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EXAMPLES OF COMPACT EINSTEIN KÄHLER MANIFOLDS WITH POSITIVE RICCI TENSOR

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Let (P, J, g) be a compact Kähler manifold. If (P, J, g) is Einstein Kähler, the first Chern class $c_1(P)$ of P is positive, zero or negative. It has been proved by Aubin [1] and Yau [20] that if (P, J) is a compact complex manifold with $c_1(P) < 0$ there exists a unique Einstein Kähler metric on (P, J) , and by Yau [20] that if (P, J) is a compact Kähler manifold with $c_1(P) = 0$ there exists an Einstein Kähler metric on (P, J) . In the case of $c_1(P) > 0$ it is known that there exist compact Kähler manifolds which do not admit any Einstein Kähler metric (cf. [6], [8], [19]). Up to now known obstructions to the existence of Einstein Kähler metrics on compact Kähler manifolds with positive first Chern class are (1) Matsushima's theorem ([10], [12]), that is, if (P, J, g) is an Einstein Kähler manifold, the Lie algebra of all Killing vector fields on P is a real form of the Lie algebra of all holomorphic vector fields on P and (2) Futaki invariant [6].

The purpose of this note is to give some examples of compact Einstein Kähler manifolds with positive first Chern class which are not homogeneous. We give a necessary and sufficient condition to the existence of Einstein Kähler metrics on $P^1(\mathcal{C})$ -bundles over hermitian symmetric spaces of compact type. In the category of Riemannian manifolds, compact Einstein manifolds of cohomogeneity one have been studied by Bérard Bergery [2]. In our case the $P^1(\mathcal{C})$ -bundle P is of cohomogeneity one with respect to a maximal compact subgroup of the complex Lie group of all holomorphic transformations on P and to prove our Main Theorem we use the similar method used by Bérard Bergery in [2]. We also remark that our Corollary 2 (2) to our Main Theorem generalizes the example given in Futaki [6].

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1 Main Theorem

Let M be an irreducible hermitian symmetric space of compact type.

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We denote by $H^1(M, \theta^*)$ the isomorphism classes of all holomorphic line bundles over M . It is known that $H^1(M, \theta^*)$ is isomorphic to the second cohomology group $H^2(M, \mathbf{Z}) \cong \mathbf{Z}$ ([5]). Take a generator L of $H^1(M, \theta^*)$ which has a positive Chern class $c_1(L) > 0$. Then the first Chern class $c_1(M)$ of M is given by $c_1(M) = \kappa c_1(L)$ where κ is an integer: $2 \leq \kappa \leq \dim_{\mathbf{C}} M + 1$ (cf. [5]).

Consider a product X of two irreducible hermitian symmetric spaces of compact type M_1 and M_2 and a holomorphic vector bundle $p_1^* L_1^a \oplus p_2^* L_2^b$ over X where $p_i: X \rightarrow M_i$ ($i=1, 2$) are projections, L_i ($i=1, 2$) are the generators of $H^1(M_i, \theta^*)$ and a, b are positive integers. We denote by P the $P^1(\mathbf{C})$ -bundle $P(p_1^* L_1^a \oplus p_2^* L_2^b)$ over X . It is not difficult to see that the first Chern class $c_1(P)$ of P is positive if $a < \kappa_1$ and $b < \kappa_2$ where κ_i ($i=1, 2$) are positive integers given by $c_1(M_i) = \kappa_i c_1(L_i)$ (cf. [15] proof of theorem (5.56)).

Main Theorem. *For irreducible hermitian symmetric spaces of compact type M_1 of complex m -dimension and M_2 of complex n -dimension, and positive integers a, b with $a < \kappa_1$ and $b < \kappa_2$, there exists an Einstein Kähler metric on the compact complex manifold P if and only if*

$$\int_{-1}^1 (\kappa_1 - ax)^m (\kappa_2 + bx)^n x dx = 0.$$

Corollary 1. *For irreducible hermitian symmetric spaces of compact type $M = M_1 = M_2$ and a positive integer $a = b$ with $a < \kappa$, there exists an Einstein Kähler metric on the $P^1(\mathbf{C})$ -bundle P over $M \times M$.*

Corollary 2.

(1) *For $M = M_1 = M_2$ and positive integers a, b such that $a, b < \kappa$ and $a \neq b$, the $P^1(\mathbf{C})$ -bundle P over $M \times M$ has the first positive Chern class but P does not admit any Einstein Kähler metric.*

(2) *For $M_1 = P^1(\mathbf{C})$, $M_2 \neq P^1(\mathbf{C})$ and a positive integer b with $b < \kappa_2$, the $P^1(\mathbf{C})$ -bundle P over $P^1(\mathbf{C}) \times M_2$ has the positive first Chern class but P does not admit any Einstein Kähler metric.*

2 Orbits on $P^1(\mathbf{C})$ -bundles over a Kähler \mathbf{C} -space

Let X be a Kähler \mathbf{C} -space, that is, a simply connected compact complex homogeneous space with a Kähler metric. By a result of H.C. Wang [18], X can be written as $X = G/U$ where G is a simply connected complex semi-simple Lie group and U is a parabolic subgroup of G . Let $\rho: U \rightarrow \mathbf{C}^*$ be a holomorphic representation of U and ξ_ρ the homogeneous holomorphic line bundle on X associated to ρ , that is, ξ_ρ is obtained from the product $G \times \mathbf{C}^*$ by identifying (gu, w) with $(g, \rho^{-1}(u)w)$ where $g \in G$, $u \in U$ and $w \in \mathbf{C}^*$. It is known that every holomorphic line bundle on a Kähler \mathbf{C} -space X is homogeneous (cf. Ise [7]).

For a holomorphic line bundle ξ on X , we consider a $P^1(\mathbb{C})$ -bundle $P(1 \oplus \xi)$ over X , where 1 denotes the trivial line bundle on X . Then G acts on $P(1 \oplus \xi)$ in the natural way.

Proposition 2.1. *If ξ is a non-trivial holomorphic line bundle on X , the $P^1(\mathbb{C})$ -bundle $P=P(1 \oplus \xi)$ is a disjoint union of three G -orbits. One of orbits is open in P and it is isomorphic to the principal \mathbb{C}^* -bundle associated to ξ . The other two orbits are isomorphic to X .*

Proof. The equivalence class of $(g, (w_1, w_2)) \in G \times \mathbb{C}^2$ is denoted by $[g, (w_1, w_2)] \in 1 \oplus \xi$. Let $p: 1 \oplus \xi \rightarrow P$ denote the canonical projection. Consider the G -orbit of the point $p[e, (1,1)]$ where e is the identity of G . We shall show that the orbit $G \cdot p[e, (1,1)]$ is isomorphic to the principal \mathbb{C}^* -bundle associated to the line bundle ξ . Let $\rho: U \rightarrow \mathbb{C}^*$ denote the holomorphic representation such that $\xi = \xi_\rho$. Then the principal \mathbb{C}^* -bundle associated to ξ is obtained from the product $G \times \mathbb{C}^*$ by identifying (gu, w) with $(g, \rho^{-1}(u)w)$ where $g \in G, u \in U$ and $w \in \mathbb{C}^*$, and the principal \mathbb{C}^* -bundle is denoted by $G \times_\rho \mathbb{C}^*$. The equivalence class of $(g, w) \in G \times \mathbb{C}^*$ is denoted by $[g, w] \in G \times_\rho \mathbb{C}^*$. We define a map $\varphi: G \cdot p[e, (1,1)] \rightarrow G \times_\rho \mathbb{C}^*$ by $\varphi(gp[e, (1,1)]) = [g, 1]$. It is not difficult to see that φ is an injective holomorphic map. Since ρ is not trivial, $\rho: U \rightarrow \mathbb{C}^*$ is surjective and thus we see that φ is surjective. Moreover for each element $p[g, (w_1, w_2)]$ ($w_1 \neq 0, w_2 \neq 0$) there is an element $u \in U$ such that $\rho(u) = w_1^{-1}w_2 \in \mathbb{C}^*$. Thus $p[g, (w_1, w_2)] = p[gu, (1,1)]$. By the same way we see that the orbits $G \cdot p[e, (1,0)]$ and $G \cdot p[e, (0,1)]$ are isomorphic to $X = G/U$. Thus the orbit $G \cdot p[e, (1,1)]$ is open in $P(1 \oplus \xi)$. q.e.d.

For a holomorphic line bundle $\xi = \xi_\rho$ on X let \tilde{U} be the isotropy subgroup of G at $p[e, (1,1)] \in P(1 \oplus \xi)$. Then $\tilde{U} = \{g \in U \mid \rho(g) = 1\}$ and $\dim_{\mathbb{C}} \tilde{U} = \dim_{\mathbb{C}} U - 1$ if ξ is non-trivial. The natural $\mathbb{C}^* \times \mathbb{C}^*$ -action on $1 \oplus \xi$ induces a \mathbb{C}^* -action on $P(1 \oplus \xi)$. Note that $G \times \mathbb{C}^*$ -orbits in $P(1 \oplus \xi)$ coincide with G -orbit and that the \mathbb{C}^* -action on the orbit $G \cdot p[e, (1,1)]$ corresponds to the right $\mathbb{C}^* \simeq U/\tilde{U}$ -action on the principal fiber bundle G/\tilde{U} over X .

Let G_u denote a maximal compact subgroup of G and $V = G_u \cap U$. Then G_u/V is diffeomorphic to G/U . Put $\tilde{V} = \{g \in V \mid \rho(g) = 1\}$. If $\rho: U \rightarrow \mathbb{C}^*$ is non-trivial, $\dim_{\mathbb{R}} \tilde{V} = \dim_{\mathbb{R}} V - 1$.

Proposition 2.2. *Let $\rho: U \rightarrow \mathbb{C}^*$ be non-trivial. Then the principal \mathbb{C}^* -bundle $G \times_\rho \mathbb{C}^*$ over X is $G_u \times S^1$ -equivariantly diffeomorphic to $G_u/\tilde{V} \times \mathbb{R}_+$ where $G_u \times S^1$ acts on \mathbb{R}_+ trivially.*

Proof. For $g \in G$, there exist elements $k \in G_u$ and $u \in U$ such that $g = ku$, since $G_u/V = G/U$. Since each element of $G \times_\rho \mathbb{C}^*$ may be written as $[g, 1] \in G \times_\rho \mathbb{C}^*$, we have $[g, 1] = [k, \rho(u)]$. Let $G_u \times_\rho \mathbb{C}^*$ denote the space obtained from

the product $G_u \times \mathbf{C}^*$ by identifying (kv, w) with $(k, \rho^{-1}(v)w)$ where $k \in G_u, v \in V$ and $w \in \mathbf{C}^*$. The equivalence class of $(k, w) \in G_u \times \mathbf{C}^*$ is also denoted by $[k, w]$. Then the map $[g, 1] \mapsto [k, \rho(g)]: G \times_{\rho} \mathbf{C}^* \rightarrow G_u \times_{\rho} \mathbf{C}^*$ is a $G_u \times S^1$ -equivariantly diffeomorphism. Put $\rho(u) = re^{i\theta}$ ($r \in \mathbf{R}_+$). Then r is uniquely determined by the class $[g, 1] \in G \times_{\rho} \mathbf{C}^*$. In fact, if $g = ku = k_1u_1$ ($k, k_1 \in G_u, u, u_1 \in U$), $k^{-1}k_1 = uu_1^{-1} \in G_u \cap U = V$. Since $\rho(uu_1^{-1}) \in S^1 = \{e^{i\theta} \mid \theta \in \mathbf{R}\}$, $\rho(u_1) = \rho(u_1u^{-1})\rho(u) = re^{i\theta_1}$ for some $\theta_1 \in \mathbf{R}$. Define a map $\psi: G_u \times_{\rho} \mathbf{C}^* \rightarrow G_u/\tilde{V} \times \mathbf{R}_+$ by $\psi([k, w]) = (kv \tilde{V}, r)$ where $w = re^{i\theta}$ and $\rho(v) = e^{i\theta}$ ($v \in V$). Then it is easy to see that ψ is a $G_u \times S^1$ -equivariantly diffeomorphism. q.e.d.

For a compact complex manifold Y let $\text{Aut}_0(Y)$ denote the connected component of the identity of the group of all holomorphic automorphisms of Y .

Proposition 2.3. *Let ξ be a non-trivial holomorphic line bundle on a Kähler C-space $X = G/U$. Then the complex Lie group $\text{Aut}_0(P(1 \oplus \xi))$ is reductive if and only if $H^0(X, \xi) = H^0(X, \xi^{-1}) = (0)$. Moreover in this case the Lie algebra of $\text{Aut}_0(P(1 \oplus \xi))$ coincides with the Lie algebra of $\text{Aut}_0(X) \times \mathbf{C}^*$.*

Proof. Let $\pi: P(1 \oplus \xi) \rightarrow X$ be the natural projection. By a theorem of Blanchard [4], the projection π induces a Lie group homomorphism, denoted also by π ,

$$\pi: \text{Aut}_0(P(1 \oplus \xi)) \rightarrow \text{Aut}_0(X).$$

It is known that the Lie algebra of $\text{Ker } \pi$ is isomorphic to $H^0(X, \text{End}(1 \oplus \xi))$ and thus it is isomorphic to

$$\left\{ \begin{pmatrix} w_1 & s_1 \\ s_2 & w_2 \end{pmatrix} \mid s_1 \in H^0(X, \xi), s_2 \in H^0(X, \xi^{-1}), w_1, w_2 \in \mathbf{C} \right\} / \left\{ \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix} \mid w \in \mathbf{C} \right\}$$

(cf. [8]). By a Borel-Weil theorem (cf. for example [7]), for a non-trivial holomorphic line bundle ξ , if $H^0(X, \xi) \neq 0, H^0(X, \xi^{-1}) = 0$. Thus if one of $H^0(X, \xi), H^0(X, \xi^{-1})$ is non-zero, $\text{Aut}_0(P(1 \oplus \xi))$ is not reductive. Conversely, if $H^0(X, \xi) = H^0(X, \xi^{-1}) = (0)$, $\dim_{\mathbf{C}} \text{Ker } \pi = 1$. Note also that $\pi: \text{Aut}_0(P(1 \oplus \xi)) \rightarrow \text{Aut}_0(X)$ is surjective. The Lie algebra of $\text{Aut}_0(P(1 \oplus \xi))$ always contains the Lie algebra of $\text{Aut}_0(X) \times \mathbf{C}^*$. Thus the Lie algebra of $\text{Aut}_0(P(1 \oplus \xi))$ coincides with the Lie algebra of $\text{Aut}_0(X) \times \mathbf{C}^*$, which is reductive, since $\text{Aut}_0(X)$ is a complex semi-simple Lie group. q.e.d.

