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Examples of Compact Lefschetz Solvmanifolds

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Introduction.

Let (M^{2m}, ω) be a compact symplectic manifold. A symplectic manifold (M, ω) is called a Lefschetz manifold if the mapping $\wedge\omega^{m-1} : H^1_{DR}(M) \to H^{2m-1}_{DR}(M)$ is an isomorphism. We also say that (M, ω) has the Hard Lefschetz property, if the mapping $\wedge\omega^k$: $H^{m-k}_{DR}(M) \to H^{m+k}_{DR}(M)$ is an isomorphism for each $k \leq m$. By a solvmanifold we mean a homogeneous space G/Γ, where G is a simply-connected solvable Lie group and Γ is a lattice, that is, a discrete co-compact subgroup of G. A solvable Lie algebra $\mathfrak{g}$ is called completely solvable if $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ has only real eigenvalues for each $X \in \mathfrak{g}$. Benson and Gordon [BG1] have proved that no non-toral compact nilmanifolds are Lefschetz manifolds for any symplectic structure to show that a non-toral compact nilmanifold does not admit any Kähler structure. Moreover, they conjecture the following :

BENSON-GORDON CONJECTURE [BG2]. *Let G be a simply-connected completely solvable Lie group and Γ a lattice of G. Then a compact solvmanifold G/Γ admits a Kähler structure if and only if it is a torus.*

The authors of [AFLM] and [FLS] have constructed examples of 6-dimensional compact Lefschetz solvmanifolds with the Hard Lefschetz property and without the Hard Lefschetz property (See Example 5.1 and 5.4). More precisely, let G_6 be the simply-connected completely solvable Lie group defined by

$$G_6 = \left\{ \left. \begin{pmatrix} e^t & 0 & xe^t & 0 & 0 & 0 & y_1 \\ 0 & e^{-t} & 0 & xe^{-t} & 0 & 0 & y_2 \\ 0 & 0 & e^t & 0 & 0 & 0 & y_3 \\ 0 & 0 & 0 & e^{-t} & 0 & 0 & y_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| t, x, y_1, y_2, y_3, y_4 \in \mathbf{R} \right\} .$$

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G_6 may be described as the semi-direct product $G_6 = \mathbf{R}^2 \ltimes_\varphi \mathbf{R}^4$, where $\varphi(t,x)$ is the automorphism of $\mathbf{R}^4$ given by the matrix

$$\varphi(t,x) = \begin{pmatrix} e^t & 0 & xe^t & 0 \\ 0 & e^{-t} & 0 & xe^{-t} \\ 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix}.$$

Fernández, León and Saralegui [FLS] have proved that G_6 admits a lattice Γ and has a symplectic structure. Furthermore, they have proved that G_6/Γ is a compact Lefschetz solvmanifold without the Hard Lefschetz property.

In the case of nilpotent Lie groups, a necessary and sufficient condition for the existence of a lattice is known. More precisely, let N be a simply-connected nilpotent Lie group and $\mathfrak{n}$ its Lie algebra. Then N admits a lattice if and only if $\mathfrak{n}$ admits a basis with respect to which the structure constant of Lie algebra are rational. However, in the case of solvable Lie groups, no such necessary and sufficient conditions are known. Recently, Tralle [T] proved that the completely solvable Lie group G^{bg} constructed in the paper of Benson and Gordon [BG2] has no lattices.

The purpose of this paper is to construct examples of higher dimensional completely solvable Lie groups which admit lattices and compact Lefschetz solvmanifolds. We also construct compact symplectic solvmanifolds with the Hard Lefschetz property. We consider Lie subgroups G of the affine transformation group given by

$$G = \left\{ \left. \begin{pmatrix} & & & 0 & 0 & y_1 \\ & \varphi(t,x) & & \vdots & \vdots & \vdots \\ & & & 0 & 0 & y_{2m} \\ 0 & \cdots & 0 & 1 & 0 & x \\ 0 & \cdots & 0 & 0 & 1 & t \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix} \right| t \in \mathbf{R}^l,\, x \in \mathbf{R}^n,\, y_i \in \mathbf{R} \right\},$$

where $\varphi(t,x) \in \mathrm{Aut}(\mathbf{R}^{2m})$. Note that G may be described as a semi-direct product $\mathbf{R}^{n+l} \ltimes_\varphi \mathbf{R}^{2m}$, where the group structure is defined by

$$(\mathbf{t}_1, \mathbf{x}_1, \mathbf{y}_1) * (\mathbf{t}_2, \mathbf{x}_2, \mathbf{y}_2) = (\mathbf{t}_1 + \mathbf{t}_2, \mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}_1 + \varphi(\mathbf{t}_1, \mathbf{x}_1)\mathbf{y}_2).$$

for $\mathbf{t}_i \in \mathbf{R}^l$, $\mathbf{x}_i \in \mathbf{R}^n$, and $\mathbf{y}_i \in \mathbf{R}^{2m}$.

In section 3, we prove

PROPOSITION 1. *Let A_i, B_j be the matrices given by*

$$A_i = \sum_{k=1}^{m} a_i^k (E_{2k-1,2k-1} - E_{2k,2k}) \quad i = 1, \cdots, l,$$

$$B_j = \sum_{k<h} b_j^{kh} (E_{2k-1,2h-1} + E_{2k,2h}) \quad j = 1, \cdots, n$$

where $a_i^k, b_j^{kh} \in \mathbf{Q}$ *and we assume that* $[A_i, B_j] = [B_i, B_j] = 0$. *We define a map*

$$\varphi_* : \mathbf{R}^{n+l} \to End(\mathbf{R}^{2m})$$

by

$$\varphi_*(t_1, \cdots, t_l, x_1, \cdots, x_l) = \sum_{i=1}^{l} t_i A_i + \sum_{j=1}^{n} x_j B_j .$$

