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Example of σ-Transition Matrices Defining the Horrocks-Mumford Bundle

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Introduction.

At the present state, the only examples of rank r indecomposable vector bundles on $\boldsymbol{P}^n$ with $2\leqq r\leqq n-2$ are essentially the following: The rank 2 vector bundle on $\boldsymbol{P}^4$ constructed by Horrocks and Mumford [2]; the rank 2 on $\boldsymbol{P}^5$ in characteristic 2 by Tango [7]; the rank 3 on $\boldsymbol{P}^5$ in characteristic not equal to 2 by Horrocks [1]. About these bundles, especially about the Horrocks-Mumford bundle, many interesting facts have been discovered.

We here propose to focus our attention on systems of frames of these bundles and transition matrices with respect to them.

Recently, Tango [8] provided a theory of "σ-transition matrices", where his σ-transition matrices are transition matrices, defining a vector bundle on a projective space $\boldsymbol{P}^n$, over the standard covering of $\boldsymbol{P}^n$ carrying natural symmetry. Moreover, he showed that, for the Tango bundle and the Horrocks bundle above, there exist σ-transition matrices defining these bundles, respectively, and he actually computed these matrices. We note that arbitrary bundles on $\boldsymbol{P}^n$ are not necessarily defined by σ-transition matrices.

The purpose of this article is to find out rational sections of the Horrocks-Mumford bundle which give a system of frames over the standard covering of $\boldsymbol{P}^4$ and have the natural symmetry. Moreover, we shall write down the rational sections explicitly. It turns out that our frames are connected to each other by σ-transition matrices, and the Horrocks-Mumford bundle is also defined by σ-transition matrices.

Sasakura [5, 6] gave a theory of "configuration of divisors and reflexive sheaves". Using this theory, from a quite simple data consisting of transition matrices and divisors, he succeeded in reconstructing the Horrocks-Mumford bundle. In order to do this, he also found rational sections of the bundle, and computed transition matrices of the bundle

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with respect to these sections. We note that his rational sections, however, do not give frames over the standard covering of $\boldsymbol{P}^4$.

Throughout this article, we shall work over a field k with arbitrary characteristic. We use the letters α, β, γ, i, j, m and n as indices running the ranges $0 \leqq \alpha, \beta, \gamma, m \leqq 4$, $1 \leqq i, j \leqq 4$, and $1 \leqq n \leqq 6$, and consider these indices except n to be elements of $\boldsymbol{Z}/5\boldsymbol{Z}$.

I would like to thank Professor Sasakura for many stimulating conversations.

§1. Preliminaries.

Let V be a 5-dimensional vector space over k with basis $(v_m)_m$, and let V^* be its dual space with dual basis $(v_m^*)_m$ of $(v_m)_m$.

Let $\boldsymbol{P}$ be a projective space consisting of 1-dimensional subspaces of V, let $(x_\alpha)_\alpha$ be the homogeneous coordinate of $\boldsymbol{P}$ corresponding to the basis $(v_m^*)_m$ via a canonical isomorphism $\Gamma(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(1)) \cong V^*$, and let $(U_\alpha)_\alpha$ be a natural affine open covering of $\boldsymbol{P}$ defined by the homogeneous coordinate $(x_\alpha)_\alpha$, called the *standard covering* of $\boldsymbol{P}$.

Now, we look at the Euler sequence over $\boldsymbol{P}$:

$$0 \longrightarrow \mathcal{O}_{\boldsymbol{P}}(-1) \longrightarrow V_{\boldsymbol{P}} \longrightarrow T_{\boldsymbol{P}}(-1) \longrightarrow 0 ,$$

where we set $V_{\boldsymbol{P}} := V \otimes_k \mathcal{O}_{\boldsymbol{P}}$ and $T_{\boldsymbol{P}}$ is the tangent bundle of $\boldsymbol{P}$. Considering each v_m as a global section of $V_{\boldsymbol{P}}$, we denote by ∂_m the image of v_m under the map in the sequence above, and denote $T_{\boldsymbol{P}}(-1)$ by Q. We see that each ∂_m is a global section of Q and its zero locus coincides with a point of $\boldsymbol{P}$ represented by v_m.

