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Exactly solvable 'discrete' quantum mechanics; shape invariance, Heisenberg solutions, annihilation-creation operators and coherent states

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Abstract

Various examples of exactly solvable 'discrete' quantum mechanics are explored explicitly with emphasis on shape invariance, Heisenberg operator solutions, annihilation-creation operators, the dynamical symmetry algebras and coherent states. The eigenfunctions are the (q-)Askey-scheme of hypergeometric orthogonal polynomials satisfying difference equation versions of the Schrödinger equation. Various reductions (restrictions) of the symmetry algebra of the Askey-Wilson system are explored in detail.

1 Introduction

General theory of exactly solvable 'discrete' quantum mechanics of one degree of freedom systems is presented with all known examples. The 'discrete' quantum mechanics is a simple extension or deformation of quantum mechanics in which the momentum operator p appears in the Hamiltonian in the exponentiated forms $e^{\pm\gamma p}$, $\gamma \in \mathbb{R}$, in stead of polynomials in ordinary quantum mechanics. The corresponding Schrödinger equations are difference equations with imaginary shifts, in stead of differential. The eigenfunctions of the exactly solvable 'discrete' quantum mechanics of one degree of freedom systems consist of the (q-)Askey-scheme of hypergeometric orthogonal polynomials [1, 2], which are deformations of the classical orthogonal polynomials, like the Hermite, Laguerre, Jacobi polynomials, etc [3], constituting the eigenfunctions of exactly solvable ordinary quantum mechanics [4, 5]. These eigenpolynomials are orthogonal with respect to absolutely continuous measure functions, which are just the square of the ground state wavefunctions; a familiar situation in quantum mechanics. For another type of orthogonal polynomials with discrete measures [1, 2, 6], see [7] for a unified theory. Like most exactly solvable quantum mechanics, every example of exactly solvable 'discrete' quantum mechanics is endowed with dynamical symmetry, *shape invariance* [8], which allows to determine the entire energy spectrum and the corresponding eigenfunctions when combined with Crum's theorem [9] or the factorisation method [4, 5]. In other words, shape invariance guarantees exact solvability in the Schrödinger picture [10, 11, 12]. As expected, exact solvability in the Heisenberg picture also holds for all these examples. The explicit forms of Heisenberg operator solutions give rise to the explicit expressions of annihilation/creation operators as the positive/negative frequency parts [13]. The annihilation/creation operators together with the Hamiltonian constitute the dynamical symmetry algebra. In some cases, the algebras are simple and tangible, like the oscillator algebra and its q-deformations [14], or $\mathfrak{su}(1,1)$.

The present paper is to supplement or to complete some results in previous publications [10, 11, 12, 13]. The 'discrete' quantum mechanics of the Meixner-Pollaczek, the continuous Hahn, the continuous dual Hahn, the Wilson and the Askey-Wilson polynomials discussed in [10, 11, 12, 13] are only for restricted parameter ranges; for example the angle was $\phi = \pi/2$ for the Meixner-Pollaczek polynomial and all the parameters were restricted real for the continuous Hahn, the continuous dual Hahn, the Wilson and the Askey-Wilson polynomials. This is due to a historical reason that these polynomials with the restricted parameter ranges were first recognised by the present authors as describing the classical equilibrium positions [15, 16, 10, 11, 12, 17] of multi-particle exactly solvable dynamical systems of Ruijsenaars-Schneider-van Diejen type [18, 19]. It is a deformation of the classical results dating as far back as Stieltjes [20], [21, 22] that the classical equilibrium positions of multi-particle exactly solvable dynamical systems of Calogero-Sutherland type [23, 24] are described by the zeros of the classical orthogonal polynomials (the Hermite, Laguerre and Jacobi). The 'discrete' quantum mechanics was constructed [10, 11, 12] based on the analogy that these orthogonal polynomials would constitute the eigenfunctions of certain quantum mechanical systems in the same way as the classical orthogonal polynomials (the Hermite, Laguerre and Jacobi) do. As will be shown in detail in the main text, these orthogonal polynomials enjoy the exact solvability and related properties for the full ranges of the parameters. Attempts to further deform these exactly solvable systems have yielded several examples [25, 26, 27] of the so-called quasi-exactly solvable systems [28, 29]. Another objective of the present paper is to explore in detail the properties of the systems obtained by restricting the Askey-Wilson system, treated in §5.2–§5.8.2. Some of these have interesting and useful forms of the dynamical symmetry algebras or the explicit forms of coherent state, etc, as evidenced by the q-oscillator algebras realised by the continuous (big) q-Hermite polynomial [14]. Aspects of ordinary theory of orthogonal polynomials are not particularly emphasised.

This paper is organised as follows. In section two, the general setting of the 'discrete' quantum mechanics is recapitulated with appropriate notation. Starting with the parameters in the potential function and the Hamiltonian, various concepts and solution methods are briefly surveyed. Sections three to five are the main body of the paper, discussing various examples of exactly solvable 'discrete' quantum mechanics. They are divided into three groups according to the sinusoidal coordinate $\eta(x)$. Section three is for the polynomials in $\eta(x) = x$. Section four is for the polynomials in $\eta(x) = x^2$. Section five is for the polynomials in $\eta(x) = \cos x$. Very roughly speaking, polynomials in section three are the deformation of the Hermite polynomial; those in section four are the deformation of the Laguerre polynomial and those in sections five are the deformation of the Jacobi polynomial from the point of view of the sinusoidal coordinates, but not from the energy spectrum. Section six is for a summary and comments. Appendix A provides a diagrammatic proof of the hermiticity (self-adjointness) of the Hamiltonians of 'discrete' quantum mechanics. Appendix B is a collection of the definition of basic symbols and functions used in this paper for self-containedness.

2 General setting

The dynamical variables are the coordinate x ($x \in \mathbb{R}$) and the conjugate momentum p, which is realised as a differential operator $p = -id/dx$. The other parameters are symbolically denoted as $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$ on top of q ($0 < q < 1$) and ϕ ($\phi \in \mathbb{R}$). For the q-systems, the parameters are denoted as $q^{\boldsymbol{\lambda}} = (q^{\lambda_1}, q^{\lambda_2}, \ldots)$. Complex conjugation is denoted by

* and the absolute value $|f(x)|$ is $|f(x)| = \sqrt{f(x)f(x)^*}$. Here $f(x)^*$ means $(f(x))^*$ and $f(x)^*|_{x\to x+a} = f(x+a^*)^*$, since x is real.

Hamiltonian The Hamiltonian has a general form

$$\mathcal{H} \stackrel{\text{def}}{=} \sqrt{V(x)}\, e^{\gamma p} \sqrt{V(x)^*} + \sqrt{V(x)^*}\, e^{-\gamma p} \sqrt{V(x)} - V(x) - V(x)^*, \tag{2.1}$$

in which γ is a real constant. It is either 1 or $\log q$. The potential function V depends on the parameters, $V(x) = V(x\,;\boldsymbol{\lambda})$, whereas the q and ϕ dependence is not explicitly indicated. The parameter dependence of the Hamiltonian $\mathcal{H} = \mathcal{H}(\boldsymbol{\lambda})$ is not explicitly indicated in most cases.

The eigenvalue problem or the time-independent Schrödinger equation is a difference equation in stead of differential in ordinary quantum mechanics:

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x) \quad (n = 0, 1, 2, \ldots), \quad \mathcal{E}_0 < \mathcal{E}_1 < \mathcal{E}_2 < \cdots\,, \tag{2.2}$$

in which $\phi_n(x) = \phi_n(x\,;\boldsymbol{\lambda})$ is the eigenfunction belonging to the energy eigenvalue $\mathcal{E}_n = \mathcal{E}_n(\boldsymbol{\lambda})$. The difference equation has inherent non-uniqueness of solutions; if $\phi(x)$ is a solution so is $\phi(x)Q(x)$ when $Q(x)$ is any periodic function with the period $i\gamma$. This non-uniqueness problem is resolved when the Hilbert space of the state vectors is specified. See Appendix A.

Factorisation Factorisation of the Hamiltonian is an important property

$$\mathcal{H} = T_+ + T_- - V(x) - V(x)^* = (S_+^\dagger - S_-^\dagger)(S_+ - S_-) = \mathcal{A}^\dagger\mathcal{A}, \tag{2.3}$$

in which various quantities $S_\pm = S_\pm(\boldsymbol{\lambda})$, $T_\pm = T_\pm(\boldsymbol{\lambda})$, $\mathcal{A} = \mathcal{A}(\boldsymbol{\lambda})$ are defined as († denote the hermitian conjugation with respect to the chosen inner product (2.75) and (A.1)–(A.3)):

$$S_+ \stackrel{\text{def}}{=} e^{\gamma p/2}\sqrt{V(x)^*},\ \ S_- \stackrel{\text{def}}{=} e^{-\gamma p/2}\sqrt{V(x)},\ \ S_+^\dagger \stackrel{\text{def}}{=} \sqrt{V(x)}\, e^{\gamma p/2},\ \ S_-^\dagger \stackrel{\text{def}}{=} \sqrt{V(x)^*}\, e^{-\gamma p/2}, \tag{2.4}$$

$$T_+ \stackrel{\text{def}}{=} S_+^\dagger S_+ = \sqrt{V(x)}\, e^{\gamma p}\sqrt{V(x)^*}, \quad T_- \stackrel{\text{def}}{=} S_-^\dagger S_- = \sqrt{V(x)^*}\, e^{-\gamma p}\sqrt{V(x)}, \tag{2.5}$$

$$\mathcal{A} \stackrel{\text{def}}{=} i(S_+ - S_-), \quad \mathcal{A}^\dagger \stackrel{\text{def}}{=} -i(S_+^\dagger - S_-^\dagger). \tag{2.6}$$

Ground state wavefunction The ground state wavefunction $\phi_0(x) = \phi_0(x\,;\boldsymbol{\lambda})$ is annihilated by the $\mathcal{A}$ operator

$$\mathcal{A}\phi_0(x) = 0 \ \Rightarrow \mathcal{H}\phi_0(x) = 0 \ \Rightarrow \mathcal{E}_0 = 0, \tag{2.7}$$

which is a zero mode of the Hamiltonian. The above equation reads explicitly as

$$\sqrt{V(x+\tfrac{i\gamma}{2})^*}\,\phi_0(x-\tfrac{i\gamma}{2})=\sqrt{V(x+\tfrac{i\gamma}{2})}\,\phi_0(x+\tfrac{i\gamma}{2}). \tag{2.8}$$

Among possible solutions, we choose a real and nodeless ϕ_0. As will be shown in Appendix A, the requirement of the hermiticity (self-adjointness) of the Hamiltonian $\mathcal{H}$ selects a unique solution ϕ_0, which is given explicitly in each subsection (3.10), (3.25), (4.7), (4.23), (5.11), (5.38), (5.59), (5.79), (5.97), (5.127) and (5.146).

Similarity transformed Hamiltonian The similarity transformed Hamiltonian $\widetilde{\mathcal{H}}=\widetilde{\mathcal{H}}(\boldsymbol{\lambda})$ in terms of the ground state wavefunction ϕ_0 (2.8) is

$$\begin{aligned}\widetilde{\mathcal{H}}&\stackrel{\text{def}}{=}\phi_0(x)^{-1}\circ\mathcal{H}\circ\phi_0(x)=\widetilde{T}_++\widetilde{T}_--V(x)-V(x)^*\\&=V(x)\,e^{\gamma p}+V(x)^*\,e^{-\gamma p}-V(x)-V(x)^*,\end{aligned} \tag{2.9}$$

in which $\widetilde{T}_\pm$ are defined as

$$\widetilde{T}_+\stackrel{\text{def}}{=}\phi_0(x)^{-1}\circ T_+\circ\phi_0(x)=V(x)\,e^{\gamma p},\quad \widetilde{T}_-\stackrel{\text{def}}{=}\phi_0(x)^{-1}\circ T_-\circ\phi_0(x)=V(x)^*\,e^{-\gamma p}. \tag{2.10}$$

It acts on the polynomial part of the eigenfunction. Let us write the excited state eigenfunction $\phi_n(x)=\phi_n(x\,;\boldsymbol{\lambda})$ as

$$\phi_n(x\,;\boldsymbol{\lambda})=\phi_0(x\,;\boldsymbol{\lambda})P_n(\eta(x)\,;\boldsymbol{\lambda}), \tag{2.11}$$

in which $P_n(\eta)=P_n(\eta\,;\boldsymbol{\lambda})$ is a polynomial in the *sinusoidal coordinate* $\eta(x)$ [13]. Here $\eta(x)$ is a real function of x. The sinusoidal coordinate $\eta(x)$ discussed in this paper has no $\boldsymbol{\lambda}$-dependence in contrast to the cases studied in [7]. Then $\widetilde{\mathcal{H}}$ acts on $P_n(\eta)$:

$$\widetilde{\mathcal{H}}(\boldsymbol{\lambda})P_n(\eta(x)\,;\boldsymbol{\lambda})=\mathcal{E}_n(\boldsymbol{\lambda})P_n(\eta(x)\,;\boldsymbol{\lambda}). \tag{2.12}$$

For all the examples discussed in this paper, $\widetilde{\mathcal{H}}$ is *lower triangular* in the special basis

$$1,\ \eta(x),\ \eta(x)^2,\ \ldots,\eta(x)^n,\ \ldots, \tag{2.13}$$

spanned by the sinusoidal coordinate $\eta(x)$ $(\eta(x)=x,x^2,\cos x)$ [10, 11, 12, 13]:

$$\widetilde{\mathcal{H}}(\boldsymbol{\lambda})\eta(x)^n=\mathcal{E}_n(\boldsymbol{\lambda})\eta(x)^n+\text{lower orders in }\eta(x). \tag{2.14}$$

Shape invariance The factorised Hamiltonian (2.3) has the dynamical symmetry called *shape invariance* [8] if the following relation holds:

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger = \kappa\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}), \tag{2.15}$$

in which κ is a real positive parameter and $\boldsymbol{\delta}$ denotes the shift of the parameters and $\mathcal{E}_1(\boldsymbol{\lambda})$ is the eigenvalue of the first excited state. This relation is satisfied by all the examples discussed in this paper. Shape invariance means that the original Hamiltonian $\mathcal{H}(\boldsymbol{\lambda})$ and the *associated* Hamiltonian $\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger$ in Crum's [9] sense (or the susy partner Hamiltonian in the so-called supersymmetric quantum mechanics [4, 5]) have the same *shape* up to a multiplicative factor κ and an additive constant $\mathcal{E}_1(\boldsymbol{\lambda})$. In terms of the potential function $V(x\,;\boldsymbol{\lambda})$, the above relation reads explicitly as

$$V(x-\tfrac{i\gamma}{2}\,;\boldsymbol{\lambda})V(x+\tfrac{i\gamma}{2}\,;\boldsymbol{\lambda})^* = \kappa^2\,V(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})V(x+i\gamma\,;\boldsymbol{\lambda}+\boldsymbol{\delta})^*, \tag{2.16}$$

$$V(x+\tfrac{i\gamma}{2}\,;\boldsymbol{\lambda}) + V(x+\tfrac{i\gamma}{2}\,;\boldsymbol{\lambda})^* = \kappa\bigl(V(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta}) + V(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})^*\bigr) - \mathcal{E}_1(\boldsymbol{\lambda}). \tag{2.17}$$

Among many consequences of shape invariance, we list the most salient ones. All the eigenvalues are generated by $\mathcal{E}_1(\boldsymbol{\lambda})$ and the corresponding eigenfunctions are generated from the known form of the ground state eigenfunction ϕ_0 (2.7) together with the multiple action of the successive $\mathcal{A}^\dagger$ operator [10, 11, 12]:

$$\mathcal{E}_n(\boldsymbol{\lambda}) = \sum_{s=0}^{n-1}\kappa^s\mathcal{E}_1(\boldsymbol{\lambda}+s\boldsymbol{\delta}), \tag{2.18}$$

$$\phi_n(x\,;\boldsymbol{\lambda}) \propto \mathcal{A}(\boldsymbol{\lambda})^\dagger\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\mathcal{A}(\boldsymbol{\lambda}+2\boldsymbol{\delta})^\dagger\cdots\mathcal{A}(\boldsymbol{\lambda}+(n-1)\boldsymbol{\delta})^\dagger\phi_0(x\,;\boldsymbol{\lambda}+n\boldsymbol{\delta}). \tag{2.19}$$

The latter is related to a Rodrigues type formula for the eigenpolynomials. We illustrate the shape invariance and Crum's scheme in Fig.1 at the end of this section. The Hilbert space belonging to the Hamiltonian $\mathcal{H}(\boldsymbol{\lambda})$ is denoted as $\mathsf{H}_{\boldsymbol{\lambda}}$.

Closure relation Another important symmetry concept of exactly solvable quantum mechanics is the *closure relation* [13, 7]:

$$[\mathcal{H},[\mathcal{H},\eta]\,] = \eta\,R_0(\mathcal{H}) + [\mathcal{H},\eta]\,R_1(\mathcal{H}) + R_{-1}(\mathcal{H}). \tag{2.20}$$

Here $\eta(x)$ is the sinusoidal coordinate and $R_i(\mathcal{H})$ is a polynomial in $\mathcal{H}$. At the classical mechanics level, it is easy to see that the closure relation means that $\eta(x)$ undergoes a sinusoidal motion with frequency $\sqrt{\mathcal{R}_0(\mathcal{E})}$. The closure relation (2.20) is satisfied by all the examples discussed in this paper and the explicit forms of $R_i(\mathcal{H})$, $i=-1,0,1$ and $\mathcal{E}_n(\boldsymbol{\lambda})$ are given in each subsection. The closure relation (2.20) enables us to express any multiple commutator $[\mathcal{H},[\mathcal{H},\cdots,[\mathcal{H},\eta(x)]\cdots]]$ as a linear combination of the operators $\eta(x)$ and $[\mathcal{H},\eta(x)]$ with coefficients depending on the Hamiltonian $\mathcal{H}$ only. As we will see shortly, the exact Heisenberg operator solution and the annihilation/creation operators are obtained as a consequence [13, 7].