Let $\varphi(\mathbf{t}, \mathbf{x}) = exp(\varphi_*(\mathbf{t}, \mathbf{x}))$ *and we define a group structure of* $\mathbf{R}^{n+l} \times \mathbf{R}^{2m}$ *by*

$$(\mathbf{t}_1, \mathbf{x}_1, \mathbf{y}_1) * (\mathbf{t}_2, \mathbf{x}_2, \mathbf{y}_2) = (\mathbf{t}_1 + \mathbf{t}_2, \mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}_1 + \varphi(\mathbf{t}_1, \mathbf{x}_1)\mathbf{y}_2)$$

for $\mathbf{t}_i \in \mathbf{R}^l, \mathbf{x}_i \in \mathbf{R}^n$ *and* $\mathbf{y}_i \in \mathbf{R}^{2m}$. *Then* $\mathbf{R}^{n+l} \ltimes_\varphi \mathbf{R}^{2m} = (\mathbf{R}^{n+l} \times \mathbf{R}^{2m}, *)$ *is a completely solvable Lie group which has a lattice* Γ.

In this paper, we always assume that for each k, there exists an i such that $a_i^k \neq 0$.

If the dimension of $G = \mathbf{R}^{n+l} \ltimes_\varphi \mathbf{R}^{2m}$ is odd, then we consider the direct product of G and a 1-dimensional vector space. We denote the direct product and a corresponding compact solvmanifold by $G \times \mathbf{R}^1$ and $G/\Gamma \times S^1$ respectively.

THEOREM 2. *Let* $M = G/\Gamma$ *or* $M = G/\Gamma \times S^1$ *be a compact solvmanifold constructed as in Proposition* 1. *If* M *admits a symplectic structure*, *then* M *is a compact Lefschetz solvmanifold for any symplectic structure.*

In section 5, we shall give some examples of compact Lefschetz solvmanifolds. In section 6, we consider a completely solvable Lie group $G = \mathbf{R}^{n+1} \ltimes_\varphi \mathbf{R}^{2m}$ which is constructed by $A = \sum_{k=1}^{m} (E_{2k-1,2k-1} - E_{2k,2k})$, $B_i = P^{2i-1}$ $(i = 1, \cdots, n)$, where P is defined by $P = \sum_{k=1}^{m-1} (E_{2k-1,2k+1} + E_{2k,2k+2})$. Then the matrix form of G has the following linear part:

$$\begin{aligned}\varphi(t, \mathbf{x}) &= \exp(\varphi_*(t, \mathbf{x})) \\ &= \sum_{k \le h} f_{kh}(x_1, \cdots, x_n)(e^t E_{2k-1,2h-1} + e^{-t} E_{2k,2h}) ,\end{aligned}$$

where $t \in \mathbf{R}, \mathbf{x} = (x_1, \cdots, x_n) \in \mathbf{R}^n$ and $f_{kh}(x_1, \cdots, x_n)$ are the following polynomials:

$$f_{kh}(x_1, \cdots, x_n) = \sum_{\substack{k_1 + 3k_2 + \cdots + (2n-1)k_n = h-k \\ k_1, \cdots, k_n > 0}} \frac{1}{k_1! \cdots k_n!} {x_1}^{k_1} \cdots {x_n}^{k_n} .$$

Thus, for each n, we can construct a compact solvmanifold by Proposition 1. We show that M has a symplectic structure and is a compact Lefschetz solvmanifold without the Hard Lefschetz property.

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1. Definitions and duality on $\mathcal{H}^*(M)$.

Let (M^{2m}, ω) be a symplectic manifold and $\Omega^*(M)$ the space of differential forms on M. We define $L_\omega = L : \Omega^k(M) \to \Omega^{k+2}(M)$ by $L(\alpha) = \alpha \wedge \omega$. Since ω is a closed, $Ld = dL$. Hence, L induces a linear mapping $L : H^k_{DR}(M) \to H^{k+2}_{DR}(M)$, $L[\alpha] = [L(\alpha)]$.

DEFINITION 1.1. Let (M^{2m}, ω) be a compact symplectic manifold. If the Lefschetz mapping $L^{m-1} : H^1_{DR}(M) \to H^{2m-1}_{DR}(M)$ is an isomorphism, then (M^{2m}, ω) is called a Lefschetz manifold. Moreover, for each $k \le m$, the Lefschetz mapping $L^k : H^{m-k}_{DR}(M) \to H^{m+k}_{DR}(M)$ is an isomorphism, we say that (M, ω) has the Hard Lefschetz property.

REMARK. Benson and Gordon have proved that a non-toral compact nilmanifold is not a Lefschetz manifold for any symplectic structure to show that a non-toral compact nilmanifold does not admit any Kähler structure.

Moreover, we define a star operator

$$* : \Omega^k(M) \to \Omega^{2m-k}(M) \quad \text{for } k = 0, \cdots, 2m$$

by requiring

$$\beta \wedge *\alpha = \bigwedge^k(\mathbf{G})(\beta, \alpha) v_M \quad \text{for } \beta, \alpha \in \Omega^k(M)\,,$$

where $v_M = \omega^m/m!$ and $\mathbf{G}$ is the skew-symmetric bivector field dual to ω. We also define $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ to be $d^* = (-1)^k * d *$.

DEFINITION 1.2. For a symplectic manifold (M, ω), a k-form $\alpha \in \Omega^k(M)$ is called ω-harmonic or simply, harmonic, if it satisfies

$$d^*\alpha = d\alpha = 0\,.$$

Let $\mathcal{H}^k_\omega(M) = \mathcal{H}^k(M)$ denotes the space of all harmonic k-forms on M. We define symplectic harmonic k-cohomology group $H^k_{\omega\text{-}hr}(M) = H^k_{hr}(M) = \mathcal{H}^k(M)/(B^k(M) \cap \mathcal{H}^k(M))$. Mathieu proved the following:

MATHIEU'S THEOREM. *Let (M^{2m}, ω) be a symplectic manifold of dimension $2m$. Then the following two assertions are equivalent*:

(a) *For any k, $L^k : H^{m-k}_{DR}(M) \to H^{m+k}_{DR}(M)$ is surjective.*
(b) *For any k, $H^k_{DR}(M) = H^k_{hr}(M)$.*

Using the following propositions, Yan [Yn] gave a simpler, more direct, proof of Mathieu's Theorem.