Moreover, we use the following notations:

$$v_{\alpha,m} := v_{\alpha+m} \quad \text{in} \quad \Gamma(U_\alpha, V_{\boldsymbol{P}})$$

$$y_{\alpha,i} := \frac{x_{\alpha+i}}{x_\alpha} \quad \text{in} \quad \Gamma(U_\alpha, \mathcal{O}_{\boldsymbol{P}})$$

$$\partial_{\alpha,i} := \partial_{\alpha+i} \quad \text{in} \quad \Gamma(U_\alpha, Q)$$

and

$$U_{\beta,\gamma} := U_\beta \cap U_\gamma ,$$

with $0 \leqq \alpha, \beta, \gamma, m \leqq 4$, $1 \leqq i, j \leqq 4$. We briefly write y_i for $y_{\alpha,i}$ when no confusion arises.

DEFINITION. Let W be a rank r vector bundle over $\boldsymbol{P}$. According to a theorem of Quillen [4], each restriction $W|_{U_\alpha}$ is a free vector bundle.

So, one can find a free basis $(w_{\alpha,l})_l$ of W over U_α for each α, where the index l is running the range $1 \leqq l \leqq r$. Then, we call $(w_{\alpha,l})_l$ a *frame* of W over U_α, and $\{(w_{\alpha,l})_l\}_\alpha$ a *system of frames* of W.

It follows that $\{(v_{\alpha,m})_m\}_\alpha$ and $\{(\partial_{\alpha,i})_i\}_\alpha$ are respectively systems of frames of V_P and Q.

Putting

$$\partial_{\alpha,i,j} := \partial_{\alpha+i} \wedge \partial_{\alpha+j} \qquad \text{in} \quad \Gamma(U_\alpha, \overset{2}{\wedge} Q) ,$$

we define $(\delta_{\alpha,n})_n$ to be

$$(\partial_{\alpha,1,2}, \partial_{\alpha,1,3}, \partial_{\alpha,1,4}, \partial_{\alpha,2,3}, \partial_{\alpha,2,4}, \partial_{\alpha,3,4}) .$$

It follows that $\{(\delta_{\alpha,n})_n\}_\alpha$ is a system of frames of $\overset{2}{\wedge} Q$. For the dual bundle $\overset{2}{\wedge} Q^*$, defining $(\delta^*_{\alpha,n})_n$ to be the dual basis of $(\delta_{\alpha,n})_n$ over U_α, we have that $\{(\delta^*_{\alpha,n})_n\}_\alpha$ is a system of frames of $\overset{2}{\wedge} Q^*$.

Putting

$$\delta^+_{\alpha,n} := \delta_{\alpha,n} \oplus 0 \qquad \text{in} \quad \Gamma(U_\alpha, \overset{2}{\oplus} \overset{2}{\wedge} Q)$$
$$\delta^-_{\alpha,n} := 0 \oplus \delta_{\alpha,n} \qquad \text{in} \quad \Gamma(U_\alpha, \overset{2}{\oplus} \overset{2}{\wedge} Q) ,$$

we write $(\delta^+_{\alpha,n}; \delta^-_{\alpha,n})_n$ for

$$(\delta^+_{\alpha,1}, \cdots, \delta^+_{\alpha,6}, \delta^-_{\alpha,1}, \cdots, \delta^-_{\alpha,6}) ,$$

and so on. We moreover put

$$\Delta_\alpha := \partial_{\alpha+1} \wedge \partial_{\alpha+2} \wedge \partial_{\alpha+3} \wedge \partial_{\alpha+4} \qquad \text{in} \quad \Gamma(U_\alpha, \overset{4}{\wedge} Q) .$$

Since Δ_α is a free basis of $\overset{4}{\wedge} Q$ over U_α, we have

LEMMA 1. (a) *$\{\Delta_\alpha\}_\alpha$ is a system of frames of $\overset{4}{\wedge} Q$;*

(b) *$\{(\delta^*_{\alpha,n} \otimes \Delta_\alpha)_n\}_\alpha$ is a system of $\overset{2}{\wedge} Q^* \otimes \overset{4}{\wedge} Q$;*

(c) *$\{(\delta^+_{\alpha,n}; \delta^-_{\alpha,n})_n\}_\alpha$, $\{(\delta^{*+}_{\alpha,n} \otimes \Delta_\alpha; \delta^{*-}_{\alpha,n} \otimes \Delta_\alpha)_n\}_\alpha$ are systems of frames of $\overset{2}{\oplus} \overset{2}{\wedge} Q$, $\overset{2}{\oplus} \overset{2}{\wedge} Q^* \otimes \overset{4}{\wedge} Q$, respectively;*

(d) *$\{(v^*_{\alpha,m} \otimes \Delta_\alpha)_m\}_\alpha$ is a system of frames of $V^*_P \otimes \overset{4}{\wedge} Q$.*

From now on, for the vector bundles above, we shall always fix the systems of frames above.