Corollary 2.4. *Let ξ be a non-trivial holomorphic line bundle on a Kähler C-space. Then $P(1 \oplus \xi)$ is almost homogeneous but not homogeneous.*

Proof. By proposition 2.1, $P(1 \oplus \xi)$ is almost homogeneous. If $\text{Aut}_0(P(1 \oplus \xi))$ acts transitively on the simply connected compact projective manifold $P(1 \oplus \xi)$,

the Lie group $\text{Aut}_0(P(1 \oplus \xi))$ is a semi-simple complex Lie group (cf. Takeuchi [16] p. 174). Since $\text{Aut}_0(P(1 \oplus \xi))$ is not semi-simple by Proposition 2.3, this is a contradiction. q.e.d.

3 $G_u \times S^1$ -invariant Kähler metrics on the open orbit

We consider a $G_u \times S^1$ -invariant Kähler metric on the open orbit $G \cdot p[e, (1,1)] \cong G \times_{\rho} \mathbf{C}^*$ in $P(1 \oplus \xi)$. Let $\mathfrak{g}_u, \mathfrak{v}, \tilde{\mathfrak{v}}$ be the Lie algebra of G_u, V, \tilde{V} respectively. Since G_u is a compact semi-simple Lie group, the Killing form of \mathfrak{g}_u is negative definite. Let \langle, \rangle denote the $\text{Ad}(G_u)$ -invariant inner product on \mathfrak{g}_u induced from the Killing form and let $\mathfrak{m} \subset \mathfrak{g}_u$ be the orthogonal complement of \mathfrak{v} with respect to the inner product \langle, \rangle . Then $\mathfrak{g}_u = \mathfrak{v} + \mathfrak{m}$ and $[\mathfrak{v}, \mathfrak{m}] \subset \mathfrak{m}$. Let \mathfrak{c}_p be the orthogonal complement of $\tilde{\mathfrak{v}}$ in \mathfrak{v} with respect to the inner product \langle, \rangle . Then we have

$$(3.1) \quad [\mathfrak{c}_p, \tilde{\mathfrak{v}}] = (0).$$

In fact, we can write $\mathfrak{v} = \mathfrak{c} + \mathfrak{v}_s$ where \mathfrak{c} is the center of \mathfrak{v} and \mathfrak{v}_s is the semi-simple part of \mathfrak{v} . Note that $\langle \mathfrak{c}, \mathfrak{v}_s \rangle = (0)$ and $\tilde{\mathfrak{v}} \supset \mathfrak{v}_s$. Thus $\mathfrak{c}_p \subset \mathfrak{c}$ and hence $[\mathfrak{c}_p, \tilde{\mathfrak{v}}] = (0)$. Moreover if the holomorphic representation $\rho: U \rightarrow \mathbf{C}^*$ corresponds to the weight Λ , then $\sqrt{-1}\Lambda$ generates \mathfrak{c}_p and thus \mathfrak{c}_p generates a closed subgroup of G_u , that is, a circle group S^1 .

Put $\mathfrak{p} = \mathfrak{c}_p + \mathfrak{m}$. Then we have orthogonal decompositions of $\mathfrak{g}_u, \mathfrak{p}$ and \mathfrak{v} with respect to \langle, \rangle :

$$(3.2) \quad \mathfrak{g}_u = \tilde{\mathfrak{v}} + \mathfrak{p}, \quad \mathfrak{p} = \mathfrak{c}_p + \mathfrak{m}, \quad \mathfrak{v} = \tilde{\mathfrak{v}} + \mathfrak{c}_p.$$

Moreover we have

$$(3.3) \quad [\mathfrak{v}, \mathfrak{c}_p] = (0), \quad [\mathfrak{v}, \mathfrak{m}] \subset \mathfrak{m}.$$

Let \mathbf{R}_+ be the subgroup of \mathbf{C}^* defined by $\{r > 0 \mid re^{i\theta} \in \mathbf{C}^*\}$. Since the open orbit $G \times_{\rho} \mathbf{C}^*$ in $P(1 \oplus \xi)$ is also a $G \times \mathbf{C}^*$ -orbit in $P(1 \oplus \xi)$ and $G \times_{\rho} \mathbf{C}^*$ is diffeomorphic to $G_u / \tilde{V} \times \mathbf{R}_+$, the Lie subgroup $G_u \times \mathbf{R}_+$ of $G \times \mathbf{C}^*$ also acts on $G \times_{\rho} \mathbf{C}^*$ transitively. Take a basis $\{\tilde{H}\}$ of the Lie algebra of \mathbf{R}_+ . Then $\mathfrak{g}_u + \mathbf{R}\tilde{H} = \tilde{\mathfrak{v}} + \mathfrak{p} + \mathbf{R}\tilde{H}$ and $\text{Ad}(\tilde{V})(\mathfrak{p} + \mathbf{R}\tilde{H}) \subset \mathfrak{p} + \mathbf{R}\tilde{H}$. We identify $\mathfrak{p} + \mathbf{R}\tilde{H}$ with the tangent space $T_o(G \times_{\rho} \mathbf{C}^*)$ at the origin $o = [e, 1]$ of $G \times_{\rho} \mathbf{C}^*$. Since the complex structure J on $G \times_{\rho} \mathbf{C}^*$ is invariant by the action of $G \times \mathbf{C}^*$, it induces a linear isomorphism $I: \mathfrak{p} + \mathbf{R}\tilde{H} \rightarrow \mathfrak{p} + \mathbf{R}\tilde{H}$ which satisfies $I^2 = -id$ and $I \circ \text{Ad}(g) = \text{Ad}(g) \circ I$ for every $g \in \tilde{V}$. Note that at the origin o of $G \times_{\rho} \mathbf{C}^*$ the orbit of the right S^1 -action coincides with the orbit of the left S^1 -action defined by \mathfrak{c}_p and that the complex structure of the fiber \mathbf{C}^* is induced from the natural complex structure of \mathbf{C} . Therefore we have

$$(3.4) \quad I\mathfrak{c}_p = \mathbf{R}\tilde{H}.$$

Moreover, since the complex structure on $P(1 \oplus \xi)$ is compatible with the invariant complex structure on $G/U = G_u/V$,

$$(3.5) \quad Im = m.$$

To investigate a $G_u \times S^1$ -invariant hermitian metric on the open orbit $G \times_p \mathcal{C}^*$, we consider a $G_u \times \mathbf{R}_+$ -invariant hermitian metric on $G \times_p \mathcal{C}^* = G_u/\tilde{V} \times \mathbf{R}_+$ for the moment. Note that there is a natural one-to-one correspondence between $G_u \times \mathbf{R}_+$ -invariant hermitian metrics on $G_u/\tilde{V} \times \mathbf{R}_+$ and the $\text{Ad}(\tilde{V})$ -invariant hermitian inner products on $\mathfrak{p} + \mathbf{R}\tilde{H}$ (cf. [11]).

From now on we assume that

$$(3.6) \quad [\tilde{\mathfrak{v}}, m] = m.$$

Let B be an $\text{Ad}(\tilde{V})$ -invariant hermitian inner product on $\mathfrak{p} + \mathbf{R}\tilde{H}$. Then B has the following properties:

$$(3.7) \quad \begin{aligned} (a) \quad & B(\mathfrak{c}_p, \tilde{H}) = (0) & (b) \quad & B(\mathfrak{c}_p, m) = (0) \\ (c) \quad & B(\tilde{H}, m) = (0). \end{aligned}$$

In fact, (a) follows from (3.4). To see (b), $B(\mathfrak{c}_p, m) = B(\mathfrak{c}_p, [\tilde{\mathfrak{v}}, m]) = B([\tilde{\mathfrak{v}}, \mathfrak{c}_p], m) = (0)$ by (3.1). Now (c) follows from (b) and (3.5).

We decompose $\tilde{\mathfrak{v}}$ -module m into irreducible component m_j ; $m = \sum_j m_j$. By (3.6) we have

$$(3.8) \quad [\tilde{\mathfrak{v}}, m_j] = m_j \quad \text{for every } j.$$

From now on we also assume that

$$(3.9) \quad [\mathfrak{v}, m_j] = m_j \quad \text{for every } j,$$

$$(3.10) \quad Im_j = m_j \quad \text{for every } j \text{ and}$$

$$(3.11) \quad \text{each multiplicity of irreducible components of } m \text{ as } \tilde{\mathfrak{v}}\text{-module is } 1.$$

Now the hermitian inner product B can be written uniquely as

$$(3.12) \quad B = d(\langle \cdot, \cdot \rangle_{\mathfrak{c}_p} + \langle I \circ, I \circ \rangle_{\mathbf{R}\tilde{H}}) + \sum_j c_j \langle \cdot, \cdot \rangle_{m_j}$$

where d, c_j are positive real numbers, $\langle \cdot, \cdot \rangle_{\mathfrak{c}_p}$ and $\langle \cdot, \cdot \rangle_{m_j}$ denote the inner products on \mathfrak{c}_p and m_j induced from $\langle \cdot, \cdot \rangle$ respectively, and $\langle I \circ, I \circ \rangle_{\mathbf{R}\tilde{H}}$ denotes the inner product on $\mathbf{R}\tilde{H}$ defined by $\langle IX, IY \rangle$ for $X, Y \in \mathbf{R}\tilde{H}$. Note that $\langle \cdot, \cdot \rangle_{\mathfrak{c}_p}$, $\langle I \circ, I \circ \rangle_{\mathbf{R}\tilde{H}}$ and $\langle \cdot, \cdot \rangle_{m_j}$ are $\text{Ad}(\tilde{V})$ -invariant symmetric bilinear form on $\mathfrak{p} + \mathbf{R}\tilde{H}$. Let $\beta_0, \beta_1, \alpha_j$ be the $G_u \times \mathbf{R}_+$ -invariant symmetric tensors on $G_u/\tilde{V} \times \mathbf{R}_+$ corresponding to $\langle \cdot, \cdot \rangle_{\mathfrak{c}_p}$, $\langle I \circ, I \circ \rangle_{\mathbf{R}\tilde{H}}$, $\langle \cdot, \cdot \rangle_{m_j}$ respectively. Then the $G_u \times \mathbf{R}_+$ -invariant hermitian metric g_B corresponding to B is given by

$$g_B = d(\beta_0 + \beta_1) + \sum_j c_j \alpha_j.$$

Lemma 3.1. *The $G_u \times \mathbf{R}_+$ -invariant symmetric tensors β_0, β_1 on $G_u/\tilde{V} \times \mathbf{R}_+$ are invariant by the right S^1 -action.*

Proof. (cf. [9] §2) Let $\tilde{\gamma}$ be \mathfrak{c}_p -valued left invariant 1-form on G_u , defined by

$\tilde{\gamma}(Y)$ = the \mathfrak{c}_p -component of $Y \in \mathfrak{g}_u$ with respect to the decomposition $\mathfrak{g}_u = \mathfrak{b} + \mathfrak{c}_p + \mathfrak{m}$.

Then there is a unique G_u -invariant connection, called the canonical connection, on the principal S^1 -bundle G_u/\tilde{V} over G_u/V such that the connection form γ is given by $\pi_1 * \gamma = \tilde{\gamma}$ where $\pi_1: G_u \rightarrow G_u/\tilde{V}$ is the canonical projection. Using the connection form γ , the symmetric tensor β_0 on $G_u/\tilde{V} \times \mathbf{R}_+$ can be written as $\beta_0 = \langle \gamma, \gamma \rangle$, that is, $\beta_0(X, Y) = \langle \gamma(X), \gamma(Y) \rangle$ for $X, Y \in T_p(G_u/\tilde{V} \times \mathbf{R}_+)$, $p \in G_u/\tilde{V} \times \mathbf{R}_+$. In particular, β_0 is invariant by the right S^1 -action. We also have $\beta_1 = \langle \gamma \circ J, \gamma \circ J \rangle$. Since the right S^1 -action is holomorphic, β_1 is also invariant by the right S^1 -action. q.e.d.

Let $\tilde{\alpha}_j$ denote the G_u -invariant symmetric tensor on $X = G_u/V$ corresponding to $\text{Ad}(V)$ -invariant symmetric bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{m}_j}$ on \mathfrak{m} . Let $\pi: G \times_p \mathbf{C}^* \rightarrow G_u/V$ denote the canonical projection. Then we have $\alpha_j = \pi^* \tilde{\alpha}_j$. In particular, α_j is also invariant by the right S^1 -action.

We now consider a $G_u \times S^1$ -invariant hermitian metric g on $G \times_p \mathbf{C}^* \simeq G_u/\tilde{V} \times \mathbf{R}_+$. Let \tilde{X} denote the vector field on $G_u/\tilde{V} \times \mathbf{R}_+$ induced by $X \in \mathfrak{g}_u$.

Proposition 3.2. *A $G_u \times S^1$ -invariant hermitian metric g on $G \times_p \mathbf{C}^*$ can be written as*

$$(3.13) \quad g = F^2(\beta_0 + \beta_1) + \sum_j H_j^2 \alpha_j$$

where F, H_j are $G_u \times S^1$ -invariant positive valued C^∞ functions on $G \times_p \mathbf{C}^*$.

Proof. We denote by \tilde{o} the origin of G_u/\tilde{V} and identify the tangent space $T_{(\tilde{o}, r)}(G_u/\tilde{V} \times \mathbf{R}_+)$ at (\tilde{o}, r) with $\mathfrak{c}_p + \mathfrak{m} + \mathbf{R} \frac{\partial}{\partial r}$. Then

$$(3.14) \quad g_{(\tilde{o}, r)}(u, \frac{\partial}{\partial r}) = 0 \quad \text{for } u \in T_{\tilde{o}}(G_u/\tilde{V}).$$

In fact, if $u \in \mathfrak{m}$, then $u = \sum_i [\tilde{X}_i, \tilde{Y}_i]_{\tilde{o}}$ for some $X_i \in \mathfrak{b}, Y_i \in \mathfrak{m}$ by our assumption (3.6). Since $(\tilde{X}_i)_{\tilde{o}} = 0$ and $[\tilde{X}_j, \frac{\partial}{\partial r}] = 0$, we have $g_{(\tilde{o}, r)}(u, \frac{\partial}{\partial r}) = \sum_i g_{(\tilde{o}, r)}([\tilde{X}_i, \tilde{Y}_i]_{\tilde{o}}, \frac{\partial}{\partial r}) = -\sum_i g_{(\tilde{o}, r)}(Y_i, [\tilde{X}_i, \frac{\partial}{\partial r}]_{(\tilde{o}, r)}) = 0$. Since the orbits of the left and right S^1 -actions at the point $(\tilde{o}, r) \in G_u/\tilde{V} \times \mathbf{R}_+$ coincide, we have $I_{\mathfrak{c}_p} = \mathbf{R} \frac{\partial}{\partial r}$.

Therefore $g_{(\tilde{\sigma}, r)}(u, \frac{\partial}{\partial r})=0$ if $u \in \mathfrak{c}_p$.

