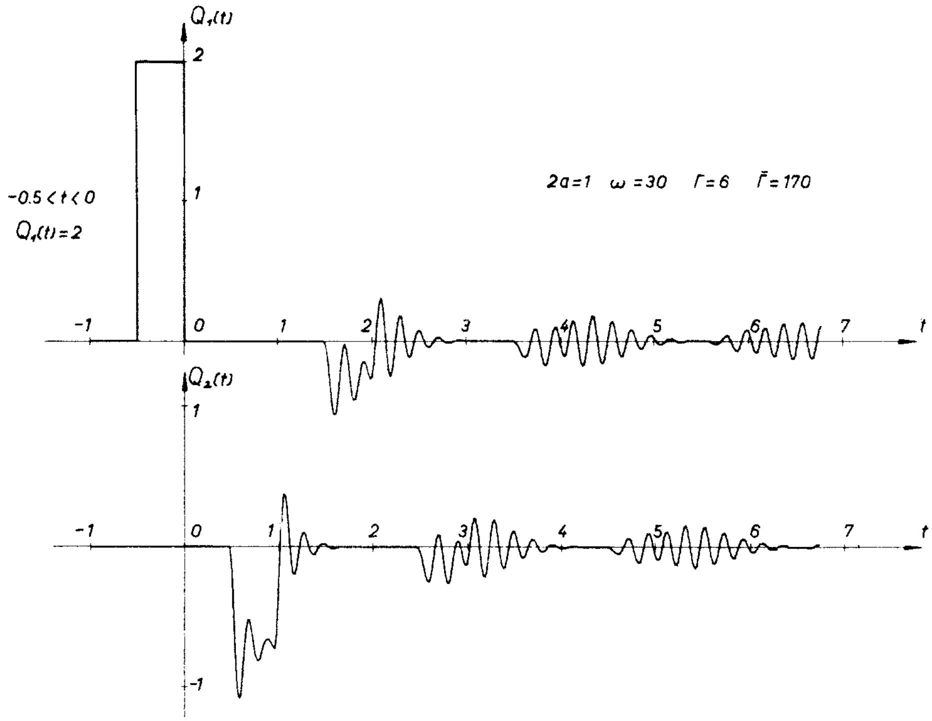

FIG. 4. In this figure the time development due to Eqs. (4.8) of a rectangular pulse is shown. Note that, since the parameters are far from the instability region, dissipative behavior of the oscillators occurs.

to (A1) in the form

$$y(t)=\frac{1}{2\pi i}\int_c ds\, e^{ts}H^{-1}(s)p(s)\,, \quad t>0$$

$$H(s)=\begin{pmatrix} s & -1 \\ \bar{\omega}^2-\frac{\Gamma}{a}e^{-2as} & s+2\Gamma\end{pmatrix}, \tag{A2}$$

$$p(s)=A_0y(0)+(A_0s+B_0)\int_{-2a}^{0}dt\, e^{-st}y(t)\,,$$

which holds for any path parallel to the imaginary axis which exceeds the least upper bound of the real parts of the zeros of $\det H(s)=h(s)$.

Our aim is to show that

$$y(t)=\sum_{i=1}^{\infty}e^{s_i t}p_i\,, \tag{A3}$$

where $e^{s_i t}p_i$ denotes the residue of $e^{st}H^{-1}(s)p(s)$ at the point s_i at which $h(s)$ becomes zero. Naturally, an expression like (A3) is expected, if we get a negligible contribution by closing the path (C) in (A2) at infinity. Since $h(s)$ is an analytic function of s, there are no accumulation points of zeros of $h(s)$ in any finite region. To get information about the zeros for large $|s|$ we follow Ref. 12 and substitute

$$h_1(s)=e^{2as}h(s), \quad z=s+\frac{1}{a}\ln s\,. \tag{A4}$$

$h_1(s)$ is transformed into

$$h_1(s)=e^{2az}[1+w(z)]-\frac{\Gamma}{a}\,, \tag{A5}$$

where the nondominant terms are collected into $w(z)$. Here it is seen that asymptotically

$$2az=\ln\left(\frac{\Gamma}{a}\right)+O(1)\,. \tag{A6}$$

Taking the real and imaginary parts of (A6), we obtain, since $\arg s$ goes to $\pm\pi/2$, for $x=\mathrm{Re}\,s$, $y=\mathrm{Im}\,s$,

$$\begin{aligned} x&=\frac{1}{2a}\left(\ln\left|\frac{\Gamma}{a}\right|-\ln\left|2k\pi\mp\frac{\pi}{2}\right|\frac{1}{a}\right)+O(1)\,,\\ y&=\frac{1}{2a}\left(2k\pi\mp\frac{\pi}{2}\right)+O(1)\,,\end{aligned} \tag{A7}$$

where k denotes a positive integer.