DEFINITION. Let W be a rank r vector bundle on $\boldsymbol{P}$, and let $\{(w_{\alpha,l})_l\}_\alpha$ be a system of frames of W. For any β and γ, there exists a unique matrix, denoted by $M_{W,\beta,\gamma}$, satisfying

$$(w_{\gamma,1}, \cdots, w_{\gamma,r}) = (w_{\beta,1}, \cdots, w_{\beta,r}) M_{W,\beta,\gamma} ,$$

where the entries of $M_{W,\beta,\gamma}$ are regular functions over $U_{\beta,\gamma}$. Then, we call $M_{W,\beta,\gamma}$ a *transition matrix* defining W with respect to $(w_{\beta,l})_l$ and $(w_{\gamma,l})_l$ over $U_{\beta,\gamma}$, and $\{M_{W,\beta,\gamma}\}_{\beta,\gamma}$ a *system of transition matrices* defining W with respect to $\{(w_{\alpha,l})_l\}_\alpha$.

Let τ be a permutation on the indices of the homogeneous coordinate $(x_\alpha)_\alpha$, and we consider a natural action of τ on $\Gamma(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(1))$. Using this action, one can define an action of τ on the matrices whose entries are rational functions over $\boldsymbol{P}$.

DEFINITION (see [8]). For a permutation σ with length 5, a system of transition matrices $\{M_{W,\beta,\gamma}\}_{\beta,\gamma}$ is called a *system of σ-transition matrices* if, for any element τ of the cyclic group generated by σ, if follows

$$\tau(M_{W,\beta,\gamma}) = M_{W,\tau(\beta),\tau(\gamma)} .$$

We call $M_{W,\sigma(0),0}$ the *initial data* of the system of σ-transition matrices because we have

$$M_{W,\beta,\gamma} = \sigma^m(\sigma^{i-1}M \ \sigma^{i-2}M \ \cdots \ \sigma M \ M)$$

where $\gamma = \sigma^m(0)$, $\beta = \sigma^i(\gamma)$ and we put $M := M_{W,\sigma(0),0}$.

REMARK. Tango [8] showed that there exist systems of σ-transition matrices defining the Tango bundle [7] and the Horrocks bundle [1] by significant computation from a geometric viewpoint.

REMARK. We always express the entries of $M_{W,\beta,\gamma}$ as rational functions over U_β.

For example, we have

$$W_{V_{\boldsymbol{P}},\alpha,\alpha+1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} .$$

From the Euler relation:

$$\sum_m x_m \partial_m = 0 ,$$

it follows that

$$(*) \qquad \partial_{\alpha+1,4} = -\sum_i y_{\alpha,i}\partial_{\alpha,i} .$$

Thus, we get

$$M_{Q,\alpha,\alpha+1} = \begin{pmatrix} 0 & 0 & 0 & -y_1 \\ 1 & 0 & 0 & -y_2 \\ 0 & 1 & 0 & -y_3 \\ 0 & 0 & 1 & -y_4 \end{pmatrix} .$$

Moreover, we have

LEMMA 2.

(a) $$M_{\overset{2}{\wedge} Q,\alpha,\alpha+1} = \begin{pmatrix} 0 & 0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_1 \\ 1 & 0 & -y_3 & 0 & y_2 & 0 \\ 0 & 1 & -y_4 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 1 & -y_4 & y_3 \end{pmatrix} ;$$

(b) $$M_{\overset{2}{\oplus}\overset{2}{\wedge} Q,\alpha,\alpha+1} = \begin{pmatrix} M_{\overset{2}{\wedge} Q,\alpha,\alpha+1} & 0 \\ 0 & M_{\overset{2}{\wedge} Q,\alpha,\alpha+1} \end{pmatrix} .$$

PROOFS. (a) The required result follows directly from the relation (*) above. For example, we have

$$\begin{aligned} \delta_{\alpha+1,3} &= \partial_{\alpha+1,1}\wedge\partial_{\alpha+1,4} = \partial_{\alpha,2}\wedge\Big(-\sum_i y_{\alpha,i}\partial_{\alpha,i}\Big) \\ &= y_{\alpha,1}\delta_{\alpha,1} - y_{\alpha,3}\delta_{\alpha,4} - y_{\alpha,4}\delta_{\alpha,5} . \end{aligned}$$

(b) Obvious.