Since G_u acts on \mathbf{R}_+ trivially, for each point $(p, r) \in G_u/\tilde{V} \times \mathbf{R}_+$

$$(3.15) \quad g_{(p,r)}(u, \frac{\partial}{\partial r}) = 0 \quad \text{for } u \in T_p(G_u/\tilde{V}).$$

Now it is easy to see that g can be written as

$$g = F_0^2 \beta_0 + F_1^2 \beta_1 + \sum_j H_j^2 \alpha_j$$

where F_0, F_1 and H_j are positive valued C^∞ -functions on $G_u/\tilde{V} \times \mathbf{R}_+$. Since g, β_0, β_1 and α_j are $G_u \times S^1$ -invariant, so are F_0, F_1 and H_j . Moreover we have $F_0 = F_1$, since $\beta_1(X, Y) = \beta_0(JX, JY)$ and g is hermitian. q.e.d.

Now we consider conditions that a $G_u \times S^1$ -invariant hermitian metric g on $G \times_p \mathbf{C}^*$ of the form (3.13) to be Kähler. For $X \in \mathfrak{c}_p$ let X^* denote the vector field on $G_u/\tilde{V} \times \mathbf{R}_+$ induced by the right action of $S^1 = \{\exp tX \mid t \in \mathbf{R}\}$. For a fixed non-zero $X \in \mathfrak{c}_p$, define 1-forms θ_0 and θ_1 on $G_u/\tilde{V} \times \mathbf{R}_+$ by

$$(3.16) \quad \theta_0(A) = \beta_0(X^*, A)$$

$$(3.17) \quad \theta_1(A) = -\beta_1(JX^*, A)$$

where A is a C^∞ -vector field on $G_u/\tilde{V} \times \mathbf{R}_+$. Then θ_0 and θ_1 are $G_u \times S^1$ -invariant forms.

Lemma 3.3. *At the origin $o \in G \times_p \mathbf{C}^*$, we have*

$$(1) \quad d\theta_1 = 0$$

$$(2) \quad d\theta_0(Y, Z) = \begin{cases} -\langle X, [Y, Z] \rangle & \text{if } Y, Z \in \mathfrak{m} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since θ_0 and θ_1 are G_u -invariant, $L_{\tilde{Y}}\theta_0 = L_{\tilde{Y}}\theta_1 = 0$ for $Y \in \mathfrak{p}$. For $Y, Z \in \mathfrak{p}$, $(d\theta_i)(\tilde{Y}, \tilde{Z}) = \tilde{Y}\theta_i(\tilde{Z}) - \tilde{Z}\theta_i(\tilde{Y}) - \theta_i([\tilde{Y}, \tilde{Z}]) = -\theta_i([\tilde{Z}, \tilde{Y}]) = \theta_i([\tilde{Z}, \tilde{Y}])$, $i = 0, 1$. Thus $d\theta_i(Y, Z) = 0$ and $d\theta_0(Y, Z) = -\langle X, [Y, Z] \rangle$. For $Y \in \mathfrak{p}$, $d\theta_i(\tilde{Y}, \frac{\partial}{\partial r}) = \tilde{Y}\theta_i(\frac{\partial}{\partial r}) - \frac{\partial}{\partial r}\theta_i(\tilde{Y}) - \theta_i([\tilde{Y}, \frac{\partial}{\partial r}]) = -\frac{\partial}{\partial r}\theta_i(\tilde{Y}) = -\theta_i([\frac{\partial}{\partial r}, \tilde{Y}]) = 0$. Therefore $d\theta_i(Y, \tilde{H}) = 0$ for $Y \in \mathfrak{p}$. q.e.d.

Let ω be the Kähler form on $G \times_p \mathbf{C}^*$ of a hermitian metric g , that is, $\omega(A, B) = g(A, JB)$, and let ω_j be the 2-form on $G_p \times \mathbf{C}^*$ corresponding to the J -invariant symmetric forms α_j . The Kähler form ω on $G \times_p \mathbf{C}^*$ corresponding to the hermitian metric g of the form (3.13) is given by

$$(3.18) \quad \omega = \frac{F^2}{\beta_0(X^*, X^*)} \theta_0 \wedge \theta_1 + \sum_j H_j^2 \omega_j.$$

Now we define a vector field H on $G \times_{\rho} \mathfrak{C}^*$ by

$$(3.19) \quad H = -\frac{1}{g(X^*, X^*)^{1/2}} JX^* .$$

Proposition 3.4. *Assume that every 2-form ω_j is d -closed. Then a hermitian metric g on $G \times_{\rho} \mathfrak{C}^*$ of the form (3.13) is Kähler if and only if*

$$(3.20) \quad -\frac{F}{\langle X, X \rangle^{1/2}} \langle X, [A, IB] \rangle + \sum_j d(H_j^2)(H) \langle A, B \rangle|_{\mathfrak{m}_j} = 0$$

where $A, B \in \mathfrak{m}, 0 \neq X \in \mathfrak{c}_p$.

Proof. Since $dF = -(JX^*)F \frac{1}{\beta_0(X^*, X^*)} \theta_1, d\theta_1 = 0$ and $d\omega_j = 0, d\omega = \frac{F^2}{\beta_0(X^*, X^*)} d\theta_0 \wedge \theta_1 + \sum_j d(H_j^2) \wedge \omega_j$. For $A, C \in \mathfrak{m}, (d\theta_0 \wedge \theta_1)(\bar{A}, \bar{C}, JX^*) = -\theta_0([\bar{A}, \bar{C}])\beta_0(X^*, X^*)$. Note also that $(d\theta_0 \wedge \theta_1)(\bar{A}, \bar{B}, \bar{C}) = (\theta_1 \wedge \omega_j)(\bar{A}, \bar{B}, \bar{C}) = 0$ for $A, B, C \in \mathfrak{m}, (d\theta_0 \wedge \theta_1)(\bar{A}, X^*, JX^*) = (\theta_1 \wedge \omega_j)(\bar{A}, X^*, JX^*) = 0$ for $A \in \mathfrak{m}, X \in \mathfrak{c}_p$ and $(d\theta_0 \wedge \theta_1)(\bar{A}, \bar{B}, X^*) = (\theta_1 \wedge \omega_j)(\bar{A}, \bar{B}, X^*) = 0$ for $A, B \in \mathfrak{m}, X \in \mathfrak{c}_p$. Thus we have $d\omega = 0$ if and only if, at $(\bar{o}, r) \in G_u / \tilde{V} \times \mathbf{R}_+$,

$$(3.21) \quad d\omega(\bar{A}, \bar{C}, JX^*) = 0 \quad \text{for } A, C \in \mathfrak{m} \text{ and } X \in \mathfrak{c}_p .$$

$$\begin{aligned} \text{Since } d\omega(\bar{A}, \bar{C}, JX^*) &= -F^2\theta_0([\bar{A}, \bar{C}]) + \sum_j d(H_j^2)(JX^*)\omega_j(\bar{A}, \bar{C}) \\ &= -F^2\beta_0(X^*, [\bar{A}, \bar{C}]) - g(X^*, X^*)^{1/2} \sum_j d(H_j^2)(H)\omega_j(\bar{A}, \bar{C}) \\ &= -F^2\beta_0(X^*, [\bar{A}, \bar{C}]) - F\beta_0(X^*, X^*)^{1/2} \sum_j d(H_j^2)(H)\omega_j(\bar{A}, \bar{C}), \end{aligned}$$

we see that (3.21) holds if and only if

$$F\beta_0(X^*, [\bar{A}, \bar{C}]) / (\beta_0(X^*, X^*)^{1/2}) + \sum_j d(H_j^2)(H)\alpha_j(\bar{A}, J\bar{C}) = 0$$

for $A, C \in \mathfrak{m}$ and $X \in \mathfrak{c}_p$. Therefore $d\omega = 0$ if and only if

$$F\langle X, [A, C] \rangle / (\langle X, X \rangle^{1/2}) + \sum_j d(H_j^2)(H) \langle A, IC \rangle|_{\mathfrak{m}_j} = 0$$

for $A, C \in \mathfrak{m}, X \in \mathfrak{c}_p$. Since $Im_j = \mathfrak{m}_j$, we get our claim by putting $B = IC$. q.e.d.

4 Extensive conditions of a $G_u \times S^1$ -invariant metric

Now we consider conditions of a $G_u \times S^1$ -invariant Kähler metric on the open orbit $G \times_{\rho} \mathfrak{C}^*$ which can be extended to a Kähler metric on $P(1 \oplus \xi)$. For a Kähler manifold (Y, J, g) let ∇ denote the Riemannian connection.

Lemma 4.1. *For a holomorphic Killing vector field X on Y and a Killing vector field A on Y such that $[A, X] = 0$, we have $g(\nabla_{JX} JX, A) = 0$.*

Proof. Since A is a Killing vector field, $Ag(X, X) = 2g([A, X], X) = 0$. Thus $g(\nabla_A X, X) = \frac{1}{2} Ag(X, X) = 0$. Since X is also Killing, $g(\nabla_X X, A) + g(X, \nabla_A X) = 0$. Therefore $g(\nabla_X X, A) = 0$. Since g is a Kähler metric and X is holomorphic, $\nabla_{JX} JX = J\nabla_{JX} X = J\nabla_X JX = -\nabla_X X$, and hence we get $g(\nabla_{JX} JX, A) = 0$. q.e.d.

Now we consider a $G_u \times S^1$ -invariant Kähler metric g on the open orbit $G \times_{\rho} \mathbf{C}^*$ of the form (3.13). Let H be the vector field on $G \times_{\rho} \mathbf{C}^*$ defined by (3.19).

Lemma 4.2. *On the open orbit $G \times_{\rho} \mathbf{C}^*$, we have*

$$(4.1) \quad \nabla_H H = 0.$$

Proof. By Lemma 4.1, we have $g(\nabla_{J_{K^*}} JX^*, \bar{A}) = 0$ for a Killing vector field \bar{A} on $G \times_{\rho} \mathbf{C}^*$ where $A \in \mathfrak{g}_u$. Since

$$\nabla_H H = \frac{1}{g(X^*, X^*)} \nabla_{JX^*} JX^* + \frac{1}{g(X^*, X^*)^{1/2}} (JX^*) (g(X^*, X^*)^{1/2}) JX^*$$

and $g(JX^*, \bar{A}) = 0$, we have $g(\nabla_H H, \bar{A}) = 0$. Since $g(H, H) = 1, g(\nabla_H H, H) = 0$. Therefore we have $\nabla_H H = 0$, q.e.d.

Let $\rho: U \rightarrow \mathbf{C}^*$ be the holomorphic representation corresponding to the weight Λ and identify $\sqrt{-1}\Lambda$ with an element of \mathfrak{c}_p . From now on denote by X_0 the element of \mathfrak{c}_p defined by $\Lambda(X_0) = \sqrt{-1}$. Then the right S^1 -action $\{\exp tX_0 | t \in \mathbf{R}\}$ on $P(1 \oplus \xi_{\rho})$ corresponds to the natural S^1 -action on $P(1 \oplus \xi_{\rho})$ induced by the S^1 -action on each fiber $P^1(\mathbf{C})$. We also define a symmetric tensor β_0 on $G_u / \tilde{V} \times \mathbf{R}_+$ by $\tilde{\beta}_0 = (1 / \langle X_0, X_0 \rangle) \beta_0$ and a function \tilde{F} on $G_u / \tilde{V} \times \mathbf{R}_+$ by $\tilde{F} = \langle X_0, X_0 \rangle^{1/2} F$ for a C^∞ function F on $G_u / \tilde{V} \times \mathbf{R}_+$. Then $\tilde{F}^2 \tilde{\beta}_0 = F^2 \beta_0$. Let r be the canonical coordinate of \mathbf{R}_+ as before. Thus we have $JX_0^* = -r(\partial/\partial r)$ on $G_u / \tilde{V} \times \mathbf{R}_+$. Thus a $G_u \times S^1$ -invariant hermitian metric g on $G_u / \tilde{V} \times \mathbf{R}_+$ of the form (3.13) can be written as

$$(4.2) \quad g = (\tilde{F}/r)^2 dr^2 + \tilde{F}^2 \tilde{\beta}_0 + \sum_j H_j^2 \alpha_j.$$

Now we consider a $G_u \times S^1$ -invariant Kähler metric g_0 on $P(1 \oplus \xi_{\rho})$. We know that there is a $G_u \times S^1$ -invariant Kähler metric on $P(1 \oplus \xi_{\rho})$, since $P(1 \oplus \xi_{\rho})$ is a Kähler manifold and the compact Lie group $G_u \times S^1$ acts on $P(1 \oplus \xi_{\rho})$ as a holomorphic transformation group. Note that the functions \tilde{F} and H_j can be regarded as functions on \mathbf{R}_+ , since they are $G_u \times S^1$ -invariant.

Lemma 4.3. *For a $G_u \times S^1$ -invariant Kähler metric g_0 on $P(1 \oplus \xi)$, let its restriction g_0 to the open orbit $G_u / \tilde{V} \times \mathbf{R}_+$ be of the form (4.2). Then the function \tilde{F} extends to a C^∞ -function $\tilde{F}: [0, \infty) \rightarrow \mathbf{R}$ such that $\tilde{F}(0) = 0, \tilde{F}'(0) > 0$ and*

$\tilde{F}(r)$ is an odd function at $r=0$, that is, $\tilde{F}(r)=-\tilde{F}(-r)$, and the functions H_j extend to C^∞ functions $H_j: [0, \infty) \rightarrow \mathbf{R}_+$ such that $H_j(0)>0$ and H_j are even functions at $r=0$.

Proof. Note that the intersection of the open orbit $G_u/\tilde{V} \times \mathbf{R}_+$ and a fiber $P^1(\mathbf{C})$ is identified with \mathbf{C}^* and that the right S^1 -action on $G_u/\tilde{V} \times \mathbf{R}_+$ induces a natural S^1 -action on \mathbf{C}^* . On the intersection \mathbf{C}^* , the metric g_0 is given by

$$(4.3) \quad g_{0|P^1(\mathbf{C})} = (\tilde{F}(r)/r)^2 dr^2 + \tilde{F}(r)^2 d\theta^2$$

by using polar coordinates (r, θ) on \mathbf{C}^* , and thus it is written as

$$g_{0|P^1(\mathbf{C})} = (\tilde{F}(r)/r)^2(dx^2 + dy^2) \quad \text{on } \mathbf{C}^*$$

by using a canonical coordinate $z = x + \sqrt{-1}y$ on \mathbf{C} . Therefore a metric $(\tilde{F}(r)/r)^2 dr^2 + \tilde{F}(r)^2 d\theta^2$ extends to a metric on \mathbf{C} if and only if \tilde{F} extends to a C^∞ function $\tilde{F}: [0, \infty) \rightarrow \mathbf{R}$ such that $\tilde{F}(0)=0, \tilde{F}'(0)>0$ and \tilde{F} is an odd function at $r=0$ (cf. [3] Proposition 4.6). By the same way we see that H_j extend to C^∞ functions $H_j: [0, \infty) \rightarrow \mathbf{R}_+$ such that $H_j(0)>0$ and H_j are even functions at $r=0$. q.e.d.