Let us consider the closure relation (2.20) as an algebraic constraint on $\eta(x)$ and the Hamiltonian, for given constants $\{r_i^{(j)}\}$. The l.h.s. consists of $e^{2\gamma p}$, $e^{\gamma p}$, 1, $e^{-\gamma p}$, $e^{-2\gamma p}$, then R_i can be parametrised as

$$R_0(y)=r_0^{(2)}y^2+r_0^{(1)}y+r_0^{(0)},\quad R_1(y)=r_1^{(1)}y+r_1^{(0)},\quad R_{-1}(y)=r_{-1}^{(2)}y^2+r_{-1}^{(1)}y+r_{-1}^{(0)}. \tag{2.21}$$

The similarity transformation of (2.20)

$$[\widetilde{\mathcal{H}},[\widetilde{\mathcal{H}},\eta]\,]=\eta\,R_0(\widetilde{\mathcal{H}})+[\widetilde{\mathcal{H}},\eta]\,R_1(\widetilde{\mathcal{H}})+R_{-1}(\widetilde{\mathcal{H}}) \tag{2.22}$$

gives rise to the following five conditions:

$$\eta(x-2i\gamma)-2\eta(x-i\gamma)+\eta(x)=r_0^{(2)}\eta(x)+r_{-1}^{(2)}+r_1^{(1)}\big(\eta(x-i\gamma)-\eta(x)\big), \tag{2.23}$$

$$\eta(x+2i\gamma)-2\eta(x+i\gamma)+\eta(x)=r_0^{(2)}\eta(x)+r_{-1}^{(2)}+r_1^{(1)}\big(\eta(x+i\gamma)-\eta(x)\big), \tag{2.24}$$

$$\begin{aligned}&\big(\eta(x-i\gamma)-\eta(x)\big)\big(V(x-i\gamma)+V(x+i\gamma)^*-V(x)-V(x)^*\big)\\&\quad=-\big(r_0^{(2)}\eta(x)+r_{-1}^{(2)}\big)\big(V(x-i\gamma)+V(x+i\gamma)^*+V(x)+V(x)^*\big)\\&\qquad-r_1^{(1)}\big(\eta(x-i\gamma)-\eta(x)\big)\big(V(x-i\gamma)+V(x+i\gamma)^*\big)\\&\qquad+r_0^{(1)}\eta(x)+r_{-1}^{(1)}+r_1^{(0)}\big(\eta(x-i\gamma)-\eta(x)\big),\end{aligned} \tag{2.25}$$

$$\begin{aligned}&\big(\eta(x+i\gamma)-\eta(x)\big)\big(V(x-i\gamma)^*+V(x+i\gamma)-V(x)^*-V(x)\big)\\&\quad=-\big(r_0^{(2)}\eta(x)+r_{-1}^{(2)}\big)\big(V(x-i\gamma)^*+V(x+i\gamma)+V(x)^*+V(x)\big)\\&\qquad-r_1^{(1)}\big(\eta(x+i\gamma)-\eta(x)\big)\big(V(x-i\gamma)^*+V(x+i\gamma)\big)\\&\qquad+r_0^{(1)}\eta(x)+r_{-1}^{(1)}+r_1^{(0)}\big(\eta(x+i\gamma)-\eta(x)\big),\end{aligned} \tag{2.26}$$

$$\begin{aligned}&2\big(\eta(x)-\eta(x-i\gamma)\big)V(x)V(x+i\gamma)^*+2\big(\eta(x)-\eta(x+i\gamma)\big)V(x)^*V(x+i\gamma)\\&\quad=\big(r_0^{(2)}\eta(x)+r_{-1}^{(2)}\big)\big(V(x)V(x+i\gamma)^*+V(x)^*V(x+i\gamma)+\big(V(x)+V(x)^*\big)^2\big)\\&\qquad+r_1^{(1)}\big(\eta(x-i\gamma)-\eta(x)\big)V(x)V(x+i\gamma)^*+r_1^{(1)}\big(\eta(x+i\gamma)-\eta(x)\big)V(x)^*V(x+i\gamma)\\&\qquad-\big(r_0^{(1)}\eta(x)+r_{-1}^{(1)}\big)\big(V(x)+V(x)^*\big)+r_0^{(0)}\eta(x)+r_{-1}^{(0)}.\end{aligned} \tag{2.27}$$

For real $\{r_i^{(j)}\}$ (this is indeed the case for all the examples discussed in this paper), (2.24) and (2.26) are the complex conjugate of (2.23) and (2.25), respectively.

In contrast to the cases of the orthogonal polynomials with discrete measures discussed in section 4 of [7], the determination of $\eta(x)$ and the possible forms of $V(x)$ is not straightforward due to the ambiguities of periodic functions with $i\gamma$ period. Here we mention only the basic results. It is easy to see that (2.23)–(2.26) require $r_0^{(2)} = r_1^{(1)}$ and $r_0^{(1)} = 2r_1^{(0)}$, which is consistent with the hermitian conjugation of (2.20). With these constraints, the first condition (2.23) reads with $x \to x + i\gamma$

$$\eta(x-i\gamma) - (2+r_1^{(1)})\eta(x) + \eta(x+i\gamma) = r_{-1}^{(2)}. \tag{2.28}$$

Following the arguments given in section 4 and appendix A of [7], we deduce from (2.25) and (2.27) the general relationship

$$\begin{aligned}
&\big(\eta(x-i\gamma)-\eta(x)\big)\big(\eta(x+i\gamma)-\eta(x)\big)\,\big(V(x)+V(x)^*\big)\\
&\quad = -r_1^{(0)}\eta(x)^2 - r_{-1}^{(1)}\eta(x) - C_1(x),
\end{aligned} \tag{2.29}$$

$$\begin{aligned}
&\big(\eta(x-2i\gamma)-\eta(x)\big)\big(\eta(x-i\gamma)-\eta(x+i\gamma)\big)V(x)V(x+i\gamma)^*\\
&\quad = \frac{\big(r_1^{(0)}\eta(x-i\gamma)\eta(x) + r_{-1}^{(1)}\eta(x-i\gamma) + C_1(x)\big)\big(r_1^{(0)}\eta(x-i\gamma)\eta(x) + r_{-1}^{(1)}\eta(x) + C_1(x)\big)}{\big(\eta(x-i\gamma)-\eta(x)\big)^2}\\
&\qquad - r_0^{(0)}\eta(x-i\gamma)\eta(x) - r_{-1}^{(0)}\big(\eta(x-i\gamma)+\eta(x)\big) + C_2(x),
\end{aligned} \tag{2.30}$$

in which $C_j(x)$ $(j=1,2)$ is an arbitrary function satisfying the periodicity $C_j(x+i\gamma) = C_j(x)$. The hermiticity of the Hamiltonian $\mathcal{H}$ would restrict $C_j(x)$ severely. Further analysis of the closure relation (2.23)–(2.27) will be published elsewhere.

Like the cases of discrete measures [7], the dual closure relation

$$[\eta,[\eta,\mathcal{H}]\,] = \mathcal{H}\,R_0^{\text{dual}}(\eta) + [\eta,\mathcal{H}]\,R_1^{\text{dual}}(\eta) + R_{-1}^{\text{dual}}(\eta) \tag{2.31}$$

holds and R_i^{dual} are given by

$$R_1^{\text{dual}}(\eta(x)) = \big(\eta(x-i\gamma)-\eta(x)\big) + \big(\eta(x+i\gamma)-\eta(x)\big), \tag{2.32}$$

$$R_0^{\text{dual}}(\eta(x)) = -\big(\eta(x-i\gamma)-\eta(x)\big)\big(\eta(x+i\gamma)-\eta(x)\big), \tag{2.33}$$

$$R_{-1}^{\text{dual}}(\eta(x)) = \big(V(x)+V(x)^*\big)R_0^{\text{dual}}(\eta(x)). \tag{2.34}$$

Eqs. (2.28) and (2.29) imply $R_1^{\text{dual}}(y) = r_1^{(1)}y + r_{-1}^{(2)}$ and $R_{-1}^{\text{dual}}(\eta(x)) = r_1^{(0)}\eta(x)^2 + r_{-1}^{(1)}\eta(x) + C_1(x)$.

Auxiliary function φ In all the examples discussed in this paper, the ground state wavefunction with shifted x and parameters $\phi_0(x-\frac{i\gamma}{2}\,;\boldsymbol{\lambda}+\boldsymbol{\delta})$ is related to its original value $\phi_0(x\,;\boldsymbol{\lambda})$ via a real auxiliary function φ:

$$\phi_0(x-\tfrac{i\gamma}{2}\,;\boldsymbol{\lambda}+\boldsymbol{\delta})=\sqrt{V(x\,;\boldsymbol{\lambda})}\,\varphi(x-\tfrac{i\gamma}{2})\phi_0(x\,;\boldsymbol{\lambda}). \tag{2.35}$$

The auxiliary function $\varphi(x)$ discussed in this paper has no $\boldsymbol{\lambda}$-dependence in contrast to the cases studied in [7]. It is easy to see that (2.35) implies (2.8). The explicit forms of $\varphi(x)$ are given at the beginning of each section (3.1), (4.1), (5.1).

'Similarity' transformation II 'Similarity' transformed Hamiltonian or that of $S_\pm$, $S_\pm^\dagger$ operators (2.4) take simpler forms with the help of the auxiliary function φ (2.35):

$$\phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1}\circ S_\pm(\boldsymbol{\lambda})\circ\phi_0(x\,;\boldsymbol{\lambda})=\varphi(x)^{-1}\,e^{\pm\gamma p/2}, \tag{2.36}$$

$$\phi_0(x\,;\boldsymbol{\lambda})^{-1}\circ S_\pm(\boldsymbol{\lambda})^\dagger\circ\phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})=\begin{cases}V(x\,;\boldsymbol{\lambda})\,e^{\gamma p/2}\,\varphi(x)\,,\\ V(x\,;\boldsymbol{\lambda})^*\,e^{-\gamma p/2}\,\varphi(x)\,.\end{cases} \tag{2.37}$$

Note that the parameter shifts $\pm\boldsymbol{\delta}$ are properly incorporated.

Forward/Backward shift operators With (2.36)–(2.37) the 'similarity' transformed $\mathcal{A}$ and $\mathcal{A}^\dagger$ operators are obtained. They are called the forward/backward shift operators:

$$\widetilde{\mathcal{H}}(\boldsymbol{\lambda})=\mathcal{B}(\boldsymbol{\lambda})\mathcal{F}(\boldsymbol{\lambda}), \tag{2.38}$$

$$\mathcal{F}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1}\circ\mathcal{A}(\boldsymbol{\lambda})\circ\phi_0(x\,;\boldsymbol{\lambda})=i\,\varphi(x)^{-1}\big(e^{\gamma p/2}-e^{-\gamma p/2}\big), \tag{2.39}$$

$$\mathcal{B}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\phi_0(x\,;\boldsymbol{\lambda})^{-1}\circ\mathcal{A}(\boldsymbol{\lambda})^\dagger\circ\phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})=-i\big(V(x\,;\boldsymbol{\lambda})\,e^{\gamma p/2}-V(x\,;\boldsymbol{\lambda})^*\,e^{-\gamma p/2}\big)\varphi(x). \tag{2.40}$$

The action of the forward shift operator $\mathcal{F}(\boldsymbol{\lambda})$ and the backward shift operator $\mathcal{B}(\boldsymbol{\lambda})$ on the polynomial $P_n(\eta\,;\boldsymbol{\lambda})$ are:

$$\mathcal{F}(\boldsymbol{\lambda})P_n(\eta\,;\boldsymbol{\lambda})=f_n(\boldsymbol{\lambda})P_{n-1}(\eta\,;\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{2.41}$$

$$\mathcal{B}(\boldsymbol{\lambda})P_n(\eta\,;\boldsymbol{\lambda}+\boldsymbol{\delta})=b_n(\boldsymbol{\lambda})P_{n+1}(\eta\,;\boldsymbol{\lambda}), \tag{2.42}$$

in which $f_n(\boldsymbol{\lambda})$ and $b_n(\boldsymbol{\lambda})$ are real constants related to $\mathcal{E}_n(\boldsymbol{\lambda})$:

$$f_n(\boldsymbol{\lambda})b_{n-1}(\boldsymbol{\lambda})=\mathcal{E}_n(\boldsymbol{\lambda}). \tag{2.43}$$

For the cases studied in [7] $b_n(\boldsymbol{\lambda})$ is actually independent of n, but here it depends on n. In terms of the forward and backward shift operators, the shape invariance condition (2.15) reads

$$\mathcal{F}(\boldsymbol{\lambda})\mathcal{B}(\boldsymbol{\lambda}) = \kappa\mathcal{B}(\boldsymbol{\lambda}+\boldsymbol{\delta})\mathcal{F}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}). \tag{2.44}$$

Corresponding to (2.19), a Rodrigues type formula for the eigenpolynomials is

$$P_n(\eta\,;\boldsymbol{\lambda}) = \frac{\mathcal{B}(\boldsymbol{\lambda})}{b_{n-1}(\boldsymbol{\lambda})}\,\frac{\mathcal{B}(\boldsymbol{\lambda}+\boldsymbol{\delta})}{b_{n-2}(\boldsymbol{\lambda}+\boldsymbol{\delta})}\,\frac{\mathcal{B}(\boldsymbol{\lambda}+2\boldsymbol{\delta})}{b_{n-3}(\boldsymbol{\lambda}+2\boldsymbol{\delta})}\cdots\frac{\mathcal{B}(\boldsymbol{\lambda}+(n-1)\boldsymbol{\delta})}{b_0(\boldsymbol{\lambda}+(n-1)\boldsymbol{\delta})}\cdot P_0(\eta\,;\boldsymbol{\lambda}+n\boldsymbol{\delta}), \tag{2.45}$$

where $P_0(\eta\,;\boldsymbol{\lambda}+n\boldsymbol{\delta}) = 1$ for all the examples given in this paper. With these quantities the action of $\mathcal{A}(\boldsymbol{\lambda})$ and $\mathcal{A}(\boldsymbol{\lambda})^\dagger$ on the eigenfunction ϕ_n can be simply expressed as

$$\mathcal{A}(\boldsymbol{\lambda})\phi_n(x\,;\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})\phi_{n-1}(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{2.46}$$

$$\mathcal{A}(\boldsymbol{\lambda})^\dagger\phi_n(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta}) = b_n(\boldsymbol{\lambda})\phi_{n+1}(x\,;\boldsymbol{\lambda}). \tag{2.47}$$

Three term recurrence relation The polynomial part of the eigenfunction $P_n(\eta)$ is an orthogonal polynomial with the measure $\phi_0(x)^2$. It satisfies three term recurrence relations [1, 2]. Let us first write the relation for the monic polynomial $P_n^{\text{monic}}(\eta) = \eta^n+$ lower degree in η:

$$P_n(\eta) = c_n P_n^{\text{monic}}(\eta), \tag{2.48}$$

$$P_{n+1}^{\text{monic}}(\eta) - (\eta - a_n^{\text{rec}})P_n^{\text{monic}}(\eta) + b_n^{\text{rec}}P_{n-1}^{\text{monic}}(\eta) = 0 \quad (n \geq 0), \tag{2.49}$$

with $P_{-1}^{\text{monic}}(\eta) = 0$. For $P_n(\eta)$ it reads

$$\eta P_n(\eta) = A_n P_{n+1}(\eta) + B_n P_n(\eta) + C_n P_{n-1}(\eta), \tag{2.50}$$

$$A_n = \frac{c_n}{c_{n+1}}, \quad B_n = a_n^{\text{rec}}, \quad C_n = \frac{c_n}{c_{n-1}}b_n^{\text{rec}}. \tag{2.51}$$

Sometimes we write the parameter dependence explicitly as $P_n(\eta) = P_n(\eta\,;\boldsymbol{\lambda})$, $a_n^{\text{rec}} = a_n^{\text{rec}}(\boldsymbol{\lambda})$, $b_n^{\text{rec}} = b_n^{\text{rec}}(\boldsymbol{\lambda})$, $c_n = c_n(\boldsymbol{\lambda})$, $A_n = A_n(\boldsymbol{\lambda})$, $B_n = B_n(\boldsymbol{\lambda})$, $C_n = C_n(\boldsymbol{\lambda})$, $f_n(\boldsymbol{\lambda})$ and $b_n(\boldsymbol{\lambda})$. They are given in each subsection.

Heisenberg operator and Annihilation-Creation operators The exact Heisenberg operator solution for $\eta(x)$ is easily obtained [13] from the closure relation (2.20):

$$e^{it\mathcal{H}}\eta(x)e^{-it\mathcal{H}} = a^{(+)}e^{i\alpha_+(\mathcal{H})t} + a^{(-)}e^{i\alpha_-(\mathcal{H})t} - R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1}, \tag{2.52}$$

$$\alpha_\pm(\mathcal{H}) \stackrel{\text{def}}{=} \tfrac{1}{2}\bigl(R_1(\mathcal{H}) \pm \sqrt{R_1(\mathcal{H})^2 + 4R_0(\mathcal{H})}\,\bigr), \tag{2.53}$$

$$R_1(\mathcal{H}) = \alpha_+(\mathcal{H}) + \alpha_-(\mathcal{H}), \quad R_0(\mathcal{H}) = -\alpha_+(\mathcal{H})\alpha_-(\mathcal{H}), \tag{2.54}$$

$$a^{(\pm)} \stackrel{\text{def}}{=} \pm\Bigl([\mathcal{H}, \eta(x)] - \bigl(\eta(x) + R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1}\bigr)\alpha_\mp(\mathcal{H})\Bigr)\bigl(\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})\bigr)^{-1} \tag{2.55}$$

$$= \pm\bigl(\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})\bigr)^{-1}\Bigl([\mathcal{H}, \eta(x)] + \alpha_\pm(\mathcal{H})\bigl(\eta(x) + R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1}\bigr)\Bigr). \tag{2.56}$$

The positive/negative frequency parts of the Heisenberg operator solution, $a^{(\pm)}$ are the annihilation creation operators

$$a^{(+)\dagger} = a^{(-)}, \quad a^{(+)}\phi_n(x) = A_n\phi_{n+1}(x), \quad a^{(-)}\phi_n(x) = C_n\phi_{n-1}(x). \tag{2.57}$$

Since

$$\alpha_\pm(\mathcal{E}_n) = \mathcal{E}_{n\pm1} - \mathcal{E}_n, \tag{2.58}$$

we obtain

$$a^{(\pm)}\phi_n(x) = \frac{\pm1}{\mathcal{E}_{n+1} - \mathcal{E}_{n-1}}\Bigl([\mathcal{H}, \eta(x)] + (\mathcal{E}_n - \mathcal{E}_{n\mp1})\eta(x) + \frac{R_{-1}(\mathcal{E}_n)}{\mathcal{E}_{n\pm1} - \mathcal{E}_n}\Bigr)\phi_n(x). \tag{2.59}$$

Commutation relations of $a^{(\pm)}$ and $\mathcal{H}$ Simple commutation relations

$$[\mathcal{H}, a^{(\pm)}] = a^{(\pm)}\alpha_\pm(\mathcal{H}) \tag{2.60}$$

follow from (2.55) and (2.20). When applied to ϕ_n, we obtain with the help of (2.58),

$$[\mathcal{H}, a^{(\pm)}]\phi_n = (\mathcal{E}_{n\pm1} - \mathcal{E}_n)a^{(\pm)}\phi_n. \tag{2.61}$$

Commutation relations of $a^{(\pm)}$ are expressed in terms of the coefficients of the three term recurrence relation by (2.57):

$$a^{(-)}a^{(+)}\phi_n = A_nC_{n+1}\phi_n = b^{\text{rec}}_{n+1}\phi_n, \quad a^{(+)}a^{(-)}\phi_n = C_nA_{n-1}\phi_n = b^{\text{rec}}_n\phi_n, \tag{2.62}$$

$$\Rightarrow \quad [a^{(-)}, a^{(+)}]\phi_n = (b^{\text{rec}}_{n+1} - b^{\text{rec}}_n)\phi_n. \tag{2.63}$$

These relation simply mean the operator relations

$$a^{(-)}a^{(+)} = f(\mathcal{H}), \tag{2.64}$$

$$a^{(+)}a^{(-)} = g(\mathcal{H}), \tag{2.65}$$

in which f and g are analytic functions of $\mathcal{H}$. In other words, $\mathcal{H}$ and $a^{(\pm)}$ form a so-called quasi-linear algebra [30]. This is because the definition of the annihilation/creation operators depend only on the closure relation (2.20), without any other inputs. The situation is quite different from those of the wide variety of proposed annihilation/creation operators for various quantum systems [31], most of which were introduced within the framework of 'algebraic theory of coherent states'. In all these cases there is no guarantee for symmetry relations like (2.64), (2.65).