Let a^+, a^- be maps from V_P to $\overset{2}{\wedge} Q$ defined by

$$\begin{aligned} a^+(v_m) &:= \partial_{m+2}\wedge\partial_{m+3} \\ a^-(v_m) &:= \partial_{m+1}\wedge\partial_{m+4} , \end{aligned}$$

and let a be the map $a^+ \oplus a^-$:

$$a : \ V_P \xrightarrow{a^+\oplus a^-} \overset{2}{\oplus}\overset{2}{\wedge} Q .$$

Let c be a natural isomorphism from $\overset{2}{\wedge} Q$ to $\overset{2}{\wedge} Q^* \otimes \overset{4}{\wedge} Q$ defined by

$$c(\delta)(\delta') := \delta\wedge\delta' , \qquad \delta, \delta' \in \overset{2}{\wedge} Q ,$$

via a canonical isomorphism $\mathrm{Hom}(\overset{2}{\wedge} Q, \overset{4}{\wedge} Q) \cong \overset{2}{\wedge} Q^* \otimes \overset{4}{\wedge} Q$, and let b be the composition of $a^*\otimes\overset{4}{\wedge}Q$ with $\begin{pmatrix}0 & c\\ -c & 0\end{pmatrix}$:

$$b:\ \overset{2}{\bigoplus}\overset{2}{\wedge}Q \xrightarrow{\begin{pmatrix}0 & c\\ -c & 0\end{pmatrix}} \overset{2}{\bigoplus}\overset{2}{\wedge}Q^*\otimes\overset{4}{\wedge}Q \xrightarrow{a^*\otimes\overset{4}{\wedge}Q} V_P^*\otimes\overset{4}{\wedge}Q\,.$$

DEFINITION. Let $W^{(k)}$ be a rank $r^{(k)}$ vector bundle on $\boldsymbol{P}$, let $\{(w^{(k)}_{\alpha,l^{(k)}})_{l^{(k)}}\}_\alpha$ be a system of frames of $W^{(k)}$, with $k=1, 2$, respectively, and let g be a homomorphism from $W^{(1)}$ to $W^{(2)}$. For any α, there exists a unique matrix, denoted by $M_{g,\alpha}$, satisfying

$$(g(w^{(1)}_{\alpha,1}),\ \cdots,\ g(w^{(1)}_{\alpha,r^{(1)}}))=(w^{(2)}_{\alpha,1},\ \cdots,\ w^{(2)}_{\alpha,r^{(2)}})M_{g,\alpha}\ ,$$

where the entries of $M_{g,\alpha}$ are regular functions over U_α. Then, we call $M_{g,\alpha}$ a *representation matrix* of g with respect to $(w^{(1)}_{\alpha,l^{(1)}})_{l^{(1)}}$ and $(w^{(2)}_{\alpha,l^{(2)}})_{l^{(2)}}$ over U_α, and $\{M_{g,\alpha}\}_\alpha$ a *system of representation matrices* of g with respect to $\{(w^{(1)}_{\alpha,l^{(1)}})_{l^{(1)}}\}_\alpha$ and $\{(w^{(2)}_{\alpha,l^{(2)}})_{l^{(2)}}\}_\alpha$.

LEMMA 3. (a)

$${}^\tau M_{a,\alpha}=\begin{pmatrix}0&0&0&1&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&y_1&0&0&-y_3&-y_4&0\\ 0&0&y_1&0&y_2&y_3&0&-1&0&0&0&0\\ y_2&y_3&y_4&0&0&0&0&0&0&0&-1&0\\ 1&0&0&0&0&0&0&-y_1&0&-y_2&0&y_4\end{pmatrix};$$