We now consider a geodesic $c(t)$ of the compact Kähler manifold $(P(1 \oplus \xi), g_0)$ through the origin $c(t_0)=(\tilde{\delta}, 1) \in G_u/\tilde{V} \times \mathbf{R}_+$ with $\dot{c}(t_0)=H_{c(t_0)}$, parametrized by arc length. Since $\nabla_H H = 0$, $c(t)$ is the integral curve of H through $(\tilde{\delta}, 1)$, that is,

$$(4.4) \quad \dot{c}(t) = H_{c(t)}.$$

Note also that

$$(4.5) \quad H = -(1/\tilde{F}(r))JX_0^* = (r/\tilde{F}(r))(\partial/\partial r).$$

We set $\dot{c}(t)=(dr/dt)(\partial/\partial r)$. Then $c(t)$ satisfies an ordinary differential equation

$$(4.6) \quad dr/dt = r/\tilde{F}(r).$$

By Lemma 4.3, the function $\tilde{F}(r)/r$ extends to a C^∞ function $\tilde{f}(r): [0, \infty) \rightarrow \mathbf{R}_+$ such that $\tilde{f}(r)$ is even at $r=0$. Thus $p_0(r) = \int_0^r \tilde{f}(u)du: [0, \infty) \rightarrow \mathbf{R}^\infty$ is a monotone increasing C^∞ function and is odd at $r=0$, and we have $t=p_0(r)$.

Let L_0 denote the length of the geodesic $c(t)$ of $P(1 \oplus \xi)$ between two singular orbits of $G_u \times S^1$. By taking the inverse function $r=q_0(t)$ of $t=p_0(r)$, we define C^∞ functions $f_0, h_j^0: (0, L_0) \rightarrow \mathbf{R}_+$ by

$$(4.7) \quad \begin{cases} f_0(t) = \tilde{F}(q_0(t)) \\ h_j^0(t) = H_j(q_0(t)). \end{cases}$$

By using a similar argument for a neighborhood of $c(L_0)$, we see that the functions f_0, h_j^0 extend to C^∞ functions $f_0, h_j^0: [0, L_0] \rightarrow \mathbf{R}$ which satisfy $f_0(0)=f_0(L_0)=0, f_0'(0)=1=-f_0'(L_0), f_0^{(2k)}(0)=f_0^{(2k)}(L_0)=0$ for each positive integer $k, h_j^0(0)>0, h_j^0(L_0)>0$ and $(h_j^0)^{(2k-1)}(0)=(h_j^0)^{(2k-1)}(L_0)=0$ for each positive integer k . Therefore we get the first part of the following theorem.

Theorem 4.4 (cf. [2] Section 4).

(1) Let g_0 be a $G_u \times S^1$ -invariant Kähler metric on $P(1 \oplus \xi)$. Then the metric g_0 is given by

$$g_0 = dt^2 + f_0^2(t)\beta_0 + \sum_j h_j^0(t)^2\alpha_j$$

on the open orbit $G \times_p \mathbf{C}^*$, where f_0, h_j^0 are C^∞ functions on $[0, L_0]$ such that

$$(4.8) \quad \begin{cases} f_0, h_j^0 \text{ are positive valued on } (0, L_0), f_0(0) = f_0(L_0) = 0, \\ f_0'(0) = 1 = -f_0'(L_0), f_0^{(2k)}(0) = f_0^{(2k)}(L_0) = 0 \text{ for each} \\ \text{positive integer } k, h_j^0(0) > 0, h_j^0(L_0) > 0 \text{ and } (h_j^0)^{(2k-1)}(0) \\ = (h_j^0)^{(2k-1)}(L_0) = 0 \text{ for each positive integer } k. \end{cases}$$

(2) Conversely let $f(s), h_j(s)$ be C^∞ functions on $[0, L]$ which satisfy the properties (4.8). Then the metric

$$g = ds^2 + f(s)^2\beta_0 + \sum_j h_j(s)^2\alpha_j$$

is defined on the open orbit $G \times_p \mathbf{C}^*$ and extends to a C^∞ metric on $P(1 \oplus \xi)$.

Proof. We prove the second part. At first we consider the ordinary differential equation

$$(4.9) \quad dr/ds = (1/f(s))r.$$

A solution of (4.9) is given by

$$r = q(s) = \exp \int_{s_0}^s (1/f(u))du$$

where $s_0 \in (0, L)$ is the point corresponding to $r=1$. By our assumption on $f(s)$ at $s=0, f(s)=s(1+s^2f_1(s))$ where $f_1(s)$ is a C^∞ function on $[0, L)$ and $f_1^{(2k-1)}(0)=0$ for every positive integer k . Since

$$\exp \int_{s_0}^s (1/f(u))du = \frac{s}{s_0} \exp \left(- \int_{s_0}^s \frac{uf_1(u)}{1+u^2f_1(u)} du \right),$$

the solution $r=sq_1(s)$ of the equation (4.9) extends to a C^∞ function on $[0, L)$ such that $q_1(0)>0$ and $q_1^{(2k-1)}(0)=0$ for each positive integer k . Note also that $r=sq_1(s)$ is a monotone increasing function. If we put $r_1=1-r$, the equation (4.9) is written as

$$dr_1/ds = -(1-f(s))r_1,$$

and, from our assumption on $f(s)$ at $s=L$, we see that the solution r_1 of the equation is of the form

$$r_1 = (L-s)\tilde{q}_1(s)$$

where $\tilde{q}_1(s)$ is a C^∞ function on $(0, L]$ such that $\tilde{q}_1(L)>0$ and $\tilde{q}_1^{(2j-1)}(L)=0$ for each positive integer k . Let $s=p(r): [0, \infty) \rightarrow [0, L]$ be the inverse function of $r=q(s)$. Then the metric g can be written in the form (4.2). Moreover, since $s=p(r)$ and $t=p_0(r)$ are monotone increasing C^∞ functions on $[0, \infty)$, s is a C^∞ function of t defined on $[0, L_0)$ such that $s(0)=0, (ds/dt)(0)>0$ and $d^{2k-1}s/dt^{2k-1}(0)=0$ for each positive integer k . Similarly we see that s is a C^∞ function of t on $(0, L_0]$, and hence $s=s(t): [0, L_0] \rightarrow [0, L]$ is an onto diffeomorphism which satisfies

$$\begin{aligned} ds/dt &= f(s)/f_0(t) \quad \text{and} \\ d^{2k}s/dt^{2k}(0) &= d^{2k}s/dt^{2k}(L_0) = 0 \quad \text{for each positive integer } k. \end{aligned}$$

Thus $h_j(s)=h_j(s(t))$ satisfies $d^{2k-1}h_j/dt^{2k-1}(0)=d^{2k-1}h_j/dt^{2k-1}(L_0)=0$ for each integer k , and hence it is C^∞ at neighborhoods of singular orbits, since the square of the distance from a point on a Riemannian manifold is C^∞ at a neighborhood of the point. Now the metric g can be written as

$$\begin{aligned} g &= (ds/dt)^2 dt^2 + (f(s)/f_0(t))^2 f_0(t)^2 \tilde{\beta}_0 + \sum_j h_j(s)^2 \alpha_j \\ &= (ds/dt)^2 (dt^2 + f_0(t)^2) \tilde{\beta}_0 + \sum_j h_j(s(t))^2 \alpha_j \\ &= (ds/dt)^2 (g_0 - \sum_j h_j^0(t)^2 \alpha_j) + \sum_j h_j(s(t))^2 \alpha_j. \end{aligned}$$

Since ds/dt is an even function at $t=0$ and $t=L_0, ds/dt(0)>0$ and $ds/dt(L_0)>0$, we see that g extends to a C^∞ Riemannian metric g on $P(1 \oplus \xi)$. q.e.d.

REMARK. If the metric g on the open orbit $G \times_{\rho} \mathcal{C}^*$ is Kähler, so is the extended metric g on $P(1 \oplus \xi)$.

5 Computations of Ricci curvature

We now compute the Ricci tensor of a $G_u \times S^1$ -invariant Kähler metric g on the open orbit $G \times_{\rho} \mathcal{C}^*$ in the projective bundle $P(1 \oplus \xi)$. We assume that the metric g is of the form

$$(5.1) \quad g = ds^2 + g_s = ds^2 + f(s)^2 \tilde{\beta}_0 + \sum_j h_j(s)^2 \alpha_j.$$

To calculate the curvature of the metric $g=ds^2+g_s$ on $G_u/\tilde{V} \times (0, L)$ we use the notion of a Riemannian submersion according to Bérard Bergery [2]. Note that the vector field H is given by the vector field $\partial/\partial s$. Let ∇ be the

Riemannian connection of g as before and $\hat{\nabla}$ that of g_s in each fiber of the Riemannian submersion $G_u/\tilde{V} \times (0, L) \rightarrow (0, L)$. We recall that, by definition, $T_x Y$ is the horizontal part of $\nabla_x Y$ for vertical vector fields X and Y , $T_x H$ is the vertical part of $\nabla_x H$ and if we put $T_H H = T_H X = 0$, we obtain a tensor T of type $(1, 2)$ on $G_u/\tilde{V} \times (0, L)$. Now the formulas of O'Neill is given by

$$(5.2) \quad \begin{cases} \nabla_x Y = \hat{\nabla}_x Y + T_x Y \\ \nabla_x H = T_x H \\ \nabla_H X \text{ and } \nabla_x H \text{ are vertical} \\ \nabla_H H = 0 \end{cases}$$

for vertical vector fields X and Y . Note that the tensor A of O'Neill [14] is zero, since the base space $(0, L)$ of the Riemannian submersion is 1-dimensional. Note also that

$$(5.3) \quad g(T_x Y, H) = -g(T_x H, Y), \quad T_x Y = T_Y X, \quad g(T_x H, Y) = g(T_Y H, X).$$

If X and Y are vertical vector fields which commute with H , that is, $[X, H] = [Y, H] = 0$, we have

$$(5.4) \quad g(T_x Y, H) = -\frac{1}{2} Hg(X, Y) = -g(T_x H, Y).$$

By the formulas of O'Neill if X, Y, Z, V are vertical vectors and \hat{R} is the curvature tensor of the metric g_s on G_u/\tilde{V} , we obtain the followings for the curvature R of $g = ds^2 + g_s$:

$$(5.5) \quad \begin{cases} g(R(X, Y)Z, V) = g(\hat{R}(X, Y)Z, V) - g(T_x Z, T_Y V) + g(T_x V, T_Y Z) \\ g(R(X, Y)Z, H) = g((\nabla_Y T)_x Z, H) - g((\nabla_x T)_Y Z, H) \\ g(R(X, H)Y, H) = g((\nabla_H T)_x Y, H) - g(T_x H, T_Y H). \end{cases}$$

To calculate the Ricci tensor r of the metric $g = ds^2 + g_s$, we take an orthogonal basis $(X_i)_{i=1, \dots, n-1}$ of the tangent space of an orbit G_u/\tilde{V} with respect to g_s and introduce the following notations:

- the principal normal vector $N = \sum_i T_{X_i} X_i$,
- the norm $\|T\|$ of T , $\|T\|^2 = \sum_i g(T_{X_i} H, T_{X_i} H)$ and
- $\hat{\delta}T(X) = -\sum_i (\nabla_{X_i} T)_{X_i} X$ for a vertical vector X .

(Note that all these notations are independent of the choice of the basis.) We also denote by \hat{r} the Ricci tensor of the metric g_s on each orbit. Then the Ricci tensor r of the metric g is given by the following formulas.

Proposition 5.1 (Bérard Bergery [2]). *If X and Y are vertical,*

$$(5.6) \quad r(X, Y) = \hat{\nu}(X, Y) - g(N, T_X Y) + g((\nabla_H T)_X Y, H)$$

$$(5.7) \quad r(X, H) = g(\hat{\delta}T(X), H)$$

$$(5.8) \quad r(H, H) = Hg(N, H) - \|T\|^2.$$

Lemma 5.2 (cf. [2] Proposition 3.18). *For a $G_u \times S^1$ -invariant Kähler metric g on the open orbit $G \times_p \mathbb{C}^*$ of the form (5.1), we have*

$$(5.9) \quad r(X, H) = 0 \quad \text{for all vertical vectors } X.$$

Proof. Since the Ricci tensor r is invariant by the complex structure J on $G \times_p \mathbb{C}^*$ and by the action of $G_u \times S^1$, we get our claim by the same way as the proof of Proposition 3.2. q.e.d.

Lemma 5.3. *If vertical vector fields X, Y commute with H , we have*

$$(5.10) \quad g((\nabla_H T)_X Y, H) = -\frac{1}{2} H \cdot H \cdot g(X, Y) + 2g(T_X H, T_Y H).$$

Proof. $g((\nabla_H T)_X Y, H) = g(\nabla_H(T_X Y), H) - g(T_{\nabla_H X} Y, H) - g(T_X(\nabla_H Y), H)$
 $= Hg(T_X Y, H) - g(T_Y(\nabla_H X), H) - g(T_X(\nabla_H Y), H)$
 $= -\frac{1}{2} H \cdot H \cdot g(X, Y) + g(\nabla_H X, T_Y H) + g(\nabla_H Y, T_X H)$ by (5.3), (5.4)
 $= -\frac{1}{2} H \cdot H \cdot g(X, Y) + 2g(T_X H, T_Y H)$, since $[X, H] = [Y, H] = 0$.
q.e.d.

From now on we assume that the Kähler C -space X is a product of two irreducible hermitian symmetric spaces of compact type M_1 and M_2 and that the projective bundle $P(1 \oplus \xi)$ is induced from a vector bundle $1 \oplus \xi$ where ξ is a line bundle given by $p_1^* L_1^{-a} \otimes p_2^* L_2^b$ for some positive integers a and b . Then our assumptions (3.6), (3.9), (3.10) and (3.11) are satisfied by taking canonical decompositions of symmetric spaces: $(g_i)_u = \mathfrak{v}_i + \mathfrak{m}_i$ ($i=1, 2$). Thus a $G_u \times S^1$ -invariant hermitian metric g on the open orbit $G \times_p \mathbb{C}^*$ is given by the form

$$(5.11) \quad g = ds^2 + f(s)^2 \tilde{\beta}_0 + h_1(s)^2 \alpha_1 + h_2(s)^2 \alpha_2$$

where α_i ($i=1, 2$) are symmetric tensors induced from the invariant metrics on M_i corresponding to the inner product $\langle , \rangle = -\text{Killing form}$.

As in section 4 let $X_0 \in \mathfrak{c}_p$ be the element defined by $\Lambda(X_0) = \sqrt{-1}$. Then $\tilde{\beta}_0(X_0, X_0) = 1$. We put $m = \dim_{\mathbb{C}} M_1$ and $n = \dim_{\mathbb{C}} M_2$. Take an orthonormal basis $\{B_1, \dots, B_{2m}, C_1, \dots, C_{2n}\}$ of $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$ with respect to the inner product \langle , \rangle such that $B_j \in \mathfrak{m}_1$ and $C_j \in \mathfrak{m}_2$.