PROPOSITION 1.3 (Duality on forms ([Yn])).

$$L^k : \Omega^{m-k}(M) \to \Omega^{m+k}(M)$$

is an isomorphism.

PROPOSITION 1.4 (Duality on harmonic forms ([Yn])).

$$L^k : \mathcal{H}^{m-k}(M) \to \mathcal{H}^{m+k}(M)$$

is an isomorphism.

PROPOSITION 1.5 ([Yn]). *Put* $P^{m-k}(M) = \{v \in H_{DR}^{m-k}(M) \mid L^{k+1}v = 0\}$. *Then*

$$P^{m-k}(M) \subset H_{hr}^{m-k}(M)\,.$$

In particular, Yan proved the following:

THEOREM 1.6. *Let* (M^{2m}, ω) *be a symplectic manifold. If* $H_{DR}^{m-k}(M) = H_{hr}^{m-k}(M)$ *and* $L^k : H_{DR}^{m-k}(M) \to H_{DR}^{m+k}(M)$ *is surjective, then* $H_{DR}^{m+k}(M) = H_{hr}^{m+k}(M)$ *and* $H_{DR}^{m-k+2}(M) = H_{hr}^{m-k+2}(M)$.

PROOF. Let $\alpha \in H_{DR}^{m-k+2}(M)$. Since $L^{k-1}\alpha \in H_{DR}^{m+k}(M)$, there exists a $\beta \in H_{DR}^{m-k}(M) = H_{hr}^{m-k}(M)$ such that

$$L^{k-1}\alpha = L^k\beta\,.$$

Then we have $L^{k-1}(\alpha - \beta \wedge \omega) = 0$, which implies that $\alpha - \beta \wedge \omega \in P^{m-k+2}(M)$. Since $\alpha = (\alpha - \beta \wedge \omega) + \beta \wedge \omega$, we have $H_{DR}^{m-k+2}(M) = H_{hr}^{m-k+2}(M)$ by Proposition 1.5. Using Proposition 1.4, we have $H_{DR}^{m+k}(M) = H_{hr}^{m+k}(M)$. □

COROLLARY 1.7. *If* (M, ω) *is a compact Lefschetz manifold, then we have* $H_{DR}^3(M) = H_{hr}^3(M)$.

2. Harmonic cohomology groups on G/Γ.

Now we consider the case of compact symplectic solvmanifolds. Let $\mathfrak{g}$ be a Lie algebra and put $\mathfrak{g}_0 = \mathfrak{g}$ and let $\mathfrak{g}_{i+1} = [\mathfrak{g}_i, \mathfrak{g}_i]$. A Lie algebra $\mathfrak{g}$ is called $(r+1)$-step solvable if $\mathfrak{g}_r \neq 0$, $\mathfrak{g}_{r+1} = 0$. A Lie group G is called solvable if the Lie algebra $\mathfrak{g}$ is solvable. If G is a simply-connected solvable Lie group and Γ is a lattice of G, that is, a discrete subgroup of G such that G/Γ is compact, then we say that G/Γ is a compact solvmanifold.

DEFINITION 2.1. A solvable Lie algebra $\mathfrak{g}$ is called completely solvable if $\mathrm{ad(X)} : \mathfrak{g} \to \mathfrak{g}$ has only real eigenvalues for each $X \in \mathfrak{g}$. A solvable Lie group G is called completely solvable if its Lie algebra is completely solvable.

Hattori [H] proved that the Chevalley-Eilenberg cohomology of completely solvable Lie algebra $H^*(\mathfrak{g})$ is isomorphic to the de Rham cohomology $H_{DR}^*(G/\Gamma)$, where G is the simply-connected Lie group corresponding to $\mathfrak{g}$ and Γ is a lattice of G.

For a left-G-invariant symplectic form ω on a compact solvmanifold G/Γ, we denote by $\mathcal{H}^k(\mathfrak{g})$ the space of all left-G-invariant harmonic forms on G/Γ.

PROPOSITION 2.2. *Let* (M^{2m}, ω) *be a compact solvmanifold such that* $\omega \in \bigwedge^2(\mathfrak{g}^*)$. *Then*

$$L^k : \mathcal{H}^{m-k}(\mathfrak{g}) \to \mathcal{H}^{m+k}(\mathfrak{g})$$

is an isomorphism.

PROOF. See [Ym]. □

For a left-G-invariant symplectic form ω, let $H^k_{hr}(\mathfrak{g}) = \mathcal{H}^k(\mathfrak{g})/(B^k(\mathfrak{g}) \cap \mathcal{H}^k(\mathfrak{g}))$ be a subspace of Lie algebra cohomology group $H^k(\mathfrak{g})$. Let $(M = G/\Gamma, \omega)$ be a compact symplectic completely solvable solvmanifold. By Nomizu and Hattori's theorem, there exists a left-G-invariant closed 2-form ω_0 such that $\omega - \omega_0 = d\gamma$. Note that ω_0 is also a symplectic form on M.

PROPOSITION 2.3. *Let $(M = G/\Gamma, \omega)$ be a compact symplectic completely solvable solvmanifold. Then for any q we have*

$$H^q_{\omega\text{-}hr}(M) = H^q_{\omega_0\text{-}hr}(M) = H^q_{\omega_0\text{-}hr}(\mathfrak{g}),$$

where ω_0 is a left-G-invariant closed 2-form which is cohomologous to ω.

PROOF. We apply Nomizu and Hattori's theorem (See [H] and [Ym]). □

By Proposition 2.3, we may assume that symplectic structures on $M^{2m} = G/\Gamma$ are left-G-invariant to study harmonic cohomology groups on a compact completely solvable solvmanifold M.