(b)

$$M_{b,\alpha}=\begin{pmatrix}0&0&0&-1&0&0&0&0&1&0&0&0\\ 0&-y_4&y_3&0&0&-y_1&1&0&0&0&0&0\\ 0&0&0&0&-1&0&y_3&-y_2&0&y_1&0&0\\ 0&-1&0&0&0&0&0&0&0&y_4&-y_3&y_2\\ -y_4&0&y_2&0&-y_1&0&0&0&0&0&0&1\end{pmatrix}.$$

PROOF. For the statement (a), see the definition of the map a. The statement (b) follows from (a) since we have

$$M_{b,\alpha}=M_{a^*\otimes\overset{4}{\wedge}Q,\alpha}M_{\left(\begin{smallmatrix}0&c\\-c&0\end{smallmatrix}\right),\alpha}\ ,$$

$$M_{a^*\otimes\overset{4}{\wedge}Q,\alpha}=M_{a^*,\alpha}={}^\tau M_{a,\alpha}\ ,$$

$$M_{\left(\begin{smallmatrix}0&c\\-c&0\end{smallmatrix}\right),\alpha}=\begin{pmatrix}0 & M_{c,\alpha}\\ -M_{c,\alpha} & 0\end{pmatrix}\ ,$$

and

$$M_{c,\alpha}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

§ 2. Results.

Let us consider a sequence of vector bundles over $\boldsymbol{P}$:

$$V_P \xrightarrow{a} \overset{2}{\oplus}\overset{2}{\wedge} Q \xrightarrow{b} V_P^* \otimes \overset{4}{\wedge} Q .$$

Then we have

PROPOSITION. *The sequence above is a monad. In other words, we have*:

(a) *Both a and b have maximal rank everywhere over $\boldsymbol{P}$;*

(b) $b \circ a = 0$.

PROOFS. (a) We have only to show the statement for the map a (see the definition of b). For each α, let a_α^l be the l-th row-vector of $M_{a,\alpha}$, with $l=1, \cdots, 12$. Then we find

$$\det({}^\tau a_\alpha^4, {}^\tau a_\alpha^7, {}^\tau a_\alpha^5, {}^\tau a_\alpha^2, {}^\tau a_\alpha^{12}) = \prod_i y_{\alpha,i}$$

$$\det({}^\tau a_\alpha^9, {}^\tau a_\alpha^6, {}^\tau a_\alpha^8, {}^\tau a_\alpha^{11}, {}^\tau a_\alpha^1) = 1 - \prod_i y_{\alpha,i} .$$

Thus, $M_{a,\alpha}$ has maximal rank everywhere over U_α for any α, so does our map a over $\boldsymbol{P}$.

(b) It follows from Lemma 3 that

$$M_{b,\alpha} M_{a,\alpha} = 0$$

for each α, which implies the result.

We define E to be the cohomology of the monad above:

$$E := \mathrm{Ker}(b)/\mathrm{Im}(a) .$$

It follows that E is a rank 2 vector bundle over $\boldsymbol{P}$. Moreover, E is indecomposable because the Chern polynomial of E is:

$$1 + h + 4h^2 ,$$

where h is the class of a hyperplane in $\boldsymbol{P}$.

DEFINITION (see [2], [3]). We call the twisted bundle $E\otimes\mathcal{O}_P(2)$ the *Horrocks-Mumford bundle*, denoted by F:

$$F:=E\otimes\mathcal{O}_P(2)\,.$$

EXAMPLE. Let us consider the following sections of $\overset{2}{\bigoplus}\overset{2}{\wedge}Q$ over U_α for each α:

$$\varepsilon_\alpha^+:=y_1y_2\delta_{\alpha,1}^+ + y_1y_4\delta_{\alpha,3}^+ + y_3y_4\delta_{\alpha,6}^+$$
$$\varepsilon_\alpha^-:=y_1y_3\delta_{\alpha,2}^- + y_2y_3\delta_{\alpha,4}^- + y_2y_4\delta_{\alpha,5}^-\,.$$

It follows from Lemma 3 that both ε_α^+ and ε_α^- are contained in Ker(b) but not in Im(a). Thus, these sections give non-zero sections of E over U_α, which we denote by the same symbols. We note that $(\varepsilon_\alpha^+, \varepsilon_\alpha^-)$ can not be a frame of E over U_α since they have zero-locus over U_α.