In many cases it is convenient to introduce the 'number operator' (or the 'level operator') $\mathcal{N}$

$$\mathcal{N}\phi_n \stackrel{\text{def}}{=} n\phi_n. \tag{2.66}$$

For the following types of energy spectra, the number operator $\mathcal{N}$ can be expressed as a function of the Hamiltonian $\mathcal{H}$:

$$\mathcal{E}_n = an \quad (a > 0) \quad \Rightarrow \quad \mathcal{N} = a^{-1}\mathcal{H}, \tag{2.67}$$

$$\mathcal{E}_n = n(n+b) \quad (b > 0) \quad \Rightarrow \quad \mathcal{N} = \sqrt{\mathcal{H} + \tfrac{1}{4}b^2} - \tfrac{1}{2}b, \tag{2.68}$$

$$\mathcal{E}_n = q^{-n} - 1 \quad \Rightarrow \quad q^{\mathcal{N}} = (\mathcal{H}+1)^{-1}, \tag{2.69}$$

$$\mathcal{E}_n = (q^{-n} - 1)(1 - bq^n) \quad (0 < b < 1) \quad \Rightarrow \quad q^{\mathcal{N}} = \frac{1}{2b}\big(\mathcal{H} + b + 1 - \sqrt{(\mathcal{H}+b+1)^2 - 4b}\,\big). \tag{2.70}$$

Obviously the Hamiltonian is expressed as $\mathcal{H} = \mathcal{E}_{\mathcal{N}}$. Then (2.63) can be expressed simply as

$$[a^{(-)}, a^{(+)}] = b^{\text{rec}}_{\mathcal{N}+1} - b^{\text{rec}}_{\mathcal{N}} \tag{2.71}$$

and (2.61) is rewritten as

$$[\mathcal{H}, a^{(\pm)}] = \mathcal{E}_{\mathcal{N}} a^{(\pm)} - a^{(\pm)}\mathcal{E}_{\mathcal{N}} = a^{(\pm)}(\mathcal{E}_{\mathcal{N}\pm 1} - \mathcal{E}_{\mathcal{N}}). \tag{2.72}$$

With a deformed commutator

$$[A, B]_\alpha \stackrel{\text{def}}{=} AB - \alpha BA, \tag{2.73}$$

we have

$$[a^{(-)}, a^{(+)}]_\alpha = b^{\text{rec}}_{\mathcal{N}+1} - \alpha b^{\text{rec}}_{\mathcal{N}}. \tag{2.74}$$

Orthogonality and normalisation The scalar product for the elements of the Hilbert space belonging to the Hamiltonian $\mathcal{H}$ is

$$(g, f) \stackrel{\text{def}}{=} \int dx\, g(x)^* f(x), \tag{2.75}$$

in which the integration range depends on the specific Hamiltonian or the polynomial. The orthogonality of the eigenvectors $\{\phi_n(x)\}$, $\phi_n(x) = \phi_0(x)P_n(\eta(x))$ is:

$$(\phi_n, \phi_m) = \int dx\, \phi_0(x\,;\boldsymbol{\lambda})^2 P_n(\eta(x)\,;\boldsymbol{\lambda})^* P_m(\eta(x)\,;\boldsymbol{\lambda}) = h_n(\boldsymbol{\lambda})\delta_{nm}, \tag{2.76}$$

in which $h_n(\boldsymbol{\lambda}) > 0$. The constants h_n, c_n and b_n^{rec} are related as

$$b_n^{\text{rec}} = \frac{c_{n-1}^2}{c_n^2}\frac{h_n}{h_{n-1}} \quad (n \geq 1), \qquad h_n = h_0 c_n^2 \prod_{j=1}^{n} b_n^{\text{rec}} \quad (n \geq 0). \tag{2.77}$$

Let us denote the n-th normalised eigenfunction as

$$\hat{\phi}_n(x\,;\boldsymbol{\lambda}) = N_n(\boldsymbol{\lambda})P_n(\eta(x)\,;\boldsymbol{\lambda})\hat{\phi}_0(x\,;\boldsymbol{\lambda}), \quad \hat{\phi}_0(x\,;\boldsymbol{\lambda}) = \frac{\phi_0(x\,;\boldsymbol{\lambda})}{\sqrt{h_0(\boldsymbol{\lambda})}}, \quad N_n(\boldsymbol{\lambda}) = \sqrt{\frac{h_0(\boldsymbol{\lambda})}{h_n(\boldsymbol{\lambda})}}. \tag{2.78}$$

These normalisation constants are given for each polynomial.

Coherent states There are many different nonequivalent definitions of coherent states. Here we adopt the most conventional one, as the eigenvector of the annihilation operator $a^{(-)}$, (2.57):

$$a^{(-)}\psi(\alpha, x) = \alpha\psi(\alpha, x), \quad \alpha \in \mathbb{C}. \tag{2.79}$$

It is expressed in terms of the coefficient C_n of the three term recurrence relation (2.50) and (2.51) as [13]

$$\psi(\alpha, x) = \psi(\alpha, x\,;\boldsymbol{\lambda}) = \phi_0(x\,;\boldsymbol{\lambda}) \sum_{n=0}^{\infty} \frac{\alpha^n}{\prod_{k=1}^{n} C_k} P_n(\eta(x)\,;\boldsymbol{\lambda}). \tag{2.80}$$

Thus we obtain one new coherent state for each polynomial; (3.19), (3.39), (4.16), (4.37), (5.20), (5.51), (5.71), (5.91), (5.118), (5.137) and (5.158). If the sum on the r.h.s. is expressed by a simple function, it is a generating function of the polynomial $P_n(\eta)$. In most explicit examples to be discussed in later sections, the potential functions, the Hamiltonians and thus the polynomials themselves are totally symmetric in the parameters, see for example, the Askey-Wilson polynomial §5.1. The above coherent state, being totally symmetric, gives the best candidate for a symmetric generating function. For the polynomials to be discussed in later sections, however, most of the known generating functions are not totally symmetric.

$\boldsymbol{\lambda}$-shift operators Let us fix an orthonormal basis $\{\hat{\phi}_n(x;\boldsymbol{\lambda})\}$ and define a unitary operator $\mathcal{U}$ ($\mathcal{U}^\dagger$) as

$$\mathcal{U}\hat{\phi}_n(x\,;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \hat{\phi}_n(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta}),\quad \mathcal{U}^\dagger\hat{\phi}_n(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta}) = \hat{\phi}_n(x\,;\boldsymbol{\lambda}). \tag{2.81}$$

Then we can define another set of annihilation-creation operators $\hat{a}$, $\hat{a}^\dagger$:

$$\hat{a} \stackrel{\text{def}}{=} \mathcal{U}^\dagger\mathcal{A},\quad \hat{a}^\dagger = \mathcal{A}^\dagger\mathcal{U}. \tag{2.82}$$

They satisfy $\mathcal{H} = \hat{a}^\dagger\hat{a}$ and their action on ϕ_n are derived from (2.46) and (2.47), $\hat{a}\phi_n(x\,;\boldsymbol{\lambda}) \propto \phi_{n-1}(x\,;\boldsymbol{\lambda})$, $\hat{a}^\dagger\phi_n(x\,;\boldsymbol{\lambda}) \propto \phi_{n+1}(x\,;\boldsymbol{\lambda})$. Although this kind of annihilation-creation operators have been considered in many literature [31], it should be stressed that they are formal because $\mathcal{U}$ and $\mathcal{U}^\dagger$ are formal operators. On the other hand, $a^{(\pm)}$ obtained from the Heisenberg solution are explicitly expressed in terms of difference operators (differential operators, in ordinary quantum mechanics), (2.55). Note that the construction method of $\hat{a}$ and $\hat{a}^\dagger$ is based on the shape invariance but that of $a^{(\pm)}$ is not. The latter is based on the closure relation.

The key point of the construction of $\hat{a}$ and $\hat{a}^\dagger$ is the proper shift of the parameters $\boldsymbol{\lambda}$, which is achieved by the formal operators $\mathcal{U}$ and $\mathcal{U}^\dagger$. We introduce another set of $\boldsymbol{\lambda}$-shift operators X and $X^\dagger$ explicitly in terms of difference operators through the following relations:

$$a^{(+)} = \mathcal{A}^\dagger X,\quad a^{(-)} = X^\dagger\mathcal{A}. \tag{2.83}$$

By using the shape invariance (2.15), we have

$$\mathcal{A}a^{(+)} = \mathcal{A}\mathcal{A}^\dagger X = \big(\kappa\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1\big)X = \big(\kappa\mathcal{H}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1\big)X. \tag{2.84}$$

Since $\kappa\mathcal{H}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1$ is a positive operator, we obtain

$$\begin{aligned} X &= \big(\kappa\mathcal{H}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1\big)^{-1}\mathcal{A}\,a^{(+)} \\ &= \big(\kappa\mathcal{H}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1\big)^{-1}\mathcal{A} \\ &\qquad\times\Big([\mathcal{H},\eta(x)] - \big(\eta(x) + R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1}\big)\alpha_-(\mathcal{H})\Big)\big(\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})\big)^{-1}. \end{aligned} \tag{2.85}$$

Similarly $X^\dagger$ is expressed as

$$X^\dagger = a^{(-)}\mathcal{A}^\dagger\big(\kappa\mathcal{H}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1\big)^{-1}. \tag{2.86}$$

Their action on ϕ_n are

$$X\phi_n(x\,;\boldsymbol{\lambda}) = \frac{A_n(\boldsymbol{\lambda})}{b_n(\boldsymbol{\lambda})}\phi_n(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{2.87}$$

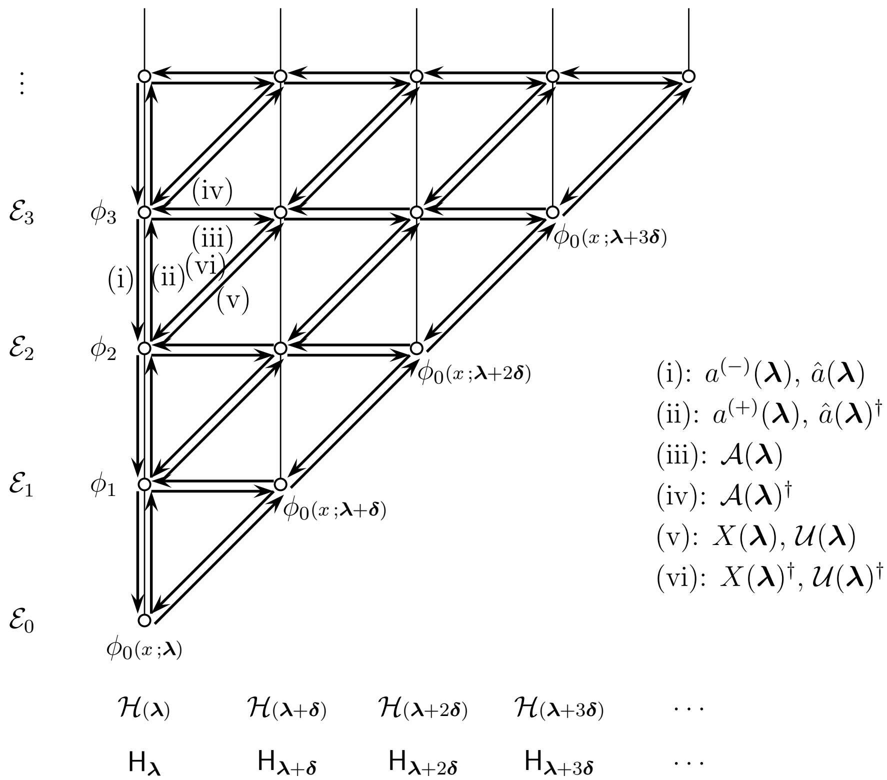

Figure 1: Shape invariance and Crum's scheme.

$$X^\dagger\phi_n(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})=\frac{C_{n+1}(\boldsymbol{\lambda})}{f_{n+1}(\boldsymbol{\lambda})}\phi_n(x\,;\boldsymbol{\lambda}), \tag{2.88}$$

and the $\boldsymbol{\lambda}$-shift without changing the level n is achieved, as expected. The $\boldsymbol{\lambda}$-shift operators for the polynomials $P_n(\eta(x)\,;\boldsymbol{\lambda})$ are given by $\phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1}\circ X\circ\phi_0(x\,;\boldsymbol{\lambda})$ and $\phi_0(x\,;\boldsymbol{\lambda})^{-1}\circ X^\dagger\circ\phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})$. The expression of X and $X^\dagger$ may be simplified for some particular cases, see §3.2, §4.2, §5.5.

Finally we illustrate the shape invariance and Crum's scheme in Fig.1. The Hilbert space belonging to the Hamiltonian $\mathcal{H}(\boldsymbol{\lambda})$ is denoted as $\mathsf{H}_{\boldsymbol{\lambda}}$. The action of various operators and their domains and images are also illustrated in Fig.1:

$$\mathcal{H}(\boldsymbol{\lambda}),\ a^{(\pm)}(\boldsymbol{\lambda}),\ \hat{a}(\boldsymbol{\lambda}),\hat{a}(\boldsymbol{\lambda})^\dagger:\mathsf{H}_{\boldsymbol{\lambda}}\to\mathsf{H}_{\boldsymbol{\lambda}}, \tag{2.89}$$

$$\mathcal{A}(\boldsymbol{\lambda}),\ \ X(\boldsymbol{\lambda}),\ \ \mathcal{U}(\boldsymbol{\lambda})\ :\mathsf{H}_{\boldsymbol{\lambda}}\to\mathsf{H}_{\boldsymbol{\lambda}+\boldsymbol{\delta}}, \tag{2.90}$$

$$\mathcal{A}(\boldsymbol{\lambda})^\dagger,\ X(\boldsymbol{\lambda})^\dagger,\ \mathcal{U}(\boldsymbol{\lambda})^\dagger:\mathsf{H}_{\boldsymbol{\lambda}+\boldsymbol{\delta}}\to\mathsf{H}_{\boldsymbol{\lambda}}. \tag{2.91}$$

3 $\eta(x)=x$

From this section to section 5, we present various formulas and results specific to each example of the exactly solvable 'discrete' quantum mechanics. These examples are divided into three groups according to the form of the sinusoidal coordinate; $\eta(x)=x$ in this section, $\eta(x)=x^2$ in section 4, $\eta(x)=\cos x$ in section 5. The names of the subsections are taken from the name of the corresponding orthogonal polynomial and the number, for example, [KS1.4] indicates the corresponding subsection of the review of Koekoek and Swarttouw [6].

In all the examples in this section, we have

$$\eta(x)=x,\quad -\infty<x<\infty,\quad \gamma=1,\quad \kappa=1,\quad \varphi(x)=1. \tag{3.1}$$

3.1 continuous Hahn [KS1.4]

In previous works [10, 11, 12, 13], the parameters a_1 and a_2 were restricted to real, positive values. Now they are complex with positive real parts.

parameters and potential functions

$$\boldsymbol{\lambda}\stackrel{\text{def}}{=}(a_1,a_2),\quad \boldsymbol{\delta}=(\tfrac12,\tfrac12);\quad \operatorname{Re}a_i>0;\quad V(x\,;\boldsymbol{\lambda})\stackrel{\text{def}}{=}(a_1+ix)(a_2+ix). \tag{3.2}$$

shape invariance and closure relation

$$\mathcal{E}_n(\boldsymbol{\lambda})=n(n+b_1-1), \tag{3.3}$$

$$R_1(y)=2,\quad R_0(y)=4y+b_1(b_1-2), \tag{3.4}$$

$$R_{-1}(y)=-i(a_1+a_2-a_3-a_4)y-i(b_1-2)(a_1a_2-a_3a_4), \tag{3.5}$$

$$b_1\stackrel{\text{def}}{=}\sum_{j=1}^{4}a_j,\quad (a_3,a_4)\stackrel{\text{def}}{=}(a_1^*,a_2^*)\ \text{ or }\ (a_2^*,a_1^*). \tag{3.6}$$

These can be rewritten as

$$\mathcal{E}_n(\boldsymbol{\lambda})=n(n+2\text{Re}(a_1+a_2)-1), \tag{3.7}$$

$$R_0(y)=4y+4\text{Re}(a_1+a_2)\big(\text{Re}(a_1+a_2)-1\big), \tag{3.8}$$

$$R_{-1}(y)=2\text{Im}(a_1+a_2)y+4\big(\text{Re}(a_1+a_2)-1\big)\text{Im}(a_1a_2). \tag{3.9}$$

eigenfunctions

$$\phi_0(x\,;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} |\Gamma(a_1+ix)\Gamma(a_2+ix)| = \sqrt{\Gamma(a_1+ix)\Gamma(a_2+ix)\Gamma(a_3-ix)\Gamma(a_4-ix)}, \tag{3.10}$$

$$\begin{aligned} P_n(\eta\,;\boldsymbol{\lambda}) &= p_n(x\,;a_1,a_2,a_3,a_4) \\ &\stackrel{\text{def}}{=} i^n\frac{(a_1+a_3)_n(a_1+a_4)_n}{n!}\,{}_3F_2\Bigl(\begin{matrix}-n,\ n+a_1+a_2+a_3+a_4-1,\ a_1+ix\\ a_1+a_3,\ a_1+a_4\end{matrix}\Bigm|1\Bigr), \end{aligned} \tag{3.11}$$

which are symmetric under $a_1 \leftrightarrow a_2$ and $a_3 \leftrightarrow a_4$ separately.