Let $\mathfrak{g}$ be a completely solvable Lie algebra and $\mathfrak{n}$ be the derived algebra : $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$. Let $\mathfrak{a}$ denote a vector space complement of $\mathfrak{n}$ in $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ and $\dim \mathfrak{a} = k$, $\dim \mathfrak{n} = l$.

For simplicity, we denote $\bigwedge^i \mathfrak{a}^* \wedge \bigwedge^j \mathfrak{n}^*$ by $\bigwedge^{i,j}$.

LEMMA 2.4 ([BG2]).

$$B^{2m-1}(\mathfrak{g}) = \bigwedge^{k,l-1}$$

PROOF. See [BG2]. □

PROPOSITION 2.5 (cf. [BG2]).

$$\dim H^1_{hr}(\mathfrak{g}) - \dim H^{2m-1}_{hr}(\mathfrak{g}) = \dim \left\{ X \in [\mathfrak{g}, \mathfrak{g}] \,\middle|\, i(X)\omega \in \bigwedge^{1,0} \right\}.$$

PROOF. Since $L^{m-1} : \mathcal{H}^1(\mathfrak{g}) \to \mathcal{H}^{2m-1}(\mathfrak{g})$ is an isomorphism, we get

$$\begin{aligned}
&\dim H^1_{hr}(\mathfrak{g}) - \dim H^{2m-1}_{hr}(\mathfrak{g}) \\
&\quad = \dim \mathcal{H}^1(\mathfrak{g}) - \dim(B^1(\mathfrak{g}) \cap \mathcal{H}^1(\mathfrak{g})) \\
&\qquad - \dim \mathcal{H}^{2m-1}(\mathfrak{g}) + \dim(B^{2m-1}(\mathfrak{g}) \cap \mathcal{H}^{2m-1}(\mathfrak{g})) \\
&\quad = \dim(B^{2m-1}(\mathfrak{g}) \cap \mathcal{H}^{2m-1}(\mathfrak{g})) - \dim(B^1(\mathfrak{g}) \cap \mathcal{H}^1(\mathfrak{g})) \\
&\quad = \dim(B^{2m-1}(\mathfrak{g}) \cap \mathcal{H}^{2m-1}(\mathfrak{g})) \\
&\quad = \dim(B^{2m-1}(\mathfrak{g}) \cap L^{m-1}(\mathcal{H}^1(\mathfrak{g}))).
\end{aligned}$$

Therefore, let $\beta \in Z^1(\mathfrak{g}) = \mathcal{H}^1(\mathfrak{g}) = \bigwedge^{1,0}$. Since ω is a nondegenerate closed 2-form, we can write $\beta = i(X)\omega$. Moreover,

$$\begin{aligned}
L^{m-1}(\beta) &= i(X)\omega \wedge \omega^{m-1} \\
&= \frac{1}{m} i(X)\omega^m.
\end{aligned}$$

Thus by Lemma 2.4, we see $L^{m-1}(\beta)$ is exact if and only if $X \in [\mathfrak{g}, \mathfrak{g}]$. $\square$

3. A construction of completely solvable Lie groups which admit a lattice.

In this section, we construct completely solvable Lie groups which admit lattices.

Let G_6 be the simply-connected completely solvable Lie group defined by

$$G_6 = \left\{ \left. \begin{pmatrix} e^t & 0 & xe^t & 0 & 0 & 0 & y_1 \\ 0 & e^{-t} & 0 & xe^{-t} & 0 & 0 & y_2 \\ 0 & 0 & e^t & 0 & 0 & 0 & y_3 \\ 0 & 0 & 0 & e^{-t} & 0 & 0 & y_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| t, x, y_1, y_2, y_3, y_4 \in \mathbf{R} \right\} .$$

The authors of [FLS] have proved that G_6 admits a lattice and has symplectic structures. Furthermore, G_6 may be described as the semi-direct product $G_6 = \mathbf{R}^2 \ltimes_\varphi \mathbf{R}^4$, where $\varphi(t, x)$ is the automorphism of $\mathbf{R}^4$ given by the matrix

$$\varphi(t, x) = \begin{pmatrix} e^t & 0 & xe^t & 0 \\ 0 & e^{-t} & 0 & xe^{-t} \\ 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix} .$$

Thus we consider $\varphi : \mathbf{R}^{n+l} \to \mathrm{Aut}(\mathbf{R}^{2m})$ and simply-connected completely solvable Lie groups $G = \mathbf{R}^{n+l} \ltimes_\varphi \mathbf{R}^{2m}$ of which matrix form are given by

$$G = \left\{ \left. \begin{pmatrix} & & & 0 & 0 & y_1 \\ & \varphi(t,x) & & \vdots & \vdots & \vdots \\ & & & 0 & 0 & y_{2m} \\ 0 & \cdots & 0 & I_n & 0 & x \\ 0 & \cdots & 0 & 0 & I_l & t \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix} \right| t \in \mathbf{R}^l, x \in \mathbf{R}^n, y_1, \cdots y_{2m} \in \mathbf{R} \right\} ,$$

where I_l, I_n are unit matrices. Let $B \in SL(2, \mathbf{Z})$ be a unimodular matrix with distinct real eigenvalues, say, $\lambda, 1/\lambda$. Take $t_0 = log\lambda$, i.e., $e^{t_0} = \lambda$. Then there exists a matrix $P \in GL(2, \mathbf{R})$ such that

$$PBP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} .$$

Let $\mathbf{R}$ act on $\mathbf{R}^2$ by

$$t \mapsto \varphi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} .$$

Thus if $t_0 n \in t_0\mathbf{Z}$, then $\left\{ P \begin{pmatrix} \mu \\ \nu \end{pmatrix} \middle| \mu, \nu \in \mathbf{Z} \right\}$ is invariant by above action.

Hence $\Gamma = (t_0\mathbf{Z} \times \mathbf{Z}) \ltimes_\varphi \left(\left\{ P\begin{pmatrix}\mu_1\\ \nu_1\end{pmatrix} \middle| \mu_1, \nu_1 \in \mathbf{Z} \right\} \times \left\{ P\begin{pmatrix}\mu_2\\ \nu_2\end{pmatrix} \middle| \mu_2, \nu_2 \in \mathbf{Z} \right\} \right)$ is a lattice of G_6.