Using Lemma 2, one can verify that both $\{x_\alpha^2\varepsilon_\alpha^+\}_\alpha$ and $\{x_\alpha^2\varepsilon_\alpha^-\}_\alpha$ form global sections of $\overset{2}{\bigoplus}\overset{2}{\wedge}Q$ and hence give global sections φ^+ and φ^- of the Horrocks-Mumford bundle F. We note that φ^+, φ^- play a key role in a computation by Sasakura [6].

THEOREM. *Let $\varepsilon_{\alpha,1}$ and $\varepsilon_{\alpha,2}$ be sections of $\overset{2}{\bigoplus}\overset{2}{\wedge}Q$ over U_α defined by*

$$\begin{aligned}\varepsilon_{\alpha,1}:=&-\delta_{\alpha,3}^+ + (y_2+y_1y_3^2)y_1y_2^2y_4\delta_{\alpha,5}^-\\ &+\{y_3+(y_2+y_1y_3^2)y_2y_4\}\delta_{\alpha,1}^- + \{y_3+(y_2+y_1y_3^2)y_2y_4\}y_3\delta_{\alpha,5}^+\\ &+(y_2+y_1y_3^2)y_2\delta_{\alpha,2}^+ + (1+y_1y_2y_3y_4)(y_2+y_1y_3^2)\delta_{\alpha,6}^-\end{aligned}$$

$$\begin{aligned}\varepsilon_{\alpha,2}:=&\,\delta_{\alpha,4}^- + (y_4+y_1^2y_2)y_1y_2y_4\delta_{\alpha,5}^-\\ &+(y_4+y_1^2y_2)y_4\delta_{\alpha,1}^- + \{y_1+(y_4+y_1^2y_2)y_3y_4\}\delta_{\alpha,5}^+\\ &+(y_4+y_1^2y_2)\delta_{\alpha,2}^+ + \{y_1+(y_4+y_1^2y_2)y_3y_4\}y_1\delta_{\alpha,6}^-\,.\end{aligned}$$

Then we have:

(a) $\{(x_\alpha^2\varepsilon_{\alpha,1}, x_\alpha^2\varepsilon_{\alpha,2})\}_\alpha$ *gives a system of frames of the Horrocks-Mumford bundle* F.

(b) $\{M_{F,\beta,\gamma}\}_{\beta,\gamma}$ *with respect to* $\{(x_\alpha^2\varepsilon_{\alpha,1}, x_\alpha^2\varepsilon_{\alpha,2})\}_\alpha$ *is a system of σ-transition matrices where we put*

$$\sigma=\begin{pmatrix}0&1&2&3&4\\4&0&1&2&3\end{pmatrix}.$$

PROOFS. (a) We shall prove that $\{(\varepsilon_{\alpha,1}, \varepsilon_{\alpha,2})\}_\alpha$ gives a system of frames of the bundle E. Using Lemma 3, one can show that $\varepsilon_{\alpha,1}$ and $\varepsilon_{\alpha,2}$ are contained in Ker(b). We must prove that the exterior product

$$\varepsilon_{\alpha,1}\wedge\varepsilon_{\alpha,2}\wedge a(v_{\alpha,0})\wedge\cdots\wedge a(v_{\alpha,4})$$

does not vanish everywhere as a section of $\overset{7}{\wedge}\overset{2}{\bigoplus}\overset{2}{\wedge}Q$ over U_α. With respect to the frame $(\delta^+_{\alpha,n};\delta^-_{\alpha,n})_n$, let $e_{\alpha,1}$, $e_{\alpha,2}$ be the coordinate of $\varepsilon_{\alpha,1}$, $\varepsilon_{\alpha,2}$, respectively:

$$(\varepsilon_{\alpha,1},\ \varepsilon_{\alpha,2})=(\delta^+_{\alpha,1};\ \delta^-_{\alpha,2})(e_{\alpha,1},\ e_{\alpha,2})\ ,$$

where we consider $e_{\alpha,1}$, $e_{\alpha,2}$ as column-vectors. Then, we have only to show that the 12×7-matrix

$$(e_{\alpha,1},\ e_{\alpha,2},\ M_{a,\alpha})$$

has maximal rank everywhere over U_α for each α. Denoting by a'^{l}_{α} the l-th row-vector of the matrix above, with $l=1,\cdots,12$, we find

$$\det({}^\tau a'^4_\alpha,\ {}^\tau a'^7_\alpha,\ {}^\tau a'^5_\alpha,\ {}^\tau a'^2_\alpha,\ {}^\tau a'^{12}_\alpha,\ {}^\tau a'^3_\alpha,\ {}^\tau a'^{10}_\alpha)=-\prod_i y_i$$

$$\det({}^\tau a'^9_\alpha,\ {}^\tau a'^6_\alpha,\ {}^\tau a'^8_\alpha,\ {}^\tau a'^{11}_\alpha,\ {}^\tau a'^1_\alpha,\ {}^\tau a'^3_\alpha,\ {}^\tau a'^{10}_\alpha)=\prod_i y_i-1\ .$$

This completes our proof.