$$c_n = \frac{(n+b_1-1)_n}{n!}, \tag{3.12}$$

$$\begin{aligned} a_n^{\text{rec}} = i\biggl(a_1 &- \frac{(n+b_1-1)(n+a_1+a_3)(n+a_1+a_4)}{(2n+b_1-1)(2n+b_1)} \\ &+ \frac{n(n+a_2+a_3-1)(n+a_2+a_4-1)}{(2n+b_1-2)(2n+b_1-1)}\biggr), \end{aligned} \tag{3.13}$$

$$b_n^{\text{rec}} = \frac{n(n+b_1-2)\prod_{j=1}^{2}\prod_{k=3}^{4}(n+a_j+a_k-1)}{(2n+b_1-3)(2n+b_1-2)^2(2n+b_1-1)}, \tag{3.14}$$

$$f_n(\boldsymbol{\lambda}) = n+b_1-1, \quad b_n(\boldsymbol{\lambda}) = n+1. \tag{3.15}$$

annihilation/creation operators and commutation relations

$$\alpha_\pm(\mathcal{H}) = 1 \pm 2\sqrt{\mathcal{H}'}, \quad \mathcal{H}' \stackrel{\text{def}}{=} \mathcal{H} + \tfrac{1}{4}(b_1-1)^2, \tag{3.16}$$

$$\mathcal{N} = \sqrt{\mathcal{H}'} - \tfrac{1}{2}(b_1-1) \quad (\text{for } b_1>1), \tag{3.17}$$

$$[\mathcal{H}, a^{(\pm)}] = a^{(\pm)}(1 \pm 2\sqrt{\mathcal{H}'}). \tag{3.18}$$

The annihilation/creation operators (2.55) and their commutation relation (2.63) are not so simplified because $b_{n+1}^{\text{rec}} - b_n^{\text{rec}} =$ (quartic polynomial in n)/(cubic polynomial in n) has a lengthy expression.

coherent state

$$\psi(\alpha, x\,;\boldsymbol{\lambda}) = \phi_0(x\,;\boldsymbol{\lambda})\sum_{n=0}^{\infty}\frac{(b_1)_{2n}\,\alpha^n}{\prod_{j=1}^{2}\prod_{k=3}^{4}(a_j+a_k)_n}\,P_n(\eta(x)\,;\boldsymbol{\lambda}). \tag{3.19}$$

The r.h.s is symmetric under $a_1 \leftrightarrow a_2$ and $a_3 \leftrightarrow a_4$ separately. We are not aware if a concise summation formula exists or not. Several non-symmetric generating functions for the continuous Hahn polynomial are given in [6].

orthogonality

$$\int_{-\infty}^{\infty}\phi_0(x\,;\boldsymbol{\lambda})^2P_n(\eta\,;\boldsymbol{\lambda})P_m(\eta\,;\boldsymbol{\lambda})dx=2\pi\,\frac{\prod_{j=1}^2\prod_{k=3}^4\Gamma(n+a_j+a_k)}{n!\,(2n+b_1-1)\Gamma(n+b_1-1)}\,\delta_{nm},\tag{3.20}$$

$$\frac{1}{h_0(\boldsymbol{\lambda})}=\frac{\Gamma(b_1)}{2\pi\prod_{j=1}^2\prod_{k=3}^4\Gamma(a_j+a_k)},\quad \frac{h_0(\boldsymbol{\lambda})}{h_n(\boldsymbol{\lambda})}=\frac{b_1+2n-1}{b_1+n-1}\frac{n!\,(b_1)_n}{\prod_{j=1}^2\prod_{k=3}^4(a_j+a_k)_n}.\tag{3.21}$$

3.2 Meixner-Pollaczek [KS1.7]

In previous works [10, 13, 32], the parameter ϕ was fixed to $\pi/2$. Here we treat the most general case $0<\phi<\pi$.

parameters and potential function

$$\boldsymbol{\lambda}\stackrel{\text{def}}{=}a,\quad \boldsymbol{\delta}=\tfrac12,\quad \phi\ \ (0<\phi<\pi);\quad a>0;\quad V(x\,;\boldsymbol{\lambda})\stackrel{\text{def}}{=}e^{i(\frac{\pi}{2}-\phi)}(a+ix).\tag{3.22}$$

shape invariance and closure relation

$$\mathcal{E}_n(\boldsymbol{\lambda})=2n\sin\phi,\tag{3.23}$$

$$R_1(y)=0,\quad R_0(y)=4\sin^2\phi,\quad R_{-1}(y)=2y\cos\phi+2a\sin 2\phi.\tag{3.24}$$

eigenfunctions

$$\phi_0(x\,;\boldsymbol{\lambda})\stackrel{\text{def}}{=}e^{(\phi-\frac{\pi}{2})x}|\Gamma(a+ix)|,\tag{3.25}$$

$$P_n(\eta\,;\boldsymbol{\lambda})=P_n^{(a)}(x\,;\phi)\stackrel{\text{def}}{=}\frac{(2a)_n}{n!}\,e^{in\phi}{}_2F_1\Bigl(\begin{matrix}-n,\,a+ix\\ 2a\end{matrix}\Bigm|1-e^{-2i\phi}\Bigr),\tag{3.26}$$

$$c_n=\frac{(2\sin\phi)^n}{n!}\quad a_n^{\text{rec}}=-\frac{n+a}{\tan\phi},\quad b_n^{\text{rec}}=\frac{n(n+2a-1)}{(2\sin\phi)^2},\tag{3.27}$$

$$f_n(\boldsymbol{\lambda})=2\sin\phi,\quad b_n(\boldsymbol{\lambda})=n+1.\tag{3.28}$$

The polynomial has the following symmetry $P_n^{(a)}(x\,;-\phi)=P_n^{(a)}(-x\,;\phi)$.

annihilation/creation operators and commutation relations

$$\alpha_\pm(\mathcal{H})=\pm2\sin\phi,\quad \mathcal{N}=\frac{1}{2\sin\phi}\,\mathcal{H},\tag{3.29}$$

$$a^{(\pm)}=\frac{\pm1}{4\sin\phi}[\mathcal{H},\eta]+\frac12\eta+\frac{\cos\phi}{4\sin^2\phi}(\mathcal{H}+2a\sin\phi),\tag{3.30}$$

$$b_{n+1}^{\text{rec}}-b_n^{\text{rec}}=\frac{n+a}{2\sin^2\phi},\tag{3.31}$$

$$[\mathcal{H}, a^{(\pm)}] = \pm 2\sin\phi \ a^{(\pm)}, \tag{3.32}$$

$$[a^{(-)}, a^{(+)}] = \frac{1}{4\sin^3\phi}(\mathcal{H} + 2a\sin\phi). \tag{3.33}$$

$$\mathfrak{su}(1,1) \text{ algebra}: \quad J^{\pm} = 2\sin\phi \ a^{(\pm)}, \quad J^3 = \frac{1}{2\sin\phi}(\mathcal{H} + 2a\sin\phi),$$
$$[J^3, J^{\pm}] = \pm J^{\pm}, \quad [J^-, J^+] = 2J^3. \tag{3.34}$$

The $\mathfrak{su}(1,1)$ or $\mathfrak{sl}(2,\mathbb{R})$ algebra reported before [13, 32] is a special case of the present one.

$\boldsymbol{\lambda}$-shift operators For the special case of $\phi = \pi/2$ the annihilation/creation operators are closely related to the $\mathcal{A}$ and $\mathcal{A}^\dagger$ operators:

$$a^{(+)} = \mathcal{A}^\dagger X, \quad X = \tfrac{1}{4}(S_+ + S_-), \tag{3.35}$$

$$a^{(-)} = X^\dagger \mathcal{A}, \quad X^\dagger = \tfrac{1}{4}(S_+^\dagger + S_-^\dagger), \tag{3.36}$$

$$\phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1}\, X(\boldsymbol{\lambda}) \ \phi_0(x\,;\boldsymbol{\lambda}) \cdot P_n(\eta\,;\boldsymbol{\lambda}) = \tfrac{1}{2} P_n(\eta\,;\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{3.37}$$

$$\phi_0(x\,;\boldsymbol{\lambda})^{-1}\, X(\boldsymbol{\lambda})^\dagger \ \phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta}) \cdot P_n(\eta\,;\boldsymbol{\lambda}+\boldsymbol{\delta}) = \tfrac{1}{4}(n+2a) P_n(\eta\,;\boldsymbol{\lambda}). \tag{3.38}$$

coherent state The coherent state gives a simple generating function, which generalises the previous result [13]:

$$\psi(\alpha, x\,;\boldsymbol{\lambda}) = \phi_0(x\,;\boldsymbol{\lambda}) \sum_{n=0}^{\infty} \frac{(2\sin\phi)^n \alpha^n}{(2a)_n} P_n(\eta(x)\,;\boldsymbol{\lambda})$$
$$= \phi_0(x\,;\boldsymbol{\lambda})\, e^{i\alpha(1-e^{2i\phi})} {}_1F_1\Bigl(\begin{matrix} a+ix \\ 2a \end{matrix} \Bigm| -4i\alpha\sin^2\phi\Bigr). \tag{3.39}$$

orthogonality

$$\int_{-\infty}^{\infty} \phi_0(x\,;\boldsymbol{\lambda})^2 P_n(\eta\,;\boldsymbol{\lambda}) P_m(\eta\,;\boldsymbol{\lambda}) dx = 2\pi \frac{\Gamma(n+2a)}{n!\,(2\sin\phi)^{2a}}\, \delta_{nm}, \tag{3.40}$$

$$\frac{1}{h_0(\boldsymbol{\lambda})} = \frac{(2\sin\phi)^{2a}}{2\pi\Gamma(2a)}, \quad \frac{h_0(\boldsymbol{\lambda})}{h_n(\boldsymbol{\lambda})} = \frac{n!}{(2a)_n}. \tag{3.41}$$

The exact solvability of the continuous Hahn and Meixner-Pollaczek polynomials for the full parameters are discussed in [27] in connection with their further deformation to give another example of quasi exactly solvable system.

4 $\eta(x) = x^2$

In all the examples in this section, we have

$$\eta(x) = x^2, \quad 0 < x < \infty, \quad \gamma = 1, \quad \kappa = 1, \quad \varphi(x) = 2x. \tag{4.1}$$

4.1 Wilson [KS1.1]

The Wilson polynomial is the most general one in this category. The parameters $a_1, \ldots, a_4$ were restricted to real positive values in previous works [10, 11, 12, 13]. The generic situation to be discussed in this paper is

$$\{a_1^*, a_2^*, a_3^*, a_4^*\} = \{a_1, a_2, a_3, a_4\} \quad \text{(as a set)}, \quad \operatorname{Re} a_i > 0 \quad (1 \leq i \leq 4). \tag{4.2}$$

parameters and potential function

$$\boldsymbol{\lambda} \stackrel{\text{def}}{=} (a_1, a_2, a_3, a_4), \ \boldsymbol{\delta} = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}); \ \ V(x\,; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{(a_1 + ix)(a_2 + ix)(a_3 + ix)(a_4 + ix)}{2ix(2ix + 1)}. \tag{4.3}$$

shape invariance and closure relation

$$\mathcal{E}_n(\boldsymbol{\lambda}) = n(n + b_1 - 1), \tag{4.4}$$

$$R_1(y) = 2, \quad R_0(y) = 4y + b_1(b_1 - 2), \quad R_{-1}(y) = -2y^2 + (b_1 - 2b_2)y + (2 - b_1)b_3, \tag{4.5}$$

$$b_1 \stackrel{\text{def}}{=} \sum_{j=1}^{4} a_j, \quad b_2 \stackrel{\text{def}}{=} \sum_{1 \leq j < k \leq 4} a_j a_k, \quad b_3 \stackrel{\text{def}}{=} \sum_{1 \leq j < k < l \leq 4} a_j a_k a_l. \tag{4.6}$$

eigenfunctions

$$\phi_0(x\,; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \left| \frac{\prod_{j=1}^{4} \Gamma(a_j + ix)}{\Gamma(2ix)} \right|, \tag{4.7}$$

$$\begin{aligned} P_n(\eta\,; \boldsymbol{\lambda}) &= W_n(x^2\,; a_1, a_2, a_3, a_4) \\ &\stackrel{\text{def}}{=} (a_1 + a_2)_n (a_1 + a_3)_n (a_1 + a_4)_n \\ &\qquad \times {}_4F_3\Big(\begin{matrix} -n, \ n + \sum_{j=1}^{4} a_j - 1, \ a_1 + ix, \ a_1 - ix \\ a_1 + a_2, \ a_1 + a_3, \ a_1 + a_4 \end{matrix} \Big| \, 1 \Big), \end{aligned} \tag{4.8}$$

which are symmetric under the permutations of (a_1, a_2, a_3, a_4).

$$c_n = (-1)^n (n + b_1 - 1)_n, \tag{4.9}$$

$$a_n^{\text{rec}} = \frac{(n + b_1 - 1) \prod_{j=2}^{4} (n + a_1 + a_j)}{(2n + b_1 - 1)(2n + b_1)} + \frac{n \prod_{2 \leq j < k \leq 4} (n + a_j + a_k - 1)}{(2n + b_1 - 2)(2n + b_1 - 1)} - a_1^2, \tag{4.10}$$

$$b_n^{\text{rec}} = \frac{n(n+b_1-2)\prod_{1\le j<k\le 4}(n+a_j+a_k-1)}{(2n+b_1-3)(2n+b_1-2)^2(2n+b_1-1)}, \tag{4.11}$$

$$f_n(\boldsymbol{\lambda}) = -n(n+b_1-1), \quad b_n(\boldsymbol{\lambda}) = -1. \tag{4.12}$$

annihilation/creation operators and commutation relations

$$\alpha_\pm(\mathcal{H}) = 1 \pm 2\sqrt{\mathcal{H}'}, \quad \mathcal{H}' = \mathcal{H} + \tfrac{1}{4}(b_1-1)^2, \tag{4.13}$$

$$\mathcal{N} = \sqrt{\mathcal{H}'} - \tfrac{1}{2}(b_1-1) \quad (\text{for } b_1 > 1), \tag{4.14}$$

$$[\mathcal{H}, a^{(\pm)}] = a^{(\pm)}(1 \pm 2\sqrt{\mathcal{H}'}). \tag{4.15}$$

The annihilation/creation operators (2.55) and their commutation relation (2.63) are not so simplified because the expression $b_{n+1}^{\text{rec}} - b_n^{\text{rec}} =$ (10-th degree polynomial in n)/(7-th degree polynomial in n) is quite complicated.

coherent state

$$\psi(\alpha, x\,;\boldsymbol{\lambda}) = \phi_0(x\,;\boldsymbol{\lambda}) \sum_{n=0}^{\infty} \frac{(-1)^n (b_1)_{2n}\, \alpha^n}{n! \prod_{1\le j<k\le 4}(a_j+a_k)_n} P_n(\eta(x)\,;\boldsymbol{\lambda}). \tag{4.16}$$

The r.h.s. is symmetric under the permutations of (a_1, a_2, a_3, a_4). It is not known to us if a concise summation formula exists or not. Several non-symmetric generating functions for the Wilson polynomial are given in [6].

orthogonality

$$\int_0^\infty \phi_0(x\,;\boldsymbol{\lambda})^2 P_n(\eta\,;\boldsymbol{\lambda}) P_m(\eta\,;\boldsymbol{\lambda}) dx = 2\pi n!\,(n+b_1-1)_n \frac{\prod_{1\le j<k\le 4}\Gamma(n+a_j+a_k)}{\Gamma(2n+b_1)}\, \delta_{nm}, \tag{4.17}$$

$$\frac{1}{h_0(\boldsymbol{\lambda})} = \frac{\Gamma(b_1)}{2\pi \prod_{1\le j<k\le 4}\Gamma(a_j+a_k)}, \quad \frac{h_0(\boldsymbol{\lambda})}{h_n(\boldsymbol{\lambda})} = \frac{b_1+2n-1}{b_1+n-1} \frac{(b_1)_n}{n! \prod_{1\le j<k\le 4}(a_j+a_k)_n}. \tag{4.18}$$

4.2 continuous dual Hahn [KS1.3]

This is a restricted case of the Wilson polynomial with $a_4 = 0$. In previous works [10, 11, 12, 13], the parameters a_1, a_2 and a_3 were real and positive. Now they are $\{a_1^*, a_2^*, a_3^*\} = \{a_1, a_2, a_3\}$, as a set and $\text{Re}\, a_i > 0$. This is dual to the continuous Hahn §3.1 in the sense that the roles of $\eta(x)$ and $\mathcal{E}_n$ are interchanged. For the continuous Hahn, $\eta(x) = x$ and $\mathcal{E}_n$ is quadratic in n, whereas $\eta(x)$ is quadratic in x and $\mathcal{E}_n = n$ for the dual Hahn. The duality has sharper meaning for polynomials with discrete orthogonality measures, see for example [7].

parameters and potential function

$$\boldsymbol{\lambda} \stackrel{\text{def}}{=} (a_1, a_2, a_3), \quad \boldsymbol{\delta} = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}); \quad V(x\,;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{(a_1+ix)(a_2+ix)(a_3+ix)}{2ix(2ix+1)}. \tag{4.19}$$

shape invariance and closure relation

$$\mathcal{E}_n(\boldsymbol{\lambda}) = n, \tag{4.20}$$

$$R_1(y) = 0, \quad R_0(y) = 1, \quad R_{-1}(y) = -2y^2 + (1-2b_1)y - b_2, \tag{4.21}$$

$$b_1 \stackrel{\text{def}}{=} a_1 + a_2 + a_3, \quad b_2 \stackrel{\text{def}}{=} a_1a_2 + a_1a_3 + a_2a_3. \tag{4.22}$$

eigenfunctions

$$\phi_0(x\,;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \left|\frac{\prod_{j=1}^{3}\Gamma(a_j+ix)}{\Gamma(2ix)}\right|, \tag{4.23}$$

$$\begin{aligned} P_n(\eta\,;\boldsymbol{\lambda}) &= S_n(x^2\,;a_1,a_2,a_3) \\ &\stackrel{\text{def}}{=} (a_1+a_2)_n(a_1+a_3)_n\ {}_3F_2\Bigl(\genfrac{}{}{0pt}{}{-n,\ a_1+ix,\ a_1-ix}{a_1+a_2,\ a_1+a_3}\Bigm| 1\Bigr), \end{aligned} \tag{4.24}$$

which are symmetric under the permutations of (a_1, a_2, a_3).

$$c_n = (-1)^n, \tag{4.25}$$

$$a_n^{\text{rec}} = (n+a_1+a_2)(n+a_1+a_3) + n(n+a_2+a_3-1) - a_1^2, \tag{4.26}$$

$$b_n^{\text{rec}} = n \prod_{1\le j<k\le 3}(n+a_j+a_k-1), \tag{4.27}$$

$$f_n(\boldsymbol{\lambda}) = -n, \quad b_n(\boldsymbol{\lambda}) = -1. \tag{4.28}$$

annihilation/creation operators and commutation relations

$$\alpha_\pm(\mathcal{H}) = \pm 1, \qquad \mathcal{N} = \mathcal{H}, \tag{4.29}$$

$$a^{(\pm)} = \pm\tfrac{1}{2}[\mathcal{H},\eta] + \tfrac{1}{2}\eta - \mathcal{H}^2 - (b_1 - \tfrac{1}{2})\mathcal{H} - \tfrac{1}{2}b_2, \tag{4.30}$$

$$b_{n+1}^{\text{rec}} - b_n^{\text{rec}} = 4n^3 + 3(2b_1-1)n^2 + \bigl(2b_1(b_1-1) + 2b_2 + 1\bigr)n + b_1b_2 - a_1a_2a_3. \tag{4.31}$$