Moreover we have

PROPOSITION 3.1. *Let A_i, B_j be the matrices given by*

$$A_i = \sum_{k=1}^{m} a_i^k (E_{2k-1,2k-1} - E_{2k,2k}) \quad i = 1, \cdots, l\,,$$
$$B_j = \sum_{k<h} b_j^{kh} (E_{2k-1,2h-1} + E_{2k,2h}) \quad j = 1, \cdots, n$$

where $a_i^k, b_j^{kh} \in \mathbf{Q}$ and assume that they satisfy that $[A_i, B_j] = [B_i, B_j] = 0$. We define a map

$$\varphi_* : \mathbf{R}^{n+l} = \mathrm{span}\{T_1, \cdots, T_l, X_1, \cdots, X_n\} \to End(\mathbf{R}^{2m} = \mathrm{span}\{Y_1, \cdots, Y_{2m}\})$$

by

$$\varphi_*\left(\sum_{i=1}^{l} t_i T_i + \sum_{j=1}^{n} x_j X_j \right) = \sum_{i=1}^{l} t_i A_i + \sum_{j=1}^{n} x_j B_j\,.$$

Then $G = \mathbf{R}^{n+l} \ltimes_\varphi \mathbf{R}^{2m}$ is a completely solvable Lie group which has a lattice.

PROOF. We construct a co-compact lattice of $G = \mathbf{R}^{n+l} \ltimes_\varphi \mathbf{R}^{2m}$. Let take

$$L_1 = \underbrace{a\,t_0\mathbf{Z} \times \cdots \times a\,t_0\mathbf{Z}}_{l \text{ times}} \times \underbrace{a^{m-1}(m-1)!\mathbf{Z} \times \cdots \times a^{m-1}(m-1)!\mathbf{Z}}_{n \text{ times}}\,,$$
$$L_2 = \left\{ P\begin{pmatrix}\mu_1\\ \nu_1\end{pmatrix} \middle| \mu_1, \nu_1 \in \mathbf{Z} \right\} \times \cdots \times \left\{ P\begin{pmatrix}\mu_m\\ \nu_m\end{pmatrix} \middle| \mu_m, \nu_m \in \mathbf{Z} \right\}\,,$$

where a is the least common multiple for denominators of a_i^k, b_j^{kh}. Since L_2 is invariant by $\varphi(t,x)$, $(t,x) \in L_1$. Then we see $\Gamma = L_1 \ltimes_\varphi L_2$ is a subgroup of $\mathbf{R}^{n+l} \ltimes_\varphi \mathbf{R}^{2m}$. It is obvious that the subgroup Γ is discrete and co-compact. □

REMARKS. (i) $\mathbf{R}^{n+l} \ltimes \mathbf{R}^{2m}$ is a generalization of 3-dimensional completely solvable Lie group G_3 which admits a lattice, where

$$G_3 = \left\{ \begin{pmatrix} e^t & 0 & 0 & y_1 \\ 0 & e^{-t} & 0 & y_2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; y_1, y_2, t \in \mathbf{R} \right\}.$$

(ii) More generally, if $\mathfrak{b}_0 = \mathrm{span}_{\mathbf{Q}}\{B_1, \cdots, B_n\}$ is a nilpotent Lie algebra over $\mathbf{Q}$, then we have that $\mathbf{R}^{n+l} \ltimes_\varphi \mathbf{R}^{2m}$ admits a lattice (cf. Raghunathan [R, Theorem 2.12 of Chapter II]).

(iii) Put $\mathfrak{a} = \mathrm{span}\{T_1, \cdots, T_l\}$, $\mathfrak{n} = \mathrm{span}\{X_1, \cdots, X_n, Y_1, \cdots, Y_{2m}\}$. Since $\mathfrak{n}$ is a nilpotent Lie algebra, $\mathfrak{a} \ltimes \mathfrak{n}$ is an l-dimensional extension of the nilpotent Lie algebra $\mathfrak{n}$ of dimension $2m + n$.

(iv) We can consider that $\mathbf{R}^{n+l} \ltimes_\varphi \mathbf{R}^{2m} / L_1 \ltimes_\varphi L_2$ is a T^{2m}-bundle over T^{n+l}.

If there exists no possibility of confusion, then we denote $L_1 \ltimes_\varphi L_2$ by $\mathbf{Z}^{n+l} \ltimes_\varphi \mathbf{Z}^{2m}$.

For $\varphi \in \mathrm{Hom}(\mathbf{R}^2, \mathrm{Aut}(\mathbf{R}^{2m_1}))$ and $\phi \in \mathrm{Hom}(\mathbf{R}^2, \mathrm{Aut}(\mathbf{R}^{2m_2}))$, we define $\varphi \oplus \phi : \mathbf{R}^2 \to Aut(\mathbf{R}^{2(m_1+m_2)})$ by

$$(\varphi \oplus \phi)(t, x)(y_1, y_2) = ((\varphi(t, x))(y_1), (\phi(t, x))(y_2)) \quad \text{for } \; y_i \in \mathbf{R}^{2m_i} .$$

Let consider $\mathbf{R}^{n+1} \ltimes_\varphi \mathbf{R}^{2m}$ as the matrices of the form

$$A = \begin{pmatrix} & & & 0 & \cdots & \cdots & 0 & y_1 \\ & \varphi(t, x_1, \cdots, x_n) & & \vdots & & & \vdots & \vdots \\ & & & 0 & \cdots & \cdots & 0 & y_{2m} \\ 0\cdots & \cdots & 0 & 1 & 0 & \cdots & 0 & x_1 \\ 0\cdots & \cdots & 0 & 0 & \ddots & & \vdots & \vdots \\ 0\cdots & \cdots & 0 & \vdots & & \ddots & 0 & x_n \\ 0\cdots & \cdots & 0 & 0 & \cdots & 0 & 1 & t \\ 0\cdots & \cdots & 0 & 0 & & \cdots & 0 & 1 \end{pmatrix} .$$