Proposition 5.4. *For an orthonormal basis $\left\{ H, \frac{1}{f} X_0, \frac{1}{h_1} B_1, \dots, \frac{1}{h_1} B_{2m}, \frac{1}{h_2} C_1, \dots, \frac{1}{h_2} C_{2n} \right\}$*

..., $\frac{1}{h_2} C_{2n}$ }, we have

$$\begin{aligned} r(H, H) &= -\left(\frac{f''}{f} + 2m\frac{h_1'}{h_1} + 2n\frac{h_2'}{h_2}\right) \\ r\left(\frac{1}{f}X_0, \frac{1}{f}X_0\right) &= \hat{r}\left(\frac{1}{f}X_0, \frac{1}{f}X_0\right) - \frac{f'}{f}\left(2m\frac{h_1'}{h_1} + 2n\frac{h_2'}{h_2}\right) - \frac{f''}{f} \\ r\left(\frac{1}{h_1}B_i, \frac{1}{h_1}B_i\right) &= \hat{r}\left(\frac{1}{h_1}B_i, \frac{1}{h_1}B_i\right) - \frac{f'h_1'}{fh_1} - \frac{h_1''}{h_1} - (2m-1)\left(\frac{h_1'}{h_1}\right)^2 - 2n\frac{h_1'h_2'}{h_1h_2} \\ r\left(\frac{1}{h_2}C_i, \frac{1}{h_2}C_i\right) &= \hat{r}\left(\frac{1}{h_2}C_i, \frac{1}{h_2}C_i\right) - \frac{f'h_2'}{fh_2} - \frac{h_2''}{h_2} - (2n-1)\left(\frac{h_2'}{h_2}\right)^2 - 2m\frac{h_1'h_2'}{h_1h_2} \\ r\left(\frac{1}{f}X_0, \frac{1}{h_1}B_i\right) &= r\left(\frac{1}{f}X_0, \frac{1}{h_2}C_i\right) = r\left(\frac{1}{h_1}B_i, \frac{1}{h_1}B_j\right) = r\left(\frac{1}{h_2}C_i, \frac{1}{h_2}C_j\right) = 0 \end{aligned}$$

for $i \neq j$ and

$$r\left(\frac{1}{h_1}B_i, \frac{1}{h_2}C_j\right) = 0 \quad \text{for each } (i, j).$$

Proof. Note that $[\tilde{Y}, H] = 0$ for $Y \in \mathfrak{p}$. Since $g(N, H) = g(T_{(1/f)X_0}(1/f)X_0, H) + \sum_i g(T_{(1/h_1)B_i}(1/h_1)B_i, H) + \sum_j g(T_{(1/h_2)C_j}(1/h_2)C_j, H) = (1/f^2)g(T_{\tilde{X}_0}\tilde{X}_0, H) + (1/h_1^2) \sum_i g(T_{\tilde{B}_i}\tilde{B}_i, H) + (1/h_2^2) \sum_j g(T_{\tilde{C}_j}\tilde{C}_j, H) = -\frac{1}{2}\{(1/f^2)Hg(\tilde{X}_0, \tilde{X}_0) + (1/h_1^2) \sum_i Hg(\tilde{B}_i, \tilde{B}_i) + (1/h_2^2) \sum_j Hg(\tilde{C}_j, \tilde{C}_j)\} = -(f'/f)\tilde{\beta}_0(\tilde{X}_0, \tilde{X}_0) - (h_1'/h_1) \sum_i \alpha_1(\tilde{B}_i, \tilde{B}_i) - (h_2'/h_2) \sum_j \alpha_2(\tilde{C}_j, \tilde{C}_j) = -(f'/f) - 2m(h_1'/h_1) - 2n(h_2'/h_2)$ by (5.4), we have

$$Hg(N, H) = \frac{f''f - (f')^2}{f^2} - 2m\frac{h_1'h_1 - (h_1')^2}{h_1^2} - 2n\frac{h_2'h_2 - (h_2')^2}{h_2^2}.$$

Note that, for $Y \in \mathfrak{p}$, $g(T_Y H, T_Y H) = \sum_k g(T_Y H, X_k)^2$ where $\{X_k\}$ is an orthonormal basis of a tangent space of an orbit G_u/\tilde{V} . Thus $g(T_{X_0}H, T_{X_0}H) = (f')^2$, $g(T_{B_i}H, T_{B_i}H) = (h_1')^2$ and $g(T_{C_i}H, T_{C_i}H) = (h_2')^2$. Therefore $\|T\|^2 = \sum_k \|T_{X_k}H\|^2 = (f'/f)^2 + 2m(h_1'/h_1)^2 + 2n(h_2'/h_2)^2$ and hence $r(H, H) = -(f''/f) - 2m(h_1'/h_1) - 2n(h_2'/h_2)$ by (5.8).

Since $g((\nabla_H T)_{(1/f)X_0}(1/f)X_0, H) = (1/f^2)g((\nabla_H T)_{X_0}X_0, H) = (1/f^2)\{-\frac{1}{2}H \cdot H \cdot g(\tilde{X}_0, \tilde{X}_0) + 2g(T_{\tilde{X}_0}H, T_{\tilde{X}_0}H)\} = (-f''f + (f')^2)/f^2$, we have, by (5.6)

$$r\left(\frac{1}{f}X_0, \frac{1}{f}X_0\right) = \hat{r}\left(\frac{1}{f}X_0, \frac{1}{f}X_0\right) - (f'/f)\left(2m\frac{h_1'}{h_1} + 2n\frac{h_2'}{h_2}\right) - \frac{f''}{f}.$$

By the same way we get two other formulas for Ricci tensor r . Since

Ricci tensor r is invariant by the complex structure J and by the action $G_u \times S^1$, we get last claims by the same way as in proof of Proposition 3.2. q.e.d.

Now to compute Ricci tensor \hat{r} we recall known facts on a hermitian symmetric space M of compact type. We write $M=G/K$ where G is the identity component of the group of all isometris of M . Let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras of G, K respectively and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{n}$ be a canonical decomposition. By identifying \mathfrak{n} with the tangent space of G/K at the origin, let I be the complex structure on \mathfrak{n} induced by the invariant complex structure J on M . By extending I to the complexification \mathfrak{n}^c of \mathfrak{n} , we have the decomposition $\mathfrak{n}^c=\mathfrak{n}^+ + \mathfrak{n}^-$, $\mathfrak{n}^+ \cap \mathfrak{n}^- = (0)$, $\bar{\mathfrak{n}}^+ = \mathfrak{n}^-$, where the bar denotes complex conjugation with respect to \mathfrak{n} . It is known that there exists an element Z in the center \mathfrak{c} of \mathfrak{k} such that $\text{ad}(Z)=I$. Moreover it is also known that $\dim \mathfrak{c}=1$ if M is irreducible. Take a Cartan subalgebra \mathfrak{h} of \mathfrak{g} containing Z . Then the centralizer of Z coincides with \mathfrak{k} . We denote by Σ the root system of \mathfrak{g}^c with respect to \mathfrak{h}^c and \mathfrak{g}_α the eigenspace of the root α . Note that $\bar{\mathfrak{g}}_\alpha = \mathfrak{g}_{-\alpha}$ where the bar denotes complex conjugation with respect to \mathfrak{g} . By setting $\Sigma^+ = \{\alpha \in \Sigma \mid \alpha(Z) = \sqrt{-1}\}$, we have

$$\mathfrak{n}^+ = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \sum_{\alpha \in \Sigma^-} \mathfrak{g}_\alpha.$$

We denote by \mathfrak{h}_0 the real subspace $\sqrt{-1}\mathfrak{h}$ of \mathfrak{h}^c and introduce a lexicographical order in the dual space \mathfrak{h}_0^* by taking a basis $\{H_1, \dots, H_l\}$ of \mathfrak{h}_0 such that $H_1 = -\sqrt{-1}Z$. We denote by Σ_0^+ the set of positive roots not belonging to Σ_Π^+ . Then

$$\Sigma_0^+ = \{\alpha \in \Sigma \mid \alpha > 0, \alpha(Z) = 0\}$$

and

$$\mathfrak{k}^c = \mathfrak{h}^c + \sum_{\alpha \in \Sigma_0^+} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$$

We also identify a linear form $\lambda \in \mathfrak{h}_0^*$ with an element $H_\lambda \in \mathfrak{h}_0$ by means of the Killing form φ on \mathfrak{g}^c ,

$$\lambda(H) = \varphi(H, H_\lambda) \quad \text{for all } H \in \mathfrak{h}_0.$$

It is also known that if M is an irreducible hermitian symmetric space there is a unique simple root α_1 belonging to Σ_Π^+ . We denote by $\Pi = \{\alpha_1, \dots, \alpha_l\}$ the set of all simple roots and by $\{\Lambda_\alpha\}_{\alpha \in \Pi}$ the fundamental weights of \mathfrak{g}^c corresponding to Π . Then Σ_0^+ is spanned by $\{\alpha_2, \dots, \alpha_l\}$ and thus the center \mathfrak{c} of \mathfrak{k} is given by $\sqrt{-1}R\Lambda_{\alpha_1}$.

Let \langle , \rangle denote the inner product of \mathfrak{h}_0 induced from the Killing form φ on \mathfrak{g}^c as before. If M is an irreducible hermitian symmetric space, the element $Z \in \mathfrak{c}$ such that $\text{ad}(Z)=I$ is given by

$$(5.12) \quad Z = \frac{2\sqrt{-1}}{\langle \alpha_1, \alpha_1 \rangle} \Lambda_{\alpha_1}.$$

Lemma 5.5. Put $\delta_{\mathfrak{n}} = \frac{1}{2} \sum_{\alpha \in \Sigma_{\mathfrak{n}}^+} \alpha$. Then $\delta_{\mathfrak{n}}$ belongs to the center of $\mathfrak{k}^{\mathbb{C}}$ and $\langle \delta_{\mathfrak{n}}, \alpha \rangle = 1/4$ for $\alpha \in \Sigma_{\mathfrak{n}}^+$.

Proof. See Murakami [13] Part II Lemma 1.1 and Corollary of Lemma 5.1, or Takeuchi [16].

It is also known that if M is irreducible there is a canonical isomorphism $Z\Lambda_{\alpha_1} \rightarrow H^2(M, Z)$ and the first Chern class $c_1(M)$ of M corresponds to $\kappa\Lambda_{\alpha_1}$ where κ is an integer given by

$$(5.13) \quad \kappa = \frac{2\langle \delta_{\mathfrak{n}}, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle}.$$

Therefore we have

$$(5.14) \quad Z = 2\sqrt{-1}\kappa\Lambda_{\alpha_1}.$$

Now we choose $E_{\alpha} \in \mathfrak{g}_{\alpha}$ for $\alpha \in \Sigma$ with the following properties:

$$[E_{\alpha}, E_{-\alpha}] = -\alpha, \varphi(E_{\alpha}, E_{-\alpha}) = -1, \bar{E}_{\alpha} = E_{-\alpha}.$$

Put $B_{\alpha} = \frac{1}{\sqrt{2}}(E_{\alpha} + E_{-\alpha})$ for $\alpha \in \Sigma_{\mathfrak{n}}^+$. Then $B_{\alpha} \in \mathfrak{n}$, $IB_{\alpha} = \frac{\sqrt{-1}}{\sqrt{2}}(E_{\alpha} - E_{-\alpha})$ and $\{B_{\alpha}, IB_{\alpha} | \alpha \in \Sigma_{\mathfrak{n}}^+\}$ is an orthonormal basis of \mathfrak{n} with respect to the inner product $\langle \cdot, \cdot \rangle$ induced from the Killing form. Note that $[B_{\alpha}, IB_{\alpha}] = \sqrt{-1}\alpha$ for $\alpha \in \Sigma_{\mathfrak{n}}^+$,

$$(5.15) \quad \langle [B_{\alpha}, IB_{\alpha}], \sqrt{-1}\Lambda_{\alpha_1} \rangle = 1/2\kappa \quad \text{for } \alpha \in \Sigma_{\mathfrak{n}}^+$$

by (5.14) and $\alpha(Z) = \sqrt{-1}$.

Now consider a product X of two irreducible hermitian symmetric spaces of compact type M_1 and M_2 and a projective bundle $P(1 \oplus p_1^*L_1^{-a} \otimes p_2^*L_2^b)$ where L_1 and L_2 are generators of the group of all holomorphic line bundles $H^1(M_1, \theta^*)$ and $H^1(M_2, \theta^*)$ respectively and a, b are positive integers. Let $\Lambda^{(1)}$ and $\Lambda^{(2)}$ be the fundamental weights corresponding to L_1 and L_2 respectively. Then the weight Λ corresponding to the holomorphic line bundle $p_1^*L_1^{-a} \otimes p_2^*L_2^b$ over $X = M_1 \times M_2$ is given by $\Lambda = -a\Lambda^{(1)} + b\Lambda^{(2)}$.

Now we take an orthonormal basis of \mathfrak{m} such that $\{B_1, \dots, B_m, IB_1, \dots, IB_m\}$ is a basis of \mathfrak{m}_1 and $\{C_1, \dots, C_n, IC_1, \dots, IC_n\}$ is a basis of \mathfrak{m}_2 which satisfy (5.15). Let κ_i be the positive integers corresponding to the first Chern class $c_1(M_i)$ of M_i as before.

Lemma 5.6.

$$(1) \quad (5.16) \quad \begin{cases} \langle \sqrt{-1}\Lambda, [B_i, IB_i] \rangle = -a/2\kappa_1 & \text{for each } i. \\ \langle \sqrt{-1}\Lambda, [C_i, IC_i] \rangle = b/2\kappa_2 & \text{for each } i. \end{cases}$$

(2) *A $G_u \times S^1$ -invariant hermitian metric g on the open orbit $G \times_{\rho} \mathcal{C}^*$ of the form (5.11) is Kähler if and only if*

$$(5.17) \quad \begin{cases} (a/2\kappa_1)f + 2h_1h'_1 = 0. \\ (-b/2\kappa_2)f + 2h_2h'_2 = 0. \end{cases}$$

Proof. At first (5.16) follows from (5.15). Since M_1 and M_2 are hermitian symmetric spaces of compact type, the assumption of Proposition 3.4 is satisfied. The condition (3.20) can be written as

$$-(f(s)/\langle X_0, X_0 \rangle)^{1/2} \cdot \langle X, X \rangle^{1/2} \langle X, [A, IB] \rangle + \sum_{j=1}^2 (d(h_j^2)/ds) \langle A, B \rangle_{m_j} = 0$$

for $A, B \in \mathfrak{m}$, $0 \neq X \in \mathfrak{c}_{\mathfrak{p}}$. Since $X_0 \in \mathfrak{c}_{\mathfrak{p}}$ is given by $\Lambda(X_0) = \sqrt{-1}$, $X_0 = \sqrt{-1} \Lambda / \langle \Lambda, \Lambda \rangle$ and thus $X_0 = \langle X_0, X_0 \rangle \sqrt{-1} \Lambda$. Now by taking an orthonormal basis of \mathfrak{m} as before, we see that the condition (3.20) is equivalent to (5.17). q.e.d.