The interesting algebra, reported in [13], with $\mathcal{H}^3$ non-linearity on the r.h.s. of (4.33) is valid for the full parameter range:

$$[\mathcal{H}, a^{(\pm)}] = \pm a^{(\pm)}, \tag{4.32}$$

$$[a^{(-)},a^{(+)}]=4\mathcal{H}^3+3(2b_1-1)\mathcal{H}^2+\bigl(2b_1(b_1-1)+2b_2+1\bigr)\mathcal{H}+b_1b_2-a_1a_2a_3. \tag{4.33}$$

$\boldsymbol{\lambda}$-shift operators

$$\begin{aligned}X&=-iS_+T_++\Bigl(x-iV(x-\tfrac{i}{2})^*-i\,\frac{\prod_{j=1}^3(2a_j-1)}{8(1+x^2)}\Bigr)S_+\\&\quad+iS_-T_-+\Bigl(x+iV(x-\tfrac{i}{2})\ +i\,\frac{\prod_{j=1}^3(2a_j-1)}{8(1+x^2)}\Bigr)S_-,\end{aligned} \tag{4.34}$$

$$\phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1}\,X(\boldsymbol{\lambda})\ \phi_0(x\,;\boldsymbol{\lambda})\cdot P_n(\eta\,;\boldsymbol{\lambda})=P_n(\eta\,;\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{4.35}$$

$$\phi_0(x\,;\boldsymbol{\lambda})^{-1}\,X(\boldsymbol{\lambda})^\dagger\ \phi_0(x\,;\boldsymbol{\lambda}+\boldsymbol{\delta})\cdot P_n(\eta\,;\boldsymbol{\lambda}+\boldsymbol{\delta})=\prod_{1\le j<k\le 3}(n+a_j+a_k)\cdot P_n(\eta\,;\boldsymbol{\lambda}). \tag{4.36}$$

coherent state

$$\psi(\alpha,x\,;\boldsymbol{\lambda})=\phi_0(x\,;\boldsymbol{\lambda})\sum_{n=0}^\infty\frac{(-1)^n\alpha^n}{n!\prod_{1\le j<k\le 3}(a_j+a_k)_n}\,P_n(\eta(x)\,;\boldsymbol{\lambda}). \tag{4.37}$$

The r.h.s. is symmetric under the permutations of (a_1,a_2,a_3). We are not aware if a concise summation formula exists or not. Several non-symmetric generating functions for the continuous dual Hahn polynomial are given in [6].

orthogonality

$$\int_0^\infty\phi_0(x\,;\boldsymbol{\lambda})^2P_n(\eta\,;\boldsymbol{\lambda})P_m(\eta\,;\boldsymbol{\lambda})dx=2\pi n!\prod_{1\le j<k\le 3}\Gamma(n+a_j+a_k)\cdot\delta_{nm}, \tag{4.38}$$

$$\frac{1}{h_0(\boldsymbol{\lambda})}=\frac{1}{2\pi\prod_{1\le j<k\le 3}\Gamma(a_j+a_k)}\,,\quad\frac{h_0(\boldsymbol{\lambda})}{h_n(\boldsymbol{\lambda})}=\frac{1}{n!\prod_{1\le j<k\le 3}(a_j+a_k)_n}\,. \tag{4.39}$$

5 $\eta(x)=\cos x$

In all the examples in this section, we have[1]

$$\eta(x)=\cos x,\quad 0<x<\pi,\quad \gamma=\log q,\quad \kappa=q^{-1},\quad \varphi(x)=2\sin x. \tag{5.1}$$

Throughout this paper q is always in the range $0<q<1$ and this will not be indicated. It is convenient to introduce a complex variable $z=e^{ix}$. Then the shift operator $e^{\gamma p}$ can be written as

$$e^{\gamma p}=e^{-i\gamma\frac{d}{dx}}=q^{z\frac{d}{dz}}, \tag{5.2}$$

[1] We have changed the sign of $\varphi(x)$ from [13].

whose action on a function of x can be expressed as $z \to qz$:

$$e^{\gamma p} f(x) = f(x - i\gamma) = q^{z\frac{d}{dz}} \check{f}(z) = \check{f}(qz), \quad \text{with } f(x) = \check{f}(z).$$

Note that $\gamma < 0$.

5.1 Askey-Wilson [KS3.1]

The Askey-Wilson polynomial is the most general one with the maximal number of parameters, four. All the other polynomials in this section are obtained by restricting the parameters $a_1,\ldots,a_4$, in one way or another. In previous publications [10, 11, 12, 13] these restricted polynomials were not discussed individually, since their exact solvability is a simple corollary of that of the Askey-Wilson. However, the simpler structure of the restricted ones would give rise to simple energy spectrum and interesting and tractable forms of the dynamical symmetry algebras and coherent states, etc., as exemplified by the continuous q-Hermite polynomial §5.5, which has $a_1 = a_2 = a_3 = a_4 = 0$. It gives a most natural realisation of the q-oscillator algebra [14].

parameters and potential function

$$q^{\boldsymbol{\lambda}} \stackrel{\text{def}}{=} (a_1, a_2, a_3, a_4), \quad \boldsymbol{\delta} = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}), \quad q; \tag{5.3}$$

$$V(x\,;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{(1-a_1z)(1-a_2z)(1-a_3z)(1-a_4z)}{(1-z^2)(1-qz^2)}, \quad z = e^{ix}. \tag{5.4}$$

The parameters have to satisfy the conditions

$$\{a_1^*, a_2^*, a_3^*, a_4^*\} = \{a_1, a_2, a_3, a_4\} \quad \text{(as a set)}, \qquad |a_i| < 1, \quad i = 1, \ldots, 4. \tag{5.5}$$

In previous works [10, 11, 12, 13] only the real parameters $a_i \in \mathbb{R}$ were discussed.

shape invariance and closure relation

$$\mathcal{E}_n(\boldsymbol{\lambda}) = (q^{-n} - 1)(1 - b_4 q^{n-1}), \tag{5.6}$$

$$R_1(y) = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 y', \quad y' \stackrel{\text{def}}{=} y + 1 + q^{-1} b_4, \tag{5.7}$$

$$R_0(y) = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 \big(y'^2 - (1 + q^{-1})^2 b_4\big), \tag{5.8}$$

$$R_{-1}(y) = -\tfrac{1}{2}(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 \big((b_1 + q^{-1} b_3) y' - (1 + q^{-1})(b_3 + q^{-1} b_1 b_4)\big), \tag{5.9}$$

$$b_1 \stackrel{\text{def}}{=} \sum_{j=1}^{4} a_j, \quad b_3 \stackrel{\text{def}}{=} \sum_{1 \le j < k < l \le 4} a_j a_k a_l, \quad b_4 \stackrel{\text{def}}{=} a_1 a_2 a_3 a_4. \tag{5.10}$$

eigenfunctions

$$\phi_0(x\,;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\Bigl|\frac{(e^{2ix}\,;q)_\infty}{\prod_{j=1}^4(a_je^{ix}\,;q)_\infty}\Bigr|, \tag{5.11}$$

$$\begin{aligned}P_n(\eta\,;\boldsymbol{\lambda})&=p_n(\cos x\,;a_1,a_2,a_3,a_4|q)\\&\stackrel{\text{def}}{=}a_1^{-n}(a_1a_2,a_1a_3,a_1a_4\,;q)_n\times{}_4\phi_3\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\ a_1a_2a_3a_4q^{n-1},\ a_1e^{ix},\ a_1e^{-ix}}{a_1a_2,\ a_1a_3,\ a_1a_4}\Bigm|q\,;q\Bigr),\end{aligned} \tag{5.12}$$

which are symmetric under the permutations of (a_1,a_2,a_3,a_4).

$$c_n=2^n(b_4q^{n-1}\,;q)_n, \tag{5.13}$$

$$\begin{aligned}a_n^{\text{rec}}=\frac12\biggl(a_1+a_1^{-1}&-\frac{(1-b_4q^{n-1})\prod_{j=2}^4(1-a_1a_jq^n)}{a_1(1-b_4q^{2n-1})(1-b_4q^{2n})}\\&-\frac{a_1(1-q^n)\prod_{2\le j<k\le 4}(1-a_ja_kq^{n-1})}{(1-b_4q^{2n-2})(1-b_4q^{2n-1})}\biggr),\end{aligned} \tag{5.14}$$

$$b_n^{\text{rec}}=\frac{(1-q^n)(1-b_4q^{n-2})\prod_{1\le j<k\le 4}(1-a_ja_kq^{n-1})}{4(1-b_4q^{2n-3})(1-b_4q^{2n-2})^2(1-b_4q^{2n-1})}, \tag{5.15}$$

$$f_n(\boldsymbol{\lambda})=q^{\frac n2}(q^{-n}-1)(1-b_4q^{n-1}),\quad b_n(\boldsymbol{\lambda})=q^{-\frac{n+1}{2}}. \tag{5.16}$$

annihilation/creation operators and commutation relations

$$\alpha_\pm(\mathcal{H})=\tfrac12(q^{-\frac12}-q^{\frac12})^2\mathcal{H}'\pm\tfrac12(q^{-1}-q)\sqrt{\mathcal{H}'^{\,2}-4q^{-1}b_4}\,,\quad \mathcal{H}'=\mathcal{H}+1+q^{-1}b_4\,, \tag{5.17}$$

$$q^{\mathcal{N}}=\frac{q}{2b_4}\Bigl(\mathcal{H}'-\sqrt{\mathcal{H}'^{\,2}-4q^{-1}b_4}\Bigr)\quad(\text{for } 0<b_4<q), \tag{5.18}$$

$$[\mathcal{H},a^{(\pm)}]=\tfrac12a^{(\pm)}\Bigl((q^{-\frac12}-q^{\frac12})^2\mathcal{H}'\pm(q^{-1}-q)\sqrt{\mathcal{H}'^{\,2}-4q^{-1}b_4}\Bigr). \tag{5.19}$$

The annihilation/creation operators (2.55) and their commutation relation (2.63) are not simplified at all. The expression $b_{n+1}^{\text{rec}}-b_n^{\text{rec}}=q^n\times$(12-th degree polynomial in q^n)/(6-th degree polynomial in q^{2n}) is very complicated.

coherent state

$$\psi(\alpha,x\,;\boldsymbol{\lambda})=\phi_0(x\,;\boldsymbol{\lambda})\sum_{n=0}^\infty\frac{2^n(b_4\,;q)_{2n}\alpha^n}{(q\,;q)_n\prod_{1\le j<k\le 4}(a_ja_k\,;q)_n}\,P_n(\eta(x)\,;\boldsymbol{\lambda}). \tag{5.20}$$

The r.h.s. is symmetric under the permutations of (a_1,a_2,a_3,a_4). We are not aware if a concise summation formula exists or not. Several non-symmetric generating functions for the Askey-Wilson polynomial are given in [6].

orthogonality

$$\int_0^\pi \phi_0(x\,;\boldsymbol{\lambda})^2 P_n(\eta\,;\boldsymbol{\lambda})P_m(\eta\,;\boldsymbol{\lambda})dx = 2\pi\,\frac{(b_4q^{n-1}\,;q)_n(b_4q^{2n}\,;q)_\infty}{(q^{n+1}\,;q)_\infty\prod_{1\le j<k\le 4}(a_ja_kq^n\,;q)_\infty}\,\delta_{nm}, \tag{5.21}$$

$$\frac{1}{h_0(\boldsymbol{\lambda})} = \frac{(q\,;q)_\infty\prod_{1\le j<k\le 4}(a_ja_k\,;q)_\infty}{2\pi(b_4\,;q)_\infty}, \quad \frac{h_0(\boldsymbol{\lambda})}{h_n(\boldsymbol{\lambda})} = \frac{1-b_4q^{2n-1}}{1-b_4q^{n-1}}\frac{(b_4\,;q)_n}{(q\,;q)_n\prod_{1\le j<k\le 4}(a_ja_k\,;q)_n}. \tag{5.22}$$

5.1.1 Askey-Wilson → Wilson

The Wilson polynomial is obtained from the Askey-Wilson polynomial by a $q\uparrow 1$ limit. Here we present a dictionary of the correspondence for future reference. Let us first introduce a new coordinate x' for the Wilson polynomial as the rescaled one of the variable x $(0<x<\pi)$ of the Askey-Wilson polynomial:

$$x' = \frac{L}{\pi}x, \quad \bigl(\Rightarrow 0<x'<L, \quad p' = \frac{\pi}{L}p\bigr), \quad \gamma = -\frac{\pi}{L}, \quad \boldsymbol{\lambda} = (a_1', a_2', a_3', a_4'), \tag{5.23}$$

in which L is related to q as $q=e^{-\pi/L}$. This entails

$$e^{\gamma p} = e^{-p'} \tag{5.24}$$

and the following limit formulas as $L\to\infty$ or $q\to 1$: (The superscript W denote the corresponding quantity for the Wilson polynomial.)

$$\lim_{L\to\infty}\frac{V(x\,;\boldsymbol{\lambda})}{(1-q)^2} = V^{\rm W}(x'\,;\boldsymbol{\lambda})^*, \tag{5.25}$$

$$\lim_{L\to\infty}\frac{\mathcal{H}(\boldsymbol{\lambda})}{(1-q)^2} = \mathcal{H}^{\rm W}(\boldsymbol{\lambda}), \quad \lim_{L\to\infty}\frac{\mathcal{E}_n(\boldsymbol{\lambda})}{(1-q)^2} = \mathcal{E}_n^{\rm W}(\boldsymbol{\lambda}), \tag{5.26}$$

$$\lim_{L\to\infty}(q\,;q)_\infty^3(1-q)^{3-\sum_j a_j'}\phi_0(x\,;\boldsymbol{\lambda}) = \phi_0^{\rm W}(x'\,;\boldsymbol{\lambda}), \quad \lim_{L\to\infty}\frac{\varphi(x)}{1-q} = \varphi^{\rm W}(x'), \tag{5.27}$$

$$\lim_{L\to\infty}\frac{P_n(\eta(x)\,;\boldsymbol{\lambda})}{(1-q)^{3n}} = P_n^{\rm W}(\eta^{\rm W}(x')\,;\boldsymbol{\lambda}), \tag{5.28}$$

$$\lim_{L\to\infty}(1-q)\mathcal{F}(\boldsymbol{\lambda}) = -\mathcal{F}^{\rm W}(\boldsymbol{\lambda}), \quad \lim_{L\to\infty}\frac{f_n(\boldsymbol{\lambda})}{(1-q)^2} = -f_n^{\rm W}(\boldsymbol{\lambda}), \tag{5.29}$$

$$\lim_{L\to\infty}\frac{\mathcal{B}(\boldsymbol{\lambda})}{(1-q)^3} = -\mathcal{B}^{\rm W}(\boldsymbol{\lambda}), \quad \lim_{L\to\infty}b_n(\boldsymbol{\lambda}) = -b_n^{\rm W}(\boldsymbol{\lambda}). \tag{5.30}$$

5.2 continuous dual q-Hahn [KS3.3]

The continuous dual q-Hahn polynomial is obtained by restricting $a_4=0$ in the Askey-Wilson polynomial §5.1. This restriction renders the energy spectrum to a simple form

$\mathcal{E}_n = q^{-n}-1$ for all the restricted polynomials in section 5 except for the continuous q-Jacobi polynomial §5.6 and the continuous q-Hahn polynomial §5.8.1. For these the commutation relations of $\mathcal{H}$ and of $a^{(\pm)}$ is the same (5.47), (5.66), (5.86), (5.104) and (5.153). They can be expressed as q-deformed commutators (5.49), (5.68), (5.88), (5.106) and (5.155). The commutation relation $[a^{(-)}, a^{(+)}]$ or its deformation becomes drastically simpler, as the number of parameters decreases.

parameters and potential function

$$q^{\boldsymbol{\lambda}} \stackrel{\text{def}}{=} (a_1, a_2, a_3), \quad \boldsymbol{\delta} = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}), \quad q; \tag{5.31}$$

$$V(x\,;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{(1-a_1z)(1-a_2z)(1-a_3z)}{(1-z^2)(1-qz^2)}, \quad z = e^{ix}. \tag{5.32}$$

The parameters have to satisfy the conditions

$$\{a_1^*, a_2^*, a_3^*\} = \{a_1, a_2, a_3\} \quad \text{(as a set)}, \qquad |a_i| < 1, \quad i = 1, 2, 3. \tag{5.33}$$

shape invariance and closure relation

$$\mathcal{E}_n(\boldsymbol{\lambda}) = q^{-n} - 1, \tag{5.34}$$

$$R_1(y) = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 y', \quad y' \stackrel{\text{def}}{=} y + 1, \quad R_0(y) = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 y'^2, \tag{5.35}$$

$$R_{-1}(y) = -\tfrac{1}{2}(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 \big((b_1 + q^{-1}b_3)y' - (1+q^{-1})b_3\big), \tag{5.36}$$

$$b_1 \stackrel{\text{def}}{=} a_1 + a_2 + a_3, \quad b_2 \stackrel{\text{def}}{=} a_1a_2 + a_1a_3 + a_2a_3, \quad b_3 \stackrel{\text{def}}{=} a_1a_2a_3. \tag{5.37}$$

eigenfunctions

$$\phi_0(x\,;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \left|\frac{(e^{2ix}\,;q)_\infty}{\prod_{j=1}^3 (a_j e^{ix}\,;q)_\infty}\right|, \tag{5.38}$$

$$\begin{aligned} P_n(\eta\,;\boldsymbol{\lambda}) &= p_n(\cos x\,; a_1, a_2, a_3|q) \\ &\stackrel{\text{def}}{=} a_1^{-n}(a_1a_2, a_1a_3\,;q)_n\, {}_3\phi_2\Big(\begin{matrix} q^{-n},\ a_1e^{ix},\ a_1e^{-ix} \\ a_1a_2,\ a_1a_3 \end{matrix}\ \Big|\ q\,;q\Big), \end{aligned} \tag{5.39}$$

which are symmetric under the permutations of (a_1, a_2, a_3).