Then, a global system of coordinates $\{t, x_1, \cdots, x_n, y_1, \cdots, y_{2m}\}$ for $\mathbf{R}^{n+1} \ltimes_\varphi \mathbf{R}^{2m}$ is given by

$$t(A) = t , \quad x_i(A) = x_i, \quad y_i(A) = y_i .$$

Moreover, we denote by $\alpha, \beta_i, \omega_i$ the left invariant 1-forms on $\mathbf{R}^{n+1} \ltimes_\varphi \mathbf{R}^{2m}$ such that

$$\alpha_e = (dt)_e , \quad (\beta_i)_e = (dx_i)_e , \quad (\omega_i)_e = (dy_i)_e .$$

Similarly, we define a global system coordinates for $\mathbf{R}^{n+l} \ltimes_\varphi \mathbf{R}^{2m}$.

Now we consider an other generalization of 3-dimensional completely solvable Lie group G_3 which admits a lattice. Let $J(\lambda, m, t)$ be an $(m \times m)$-matrix as follows.

$$J(\lambda, m, t) = \begin{pmatrix} \lambda^t & t\lambda^{t-1} & & \Large 0 \\ & \ddots & \ddots & \\ & & \ddots & t\lambda^{t-1} \\ \Large 0 & & & \lambda^t \end{pmatrix} ,$$

and $J(\lambda, m) = J(\lambda, m, 1)$. Let A_1, A_2 be $(m_i \times m_i)$-matrices $(i = 1, 2)$. We define an $(m_1 + m_2 \times m_1 + m_2)$-matrix $A_1 \oplus A_2$ by

$$A_1 \oplus A_2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} .$$

Let $B \in SL(n, \mathbf{Z})$ be a unimodular matrix with distinct positive eigenvalues, $\lambda_1, \cdots, \lambda_l$ $(l \leq n)$. Then there exists a matrix $P \in GL(n, \mathbf{R})$ such that $PBP^{-1} = J = J(\lambda_1, m_1^1) \oplus \cdots \oplus J(\lambda_1, m_{n_1}^1) \oplus \cdots \oplus J(\lambda_l, m_1^l) \oplus \cdots \oplus J(\lambda_l, m_{n_l}^l)$. (Note that J is a Jordan normal form similar to B.) Moreover let $\mathbf{R}$ act on $\mathbf{R}^n$ by

$$t \mapsto \varphi(t) = J(t) = J(\lambda_1, m_1^1, t) \oplus \cdots \oplus J(\lambda_l, m_{n_l}^l, t) .$$

Then $\left\{ P \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix} \middle| \ \mu_1, \cdots, \mu_m \in \mathbf{Z} \right\}$ is invariant by $\varphi(n)$ for $n \in \mathbf{Z}$.

Thus we now have a completely solvable Lie group

$$G = \mathbf{R}^1 \ltimes_\varphi \mathbf{R}^n = \left\{ \begin{pmatrix} & & & 0 & y_1 \\ & J(t) & & \vdots & \vdots \\ & & & 0 & y_n \\ 0 & \cdots & 0 & 1 & t \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \middle| \ t, y_1, \cdots, y_n \in \mathbf{R} \right\}$$

which admits a lattice Γ, where

$$\Gamma = \mathbf{Z} \ltimes_\varphi \left\{ P \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix} \middle| \ \mu_1, \cdots, \mu_m \in \mathbf{Z} \right\}.$$

Similarly to Proposition 3.1, we can generalize this Lie group (See Example 5.3).

4. Main results.

In this section, we use the same notations introduced in section 3.

From now on we consider a solvable Lie group $\mathbf{R}^{n+l} \ltimes \mathbf{R}^{2m}$ constructed in Proposition 3.1. Moreover, we assume for each k, there exists an i such that $a_i^k \neq 0$ and if $B_j = 0$ for each j, then we call $\mathbf{R}^{l+n} \ltimes_\varphi \mathbf{R}^{2m}$ is A-type.

For simplicity, let $\omega_K = \omega_{k_1} \wedge \cdots \wedge \omega_{k_p}$ for $K = (k_1, \cdots, k_p)$. Note that $d\omega_K$ can be written as

$$d\omega_K = -\sum_{i=1}^{l} a_i^K \alpha_i \wedge \omega_K - \sum_{j=1}^{n} \sum_H b_j^{KH} \beta_j \wedge \omega_H,$$

where $a_i^K, b_j^{KH} \in \mathbf{R}$. By a straightforward computation, we see

$$\begin{aligned} 0 = dd\omega_K &= +\sum_{i=1}^{l} a_i^K \alpha_i \wedge d\omega_K + \sum_{j=1}^{n} \sum_H b_j^{KH} \beta_j \wedge d\omega_H \\ &= -\sum_{i_1, i_2=1}^{l} a_{i_1}^K a_{i_2}^K \alpha_{i_1} \wedge \alpha_{i_2} \wedge \omega_K - \sum_{i=1}^{l} \sum_{j=1}^{n} \sum_H a_i^K b_j^{KH} \alpha_i \wedge \beta_j \wedge \omega_H \\ &\quad + \sum_{i=1}^{l} \sum_{j=1}^{n} \sum_H a_i^H b_j^{KH} \alpha_i \wedge \beta_j \wedge \omega_H - \sum_{j_1, j_2=1}^{n} \sum_H \sum_L b_{j_1}^{KH} b_{j_2}^{HL} \beta_{j_1} \wedge \beta_{j_2} \wedge \omega_L. \end{aligned}$$

Hence,

$$a_i^K b_j^{KH} - a_i^H b_j^{KH} = 0. \tag{4.1}$$

Therefore, if $a_i^K \neq 0$ and $b_j^{KH} \neq 0$, then $a_i^H \neq 0$.