Now we compute Ricci tensor \hat{r} of a metric $g_s = f(s)^2 \beta_0 + h_1(s)^2 \alpha_1 + h_2(s)^2 \alpha_2$ on G_u / \tilde{V} . Let $\mathfrak{g}_u = \tilde{\mathfrak{v}} + \mathfrak{p}$ be the decomposition as before. Then

$$\mathfrak{p} = \mathfrak{c}_{\mathfrak{p}} + \mathfrak{m}_1 + \mathfrak{m}_2, \quad [\mathfrak{m}_i, \mathfrak{m}_i] \subset \tilde{\mathfrak{v}} + \mathfrak{c}_{\mathfrak{p}} \quad (i = 1, 2)$$

and $[\mathfrak{c}_{\mathfrak{p}}, \mathfrak{m}_i] \subset \mathfrak{m}_i \quad (i=1, 2)$. We denote by \hat{R} the curvature tensor of $(G_u / \tilde{V}, g_s)$. Note also that the metric g_s corresponds to an inner product

$$(5.18) \quad \langle \cdot, \cdot \rangle_s = (f(s)^2 / \langle X_0, X_0 \rangle) \langle \cdot, \cdot \rangle_{\mathfrak{c}_{\mathfrak{p}}} + h_1(s)^2 \langle \cdot, \cdot \rangle_{\mathfrak{m}_1} + h_2(s)^2 \langle \cdot, \cdot \rangle_{\mathfrak{m}_2}$$

on \mathfrak{p} .

Lemma 5.7. *For $X, Y \in \mathfrak{p}$, we have*

$$(5.19) \quad \begin{aligned} \langle \hat{R}(X, Y)Y, X \rangle_s &= -(3/4) \langle [X, Y]_{\mathfrak{p}}, [X, Y]_{\mathfrak{p}} \rangle - \langle [[X, Y]_{\tilde{\mathfrak{v}}}, Y], X \rangle_s \\ &- (1/2) \langle Y, [X, [X, Y]_{\mathfrak{p}}] \rangle_s - (1/2) \langle X, [Y, [Y, X]_{\mathfrak{p}}] \rangle_s + \langle U(X, Y), U(X, Y) \rangle_s \\ &+ \langle U(X, X), U(Y, Y) \rangle_s \end{aligned}$$

where $Z_{\tilde{\mathfrak{v}}}, Z_{\mathfrak{p}}$ denote $\tilde{\mathfrak{v}}$ -component, \mathfrak{p} -component of $Z \in \mathfrak{g}_u$ respectively, and $U: \mathfrak{p} \times \mathfrak{p} \rightarrow \mathfrak{p}$ is a symmetric bilinear form defined by

$$\langle U(X, Y), Z \rangle_s = \frac{1}{2} \{ \langle [Z, X]_{\mathfrak{p}}, Y \rangle_s + [Z, Y]_{\mathfrak{p}}, X \rangle_s \}$$

for $X, Y, Z \in \mathfrak{p}$.

Proof. See [17] Lemma 7.1.

Proposition 5.8. *For an orthonormal basis $\left\{ \frac{1}{f} X_0, \frac{1}{h_1} B_1, \dots, \frac{1}{h_1} B_m, \frac{1}{h_1} IB_1, \dots \right\}$*

$\dots, \frac{1}{h_1} IB_m, \frac{1}{h_2} C_1, \dots, \frac{1}{h_2} C_n, \frac{1}{h_2} IC_1, \dots, \frac{1}{h_2} IC_n \}$ of \mathfrak{p} , we have

$$(5.20) \quad \hat{\rho} \left(\frac{1}{f} X_0, \frac{1}{f} X_0 \right) = 2m \left(\frac{a}{2\kappa_1} \right)^2 \frac{f^2}{4h_1^4} + 2n \left(\frac{b}{2\kappa_2} \right)^2 \frac{f^2}{4h_2^4}$$

$$(5.21) \quad \hat{\rho} \left(\frac{1}{h_1} B_i, \frac{1}{h_1} B_i \right) = \hat{\rho} \left(\frac{1}{h_1} IB_i, \frac{1}{h_1} IB_i \right) = \frac{1}{2h_1^2} - \left(\frac{a}{2\kappa_1} \right)^2 \frac{f^2}{2h_1^4}$$

$$(5.22) \quad \hat{\rho} \left(\frac{1}{h_2} C_j, \frac{1}{h_2} C_j \right) = \hat{\rho} \left(\frac{1}{h_2} IC_j, \frac{1}{h_2} IC_j \right) = \frac{1}{2h_2^2} - \left(\frac{b}{2\kappa_2} \right)^2 \frac{f^2}{2h_2^4}$$

for $i=1, \dots, m, j=1, \dots, n$.

Proof. For simplicity we put $B'_i = B_i, B'_{i+m} = IB_i$ for $i=1, \dots, m$ and $C'_j = C_j, C'_{j+n} = IC_j$ for $j=1, \dots, n$. Note that $[X_0, Y] = -(a/2\kappa_1) \langle X_0, X_0 \rangle IY$ for $Y \in \mathfrak{m}_1$ and $[X_0, Y] = (b/2\kappa_2) \langle X_0, X_0 \rangle IY$ for $Y \in \mathfrak{m}_2$. By straightforward computations, we have

$$\begin{aligned} -\frac{3}{4} \left\langle \left[\frac{1}{f} X_0, \frac{1}{h_1} B'_i \right]_{\mathfrak{p}}, \left[\frac{1}{f} X_0, \frac{1}{h_1} B'_i \right]_{\mathfrak{p}} \right\rangle_s &= -\frac{3}{4} \frac{1}{f^2} \left(\frac{a}{2\kappa_1} \langle X_0, X_0 \rangle \right)^2, \\ -\frac{1}{2} \left\langle \frac{1}{h_1} B'_i, \left[\frac{1}{f} X_0, \left[\frac{1}{f} X_0, \frac{1}{h_1} B'_i \right]_{\mathfrak{p}} \right]_{\mathfrak{p}} \right\rangle_s &= \frac{1}{2} \frac{1}{f^2} \left(\frac{a}{2\kappa_1} \langle X_0, X_0 \rangle \right)^2, \\ -\frac{1}{2} \left\langle \frac{1}{f} X_0, \left[\frac{1}{h_1} B'_i, \left[\frac{1}{h_1} B'_i, \frac{1}{f} X_0 \right]_{\mathfrak{p}} \right]_{\mathfrak{p}} \right\rangle_s &= \frac{1}{2} \frac{1}{h_1^2} \left(\frac{a}{2\kappa_1} \right)^2 \langle X_0, X_0 \rangle, \\ \left\langle U \left(\frac{1}{f} X_0, \frac{1}{h_1} B'_i \right), U \left(\frac{1}{f} X_0, \frac{1}{h_1} B'_i \right) \right\rangle_s &= \frac{1}{4} \frac{1}{f^2 h_1^2} \left(\frac{a}{2\kappa_1} \right)^2 \left\{ h_1 \langle X_0, X_0 \rangle - \frac{f^2}{h_1} \right\}^2 \end{aligned}$$

and

$$\left\langle U \left(\frac{1}{f} X_0, \frac{1}{f} X_0 \right), U \frac{1}{h_1} B'_i, \frac{1}{h_1} B'_i \right\rangle_s = 0. \quad \text{Note also that}$$

$[X_0, B'_i] = 0$. Thus by Lemma 5.6, we get

$$\left\langle \hat{R} \left(\frac{1}{f} X_0, \frac{1}{h_1} B'_i \right) \frac{1}{h_1} B'_i, \frac{1}{f} X_0 \right\rangle_s = \frac{1}{4} \left(\frac{a}{2\kappa_1} \right)^2 \frac{f^2}{h_1^4}.$$

By the same way we get

$$\left\langle \hat{R} \left(\frac{1}{f} X_0, \frac{1}{h_2} C'_j \right) \frac{1}{h_2} C'_j, \frac{1}{f} X_0 \right\rangle_s = \frac{1}{4} \left(\frac{b}{2\kappa_2} \right)^2 \frac{f^2}{h_2^4}.$$

Since $\hat{\rho} \left(\frac{1}{f} X_0, \frac{1}{f} X_0 \right) = \sum_{i=1}^{2m} \left\langle \hat{R} \left(\frac{1}{f} X_0, \frac{1}{h_1} B'_i \right) \frac{1}{h_1} B'_i, \frac{1}{f} X_0 \right\rangle_s$
 $+ \sum_{j=1}^{2n} \left\langle \hat{R} \left(\frac{1}{f} X_0, \frac{1}{h_2} C'_j \right) \frac{1}{h_2} C'_j, \frac{1}{f} X_0 \right\rangle_s$, we get (5.20).

Note that $[B_i, B_j]_{\mathfrak{p}} = 0, [IB_i, IB_j]_{\mathfrak{p}} = 0$ and $[B_i, IB_j]_{\mathfrak{p}} = [B_i, IB_j]_{\mathfrak{C}_p} = \delta_{ij} \frac{-a}{2\kappa_1} X_0$.

and $[c_p, m_i] \subset m_i$ ($i=1, 2$). By straightforward computations, we have

$$\langle \hat{R}\left(\frac{1}{h_1}B_i, \frac{1}{h_1}IB_i\right)\frac{1}{h_1}IB_i, \frac{1}{h_1}B_i \rangle_s = -\frac{3}{4}\left(\frac{a}{2\kappa_1}\right)^2\frac{f^2}{h_1^4} - \frac{1}{h_1^2}\langle [[B_i, IB_i], IB_i], B_i \rangle$$

and

$$\langle \hat{R}\left(\frac{1}{h_1}B'_i, \frac{1}{h_1}B'_j\right)\frac{1}{h_1}B'_i, \frac{1}{h_1}B'_j \rangle_s = -\frac{1}{h_1^2}\langle [[B'_i, B'_j], B'_j], B'_i \rangle$$

otherwise.

We note that if \bar{R}_1 is the curvature tensor of the hermitian symmetric space M_1 with the metric induced from the Killing form then

$$\langle \bar{R}_1(B'_i, B'_j)B'_j, B'_i \rangle = -\langle [[B'_i, B'_j], B'_j], B'_i \rangle.$$

Moreover it is known that the Ricci tensor r_1 of a hermitian symmetric space M_1 is given by

$$r_1(X, Y) = \frac{1}{2}\langle X, Y \rangle \quad \text{for } X, Y \in m_1$$

(see [11] Proposition 9.7). Obviously we have

$$\langle \hat{R}\left(\frac{1}{h_1}B'_i, \frac{1}{h_2}C'_j\right)\frac{1}{h_2}C'_j, \frac{1}{h_1}B'_i \rangle_s = 0 \quad \text{for each } (i, j).$$

Therefore we get

$$\begin{aligned} \hat{r}\left(\frac{1}{h_1}B'_i, \frac{1}{h_1}B'_i\right) &= \langle \hat{R}\left(\frac{1}{h_1}B'_i, \frac{1}{f}X_0\right)\frac{1}{f}X_0, \frac{1}{h_1}B'_i \rangle \\ &+ \sum_{j=1}^{2m} \langle \hat{R}\left(\frac{1}{h_1}B'_i, \frac{1}{h_1}B'_j\right)\frac{1}{h_1}B'_j, \frac{1}{h_1}B'_i \rangle \\ &= -\frac{1}{2}\left(\frac{a}{2\kappa_1}\right)^2\frac{f^2}{h_1^4} + \frac{1}{2h_1^2}. \end{aligned}$$

By the same way we also get (5.22). q.e.d.

By Proposition 5.4, Lemma 5.6 and Proposition 5.8, we get following theorem.

Theorem 5.9. *Let X be a product of two irreducible hermitian symmetric spaces of compact type M_1 and M_2 and let $P(1 \oplus \xi_p)$ be a projective bundle on X such that $\xi_p = p_1^*L_1^{-a} \otimes p_2^*L_2^b$ where a, b are positive integers. Then a $G_u \times S^1$ -invariant hermitian metric g on the open orbit $G \times_p \mathbb{C}^*$ of the form (5.11) is Einstein Kähler if and only if f, h_1 and h_2 satisfy the following ordinary differential equations:*

$$(5.23) \quad \left\{ \begin{array}{l} (1) \quad \frac{a}{2\kappa_1} f + 2h_1 h_1' = 0 \\ (2) \quad -\frac{b}{2\kappa_2} f + 2h_2 h_2' = 0 \\ (3) \quad -\left(\frac{f''}{f} + 2m \frac{h_1'}{h_1} + 2n \frac{h_2'}{h_2}\right) = \lambda \\ (4) \quad -\frac{f''}{f} - \frac{f'}{f} \left(2m \frac{h_1'}{h_1} + 2n \frac{h_2'}{h_2}\right) + 2m \left(\frac{a}{2\kappa_1}\right)^2 \frac{f^2}{4h_1^4} + 2n \left(\frac{b}{2\kappa_2}\right)^2 \frac{f^2}{4h_2^4} = \lambda \\ (5) \quad -\frac{h_1''}{h_1} - \frac{f' h_1'}{f h_1} - (2m-1) \left(\frac{h_1'}{h_1}\right)^2 - 2n \left(\frac{h_1' h_2'}{h_1 h_2}\right) + \frac{1}{2h_1^2} - \left(\frac{a}{2\kappa_1}\right)^2 \frac{f^2}{2h_1^4} = \lambda \\ (6) \quad -\frac{h_2''}{h_2} - \frac{f' h_2'}{f h_2} - (2n-1) \left(\frac{h_2'}{h_2}\right)^2 - 2m \left(\frac{h_1' h_2'}{h_1 h_2}\right) + \frac{1}{2h_2^2} - \left(\frac{b}{2\kappa_2}\right)^2 \frac{f^2}{2h_2^4} = \lambda \end{array} \right.$$

for some constant $\lambda > 0$.

6 A proof of Main Theorem

At first we shall solve the system of ordinary differential equations (5.23). We consider a solution such that f , h_1 and h_2 are positive valued functions on an open interval. By (5.23) (2) we see that $h_2' > 0$. From (5.23) (1) and (2) we have

$$(6.1) \quad \frac{f'}{f} = \frac{h_1'}{h_1} + \frac{h_1}{h_1} = \frac{h_2'}{h_2} + \frac{h_2}{h_2}$$

and

$$(6.2) \quad \left\{ \begin{array}{l} \frac{f'}{f} \frac{h_1'}{h_1} = \frac{h_1''}{h_1} + \left(\frac{h_1'}{h_1}\right)^2 = \frac{h_1''}{h_1} + \left(\frac{a}{2\kappa_1}\right)^2 \frac{f^2}{4h_1^4} \\ \frac{f'}{f} \frac{h_2'}{h_2} = \frac{h_2''}{h_2} + \left(\frac{h_2'}{h_2}\right)^2 = \frac{h_2''}{h_2} + \left(\frac{b}{2\kappa_2}\right)^2 \frac{f^2}{4h_2^4} \end{array} \right.$$

Thus under the equations (5.23) (1) and (2), the equations (5.23) (3) and (4) are identical.