So it is seen that all zeros of sufficiently large $|s|$ lie within a strip defined by a constant K depending on Γ and a,

$$-K<\mathrm{Re}\,s+\frac{1}{a}\ln|s|<K\,. \tag{A8}$$

Furthermore they are asymptotically spaced a distance π/a apart, so that there exists a sequence of closed contours C_l in the complex s plane with the properties that they contain the origin, they never cross each other, and they have a distance $d>0$ from the set of all zeros of $h(s)$; furthermore, between C_l and C_{l+1} exactly one zero of $h(s)$ is included. A convenient choice is given by circles with radius $|s_i|$, where s_i is a zero of $h(s)$, if one modifies these circles in the neighborhood of the zeros by small half circles surrounding the zero.

If we now denote by C_{l-} those parts of C_l which are to the left of $\mathrm{Re}\,s=C$, we get

$$y(t)=\lim_{l\to\infty}\sum_i e^{s_i t}p_i(t)+\lim_{l\to\infty}\int_{C_{l-}} ds\, e^{ts}H^{-1}(s)p(s), \qquad \text{(A11)}$$

where the sum in the first expression runs over all zeros within C_l. In the case where $C<0$, a study of the order of magnitude of $H^{-1}(s)$ shows that

$$\lim_{l\to\infty}\int_{C_{l-}} ds\, e^{st}H^{-1}(s)=0, \qquad t>0. \qquad \text{(A12)}$$

Using (A12) and the explicit form of $p(s)$ from (A2) shows that the last expression goes to zero for all $t>0$. Since by construction there is exactly one zero between C_l and C_{l+1}, the first term becomes exactly (A3).

In the special case of the existence of a zero with $\mathrm{Re}\,s_i\geq 0$ one has to shift first the path (C) to the left and to take out the contributions from that zero.

APPENDIX B

We prove that there are no zeros of $h_\pm(s)$ with positive real part if and only if $\bar{\omega}^2>\Gamma/a$. We set $2a=1$:

$$h_\pm(s)=s^2+2\Gamma s+\bar{\omega}^2\mp 2\Gamma e^{-s}.$$

The number of zeros (n) of a regular analytic function $f(s)$ inside a domain with closed contour C is given by

$$n=\frac{1}{2\pi i}\int_C\frac{f'(s)}{f(s)}ds=\frac{1}{2\pi}\int_C \arg[f(s)].$$

We make use of this theorem and apply it to $h_\pm(s)$, taking for the integration path C the imaginary axis of the complex s plan and an infinite semicircle in the right half plane (the sense of integration is anticlockwise).

If we write $s=re^{i\theta}$, we see that for large r and $-\pi/2<\theta<\pi/2$

$$h_\pm(re^{i\theta})=r^2e^{i2\theta}\left[1+O\left(\frac{1}{r}\right)\right];$$

thus when θ varies from $-\pi/2$ to $\pi/2$, the argument of $h_\pm$ varies by 2π along the infinite semicircle. Now we have to know how $\arg h_\pm(s)$ varies along the imaginary axis. We set $s=iy$; then

$$\tan\arg[h_\pm(iy)]=\frac{2\Gamma(y\pm\sin y)}{g_\pm(y)}, \qquad \text{(B1)}$$

where $g_\pm(y)=-y^2+\bar{\omega}^2\mp 2\Gamma\cos y$.

First we note that $y\pm\sin y>0$ for $y>0$ and that (B1) is antisymmetric in y, so we restrict ourselves to $y>0$. Since at $y=+\infty$ and $y=0$ the tan is zero, we have to know how often the argument of $h_\pm$ goes through $\pi/2$. This will be the case if

$$g_\pm(y)=0 \qquad \text{(B2)}$$

has *real* simple roots.

By graphical inspection one finds that $g_-(y)=0$ has always, and $g_+(s)$ for $\bar{\omega}^2>2\Gamma$, an odd number of real solutions (double roots are counted as two). For $\bar{\omega}^2<2\Gamma$, $g_+(y)$ has *no* or an *even* number of real zeros.

If the number of zeros is odd, it is easy to show that $\arg h_\pm(s)$ varies by $-\pi$, if y varies from $+\infty$ to 0, therefore by -2π along the whole axis. Thus, we conclude $\arg h_\pm(s)$ is not changed along C. Therefore there are no zeros of $h_\pm(s)=0$ with positive real part if $\bar{\omega}^2>2\Gamma$.

For $\bar{\omega}^2<2\Gamma$ one has no or an even number of zeros. No zeros means that $\arg h_+(s)$ does not change along the imaginary axis because $\tan[\arg h_+(iy)]$ does not go through $\pi/2$. Further it is easy to show that an even number of zeros is equivalent to no zeros of $g_+(y)$. In this case $\arg h_+(s)$ varies by 2π and thus there exists *one* root of $h_+(s)=0$ with positive real part. Moreover it is trivial to show that this root is *real*. Similarly one shows that there are no zeros of $h_\pm$ with $\mathrm{Re}\,s_i=0$, unless $\bar{\omega}^2=2\Gamma$.

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