LEMMA 4.1. *Let $\gamma = \sum_{IJK} c_{IJK}\alpha_I \wedge \beta_J \wedge \omega_K$ be a closed form such that $\sharp I + \sharp J =$ const, $\sharp K =$ const. If, for each K such that $c_{IJK} \neq 0$, there exists an i such that $a_i^K \neq 0$, then γ is an exact form.*

PROOF. Let $p = \max\{\sharp I \mid c_{IJK} \neq 0\}$. It suffices to show that γ can be written as follows:

$$\gamma = d\theta + \sum_{\sharp I<p} \sum_J \sum_K c'_{IJK}\alpha_I \wedge \beta_J \wedge \omega_K \,,$$

where $\gamma' = \sum_{\sharp I<p} \sum_J \sum_K c'_{IJK}\alpha_I \wedge \beta_J \wedge \omega_K$ admits the condition in Lemma 4.1. Without loss of generality, we may assume that $a_1^{K_0} \neq 0$ for some K_0. Then we have

$$\alpha_1 \wedge \omega_{K_0} = -1/a_1^{K_0}\left(d\omega_{K_0} + \sum_{i=2}^{l} a_i^{K_0}\alpha_i \wedge \omega_{K_0} + \sum_{j=1}^{n}\sum_H b_j^{K_0 H}\beta_j \wedge \omega_H\right). \tag{4.2}$$

Let $\zeta = \sum_{\sharp I<p} \sum_J c_{IJK_0}\alpha_I \wedge \beta_J \wedge \omega_{K_0} + \sum_I \sum_J \sum_{K\neq K_0} c_{IJK}\alpha_I \wedge \beta_J \wedge \omega_K$. Using the equation (4.2), we see

$$\begin{aligned}
\gamma &= \sum_{\sharp I=p} \sum_J c_{IJK_0}\alpha_I \wedge \beta_J \wedge \omega_{K_0} + \zeta \\
&= -\,d\left(\sum_{1\in I,\sharp I=p} \sum_J \frac{c_{IJK_0}}{a_1^{K_0}}\alpha_{I/\{1\}} \wedge \beta_J \wedge \omega_{K_0}\right) \\
&\quad + (-1)^{p+\sharp J} \sum_{1\in I,\sharp I=p} \sum_J \sum_H \sum_{j=1}^{n} \frac{c_{IJK_0}}{a_1^{K_0}} b_j^{K_0 H}\alpha_{I/\{1\}} \wedge \beta_J \wedge \beta_j \wedge \omega_H \\
&\quad + \sum_{1\notin I,\sharp I=p} \sum_J \tilde{c}_{IJK_0}\alpha_I \wedge \beta_J \wedge \omega_{K_0} + \zeta \,,
\end{aligned}$$

where

$$\begin{aligned}
\sum_{1\in I,\sharp I=p} \sum_J \tilde{c}_{IJK_0}\alpha_I \wedge \beta_J \wedge \omega_{K_0} = (-1)^p &\sum_{1\in I,\sharp I=p} \sum_J \sum_{i=2}^{l} \frac{c_{IJK_0}}{a_1^{K_0}}\alpha_{I/\{1\}} \wedge \alpha_i \wedge \beta_J \wedge \omega_{K_0} \\
&+ \sum_{1\notin I,\sharp I=p} \sum_J c_{IJK_0}\alpha_I \wedge \beta_J \wedge \omega_{K_0} \,,
\end{aligned}$$

and $\alpha_{I/\{1\}} = \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_{p-1}}$ for $I = (1, i_1, \cdots, i_{p-1})$.

Since $d\gamma = 0$ and $\{\alpha_I \wedge \beta_J \wedge \omega_{K_0}\}_{\sharp I=p,1\in I}$ are not components of decomposition of $d\zeta$, we have

$$a_1^{K_0}\tilde{c}_{IJK_0} = 0 \,.$$

Thus we obtain

$$\tilde{c}_{IJK_0} = 0 \,.$$

By the same argument, we have

$$\gamma = d\theta + \gamma' \,,$$

where $\gamma' = \sum_{\sharp I<p} \sum_J \sum_K c'_{IJK}\alpha_I \wedge \beta_J \wedge \omega_K$. By the equation (4.1), we see that γ' admits the condition in Lemma 4.1. $\square$

By Lemma 4.1, we have

LEMMA 4.2. (1) *If* $\alpha = \alpha_{2,0} + \alpha_{1,1} + \alpha_{0,2} \in Z^2(\mathfrak{g})$, *where* $\alpha_{i,j} \in \bigwedge^{i,j}$, *then* $d\alpha_{2,0} = d\alpha_{1,1} = d\alpha_{0,2} = 0$.
(2) $\bigwedge^{1,1} \cap Z^2(\mathfrak{g}) \subset B^2(\mathfrak{g})$.

PROOF. Since

$$d\omega_{2k-1} = -\sum_i a_i^k \alpha_i \wedge \omega_{2k-1} - \sum_{k<h,j} b_j^{kh} \beta_j \wedge \omega_{2h-1}\,,$$
$$d\omega_{2k} = \sum_i a_i^k \alpha_i \wedge \omega_{2k} - \sum_{k<h,j} b_j^{kh} \beta_j \wedge \omega_{2h}\,,$$

we have

$$\bigwedge^{0,2} \xrightarrow{d} \bigwedge^{1,2},$$
$$\bigwedge^{2,0} \xrightarrow{d} 0\,,$$
$$\bigwedge^{1,1} \xrightarrow{d} \bigwedge^{2,1}.$$

Since we assume that, for each k, there exists an i such that $a_i^k \neq 0$, we have Lemma 4.2. $\square$

THEOREM 4.3. *Let* $(G/\Gamma, \omega)$ *be a compact symplectic solvmanifold constructed as in Proposition* 3.1. *Then* $(G/\Gamma, \omega)$ *is a compact Lefschetz manifold, which implies* $H^3_{DR}(M) = H^3_{hr}(M)$ *for each symplectic form.*