From (5.23) (1) and (2) we also get

$$(6.3) \quad a\kappa_2 h_2' h_2 + b\kappa_1 h_1' h_1 = 0,$$

and we introduce a constant $\delta > 0$ by

$$(6.4) \quad \delta^2 = a\kappa_2 h_2^2 + b\kappa_1 h_1^2.$$

Now we introduce a new variable $y = y(h_2)$ by

$$(6.5) \quad h_2' = \sqrt{y(h_2)}.$$

Then we have

$$(6.6) \quad \frac{d^2h_2}{ds^2} = \frac{1}{2} \frac{dy}{dh_2} \text{ and } \frac{d^3h_2}{ds^3} = \frac{1}{2} \frac{d^2y}{dh_2^2} \frac{dh_2}{ds}.$$

By (6.1), (6.3) and (5.23) (2), the equation (5.23) (6) is written as

$$-2 \frac{h_2''}{h_2} - (2n+2) \left(\frac{h_2'}{h_2}\right)^2 + 2m \frac{a\kappa_2}{b\kappa_1} \frac{1}{h_1^2} (h_2')^2 + \frac{1}{2h_2^2} = \lambda.$$

Thus by (6.5) and (6.6) we get

$$(6.7) \quad \frac{dy}{dh_2} + 2 \left(\frac{n+1}{h_2} - m \frac{a\kappa_2}{b\kappa_1} \frac{h_2}{h_1^2} \right) y = \frac{1}{2h_2} - \lambda h_2.$$

Similarly, by (6.2), the equation (5.23) (5) is written as

$$(6.8) \quad -2 \frac{h_1''}{h_1} - (2m+2) \left(\frac{h_1'}{h_1}\right)^2 - 2n \frac{h_1'h_2'}{h_1 h_2} + \frac{1}{2h_1^2} = \lambda.$$

From (6.3), (6.4), (6.5) and (6.6) we obtain

$$(6.9) \quad \left(\frac{h_1'}{h_1}\right)^2 = \left(\frac{a\kappa_2}{b\kappa_1}\right)^2 \frac{h_2^2}{h_1^4} y$$

and

$$(6.10) \quad \frac{h_1''}{h_1} = -\frac{1}{2} \frac{a\kappa_2}{b\kappa_1} \frac{h_2}{h_1^2} \frac{dy}{dh_2} - \frac{a\kappa_2}{(b\kappa_1)^2} \frac{\delta^2}{h_1^4} y.$$

Therefore the equation (6.8) is written as

$$(6.11) \quad \frac{dy}{dh_2} + 2 \left(\frac{n+1}{h_2} - m \frac{a\kappa_2}{b\kappa_1} \frac{h_2}{h_1^2} \right) y = \frac{b\kappa_1}{a\kappa_2} \frac{h_1^2}{h_2} \lambda - \frac{1}{2} \frac{b\kappa_1}{a\kappa_2} \frac{1}{h_2}.$$

From the equations (6.7), (6.11) and (6.4), we obtain a relation

$$(6.12) \quad a\kappa_2 + b\kappa_1 = 2\lambda\delta^2.$$

Now by (5.23) (2) and (6.6), we have

$$(6.13) \quad \frac{f''}{f} = 3 \frac{h_2''}{h_2} + \frac{h_2'''}{h_2'} = \frac{3}{2} \frac{1}{h_2} \frac{dy}{dh_2} + \frac{1}{2} \frac{d^2y}{dh_2^2}.$$

Thus the equation (5.23) (3) is written as

$$(6.14) \quad \frac{d^2y}{dh_2^2} + \left(\frac{2n+3}{h_2} - \frac{2ma\kappa_2 h_2}{b\kappa_1 h_1^2} \right) \frac{dy}{dh_2} - \frac{4ma\kappa_2 \delta^2}{(b\kappa_1)^2 h_1^4} y = -2\lambda.$$

Now it is easy to see that the equation (6.14) is obtained from the equation (6.7) by differentiation and (6.4). Hence we get the following lemma.

Lemma 6.1. *The system of differential equations (5.23) is equivalent to the following system of equations:*

$$(6.15) \quad \begin{cases} \frac{a}{2\kappa_1}f + 2h_1h_1' = 0, & -\frac{b}{2\kappa_2}f + 2h_2h_2' = 0 \\ h_2' = \sqrt{y(h_2)}, & 2\lambda(a\kappa_2h_2^2 + b\kappa_1h_1^2) = a\kappa_2 + b\kappa_1 \\ \frac{dy}{dh_2} + 2\left(\frac{n+1}{h_2} - m\frac{a\kappa_2}{b\kappa_1}\frac{h_2}{h_1^2}\right)y = \frac{1}{2h_2} - \lambda h_2. \end{cases}$$

Now we consider the first order linear differential equation (6.7). Since an integral factor μ is given by

$$(6.16) \quad \mu = h_2^{2(n+1)}(\delta^2 - a\kappa_2h_2^2)^m = h_2^{2(n+1)}(b\kappa_1h_1^2)^m,$$

a solution y of the equation (6.16) is given by

$$(6.17) \quad y = \frac{1}{2h_2^{2(n+1)}(b\kappa_1h_1^2)^m} \left\{ \int h_2^{2n+1}(b\kappa_1h_1^2)^m (1 - 2\lambda h_2^2) dh_2 + C \right\}$$

where C is a constant and $a\kappa_2h_2^2 + b\kappa_1h_1^2 = \delta^2$.

Now we recall the following theorem on a compact Einstein Kähler manifold.

Theorem 6.2 (Matsushima [12]). *Let (P, J, g) be a compact Einstein Kähler manifold with positive Ricci tensor. Then the Lie algebra $\mathfrak{k}(P, g)$ of all Killing vector fields on P is a real form of the Lie algebra $\mathfrak{g}(P, J)$ of all holomorphic vector fields on P .*

Let $P(1 \oplus \xi_\rho)$ be the projective bundle on X as in Theorem 5.9 and assume that g is an Einstein Kähler metric on $P(1 \oplus \xi_\rho)$. Then we assume that g is invariant by the maximal compact Lie group $G_u \times S^1$ by Theorem 6.2, and hence g is of the form (5.11) on the open orbit $G \times_{\rho} \mathbf{C}^*$, and f, h_1, h_2 satisfy the equations (5.23) and conditions of Theorem 4.4 at the boundaries 0 and L . By (5.23) (1) and (2), we obtain

$$(6.18) \quad \begin{cases} \frac{a}{2\kappa_1}f' + 2h_1h_1'' + 2(h_1')^2 = 0, \\ -\frac{b}{2\kappa_2}f' + 2h_2h_2'' + 2(h_2')^2 = 0 \end{cases}$$

Since $f'(0) = 1, f'(L) = -1, h_1'(0) = h_1'(L) = h_2'(0) = h_2'(L) = 0$, we have

$$(6.19) \quad \begin{cases} \frac{a}{2\kappa_1} + 2h_1(0)h_1''(0) = 0, & -\frac{a}{2\kappa_1} + 2h_1(L)h_1''(L) = 0, \\ -\frac{b}{2\kappa_2} + 2h_2(0)h_2''(0) = 0, & \frac{b}{2\kappa_2} + 2h_2(L)h_2''(L) = 0. \end{cases}$$

By (6.7) and (6.8) we have

$$(6.20) \quad -4h_i''(0)h_i(0) = 2\lambda h_i^2(0) - 1, \quad -4h_i''(L)h_i(L) = 2\lambda h_i^2(L) - 1$$

for $i=1, 2$. Thus by (6.19) and (6.20), we get

$$(6.21) \quad \begin{cases} 2\lambda h_1^2(0) = 1 + (a/\kappa_1), & 2\lambda h_1^2(L) = 1 - (a/\kappa_1), \\ 2\lambda h_2^2(0) = 1 - (b/\kappa_2), & 2\lambda h_2^2(L) = 1 + (b/\kappa_2). \end{cases}$$

In particular, we obtain conditions $a < \kappa_1$ and $b < \kappa_2$, which are known as the conditions for the first Chern class of $P(1 \oplus \xi_\rho)$ being positive. Now, since $y(h_2(0)) = (h_2'(0))^2 = 0$, $y(h_2)$ is given by

$$(6.22) \quad y(h_2) = \frac{1}{2h_2^{2(n+1)}(b\kappa_1 h_1^2)^m} \int_{h_2(0)}^{h_2} h_2^{2n+1} (b\kappa_1 h_1^2)^m (1 - 2\lambda h_2^2) dh_2.$$

Since $y(h_2(L)) = 0$, we have

$$y(h_2(L)) = \frac{1}{2h_2^{2(n+1)}(L) (b\kappa_1 h_1^2(L))^m} \int_{h_2(0)}^{h_2(L)} h_2^{2n+1} (b\kappa_1 h_1^2)^m (1 - 2\lambda h_2^2) dh_2 = 0.$$

Hence, if g is an Einstein Kähler metric on $P(1 \oplus \xi_\rho)$, we have

$$(6.23) \quad \int_{\sqrt{(1-(b/\kappa_2))/2\lambda}}^{\sqrt{(1+(b/\kappa_2))/2\lambda}} h_2^{2n+1} (b\kappa_1 h_1^2)^m (1 - 2\lambda h_2^2) dh_2 = 0$$

where $2\lambda(a\kappa_2 h_2^2 + b\kappa_1 h_1^2) = a\kappa_2 + b\kappa_1$. Now we put $u = 2\lambda h_2^2 - 1$. Then (6.23) can be written as

$$\int_{-b/\kappa_2}^{b/\kappa_2} (u+1)^n (b\kappa_1 - a\kappa_2 u)^m u du = 0,$$

since $2\lambda(a\kappa_2 h_2^2 + b\kappa_1 h_1^2) = a\kappa_2 + b\kappa_1$.

Thus by setting $x = (\kappa_2/b)u$, we see that (6.23) is given by

$$\int_{-1}^1 (\kappa_2 + bx)^n (\kappa_1 - ax)^m x dx = 0.$$

Conversely, assume that (6.23) is satisfied. We define $y(h_2)$ on a neighborhood of $[\sqrt{(1-(b/\kappa_2))/2\lambda}, \sqrt{(1+(b/\kappa_2))/2\lambda}]$ by

$$y(h_2) = \frac{1}{2h_2^{2(n+1)}(b\kappa_1 h_1^2)^m} \int_{\sqrt{(1-(b/\kappa_2))/2\lambda}}^{h_2} h_2^{2n+1} (b\kappa_1 h_1^2)^m (1 - 2\lambda h_2^2) dh_2.$$

For simplicity, we put $h^0 = \sqrt{(1-(b/\kappa_2))/2\lambda}$, $h^1 = \sqrt{(1+(b/\kappa_2))/2\lambda}$. Then $y(h^0) = y(h^1) = 0$ and $y(h_2) > 0$ for $h^0 < h_2 < h^1$. Note also that $dy/dh_2(h^0) > 0$ and $dy/dh_2(h^1) > 0$. Define a function $\tilde{t}(h_2)$ on (h^0, h^1) by

$$(6.24) \quad \tilde{t}(h_2) = 1 \int_{\sqrt{1/2\lambda}}^{h_2} \frac{1}{\sqrt{y(h_2)}} dh_2.$$

Since $h_2 = h^0, h^1$ are simple roots of $y(h_2) = 0$, $\lim_{h_2 \rightarrow h^0+} \tilde{t}(h_2)$ and $\lim_{h_2 \rightarrow h^1-} \tilde{t}(h_2)$ exist. We put

$$\tilde{t}_0 = \lim_{h_2 \rightarrow h^0+} \tilde{t}(h_2) \quad \text{and} \quad \tilde{t}_1 = \lim_{h_2 \rightarrow h^1-} \tilde{t}(h_2).$$

We also define a function $t(h_2)$ on $[h^0, h^1]$ by

$$t(h_2) = \tilde{t}(h_2) - \tilde{t}_0, \quad t(h^0) = 0 \quad \text{and} \quad t(h^1) = \tilde{t}_1 - \tilde{t}_0$$

and we put $L = t(h^1)$. Then $t(h_2): [h^0, h^1] \rightarrow [0, L]$ is a monotone increasing continuous function which is C^∞ on (h^0, h^1) .

Now let $h_2(t)$ be the inverse function of $t(h_2)$. Then $dh_2/dt = \sqrt{y(h_2)}$ on $(0, L)$. We claim that $h_2(t)$ can be extended to a C^∞ function $h_2(t): [0, L] \rightarrow \mathbf{R}_+$ such that $h_2^{(2k-1)}(0) = h_2^{(2k-1)}(L) = 0$ for each positive integer k . For a sufficient small $\varepsilon > 0$, we extend $h_2(t)$ to a function $h_2(t): (-\varepsilon, L + \varepsilon) \rightarrow \mathbf{R}_+$ by $h_2(t) = h_2(-t)$ for $-\varepsilon < t < 0$ and $h_2(t + L) = h_2(L - t)$ for $0 < t < \varepsilon$. Then we see that $h_2(t): (-\varepsilon, L + \varepsilon) \rightarrow \mathbf{R}$ is continuous and is a C^∞ function except $t = 0$ and $t = L$. Since $dh_2/dt = \sqrt{y(h_2)}$ on $(0, L)$, $dh_2/dt = -\sqrt{y(h_2)}$ on $(-\varepsilon, 0)$ and $\lim_{t \rightarrow 0} \frac{dh_2}{dt} = 0$, we see that $dh_2/dt(0)$ exists and $dh_2/dt(0) = 0$. Similarly we have $dh_2/dt(L) = 0$. Thus we see that $h_2(t): (-\varepsilon, L + \varepsilon) \rightarrow \mathbf{R}_+$ is a function of class C^1 . By $dh_2/dt = \sqrt{y(h_2)}$ on $(0, L)$, we have

$$\frac{d^2 h_2}{dt^2} = \frac{1}{2} \frac{dy}{dh_2}(h_2(t)) \quad \text{on } (0, L).$$

By $dh_2/dt = -\sqrt{y(h_2)}$ on $(-\varepsilon, 0)$, we also have

$$\frac{d^2 h_2}{dt^2} = \frac{1}{2} \frac{dy}{dh_2}(h_2(t)) \quad \text{on } (-\varepsilon, 0).$$

Thus we see that $\lim_{t \rightarrow 0} \frac{d^2 h_2}{dt^2}$ exists and

$$\frac{d^2 h_2}{dt^2}(0) = \frac{1}{2} \frac{dy}{dh_2}(h^0) = \frac{1}{2} \left(\frac{1}{2h^0} - \lambda h^0 \right).$$

Similarly we see that $\lim_{t \rightarrow L} \frac{d^2 h_2}{dt^2}$ exists and

$$\frac{d^2 h_2}{dt^2}(L) = \frac{1}{2} \frac{dy}{dh_2}(h^1) = \frac{1}{2} \left(\frac{1}{2h^1} - \lambda h^1 \right).$$

Therefore $h_2(t): (-\varepsilon, L + \varepsilon) \rightarrow \mathbf{R}_+$ is of class C^2 . Now we put $\varphi(h_2) = \frac{1}{2} \frac{dy}{dh_2}$.