$$c_n = 2^n, \tag{5.40}$$

$$a_n^{\text{rec}} = \tfrac{1}{2}\big(a_1 + a_1^{-1} - a_1^{-1}(1-a_1a_2q^n)(1-a_1a_3q^n) - a_1(1-q^n)(1-a_2a_3q^{n-1})\big), \tag{5.41}$$

$$b_n^{\text{rec}} = \tfrac{1}{4}(1-q^n)\prod_{1\le j<k\le 3}(1-a_ja_kq^{n-1}), \tag{5.42}$$

$$f_n(\boldsymbol{\lambda}) = q^{\frac{n}{2}}(q^{-n}-1), \quad b_n(\boldsymbol{\lambda}) = q^{-\frac{n+1}{2}}. \tag{5.43}$$

annihilation/creation operators and commutation relations

$$\alpha_\pm(\mathcal{H}) = (q^{\mp 1} - 1)(\mathcal{H} + 1), \qquad q^{\mathcal{N}} = (\mathcal{H} + 1)^{-1}, \tag{5.44}$$

$$a^{(\pm)} = \frac{\pm 1}{q^{-1} - q}\Big([\mathcal{H}, \eta]_{q^{\pm 1}} + (1 - q^{\pm 1})\big(\eta - \tfrac{1}{2}(b_1 + q^{-1}b_3) + \tfrac{1}{2}(1 + q^{-1})b_3(\mathcal{H} + 1)^{-1}\big)\Big)(\mathcal{H} + 1)^{-1}, \tag{5.45}$$

$$b^{\text{rec}}_{n+1} - b^{\text{rec}}_n = -\tfrac{1}{4}(q^{-4} - 1)qb_3^2q^{4n} + \tfrac{1}{4}(q^{-3} - 1)b_3(b_3 + qb_1)q^{3n} - \tfrac{1}{4}(q^{-2} - 1)(b_1b_3 + qb_2)q^{2n} + \tfrac{1}{4}(q^{-1} - 1)(b_2 + q)q^n, \tag{5.46}$$

$$[\mathcal{H}, a^{(\pm)}] = (q^{\mp 1} - 1)a^{(\pm)}(\mathcal{H} + 1), \tag{5.47}$$

$$[a^{(-)}, a^{(+)}] = -\tfrac{1}{4}(q^{-4} - 1)qb_3^2(\mathcal{H} + 1)^{-4} + \tfrac{1}{4}(q^{-3} - 1)b_3(b_3 + qb_1)(\mathcal{H} + 1)^{-3} - \tfrac{1}{4}(q^{-2} - 1)(b_1b_3 + qb_2)(\mathcal{H} + 1)^{-2} + \tfrac{1}{4}(q^{-1} - 1)(b_2 + q)(\mathcal{H} + 1)^{-1}. \tag{5.48}$$

The r.h.s. of the above commutation relation is a quartic polynomial in $q^{\mathcal{N}}$. In terms of a deformed commutator we have:

$$\mathcal{H}a^{(\pm)} - q^{\mp 1}a^{(\pm)}\mathcal{H} = (q^{\mp 1} - 1)a^{(\pm)}, \quad \text{namely,} \quad [\mathcal{H}, a^{(\pm)}]_{q^{\mp 1}} = (q^{\mp 1} - 1)a^{(\pm)}. \tag{5.49}$$

The following relation

$$b^{\text{rec}}_{n+1} - q^4b^{\text{rec}}_n = -\tfrac{1}{4}(1 - q)b_3(b_3 + qb_1)q^{3n} + \tfrac{1}{4}(1 - q^2)(b_1b_3 + qb_2)q^{2n} - \tfrac{1}{4}(1 - q^3)(b_2 + q)q^n + \tfrac{1}{4}(1 - q^4) \tag{5.50}$$

means that $[a^{(-)}, a^{(+)}]_{q^4}$ is a cubic polynomial in $q^{\mathcal{N}}$.

coherent state

$$\psi(\alpha, x\,; \boldsymbol{\lambda}) = \phi_0(x\,; \boldsymbol{\lambda}) \sum_{n=0}^{\infty} \frac{2^n\alpha^n}{(q\,; q)_n \prod_{1\le j<k\le 3}(a_ja_k\,; q)_n} P_n(\eta(x)\,; \boldsymbol{\lambda}). \tag{5.51}$$

orthogonality

$$\int_0^\pi \phi_0(x\,; \boldsymbol{\lambda})^2 P_n(\eta\,; \boldsymbol{\lambda})P_m(\eta\,; \boldsymbol{\lambda})dx = 2\pi \frac{1}{(q^{n+1}\,; q)_\infty \prod_{1\le j<k\le 3}(a_ja_kq^n\,; q)_\infty} \delta_{nm}, \tag{5.52}$$

$$\frac{1}{h_0(\boldsymbol{\lambda})} = \frac{1}{2\pi}(q\,; q)_\infty \prod_{1\le j<k\le 3}(a_ja_k\,; q)_\infty\,, \quad \frac{h_0(\boldsymbol{\lambda})}{h_n(\boldsymbol{\lambda})} = \frac{1}{(q\,; q)_n \prod_{1\le j<k\le 3}(a_ja_k\,; q)_n}\,. \tag{5.53}$$

5.3 Al-Salam-Chihara [KS3.8]

This is a further restriction of the continuous dual q-Hahn polynomial §5.2 with $a_3 = 0$. The dynamical symmetry algebra is further simplified and $[a^{(-)}, a^{(+)}]$ is a quadratic polynomial in $q^{\mathcal{N}}$. The coherent state gives an explicit generating function with symmetry $a_1 \leftrightarrow a_2$ (5.71).

parameters and potential function

$$q^{\boldsymbol{\lambda}} \stackrel{\text{def}}{=} (a_1, a_2),\ \boldsymbol{\delta} = (\tfrac{1}{2}, \tfrac{1}{2}),\ q; \quad \{a_1^*, a_2^*\} = \{a_1, a_2\} \text{ (as a set)}, \quad |a_i| < 1, \quad i = 1, 2; \tag{5.54}$$

$$V(x\,;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{(1 - a_1 z)(1 - a_2 z)}{(1 - z^2)(1 - q z^2)}, \quad z = e^{ix}. \tag{5.55}$$

shape invariance and closure relation

$$\mathcal{E}_n(\boldsymbol{\lambda}) = q^{-n} - 1, \tag{5.56}$$

$$R_1(y) = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 y', \quad y' \stackrel{\text{def}}{=} y + 1, \quad R_0(y) = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 y'^2, \tag{5.57}$$

$$R_{-1}(y) = -\tfrac{1}{2}(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 (a_1 + a_2) y'. \tag{5.58}$$

eigenfunctions

$$\phi_0(x\,;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \left| \frac{(e^{2ix}\,; q)_\infty}{\prod_{j=1}^2 (a_j e^{ix}\,; q)_\infty} \right|, \tag{5.59}$$

$$P_n(\eta\,;\boldsymbol{\lambda}) = Q_n(\cos x\,; a_1, a_2|q) \stackrel{\text{def}}{=} a_1^{-n} (a_1 a_2\,; q)_n\, {}_3\phi_2\Bigl(\begin{matrix} q^{-n},\ a_1 e^{ix},\ a_1 e^{-ix} \\ a_1 a_2,\ 0 \end{matrix} \Bigm| q\,; q \Bigr), \tag{5.60}$$

which are symmetric under the permutations of (a_1, a_2).

$$c_n = 2^n, \quad a_n^{\text{rec}} = \tfrac{1}{2}(a_1 + a_2) q^n, \quad b_n^{\text{rec}} = \tfrac{1}{4}(1 - q^n)(1 - a_1 a_2 q^{n-1}), \tag{5.61}$$

$$f_n(\boldsymbol{\lambda}) = q^{\frac{n}{2}}(q^{-n} - 1), \quad b_n(\boldsymbol{\lambda}) = q^{-\frac{n+1}{2}}. \tag{5.62}$$

annihilation/creation operators and commutation relations

$$\alpha_\pm(\mathcal{H}) = (q^{\mp 1} - 1)(\mathcal{H} + 1), \qquad q^{\mathcal{N}} = (\mathcal{H} + 1)^{-1}, \tag{5.63}$$

$$a^{(\pm)} = \frac{\pm 1}{q^{-1} - q} \Bigl([\mathcal{H}, \eta]_{q^{\pm 1}} + (1 - q^{\pm 1})\bigl(\eta - \tfrac{1}{2}(a_1 + a_2)\bigr) \Bigr) (\mathcal{H} + 1)^{-1}, \tag{5.64}$$

$$b_{n+1}^{\text{rec}} - b_n^{\text{rec}} = \tfrac{1}{4}(q^{-1} - 1)\bigl(-(1 + q) a_1 a_2 q^{2n} + (a_1 a_2 + q) q^n\bigr), \tag{5.65}$$

$$[\mathcal{H}, a^{(\pm)}] = (q^{\mp 1} - 1) a^{(\pm)} (\mathcal{H} + 1), \tag{5.66}$$

$$[a^{(-)},a^{(+)}]=\tfrac{1}{4}(q^{-1}-1)\bigl(-(1+q)a_1a_2(\mathcal{H}+1)^{-2}+(a_1a_2+q)(\mathcal{H}+1)^{-1}\bigr). \tag{5.67}$$

The r.h.s. is a quadratic polynomial in $q^{\mathcal{N}}$. The deformed commutators are:

$$\mathcal{H}a^{(\pm)}-q^{\mp1}a^{(\pm)}\mathcal{H}=(q^{\mp1}-1)a^{(\pm)},\quad \text{namely},\quad [\mathcal{H},a^{(\pm)}]_{q^{\mp1}}=(q^{\mp1}-1)a^{(\pm)}. \tag{5.68}$$

Other interesting quantities are:

$$b_{n+1}^{\text{rec}}-qb_n^{\text{rec}}=\tfrac{1}{4}(1-q)(1-a_1a_2q^{2n}), \tag{5.69}$$

$$b_{n+1}^{\text{rec}}-q^2b_n^{\text{rec}}=\tfrac{1}{4}(1-q)\bigl(1+q-(a_1a_2+q)q^n\bigr). \tag{5.70}$$

These mean that $[a^{(-)},a^{(+)}]_q$ and $[a^{(-)},a^{(+)}]_{q^2}$ take simple forms and, in particular, the latter is linear in $q^{\mathcal{N}}$. As we will see in another example, the continuous q-Laguerre §5.7, these are special to the restricted Askey-Wilson polynomials with a quadratic polynomial $(1-a_1z)(1-a_2z)$ in the numerator of the potential function $V(x)$, see (5.156)–(5.157).

coherent state

$$\begin{aligned}\psi(\alpha,x\,;\boldsymbol{\lambda})&=\phi_0(x\,;\boldsymbol{\lambda})\sum_{n=0}^{\infty}\frac{2^n\alpha^n}{(q,a_1a_2\,;q)_n}\,P_n(\eta(x)\,;\boldsymbol{\lambda})\\&=\phi_0(x\,;\boldsymbol{\lambda})\,\frac{1}{(2\alpha e^{ix}\,;q)_\infty}\,{}_2\phi_1\Bigl(\begin{matrix}a_1e^{ix},\ a_2e^{ix}\\ a_1a_2\end{matrix}\Bigm|q\,;2\alpha e^{-ix}\Bigr),\end{aligned} \tag{5.71}$$

which is obviously symmetric in $a_1\leftrightarrow a_2$ and listed in [6].

orthogonality

$$\int_0^{\pi}\phi_0(x\,;\boldsymbol{\lambda})^2P_n(\eta\,;\boldsymbol{\lambda})P_m(\eta\,;\boldsymbol{\lambda})dx=2\pi\,\frac{1}{(q^{n+1},a_1a_2q^n\,;q)_\infty}\,\delta_{nm}, \tag{5.72}$$

$$\frac{1}{h_0(\boldsymbol{\lambda})}=\frac{1}{2\pi}(q,a_1a_2\,;q)_\infty\,,\quad \frac{h_0(\boldsymbol{\lambda})}{h_n(\boldsymbol{\lambda})}=\frac{1}{(q\,;q)_n(a_1a_2\,;q)_n}\,. \tag{5.73}$$

5.4 continuous big q-Hermite [KS3.18]

This is a further restriction of the Al-Salam-Chihara polynomial §5.3 with $a_2=0$. The continuous big q-Hermite gives another simple realisation of the q-oscillator algebra (5.90).

parameters and potential function

$$q^{\boldsymbol{\lambda}}\stackrel{\text{def}}{=}a,\quad \boldsymbol{\delta}=\tfrac{1}{2},\quad q;\quad -1<a<1; \tag{5.74}$$

$$V(x\,;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\frac{1-az}{(1-z^2)(1-qz^2)},\quad z=e^{ix}. \tag{5.75}$$

shape invariance and closure relation

$$\mathcal{E}_n(\boldsymbol{\lambda}) = q^{-n} - 1, \tag{5.76}$$

$$R_1(y) = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 y', \quad y' \stackrel{\text{def}}{=} y + 1, \quad R_0(y) = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 y'^2, \tag{5.77}$$

$$R_{-1}(y) = -\tfrac{1}{2}(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 a y'. \tag{5.78}$$

eigenfunctions

$$\phi_0(x\,;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \left|\frac{(e^{2ix}\,;q)_\infty}{(ae^{ix}\,;q)_\infty}\right|, \tag{5.79}$$

$$P_n(\eta\,;\boldsymbol{\lambda}) = H_n(\cos x\,;a|q) \stackrel{\text{def}}{=} a^{-n}\,{}_3\phi_2\Bigl(\begin{matrix}q^{-n},\ ae^{ix},\ ae^{-ix}\\ 0,\ 0\end{matrix}\Bigm|\,q\,;q\Bigr), \tag{5.80}$$

$$c_n = 2^n, \quad a_n^{\text{rec}} = \tfrac{1}{2} a q^n, \quad b_n^{\text{rec}} = \tfrac{1}{4}(1 - q^n), \tag{5.81}$$

$$f_n(\boldsymbol{\lambda}) = q^{\frac{n}{2}}(q^{-n} - 1), \quad b_n(\boldsymbol{\lambda}) = q^{-\frac{n+1}{2}}. \tag{5.82}$$

annihilation/creation operators and commutation relations

$$\alpha_\pm(\mathcal{H}) = (q^{\mp 1} - 1)(\mathcal{H} + 1), \qquad q^{\mathcal{N}} = (\mathcal{H} + 1)^{-1}, \tag{5.83}$$

$$a^{(\pm)} = \frac{\pm 1}{q^{-1} - q}\Bigl([\mathcal{H}, \eta]_{q^{\pm 1}} + (1 - q^{\pm 1})(\eta - \tfrac{1}{2}a)\Bigr)(\mathcal{H} + 1)^{-1}, \tag{5.84}$$

$$b_{n+1}^{\text{rec}} - b_n^{\text{rec}} = \tfrac{1}{4}(1 - q) q^n, \tag{5.85}$$

$$[\mathcal{H}, a^{(\pm)}] = (q^{\mp 1} - 1) a^{(\pm)}(\mathcal{H} + 1), \tag{5.86}$$

$$[a^{(-)}, a^{(+)}] = \tfrac{1}{4}(1 - q)(\mathcal{H} + 1)^{-1}. \tag{5.87}$$

The deformed commutator makes (5.86) simpler

$$\mathcal{H} a^{(\pm)} - q^{\mp 1} a^{(\pm)} \mathcal{H} = (q^{\mp 1} - 1) a^{(\pm)}, \quad \text{namely,} \quad [\mathcal{H}, a^{(\pm)}]_{q^{\mp 1}} = (q^{\mp 1} - 1) a^{(\pm)}. \tag{5.88}$$

The relation

$$b_{n+1}^{\text{rec}} - q b_n^{\text{rec}} = \tfrac{1}{4}(1 - q), \tag{5.89}$$

implies another realisation of the q-oscillator

$$a^{(-)} a^{(+)} - q a^{(+)} a^{(-)} = \tfrac{1}{4}(1 - q), \quad \text{namely,} \quad [a^{(-)}, a^{(+)}]_q = \tfrac{1}{4}(1 - q). \tag{5.90}$$

coherent state (2.80) reads with the help of [KS(3.18.13)]

$$\psi(\alpha, x\,;\boldsymbol{\lambda}) = \phi_0(x\,;\boldsymbol{\lambda}) \sum_{n=0}^{\infty} \frac{2^n \alpha^n}{(q\,;q)_n} P_n(\eta(x)\,;\boldsymbol{\lambda}) = \phi_0(x\,;\boldsymbol{\lambda}) \frac{(2\alpha a\,;q)_\infty}{(2\alpha e^{ix}, 2\alpha e^{-ix}\,;q)_\infty}. \tag{5.91}$$

orthogonality

$$\int_0^{\pi} \phi_0(x\,;\boldsymbol{\lambda})^2 P_n(\eta\,;\boldsymbol{\lambda}) P_m(\eta\,;\boldsymbol{\lambda}) dx = 2\pi \, \frac{1}{(q^{n+1}\,;q)_\infty}\, \delta_{nm}, \tag{5.92}$$

$$\frac{1}{h_0(\boldsymbol{\lambda})} = \frac{1}{2\pi}(q\,;q)_\infty\,, \quad \frac{h_0(\boldsymbol{\lambda})}{h_n(\boldsymbol{\lambda})} = \frac{1}{(q\,;q)_n}\,. \tag{5.93}$$

5.5 continuous q-Hermite [KS3.26]

The continuous q-Hermite polynomial has been discussed in some detail in [14] as the simplest dynamical system realising the q-oscillator algebra in two different ways (5.108) and (5.117). Here we recapitulate some formulas to make this paper complete. Like the Hermite polynomial, the continuous q-Hermite has no parameter other than q.

parameters and potential function

$$V(x\,;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{1}{(1-z^2)(1-qz^2)}, \quad z = e^{ix}. \tag{5.94}$$

shape invariance and closure relation

$$\mathcal{E}_n(\boldsymbol{\lambda}) = q^{-n} - 1, \tag{5.95}$$

$$R_1(y) = (q^{-\frac12} - q^{\frac12})^2 y', \quad y' \stackrel{\text{def}}{=} y + 1, \quad R_0(y) = (q^{-\frac12} - q^{\frac12})^2 y'^2, \quad R_{-1}(y) = 0. \tag{5.96}$$

eigenfunctions

$$\phi_0(x\,;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \big|(e^{2ix}\,;q)_\infty\big|, \tag{5.97}$$

$$P_n(\eta\,;\boldsymbol{\lambda}) = H_n(\cos x|q) \stackrel{\text{def}}{=} e^{inx}\, {}_2\phi_0\Big(\begin{matrix} q^{-n},\, 0 \\ - \end{matrix}\,\Big|\, q\,; q^n e^{-2ix}\Big), \tag{5.98}$$

$$c_n = 2^n, \quad a_n^{\text{rec}} = 0, \quad b_n^{\text{rec}} = \tfrac14(1-q^n), \tag{5.99}$$

$$f_n(\boldsymbol{\lambda}) = q^{\frac{n}{2}}(q^{-n} - 1), \quad b_n(\boldsymbol{\lambda}) = q^{-\frac{n+1}{2}}. \tag{5.100}$$

annihilation/creation operators and commutation relations

$$\alpha_\pm(\mathcal{H}) = (q^{\mp 1} - 1)(\mathcal{H} + 1), \quad q^{\mathcal{N}} = (\mathcal{H}+1)^{-1}, \tag{5.101}$$

$$a^{(\pm)} = \frac{\pm 1}{q^{-1} - q}\Big([\mathcal{H}, \eta]_{q^{\pm 1}} + (1 - q^{\pm 1})\eta\Big)(\mathcal{H}+1)^{-1}, \tag{5.102}$$

$$b_{n+1}^{\text{rec}} - b_n^{\text{rec}} = \tfrac14(1-q)q^n, \tag{5.103}$$

$$[\mathcal{H}, a^{(\pm)}] = (q^{\mp 1} - 1) a^{(\pm)}(\mathcal{H}+1), \tag{5.104}$$

$$[a^{(-)},a^{(+)}]=\tfrac{1}{4}(1-q)(\mathcal{H}+1)^{-1}. \tag{5.105}$$

The formula (5.104) can be simplified as a deformed commutator:

$$\mathcal{H}a^{(\pm)}-q^{\mp1}a^{(\pm)}\mathcal{H}=(q^{\mp1}-1)a^{(\pm)},\quad\text{namely,}\quad[\mathcal{H},a^{(\pm)}]_{q^{\mp1}}=(q^{\mp1}-1)a^{(\pm)}. \tag{5.106}$$