(b) We find that, for any element τ of the cyclic group generated by σ, it follows

$$\tau(M_{F,\alpha,\alpha+1})=M_{F,\tau(\alpha),\tau(\alpha+1)}\ ,$$

so that

$$\tau(M_{F,\beta,\gamma})=M_{F,\tau(\beta),\tau(\gamma)}$$

for any indices β and γ.

REMARK. It can be shown that, for any permutation σ on the indices with length 5, the Horrocks-Mumford bundle F has rational sections which are connected by a system of σ-transition matrices.

To compute system of σ-transition matrices $\{M_{F,\beta,\gamma}\}_{\beta,\gamma}$ defining the Horrocks-Mumford bundle F with respect to the frames obtained in Theorem, it is sufficient to compute that of the bundle E since we have

$$M_{F,\beta,\gamma}=y_{\beta,\gamma-\beta}{}^2M_{E,\beta,\gamma}\ .$$

We here compute the transition matrix $M_{F,\alpha,\alpha+1}$, where we note that $M_{F,4,0}$ is the initial data of $\{M_{F,\beta,\gamma}\}_{\beta,\gamma}$. With the same notation as in the proof of Theorem, we have the following relation:

$$
\begin{aligned}
M_{\overset{2}{\oplus}\overset{2}{\wedge} Q,\alpha,\alpha+1}&(e_{\alpha+1,1}, e_{\alpha+1,2}, M_{a,\alpha+1}) \\
&=(e_{\alpha,1}, e_{\alpha,2}, M_{a,\alpha})\begin{pmatrix} M_{E,\alpha,\alpha+1} & 0 \\ N & M_{V_P,\alpha,\alpha+1}\end{pmatrix}
\end{aligned}
$$

for some 5×2-matrix N. Choose a 7×7-submatrix of the 12×7-matrix $(e_{\alpha,1}, e_{\alpha,2}, M_{a,\alpha})$ suitable to compute its inverse (by the sweeping-out method), and multiply the relation above by the inverse on the right side (for example, take 9th, 6th, 8th, 11th, 1st, 3rd and 10th rows of the matrix). Making a calculation, denoting

$$
\begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2}\end{pmatrix} := M_{F,\alpha,\alpha+1}\ ,
$$

we have

$$
\begin{aligned}
m_{1,1}= & -\frac{y_2^3y_3y_4^4}{y_1^5}+\frac{y_2y_3y_4^6}{y_1^4}-\frac{y_2^2y_3^3y_4^4}{y_1^4}-\frac{2y_2^2y_3^2y_4^2}{y_1^3}-\frac{y_2^4y_3y_4^3}{y_1^3} \\
& -\frac{y_2y_3^4y_4^2}{y_1^2}+\frac{2y_2^2y_3y_4^5}{y_1^2}-\frac{y_2^3y_3^3y_4^3}{y_1^2}+\frac{y_3^2y_4^4}{y_1^2} \\
& -\frac{y_2y_3^3}{y_1}-\frac{y_2^2y_4^3}{y_1}-\frac{y_2^3y_3^2y_4}{y_1} \\
& +2y_2y_3^2y_4^3-y_2^2y_3^4y_4+y_2^3y_3y_4^4+y_4^5 \\
& -y_1y_2y_3y_4-y_1y_2^3y_4^2 \\
& +y_1^2y_2^2y_3^2y_4^2+2y_1^2y_2y_4^4-y_1^3y_2^2y_3+y_1^4y_3y_4^2+y_1^4y_2^2y_4^3 \\
& +y_1^5+y_1^6y_2y_3y_4
\end{aligned}
$$