Then $\varphi(h_2)$ is a C^∞ function on a neighborhood of $[h^0, h^1]$ and

$$(6.25) \quad \frac{d^2 h_2}{dt^2} = \varphi(h_2(t)) \quad \text{on } (0, L).$$

Lemma 6.3. *On $(0, L)$, we have, for each positive integer k ,*

$$(6.26) \quad \frac{dh_2^{2k+1}}{dt^{2k+1}} = \frac{d^{2k-1}\varphi}{dh_2^{2k+1}} \left(\frac{dh_2}{dt} \right)^{2k-1} + \sum_{j=1}^{k-1} \Phi_{2(k-j)-1}^{2k+1} \left(\varphi, \frac{d\varphi}{dh_2}, \dots, \frac{d^{2k-1-j}\varphi}{dh_2^{2k-1-j}} \right) \left(\frac{dh_2}{dt} \right)^{2(k-j)-1}$$

$$(6.27) \quad \frac{dh_2^{2k}}{dt^{2k}} = \frac{d^{2k-2}\varphi}{dh_2^{2k-2}} \left(\frac{dh_2}{dt} \right)^{2k-2} + \sum_{j=1}^{k-1} \Phi_{2(k-j)-2}^{2k} \left(\varphi, \frac{d\varphi}{dh_2}, \dots, \frac{d^{2k-2-j}\varphi}{dh_2^{2k-2-j}} \right) \left(\frac{dh_2}{dt} \right)^{2(k-j)-2}$$

where $\Phi_{l-1-2j}^l \left(\varphi, \frac{d\varphi}{dh_2}, \dots, \frac{d^{l-1-j}\varphi}{dh_2^{l-1-j}} \right)$ are polynomials of $\varphi, \frac{d\varphi}{dh_2}, \dots, \frac{d^{l-1-j}\varphi}{dh_2^{l-1-j}}$.

Proof. By routine computations using induction.

In particular, we see that

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{dh_2^{2k+1}}{dt^{2k+1}} &= \lim_{t \rightarrow L} \frac{dh_2^{2k+1}}{dt^{2k+1}} = 0, \\ \lim_{t \rightarrow 0} \frac{dh_2^{2k}}{dt^{2k}} &= \Phi_0^{2k} \left(\varphi(h^0), \frac{d\varphi}{dh_2}(h^0), \dots, \frac{d^{k-1}\varphi}{dh_2^{k-1}}(h^0) \right) \\ \lim_{t \rightarrow L} \frac{dh_2^{2k}}{dt^{2k}} &= \Phi_0^{2k} \left(\varphi(h^1), \frac{d\varphi}{dh_2}(h^1), \dots, \frac{d^{k-1}\varphi}{dh_2^{k-1}}(h^1) \right), \end{aligned}$$

and hence $h_2(t) : (-\varepsilon, L + \varepsilon) \rightarrow \mathbf{R}_+$ is a C^∞ function such that $h_2^{(2k-1)}(0) = h_2^{(2k-1)}(L) = 0$ for each positive integer k . We define a function f by

$$f = (4\kappa_2/b)h_2h_2'$$

and a function $h_1 > 0$ by

$$2\lambda(a\kappa_2h_2^2 + b\kappa_1h_1^2) = a\kappa_2 + b\kappa_1.$$

Then f is a C^∞ function on $[0, L]$ such that $f(0) = f(L) = 0, f'(0) = -f'(L) = 1$ and $f^{(2k)}(0) = f^{(2k)}(L) = 0$ for each positive integer k , and f, h_1, h_2 satisfy the equation (5.23). Therefore a metric $g = dt^2 + f(t)^2\tilde{\beta}_0 + h_1(t)^2\alpha_1 + h_2(t)\alpha_2$ is an Einstein Kähler metric on $P(1 \oplus \xi_\rho)$ by Theorem 4.4 and Theorem 5.9. This proves our Main Theorem.

Proof of Corollary 1. Since $\int_{-1}^1 (\kappa - ax)^m (\kappa + ax)^m x dx = 0$, we see that there exists an Einstein Kähler metric on P by our Main Theorem.

Proof of Corollary 2 (1). By our Main Theorem it is enough to see that

$$\int_{-1}^1 (\kappa + bx)^m (\kappa - ax)^m x dx \neq 0 \quad \text{for } a \neq b.$$

We may assume that $b > a$.

$$\begin{aligned} & \int_{-1}^1 (\kappa + bx)^m (\kappa - ax)^m x dx = \int_{-1}^1 (\kappa^2 + (b-a)x - abx^2)^m x dx \\ &= \sum_{j=a}^m \int_{-1}^1 \binom{m}{j} (\kappa^2 - abx^2)^{m-j} ((b-a)x)^j x dx \\ &= \sum_{k \geq 1} \int_{-1}^1 \binom{m}{2k-1} (\kappa^2 - abx^2)^{m-2k+1} (b-a)^{2k-1} x^{2k} dx \\ &= 2 \sum_{k \geq 1} \int_0^1 \binom{m}{2k-1} (\kappa^2 - abx^2)^{m-2k+1} (b-a)^{2k-1} x^{2k} dx > 0. \end{aligned} \quad \text{q.e.d.}$$

Proof of Corollary 2 (2). Since $\kappa_1=2$ and $a=1$, we have to show that

$$(6.28) \quad \int_{-1}^1 (2-x) (\kappa_2 + bx)^n x dx \neq 0 \quad \text{for } n \geq 2.$$

Put $y = \kappa_2 + bx$. Then the integral (6.28) is given by

$$\int_{\kappa_2-b}^{\kappa_2+b} \frac{1}{b^3} (2b + \kappa_2 - y) (y - \kappa_2) y^n dy.$$

Now we have

$$\begin{aligned} (6.29) \quad & \int_{\kappa_2-b}^{\kappa_2+b} (2b + \kappa_2 - y) (y - \kappa_2) y^n dy \\ &= \frac{1}{(n+1)(n+2)(n+3)} [(\kappa_2 - b)^{n+1} (2\kappa_2^2 + (2n+4)2b\kappa_2 + (n+1)(3n+8)b^2) \\ & \quad - (\kappa_2 + b)^{n+1} (2(\kappa_2^2 + 2b\kappa_2) - b^2(n^2 + 5n + 4))]. \end{aligned}$$

Case 1. $b \geq 2$.

Since $b < \kappa_2 \leq n+1$,

$$\begin{aligned} & b^2(n^2 + 5n + 4) - 2(\kappa_2^2 + 2b\kappa_2) \geq b^2(n^2 + 5n + 4) - 2(n+1)(n+1+2b) \\ &= (b^2 - 2)n^2 + (5b^2 - 2b - 2)n + (4b^2 - 4b - 2) > 0 \quad \text{if } b \geq 2. \end{aligned}$$

Thus the integration (6.29) is positive.

Case 2. $b = 1$.

We use a classification of irreducible hermitian symmetric spaces. It is also known that the integer κ of an irreducible hermitian symmetric space of compact type M is given as follows (cf. [5]):

I	$M = U(p+q)/U(p) \times U(q)$	$\kappa = p+q$	$\dim_{\mathbb{C}} M = pq$
II	$M = SO(2q)/U(q) \quad (q \geq 5)$	$\kappa = 2q-2$	$\dim_{\mathbb{C}} M = q(q-1)/2$
III	$M = Sp(q)/U(q) \quad (q \geq 3)$	$\kappa = q+1$	$\dim_{\mathbb{C}} M = q(q+1)/2$
IV	$M = SO(q+2)/(SO(2) \times SO(q)) \quad (q \geq 3)$	$\kappa = q$	$\dim_{\mathbb{C}} M = q$

$$\begin{aligned} \text{V } M &= E_6/(Spin(10) \times T^1) & \kappa &= 12 & \dim_{\mathbb{C}} M &= 16 \\ \text{VI } M &= E_7/(E_6 \times T^1) & \kappa &= 18 & \dim_{\mathbb{C}} M &= 27. \end{aligned}$$

Now, since $b=1$, (6.29) is given by

$$\begin{aligned} (6.30) \quad & \int_{\kappa_2-1}^{\kappa_2+1} (2+\kappa_2-y)(y-\kappa_2)y^n dy \\ &= \frac{1}{(n+3)(n+2)(n+1)} [(\kappa_2-1)^{n+1}(2\kappa_2^2+2(2n+4)\kappa_2+(n+1)(3n+8)) \\ & \quad -(\kappa_2+1)^{n+1}(2(\kappa_2^2+2\kappa_2)-(n^2+5n+4))]. \end{aligned}$$

Case 2.1.

If $M=U(p+q)/(U(p) \times U(q))$ and $p, q \geq 2$,

$$\begin{aligned} n^2+5n+4-2(\kappa_2^2+2\kappa_2) &= (pq)^2+5pq+4-2(p+q)^2-4(p+q) \\ &= (p^2-2)(q^2-2)+pq-4p-4q \geq 2(p^2-2)+q(p-4)-4p. \end{aligned}$$

$$\begin{aligned} \text{If } p \geq 4, \quad 2(p^2-2)+q(p-4)-4p &\geq 2(p^2-2)+2(p-4)-4p \\ &= 2(p-3)(p+2) \geq 0. \end{aligned}$$

We may also assume that $p \geq q$. If $p=3 \geq q \geq 2$,

$$n^2+5n+4-2(\kappa_2^2+2\kappa_2) = 7(q^2-2)+3q-12-4q = 7q^2-q-26 > 0.$$

Note that if $p=q=2$ then M is a quadric $Q^4(\mathbb{C})$.

Case 2.2.

If $M=SO(2q)/U(q)$ ($q \geq 5$),

$$\begin{aligned} n^2+5n+4-2(\kappa_2^2+2\kappa_2) &= (q(q-1)/2)^2+5(q(q-1)/2)+4 \\ &\quad -2(2q-2)^2-4(2q-2). \end{aligned}$$

Since $n=q(q-1)/2$, $n^2+5n+4-2(\kappa_2^2+2\kappa_2)=n^2-11n+4 > 0$ if $q \geq 6$, that is, $n \geq 15$.

For $q=5$, we have $n=10$ and thus (6.30) becomes

$$\frac{1}{13 \times 12 \times 11} (7^{11}(2^9+11 \times 38)-9^{11} \times 6) \neq 0$$

Case 2.3.

If $M=Sp(q)/U(q)$ ($q \geq 3$),

$$n^2+5n+4-2(\kappa_2^2+2\kappa_2) = (q(q+1)/2)^2+5q(q+1)/2+4-2((q+1)^2+2(q+1))$$

Put $p(x)=(x(x+1)/2)^2+5x(x+1)/2+4-2((x+1)^2+2(x+1))$.

Then $p(3)=22$ and $p'(x) > 0$ for $x > 3$ and hence $n^2+5n+4-2(\kappa_2^2+2\kappa_2) > 0$ for $q \geq 3$.

Case 2.4.

If $M=E_6/(Spin(10) \times T^1)$, $\kappa_2=12$ and $n=16$, thus

$$n^2+5n+6-2(\kappa_2^2+2\kappa_2) = 4 > 0.$$

Case 2.5.

If $M=E_7/(E_6 \times T^1)$, $\kappa_2=18$ and $n=27$, thus

$$n^2+5n+9-2(\kappa_2^2+2\kappa_2) = 3^2+5 \times 3^3+4 > 0.$$

Therefore the integral (6.30) is positive for the cases above.

Now we consider the cases $M=P^n(\mathbf{C})$ and $M=Q^n(\mathbf{C})$.

Case 2.6.

If $M=P^n(\mathbf{C})$, $\kappa_2=n+1$, and thus (6.30) is given by

$$\begin{aligned} & \frac{1}{(n+3)(n+2)(n+1)} \{n^{n+1}9(n+1)(n+2) - (n+2)^{n+1}(n+1)(n+2)\} \\ &= \frac{1}{n+3} (9n^{n+1} - (n+2)^{n+1}) = \frac{r^{n+1}}{n+3} \left(9 - \left(\frac{n+2}{n}\right)^{n+1}\right). \end{aligned}$$

We define a function $p(y)$ ($y \geq 2$) by

$$(6.31) \quad p(y) = \left(\frac{y+1}{y-1}\right)^y.$$

Then it is not difficult to see that $p(y)$ is a monotone decreasing function. Therefore we see that the integral (6.30) is positive for $n \geq 2$.

Case 2.7.

If $M=Q^n(\mathbf{C})$ ($n \geq 3$), $\kappa_2=n$ and thus (6.30) is given by

$$\frac{(n-1)^{n+1}(n^2-n-4)}{(n+3)(n+2)(n+1)} \left\{ \frac{9n^2+19n+8}{n^2-n-4} - \left(\frac{n+1}{n-1}\right)^{n+1} \right\}.$$

We claim that $\frac{9n^2+19n+8}{n^2-n-4} - \left(\frac{n+1}{n-1}\right)^{n+1} > 0$ for $n \geq 3$. Since the function $p(y)$ defined by (6.31) is monotone decreasing, it is enough to show that

$$\frac{(9n^2+19n+8)(n-1)}{(n^2-n-4)(n+1)} > 8 \quad \text{for } n \geq 3.$$

But this is obvious, since

$$(9n^2+19n+8)(n-1) - 8(n+1)(n^2-n-4) = n^3+10n^2+29n+24 > 0.$$

Thus the integral (6.30) is positive for $n \geq 3$.

q.e.d.

Finally we give an example of Einstein Kähler manifold which is not of the type in Corollary 1 of Main Theorem.

EXAMPLE 6.4. Let M_1 be the complex Grassmann manifold $G_{6,2}(\mathbf{C})$ of 2-planes in \mathbf{C}^6 and M_2 the complex projective space $P^3(\mathbf{C})$. Note that in this case $\kappa_1=6$ and $\kappa_2=9$. Consider the $P^1(\mathbf{C})$ -bundle $P(p_1^*L_1^2 \oplus p_2^*L_2^3)$ over $M_1 \times M_2$. Then the integral in Main Theorem is given by

$$\int_{-1}^1 (6-2x)^8(9+3x)^8 x dx = 0.$$

Thus $P(p_1^*L_1^2 \oplus p_2^*L_2^3)$ has an Einstein Kähler metric.

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