The following relation

$$b^{\text{rec}}_{n+1}-qb^{\text{rec}}_n=\tfrac{1}{4}(1-q), \tag{5.107}$$

means a q-oscillator algebra:

$$a^{(-)}a^{(+)}-qa^{(+)}a^{(-)}=\tfrac{1}{4}(1-q),\quad\text{namely,}\quad[a^{(-)},a^{(+)}]_q=\tfrac{1}{4}(1-q). \tag{5.108}$$

$\boldsymbol{\lambda}$-shift operators Since the theory has no parameter $\boldsymbol{\lambda}$, $\mathcal{A}^\dagger$ and $\mathcal{A}$ work as the creation and annihilation operators. Thus $a^{(+)}$ and $\mathcal{A}^\dagger$ ($a^{(-)}$ and $\mathcal{A}$) are closely related:

$$a^{(+)}=\mathcal{A}^\dagger X, \tag{5.109}$$

$$\mathcal{A}^\dagger=-i\bigl(\sqrt{V(x)}\,e^{\gamma p/2}-\sqrt{V(x)^*}\,e^{-\gamma p/2}\bigr), \tag{5.110}$$

$$X\stackrel{\text{def}}{=}-\frac{i}{2}q\bigl(z\sqrt{V(x)}\,e^{\gamma p/2}-z^{-1}\sqrt{V(x)^*}\,e^{-\gamma p/2}\bigr)(\mathcal{H}+1)^{-1}. \tag{5.111}$$

The similarity transformed quantities are:

$$\widetilde{a}^{(+)}=\widetilde{\mathcal{A}}^\dagger\widetilde{X}, \tag{5.112}$$

$$\widetilde{\mathcal{A}}^\dagger\stackrel{\text{def}}{=}\phi_0(x)^{-1}\circ\mathcal{A}^\dagger\circ\phi_0(x)=q^{-\frac{1}{2}}\Bigl(\frac{z^{-1}}{1-z^2}\,e^{\gamma p/2}+\frac{z}{1-z^{-2}}\,e^{-\gamma p/2}\Bigr), \tag{5.113}$$

$$\widetilde{X}\stackrel{\text{def}}{=}\phi_0(x)^{-1}\circ X\circ\phi_0(x)=\frac{1}{2}q^{\frac{1}{2}}\Bigl(\frac{1}{1-z^2}\,e^{\gamma p/2}+\frac{1}{1-z^{-2}}\,e^{-\gamma p/2}\Bigr)(\widetilde{\mathcal{H}}+1)^{-1}. \tag{5.114}$$

As there is no $\boldsymbol{\lambda}$ to be shifted, we have $\widetilde{\mathcal{A}}^\dagger=\mathcal{B}(\boldsymbol{\lambda})$ (2.40) and $\widetilde{\mathcal{A}}=\mathcal{F}(\boldsymbol{\lambda})$ (2.39). The $\widetilde{X}$ and $\widetilde{A}^\dagger$ operators work as

$$\widetilde{X}P_n(x)=\frac{1}{2}q^{\frac{n+1}{2}}P_n(x),\qquad\widetilde{\mathcal{A}}^\dagger P_n(x)=q^{-\frac{n+1}{2}}P_{n+1}(x) \tag{5.115}$$

and $\widetilde{X}$ satisfies the relation

$$\begin{aligned}\Bigl(2q^{-\frac{1}{2}}\widetilde{X}(\widetilde{\mathcal{H}}+1)\Bigr)^2&=\Bigl(\frac{1}{1-z^2}\,e^{\gamma p/2}+\frac{1}{1-z^{-2}}\,e^{-\gamma p/2}\Bigr)^2\\&=V(x)e^{\gamma p}+V(x)^*e^{-\gamma p}-V(x)-V(x)^*+1\\&=\widetilde{\mathcal{H}}+1.\end{aligned} \tag{5.116}$$

It is easy to verify that the shape invariance relation (2.15) itself implies a realisation of the q-oscillator algebra with $\mathcal{A}$ and $\mathcal{A}^\dagger$ [14]:

$$\mathcal{A}\mathcal{A}^\dagger-q^{-1}\mathcal{A}^\dagger\mathcal{A}=q^{-1}-1,\quad\text{namely,}\quad[\mathcal{A},\mathcal{A}^\dagger]_{q^{-1}}=q^{-1}-1. \tag{5.117}$$

coherent state (2.80) reads with the help of [KS(3.26.11)]

$$\psi(\alpha,x\,;\boldsymbol{\lambda})=\phi_0(x\,;\boldsymbol{\lambda})\sum_{n=0}^{\infty}\frac{2^n\alpha^n}{(q\,;q)_n}\,P_n(\eta(x)\,;\boldsymbol{\lambda})=\phi_0(x\,;\boldsymbol{\lambda})\,\frac{1}{(2\alpha e^{ix},2\alpha e^{-ix}\,;q)_\infty}. \tag{5.118}$$

orthogonality

$$\int_0^{\pi}\phi_0(x\,;\boldsymbol{\lambda})^2P_n(\eta\,;\boldsymbol{\lambda})P_m(\eta\,;\boldsymbol{\lambda})dx=2\pi\,\frac{1}{(q^{n+1}\,;q)_\infty}\,\delta_{nm}, \tag{5.119}$$

$$\frac{1}{h_0(\boldsymbol{\lambda})}=\frac{1}{2\pi}(q\,;q)_\infty\,,\quad \frac{h_0(\boldsymbol{\lambda})}{h_n(\boldsymbol{\lambda})}=\frac{1}{(q\,;q)_n}\,. \tag{5.120}$$

5.6 continuous q-Jacobi [KS3.10]

parameters and potential function

$$\boldsymbol{\lambda}\stackrel{\text{def}}{=}(\alpha,\beta),\quad \boldsymbol{\delta}=(1,1),\quad q;\quad \alpha,\beta\geq-\frac{1}{2}; \tag{5.121}$$

$$V(x\,;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\frac{(1-q^{\frac12(\alpha+\frac12)}z)(1-q^{\frac12(\alpha+\frac32)}z)(1+q^{\frac12(\beta+\frac12)}z)(1+q^{\frac12(\beta+\frac32)}z)}{(1-z^2)(1-qz^2)},\quad z=e^{ix}. \tag{5.122}$$

shape invariance and closure relation

$$\mathcal{E}_n(\boldsymbol{\lambda})=(q^{-n}-1)(1-q^{n+\alpha+\beta+1}), \tag{5.123}$$

$$R_1(y)=(q^{-\frac12}-q^{\frac12})^2y',\quad y'\stackrel{\text{def}}{=}y+1+q^{\alpha+\beta+1}, \tag{5.124}$$

$$R_0(y)=(q^{-\frac12}-q^{\frac12})^2\big(y'^2-(1+q)^2q^{\alpha+\beta}\big), \tag{5.125}$$

$$R_{-1}(y)=-\tfrac12(q^{-\frac12}-q^{\frac12})^2q^{\frac14}(1+q^{\frac12})(q^{\frac12\alpha}-q^{\frac12\beta})(1-q^{\frac12(\alpha+\beta)})\big(y'+(1+q)q^{\frac12(\alpha+\beta)}\big). \tag{5.126}$$

eigenfunctions

$$\phi_0(x\,;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\left|\frac{(e^{2ix}\,;q)_\infty}{(q^{\frac12(\alpha+\frac12)}e^{ix},-q^{\frac12(\beta+\frac12)}e^{ix}\,;q^{\frac12})_\infty}\right|, \tag{5.127}$$

$$\begin{aligned}P_n(\eta\,;\boldsymbol{\lambda})&=P_n^{(\alpha,\beta)}(\cos x|q)\\&\stackrel{\text{def}}{=}\frac{(q^{\alpha+1}\,;q)_n}{(q\,;q)_n}\,{}_4\phi_3\Big(\genfrac{}{}{0pt}{}{q^{-n},\ q^{n+\alpha+\beta+1},\ q^{\frac12(\alpha+\frac12)}e^{ix},\ q^{\frac12(\alpha+\frac12)}e^{-ix}}{q^{\alpha+1},\ -q^{\frac12(\alpha+\beta+1)},\ -q^{\frac12(\alpha+\beta+2)}}\ \Big|\ q\,;q\Big),\end{aligned} \tag{5.128}$$

$$c_n=\frac{2^nq^{\frac12(\alpha+\frac12)n}(q^{n+\alpha+\beta+1}\,;q)_n}{(q,-q^{\frac12(\alpha+\beta+1)},-q^{\frac12(\alpha+\beta+2)}\,;q)_n}, \tag{5.129}$$

$$a_n^{\text{rec}}=\frac12\bigg(q^{\frac12(\alpha+\frac12)}+q^{-\frac12(\alpha+\frac12)}$$

$$-\frac{(1-q^{n+\alpha+1})(1-q^{n+\alpha+\beta+1})(1+q^{n+\frac12(\alpha+\beta+1)})(1+q^{n+\frac12(\alpha+\beta+2)})}{q^{\frac12(\alpha+\frac12)}(1-q^{2n+\alpha+\beta+1})(1-q^{2n+\alpha+\beta+2})}$$

$$-\frac{q^{\frac12(\alpha+\frac12)}(1-q^{n})(1-q^{n+\beta})(1+q^{n+\frac12(\alpha+\beta)})(1+q^{n+\frac12(\alpha+\beta+1)})}{(1-q^{2n+\alpha+\beta})(1-q^{2n+\alpha+\beta+1})}\Bigg), \tag{5.130}$$

$$b_n^{\text{rec}}=(1-q^n)(1-q^{n+\alpha})(1-q^{n+\beta})(1-q^{n+\alpha+\beta}) \times\frac{(1+q^{n+\frac12(\alpha+\beta-1)})(1+q^{n+\frac12(\alpha+\beta)})^2(1+q^{n+\frac12(\alpha+\beta+1)})}{4(1-q^{2n+\alpha+\beta-1})(1-q^{2n+\alpha+\beta})^2(1-q^{2n+\alpha+\beta+1})}, \tag{5.131}$$

$$f_n(\boldsymbol{\lambda})=\frac{q^{\frac12(\alpha+\frac32)}q^{-n}(1-q^{n+\alpha+\beta+1})}{(1+q^{\frac12(\alpha+\beta+1)})(1+q^{\frac12(\alpha+\beta+2)})}, \tag{5.132}$$

$$b_n(\boldsymbol{\lambda})=q^{-\frac12(\alpha+\frac32)}q^{n+1}(q^{-(n+1)}-1)(1+q^{\frac12(\alpha+\beta+1)})(1+q^{\frac12(\alpha+\beta+2)}). \tag{5.133}$$

annihilation/creation operators and commutation relations

$$\alpha_\pm(\mathcal{H})=\tfrac12(q^{-\frac12}-q^{\frac12})^2\mathcal{H}'\pm\tfrac12(q^{-1}-q)\sqrt{\mathcal{H}'^2-4q^{\alpha+\beta+1}},\quad \mathcal{H}'\stackrel{\text{def}}{=}\mathcal{H}+1+q^{\alpha+\beta+1}, \tag{5.134}$$

$$q^{\mathcal{N}}=\tfrac12q^{-\alpha-\beta-1}\big(\mathcal{H}'-\sqrt{\mathcal{H}'^2-4q^{\alpha+\beta+1}}\big)\quad(\text{for } 0<q^{\alpha+\beta+1}<1), \tag{5.135}$$

$$[\mathcal{H},a^{(\pm)}]=\tfrac12a^{(\pm)}\Big((q^{-\frac12}-q^{\frac12})^2\mathcal{H}'\pm(q^{-1}-q)\sqrt{\mathcal{H}'^2-4q^{\alpha+\beta+1}}\Big). \tag{5.136}$$

The annihilation/creation operators (2.55) and their commutation relation (2.63) are not so simplified because $b_{n+1}^{\text{rec}}-b_n^{\text{rec}}=q^n\times$(9-th degree polynomial in q^n)/(11-th degree polynomial in q^n) has a lengthy expression.

coherent state We are not aware if a simple summation formula exists for the coherent state:

$$\psi(\alpha',x\,;\boldsymbol{\lambda})=\phi_0(x\,;\boldsymbol{\lambda})\sum_{n=0}^{\infty}\frac{(2q^{-\frac12(\alpha+\frac12)})^n(q^{\frac12(\alpha+\beta)+1}\,;q^{\frac12})_{2n}\alpha'^n}{(q^{\alpha+1},q^{\beta+1}\,;q)_n}P_n(\eta(x)\,;\boldsymbol{\lambda}). \tag{5.137}$$

orthogonality

$$\int_0^\pi\phi_0(x\,;\boldsymbol{\lambda})^2P_n(\eta\,;\boldsymbol{\lambda})P_m(\eta\,;\boldsymbol{\lambda})dx =2\pi\,\frac{(1-q^{\alpha+\beta+1})(q^{\alpha+1},q^{\beta+1},-q^{\frac12(\alpha+\beta+3)}\,;q)_n}{(1-q^{2n+\alpha+\beta+1})(q,q^{\alpha+\beta+1},-q^{\frac12(\alpha+\beta+1)}\,;q)_n}\,q^{(\alpha+\frac12)n} \times\frac{(q^{\frac12(\alpha+\beta+2)},q^{\frac12(\alpha+\beta+3)}\,;q)_\infty}{(q,q^{\alpha+1},q^{\beta+1},-q^{\frac12(\alpha+\beta+1)},-q^{\frac12(\alpha+\beta+2)}\,;q)_\infty}\,\delta_{nm}, \tag{5.138}$$

$$\frac{1}{h_0(\boldsymbol{\lambda})}=\frac{(q,q^{\alpha+1},q^{\beta+1},-q^{\frac12(\alpha+\beta+1)},-q^{\frac12(\alpha+\beta+2)}\,;q)_\infty}{2\pi(q^{\frac12(\alpha+\beta+2)},q^{\frac12(\alpha+\beta+3)}\,;q)_\infty}, \tag{5.139}$$

$$\frac{h_0(\boldsymbol{\lambda})}{h_n(\boldsymbol{\lambda})}=\frac{(1-q^{2n+\alpha+\beta+1})(q,q^{\alpha+\beta+1},-q^{\frac12(\alpha+\beta+1)}\,;q)_n}{(1-q^{\alpha+\beta+1})(q^{\alpha+1},q^{\beta+1},-q^{\frac12(\alpha+\beta+3)}\,;q)_n}\,q^{-(\alpha+\frac12)n}. \tag{5.140}$$

5.7 continuous q-Laguerre [KS3.19]

This is a further restriction ($\beta\to\infty$ or $q^\beta\to 0$) of the continuous q-Jacobi polynomial §5.6. Many formulas are drastically simplified.

parameters and potential function

$$\boldsymbol{\lambda}\stackrel{\text{def}}{=}\alpha,\quad \boldsymbol{\delta}=1,\quad q;\quad \alpha\geq-\frac12; \tag{5.141}$$

$$V(x\,;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\frac{(1-q^{\frac12(\alpha+\frac12)}z)(1-q^{\frac12(\alpha+\frac32)}z)}{(1-z^2)(1-qz^2)},\quad z=e^{ix}. \tag{5.142}$$

shape invariance and closure relation

$$\mathcal{E}_n(\boldsymbol{\lambda})=q^{-n}-1, \tag{5.143}$$

$$R_1(y)=(q^{-\frac12}-q^{\frac12})^2y',\quad y'\stackrel{\text{def}}{=}y+1,\quad R_0(y)=(q^{-\frac12}-q^{\frac12})^2y'^2, \tag{5.144}$$

$$R_{-1}(y)=-\tfrac12(q^{-\frac12}-q^{\frac12})^2q^{\frac12(\alpha+\frac12)}(1+q^{\frac12})y'. \tag{5.145}$$

eigenfunction

$$\phi_0(x\,;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\left|\frac{(e^{2ix}\,;q)_\infty}{(q^{\frac12(\alpha+\frac12)}e^{ix}\,;q^{\frac12})_\infty}\right|, \tag{5.146}$$

$$P_n(\eta\,;\boldsymbol{\lambda})=P_n^{(\alpha)}(\cos x|q)\stackrel{\text{def}}{=}\frac{(q^{\alpha+1}\,;q)_n}{(q\,;q)_n}\,{}_3\phi_2\Bigl(\begin{matrix}q^{-n},\ q^{\frac12(\alpha+\frac12)}e^{ix},\ q^{\frac12(\alpha+\frac12)}e^{-ix}\\ q^{\alpha+1},\ 0\end{matrix}\Bigm|q\,;q\Bigr), \tag{5.147}$$

$$c_n=\frac{2^nq^{\frac12(\alpha+\frac12)n}}{(q\,;q)_n},\quad a_n^{\text{rec}}=\tfrac12q^{n+\frac12(\alpha+\frac12)}(1+q^{\frac12}),\quad b_n^{\text{rec}}=\tfrac14(1-q^n)(1-q^{n+\alpha}), \tag{5.148}$$

$$f_n(\boldsymbol{\lambda})=q^{\frac12(\alpha+\frac32)}q^{-n},\quad b_n(\boldsymbol{\lambda})=q^{-\frac12(\alpha+\frac32)}q^{n+1}(q^{-(n+1)}-1). \tag{5.149}$$

annihilation/creation operators and commutation relations

$$\alpha_\pm(\mathcal{H})=(q^{\mp1}-1)(\mathcal{H}+1),\qquad q^{\mathcal{N}}=(\mathcal{H}+1)^{-1}, \tag{5.150}$$

$$a^{(\pm)}=\frac{\pm1}{q^{-1}-q}\Bigl([\mathcal{H},\eta]_{q^{\pm1}}+(1-q^{\pm1})\bigl(\eta-\tfrac12q^{\frac12(\alpha+\frac12)}(1+q^{\frac12})\bigr)\Bigr)(\mathcal{H}+1)^{-1}, \tag{5.151}$$

$$b_{n+1}^{\text{rec}}-b_n^{\text{rec}}=\tfrac14(1-q)\bigl(-(1+q)q^\alpha q^{2n}+(1+q^\alpha)q^n\bigr), \tag{5.152}$$

$$[\mathcal{H},a^{(\pm)}]=(q^{\mp1}-1)a^{(\pm)}(\mathcal{H}+1), \tag{5.153}$$

$$[a^{(-)},a^{(+)}]=\tfrac14(1-q)\bigl(-(1+q)q^{\alpha}(\mathcal{H}+1)^{-2}+(1+q^{\alpha})(\mathcal{H}+1)^{-1}\bigr). \tag{5.154}$$