$$
\begin{aligned}
m_{1,2}= & -\frac{y_2^4y_3y_4^2}{y_1^4}+\frac{y_2^2y_3y_4^4}{y_1^3}-\frac{y_2^3y_3^3y_4^2}{y_1^3} \\
& -\frac{y_2^2y_4^2}{y_1^2}-\frac{y_2^3y_3^2}{y_1^2}-\frac{y_2^5y_3y_4}{y_1^2} \\
& -\frac{y_2y_3^2y_4^2}{y_1}+\frac{2y_2^3y_3y_4^3}{y_1}-\frac{y_2^4y_3^3y_4}{y_1}+\frac{y_4^4}{y_1} \\
& -y_2y_3-2y_2^3y_4 \\
& +3y_1y_2y_4^3-y_1y_2^2y_3^2y_4+y_1y_2^4y_3y_4^2 \\
& -y_1^2y_2^4+3y_1^3y_2^2y_4^2+y_1^5y_2^3y_4
\end{aligned}
$$

$$
m_{2,1}=\frac{y_2^3y_3^2y_4^2}{y_1^5}+\frac{y_2^5y_3y_4^3}{y_1^5}
$$

$$
\begin{aligned}
&-\frac{y_2y_3^2y_4^4}{y_1^4}+\frac{y_2^2y_3^4y_4^2}{y_1^4}-\frac{y_2^3y_3y_4^5}{y_1^4}+\frac{2y_2^4y_3^3y_4^3}{y_1^4}\\
&+\frac{y_2y_3y_4^2}{y_1^3}+\frac{y_2^2y_3^3}{y_1^3}-\frac{y_2^2y_3^3y_4^5}{y_1^3}+\frac{y_2^3y_3^5y_4^3}{y_1^3}+\frac{y_2^4y_3^2y_4}{y_1^3}\\
&+\frac{y_2y_3^5}{y_1^2}-\frac{y_2^2y_3^2y_4^3}{y_1^2}+\frac{2y_2^3y_3^4y_4}{y_1^2}-\frac{y_2^4y_3y_4^4}{y_1^2}-\frac{y_3^3y_4^2}{y_1^2}\\
&-\frac{y_2y_3^4y_4^3}{y_1}+\frac{y_2^2y_3^6y_4}{y_1}-\frac{y_2^3y_3^3y_4^4}{y_1}+\frac{y_2^4y_4^2}{y_1}+\frac{y_3^2}{y_1}\\
&-y_3y_4^3-y_2^2y_4^4\\
&-y_1y_2y_3^2y_4^4+y_1y_4-y_1y_2^2y_3^4y_4^2+y_1y_2^3y_3\\
&+y_1^2y_2^2y_3^3-y_1^2y_2^3y_4^3-y_1^3y_2-y_1^3y_2^2y_3^2y_4^3\\
&-y_1^4y_2^2y_3y_4-y_1^4y_3^2-y_1^5y_2y_3^3y_4
\end{aligned}
$$

$$
\begin{aligned}
m_{2,2}=&\frac{y_2^4y_3^2}{y_1^4}+\frac{y_2^6y_3y_4}{y_1^4}\\
&-\frac{y_2^2y_3^2y_4^2}{y_1^3}+\frac{y_2^3y_3^4}{y_1^3}-\frac{y_2^4y_3y_4^3}{y_1^3}+\frac{2y_2^5y_3^3y_4}{y_1^3}\\
&+\frac{2y_2^2y_3}{y_1^2}-\frac{y_2^3y_3^3y_4^3}{y_1^2}+\frac{y_2^4y_4}{y_1^2}+\frac{y_2^4y_3^5y_4}{y_1^2}\\
&+\frac{y_2y_3^3}{y_1}-\frac{y_3y_4^2}{y_1}-\frac{y_2^2y_4^3}{y_1}+\frac{2y_2^3y_3^2y_4}{y_1}-\frac{y_2^5y_3y_4^2}{y_1}\\
&-y_2y_3^2y_4^3+y_2^2y_3^4y_4-y_2^4y_3^3y_4^2+y_2^5\\
&+1-y_1y_2y_3y_4-2y_1y_2^3y_4^2+y_1y_2^4y_3^2\\
&-2y_1^2y_2^2y_3^2y_4^2-y_1^3y_2^4y_4-y_1^4y_2^3y_3^2y_4\,,
\end{aligned}
$$

where we denote $y_{\alpha,i}$ by y_i briefly.

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