Again (5.153) can be written as q-deformed commutators:

$$\mathcal{H}a^{(\pm)}-q^{\mp1}a^{(\pm)}\mathcal{H}=(q^{\mp1}-1)a^{(\pm)},\quad\text{namely,}\quad [\mathcal{H},a^{(\pm)}]_{q^{\mp1}}=(q^{\mp1}-1)a^{(\pm)}. \tag{5.155}$$

The following mean that $[a^{(-)},a^{(+)}]_q$ and $[a^{(-)},a^{(+)}]_{q^2}$ take simple forms, see (5.69)–(5.70):

$$b^{\text{rec}}_{n+1}-qb^{\text{rec}}_n=\tfrac14(1-q)(1-q^{\alpha+1+2n}), \tag{5.156}$$

$$b^{\text{rec}}_{n+1}-q^2b^{\text{rec}}_n=\tfrac14(1-q)\bigl(1+q-(1+q^{\alpha})q^{n+1}\bigr). \tag{5.157}$$

coherent state (2.80) reads with the help of [KS(3.19.12)]

$$\begin{aligned}\psi(\alpha',x\,;\boldsymbol{\lambda})&=\phi_0(x\,;\boldsymbol{\lambda})\sum_{n=0}^{\infty}\frac{(2q^{-\frac12(\alpha+\frac12)})^n\alpha'^n}{(q^{\alpha+1}\,;q)_n}\,P_n(\eta(x)\,;\boldsymbol{\lambda})\\&=\phi_0(x\,;\boldsymbol{\lambda})\,\frac{1}{(2\alpha'e^{ix}\,;q)_\infty}\,{}_2\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{\frac12(\alpha+\frac12)}e^{ix},\ q^{\frac12(\alpha+\frac32)}e^{ix}}{q^{\alpha+1}}\Bigm|q\,;2\alpha'e^{-ix}\Bigr).\end{aligned} \tag{5.158}$$

orthogonality

$$\int_0^{\pi}\phi_0(x\,;\boldsymbol{\lambda})^2P_n(\eta\,;\boldsymbol{\lambda})P_m(\eta\,;\boldsymbol{\lambda})dx=2\pi\,\frac{(q^{\alpha+1}\,;q)_n}{(q\,;q)_n}\,q^{(\alpha+\frac12)n}\frac{1}{(q,q^{\alpha+1}\,;q)_\infty}\,\delta_{nm}, \tag{5.159}$$

$$\frac{1}{h_0(\boldsymbol{\lambda})}=\frac{1}{2\pi}(q,q^{\alpha+1}\,;q)_\infty\,,\quad\frac{h_0(\boldsymbol{\lambda})}{h_n(\boldsymbol{\lambda})}=\frac{(q\,;q)_n}{(q^{\alpha+1}\,;q)_n}\,q^{-(\alpha+\frac12)n}. \tag{5.160}$$

5.8 Comments on the two polynomials with $\eta(x)=\cos(x+\phi)$

In the review of Koekoek and Swarttouw [6], two polynomials, the continuous q-Hahn [KS3.4] and the q-Meixner-Pollaczek [KS3.9] are listed as having $\eta(x)=\cos(x+\phi)$, with non-vanishing angle ϕ appearing in the definition of polynomials. In fact, the continuous q-Hahn polynomial is the same as the Askey-Wilson polynomial §5.1 and the q-Meixner-Pollaczek polynomial is proportional to the Al-Salam-Chihara polynomial §5.3 with degree- or n-dependent coefficients. Therefore we will not treat them as independent 'discrete' quantum mechanical systems.

5.8.1 continuous q-Hahn [KS3.4]

A simple comparison of the normalised three term recurrence relation for the continuous q-Hahn polynomial (KS3.4.4) with that for the Askey-Wilson polynomial (KS3.1.5) reveals that they are one and the same polynomial after the identification of the parameters (in the notation of [6])

$$a^{\mathrm{AW}} \to a\,e^{i\phi}, \quad b^{\mathrm{AW}} \to b\,e^{i\phi}, \quad c^{\mathrm{AW}} \to c\,e^{-i\phi}, \quad d^{\mathrm{AW}} \to d\,e^{-i\phi}, \tag{5.161}$$

in which the superscript AW indicates the quantity of the Askey-Wilson polynomial.

5.8.2 q-Meixner-Pollaczek [KS3.9]

Likewise, the normalised three term recurrence relation for the q-Meixner-Pollaczek polynomial $P_n^{q\mathrm{MP}}(\eta)$ (KS3.9.4) is the same as that for the Al-Salam-Chihara polynomial $P_n^{\mathrm{ASC}}(\eta)$ (KS3.8.4) after the identification

$$a^{\mathrm{ASC}} \to a\,e^{i\phi}, \quad b^{\mathrm{ASC}} \to a\,e^{-i\phi}; \qquad a^{\mathrm{ASC}} = (b^{\mathrm{ASC}})^* \in \mathbb{C}, \quad a > 0, \tag{5.162}$$

in which the superscript ASC denotes the quantity of the Al-Salam-Chihara polynomial. These two polynomials are different only by a multiplicative constant:

$$\frac{P_n^{\mathrm{ASC}}(\eta)}{(q;q)_n} = P_n^{q\mathrm{MP}}(\eta). \tag{5.163}$$

6 Summary and Comments

Known examples of exactly solvable 'discrete' quantum mechanics of one degree of freedom are discussed in detail and in full generality. The shape invariance property, the exact solutions in the Schrödinger and Heisenberg pictures, the annihilation/creation operators together with their symmetry algebra, the coherent state as the eigenvector of the annihilation operator, the ground state wavefunction giving the orthogonality measure of the eigenpolynomial are given explicitly for each system, which is named after the corresponding orthogonal polynomial. The present paper supplements the earlier results [10, 11, 12, 13, 7]. The main focus is the polynomials obtained by restricting the Askey-Wilson polynomials. In general, they have simple and tractable symmetry algebras, some of them are the q-oscillator algebra [14]. Another main feature is the coherent states. As many as eleven new and exact coherent states are presented (3.19), (3.39), (4.16), (4.37), (5.20), (5.51), (5.71), (5.91), (5.118), (5.137), (5.158) as the eigenvectors of the annihilation operators for the 'discrete' quantum mechanical systems. These coherent states are by construction totally symmetric in the symmetric parameters of the Hamiltonians. In other words, they realise the dynamically favourable generating functions of the eigenpolynomials. Like the standard coherent state of the harmonic oscillator, these new coherent states are expected to find various applications in many branches of physical sciences, in particular, quantum optics and quantum information. It would be interesting to investigate if and to what extent these new coherent states share the remarkable properties of the standard coherent state of the harmonic oscillator.

One interesting future task is to solve the closure relation (2.23)–(2.27) algebraically to determine all the possible forms of the sinusoidal coordinate $\eta(x)$ and the potential function $V(x)$. For the ordinary quantum mechanics and for the orthogonal polynomials of discrete measures, this task was done in Appendix A of [13] and Appendix A of [7]. The present case is more complicated than these due to the presence of arbitrary periodic functions with period $i\gamma$. It is interesting to see if difference equation versions of the soliton potential, i.e. $1/\cosh^2 x$ potential in ordinary quantum mechanics, (see, for example, §3.1.3 of [13]) with $\eta(x) = \sinh x$, and the Morse potential with $\eta(x) = e^{-x}$ (see §3.1.4 of [13]) are contained as solutions or not.

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Appendix A: Diagrammatic proof of the hermiticity of the Hamiltonian

Here we give a diagrammatic proof of the hermiticity (self-adjointness) of the Hamiltonian (2.1) for the three different cases of the sinusoidal coordinates corresponding to sections 3–5. A less detailed proof of the hermiticity can be found in [26]. The hermiticity or self-adjointness of the Hamiltonian $\mathcal{H}$ means $(g, \mathcal{H}f) = (\mathcal{H}g, f)$ for a given inner product (g, f) (2.75) for arbitrary elements f and g of the appropriate Hilbert space. It is necessary and sufficient to show that in a certain dense subspace of the Hilbert space. The obvious choice for such a subspace is spanned by the ground state wavefunction ϕ_0, which is given in each subsection (3.10), (3.25), (4.7), (4.23), (5.11), (5.38), (5.59), (5.79), (5.97), (5.127), (5.146), times the eigenpolynomials $P_n(\eta(x))$. The types of the polynomials are:

(a) : polynomials in $\eta(x) = x$ for the Hamiltonians in section 3,

$$(g,f)=\int_{-\infty}^{\infty}g(x)^*f(x)dx,\quad f(x)=\phi_0(x)P(x),\quad g(x)=\phi_0(x)Q(x), \tag{A.1}$$

(b) : polynomials in $\eta(x)=x^2$ for the Hamiltonians in section 4,

$$(g,f)=\int_{0}^{\infty}g(x)^*f(x)dx,\quad f(x)=\phi_0(x)P(x^2),\quad g(x)=\phi_0(x)Q(x^2), \tag{A.2}$$

(c) : polynomials in $\eta(x)=\cos x$ for the Hamiltonians in section 5,

$$(g,f)=\int_{0}^{\pi}g(x)^*f(x)dx,\quad f(x)=\phi_0(x)P(\cos x),\quad g(x)=\phi_0(x)Q(\cos x). \tag{A.3}$$

This clearly removes the non-uniqueness of the eigenfunctions, which was mentioned in section two. For the Hamiltonian (2.3) $\mathcal{H}=T_++T_--V(x)-V(x)^*$, it is obvious that the function part $-V(x)-V(x)^*$ is hermitian by itself. When $T_+=\sqrt{V(x)}\,e^{\gamma p}\sqrt{V(x)^*}$ acts on f, the argument of f is shifted from x to $x-i\gamma$. With the compensating change of integration variable from x to $x+i\gamma$ one can formally show $(g,T_+f)=(T_+g,f)$ in a straightforward way. Similarly we have $(g,T_-f)=(T_-g,f)$ by another change of integration variable x to $x-i\gamma$. This is the 'formal hermiticity.'

In reality, the shift of integration variable, to be realised by the Cauchy integral, would involve additional integration contours:

$$(a):(-\infty,\pm i-\infty),\quad(+\infty,\pm i+\infty)\quad\text{for the Hamiltonians in section 3,} \tag{A.4}$$

$$(b):(0,\pm i),\quad(+\infty,\pm i+\infty)\quad\text{for the Hamiltonians in section 4,} \tag{A.5}$$

$$(c):(0,\pm i\log q),\quad(\pi,\pi\pm i\log q)\quad\text{for the Hamiltonians in section 5.} \tag{A.6}$$

It is easy to verify that all the singularities arising from V and V^* in cases (b) and (c) are cancelled by the zeros coming from the ground state wavefunctions ϕ_0 and ϕ_0^*, and the Cauchy integration formula applies in all cases. As can be seen from the diagrams in Fig.2 the contribution of the additional contour integrals (A.4)–(A.6) cancel with each other and the shifts of integration variables are justified and the hermiticity is established.

First, the contribution from the contours at infinity in (a) vanish identically due to the strong damping by ϕ_0 and ϕ_0^*, see (3.10) and (3.25). This establishes the hermiticity in the case (a). Next let us discuss the case (b) in detail. In this case $\gamma=1$. The integrand of $(g,T_\pm f)$ are

$$g^*T_+f=\phi_0(x)^*Q(x^2)^*\sqrt{V(x)}\sqrt{V(x+i)^*}\phi_0(x-i)P((x-i)^2)\stackrel{\text{def}}{=}F(x), \tag{A.7}$$

$$g^*T_-f=\phi_0(x)^*Q(x^2)^*\sqrt{V(x)^*}\sqrt{V(x+i)}\phi_0(x+i)P((x+i)^2)\stackrel{\text{def}}{=}G(x). \tag{A.8}$$

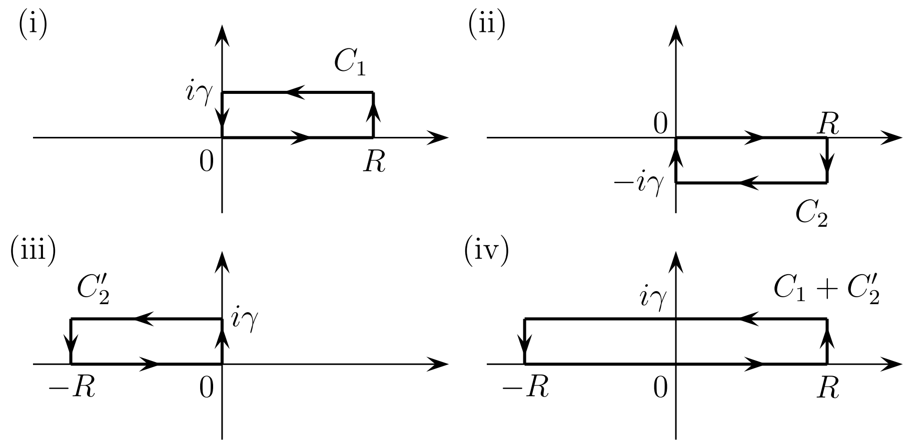

Figure 2: Integration contours in complex x plane. The endpoint $R = \infty$ for cases (a) and (b), $R = \pi$ for case (c). (For case (c), $i\gamma$ is in the lower half plane because of $\gamma = \log q < 0$.)

Due to the evenness of the eigenfunctions, $\phi_0(-x) = \phi_0(x)$, $P((-x)^2) = P(x^2)$, $Q((-x)^2) = Q(x^2)$ and $V(x)^* = V(-x^*)$, we have

$$
\begin{aligned}
G(x) &= \phi_0(-x)^* Q((-x)^2)^* \sqrt{V(-x)}\sqrt{V(-x+i)^*}\phi_0(-x-i)P((-x-i)^2) \\
&= F(-x). \qquad \text{(A.9)}
\end{aligned}
$$

On the other hand, the integrand of $(T_\pm g, f)$ are

$$
(T_+ g)^* f = \sqrt{V(x)^*}\sqrt{V(x+i)}\phi_0(x-i)^* Q((x-i)^2)^* \phi_0(x) P(x^2) = F(x+i), \qquad \text{(A.10)}
$$

$$
\begin{aligned}
(T_- g)^* f &= \sqrt{V(x)}\sqrt{V(x+i)^*}\phi_0(x+i)^* Q((x+i)^2)^* \phi_0(x) P(x^2) = G(x-i) \\
&= F(-x+i), \qquad \text{(A.11)}
\end{aligned}
$$

in which (A.9) is used for the last equality. Since the integrands are analytic in x and there is no pole within the contours, see Fig.2, we have

$$
\oint_{C_1} F(x)dx = 0, \qquad \oint_{C_2} G(x)dx = \oint_{C_2} F(-x)dx = \oint_{C_2'} F(x)dx = 0. \qquad \text{(A.12)}
$$

Combining them, we obtain

$$
\begin{aligned}
0 &= \oint_{C_1} F(x)dx + \oint_{C_2'} F(x)dx = \oint_{C_1 + C_2'} F(x)dx \\
&= \int_{-\infty}^{\infty} F(x)dx - \int_{-\infty}^{\infty} F(x+i)dx + \int_{\uparrow \text{ at} +\infty} F(x)dx + \int_{\downarrow \text{ at} -\infty} F(x)dx. \qquad \text{(A.13)}
\end{aligned}
$$

The contribution from the contours at infinity in the case (b) vanish identically due to the strong damping by ϕ_0 and ϕ_0^*, see (4.7) and (4.23). Thus (A.13) implies $\int_{-\infty}^{\infty} F(x)dx =$

$\int_{-\infty}^{\infty} F(x+i)dx$. The l.h.s. is

$$\int_0^\infty F(x)dx + \int_0^\infty G(x)dx = (g, T_+ f) + (g, T_- f). \qquad (A.14)$$

The r.h.s. is

$$\int_0^\infty F(x+i)dx + \int_0^\infty G(x-i)dx = (T_+ g, f) + (T_- g, f). \qquad (A.15)$$

Thus the hermiticity of the Hamiltonians for the case (b) is proved. The hermiticity of the Hamiltonians for the case (c) is proved in a similar way together with the evenness and the 2π periodicity of the ground state wavefunction $\phi_0(x)$, the sinusoidal coordinate $\eta(x) = \cos x$ and the potential function $V(x)$; $\phi_0(-x) = \phi_0(x)$, $\eta(-x) = \eta(x)$, $V(x)^* = V(-x^*)$, $\phi_0(x+2\pi) = \phi(x)$, $\eta(x+2\pi) = \eta(x)$, $V(x+2\pi) = V(x)$.

Appendix B: Some definitions related to the hypergeometric and q-hypergeometric functions

For self-containedness we collect several definitions related to the (q-)hypergeometric functions [6].

∘ Pochhammer symbol $(a)_n$:

$$(a)_n \stackrel{\text{def}}{=} \prod_{k=1}^{n}(a+k-1) = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}. \qquad (B.1)$$

∘ q-Pochhammer symbol $(a\,;q)_n$:

$$(a\,;q)_n \stackrel{\text{def}}{=} \prod_{k=1}^{n}(1-aq^{k-1}) = (1-a)(1-aq)\cdots(1-aq^{n-1}). \qquad (B.2)$$

∘ hypergeometric series ${}_rF_s$:

$${}_rF_s\Bigl(\begin{matrix} a_1,\,\cdots,a_r \\ b_1,\,\cdots,b_s \end{matrix}\Bigm| z\Bigr) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(a_1,\,\cdots,a_r)_n}{(b_1,\,\cdots,b_s)_n}\frac{z^n}{n!}\,, \qquad (B.3)$$

where $(a_1,\,\cdots,a_r)_n \stackrel{\text{def}}{=} \prod_{j=1}^{r}(a_j)_n = (a_1)_n\cdots(a_r)_n$.

∘ q-hypergeometric series (the basic hypergeometric series) ${}_r\phi_s$:

$${}_r\phi_s\Bigl(\begin{matrix} a_1,\,\cdots,a_r \\ b_1,\,\cdots,b_s \end{matrix}\Bigm| q\,;z\Bigr) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(a_1,\,\cdots,a_r\,;q)_n}{(b_1,\,\cdots,b_s\,;q)_n}(-1)^{(1+s-r)n}q^{(1+s-r)n(n-1)/2}\frac{z^n}{(q\,;q)_n}\,, \qquad (B.4)$$

where $(a_1,\,\cdots,a_r\,;q)_n \stackrel{\text{def}}{=} \prod_{j=1}^{r}(a_j\,;q)_n = (a_1\,;q)_n\cdots(a_r\,;q)_n$.

∘ q-gamma function $\Gamma_q(z)$:

$$\Gamma_q(z) \stackrel{\text{def}}{=} \frac{(q\,;q)_\infty}{(q^z\,;q)_\infty}(1-q)^{1-z}, \qquad \lim_{q\nearrow 1}\Gamma_q(z) = \Gamma(z). \qquad (B.5)$$

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