PROOF. By Lemma 4.2, we can assume that $\omega \in \bigwedge^{2,0} + \bigwedge^{0,2}$. Then we have Theorem 4.3 by Proposition 2.5. $\square$

PROPOSITION 4.4. *Let* G/Γ *be a compact A-type solvmanifold with a symplectic structure* ω. *Then* $(G/\Gamma, \omega)$ *has the Hard Lefschetz property.*

PROOF. Since $d\omega_K = -\sum_{i=1}^l a_i^K \alpha_i \wedge \omega_K$, if $\sum_{\sharp I+\sharp K=p+q=r} c_{IK}\alpha_I \wedge \omega_K$ is a closed form, then $\sum_{\sharp I=p} c_{IK}\alpha_I \wedge \omega_K$ is also a closed form. Moreover, it is obvious that if $d\omega_K = 0$, then $\sum_{\sharp I=p} c_{IK}\alpha_I \wedge \omega_K$ is a non-exact closed form. By Lemma 4.1, if $d\omega_K \neq 0$, then a closed form $\sum_{\sharp I=p} c_{IK}\alpha_I \wedge \omega_K$ is exact. Then for each de Rham cohomology class, we can choose a representation $\alpha = \sum_{IK} c_{IK}\alpha_I \wedge \omega_K$ such that $d\omega_K = 0$.

In the other hand, we can assume that a symplectic form ω can be written as

$$\omega = \omega_{2,0} + \sum_{kh} P_{kh}\omega_k \wedge \omega_h\,,$$

where $\omega_{2,0} \in \bigwedge^{2,0}$. Note that $\omega_k \wedge \omega_h$ is closed for each $P_{kh} \neq 0$. Then we have

$$L^k\alpha = \sum_{I'K'} c_{I'K'}\alpha_{I'} \wedge \omega_{K'} \quad d\omega_{K'} = 0\,,$$

which implies $L^k\alpha$ is not exact by the above argument. Then A-type has the Hard Lefschetz property. $\square$

5. Examples.

EXAMPLE 5.1. We consider the following matrices:

$$A = \sum_{k=1}^{m}(E_{2k-1,2k-1} - E_{2k,2k})$$
$$B = \sum_{k=1}^{m-1}((m-k)E_{2k-1,2k+1} + (m-k)E_{2k,2k+2}) .$$

We denote by $\varphi(t,x,m)$ the automorphism of $\mathbf{R}^{2m}$ induced by A and B. For example, if $m=4$, then $\varphi(t,x,4)$ can be written as follows.

$$\varphi(t,x,4) = \begin{pmatrix} e^t & 0 & 3xe^t & 0 & 3x^2e^t & 0 & x^3e^t & 0 \\ 0 & e^{-t} & 0 & 3xe^{-t} & 0 & 3x^2e^{-t} & 0 & x^3e^{-t} \\ 0 & 0 & e^t & 0 & 2xe^t & 0 & x^2e^t & 0 \\ 0 & 0 & 0 & e^{-t} & 0 & 2xe^{-t} & 0 & x^2e^{-t} \\ 0 & 0 & 0 & 0 & e^t & 0 & xe^t & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-t} & 0 & xe^{-t} \\ 0 & 0 & 0 & 0 & 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-t} \end{pmatrix} .$$

By a straightforward computation, we see

$$\begin{aligned} d\omega_{2k-1} &= -\alpha\wedge\omega_{2k-1} - (m-k)\beta\wedge\omega_{2k+1}(k=1,\cdots,m-1) \\ d\omega_{2k} &= \alpha\wedge\omega_{2k} - (m-k)\beta\wedge\omega_{2k+2}(k=1,\cdots,m-1) \\ d\omega_{2m-1} &= -\alpha\wedge\omega_{2m-1} \\ d\omega_{2m} &= \alpha\wedge\omega_{2m} . \end{aligned}$$

Then $\mathbf{R}^2 \ltimes_\varphi \mathbf{R}^{2m}/\mathbf{Z}^2 \ltimes_\varphi \mathbf{Z}^{2m}$ has the following non-degenerate closed 2-form:

$$\omega = \alpha\wedge\beta + \sum_{k=0}^{m-1}\frac{(-1)^k}{k!}\frac{(m-1)!}{(m-k-1)!}\omega_{2k+1}\wedge\omega_{2m-2k} .$$

For example, if $m=4$, then

$$\omega = \alpha\wedge\beta + \frac{1}{3}\omega_1\wedge\omega_8 - \omega_3\wedge\omega_6 + \omega_5\wedge\omega_4 - \frac{1}{3}\omega_7\wedge\omega_2$$

is a symplectic structure of $\mathbf{R}^2 \ltimes_\varphi \mathbf{R}^8$. By Theorem 4.3, we see $\mathbf{R}^2 \ltimes_\varphi \mathbf{R}^{2m}/\mathbf{Z}^2 \ltimes_\varphi \mathbf{Z}^{2m}$ is a compact Lefschetz manifold.

REMARKS. (i) Since $[\alpha\wedge\beta\wedge\omega_1\wedge\cdots\wedge\omega_{2m-2}] \notin L_\omega^{m-2}(H^2(\mathfrak{g}))$ for any symplectic form ω (See proof of Theorem 6.7), we have $\mathbf{R}^2 \ltimes_\varphi \mathbf{R}^{2m}/\mathbf{Z}^2 \ltimes_\varphi \mathbf{Z}^{2m}$ does not have the Hard Lefschetz property.

(ii) $\mathbf{R}^2 \ltimes_\varphi \mathbf{R}^{2m}$ is a 1-dimensional extension of the $(m+1)$-step nilpotent Lie algebra $\mathfrak{n} = \mathrm{span}\{X, Y_1, \cdots Y_{2m}\}$.

EXAMPLE 5.2. Let

$$\varphi(t,x) = \varphi(t,x,m_1) \oplus \cdots \oplus \varphi(t,x,m_{s'}),$$

where $m_1 = \cdots = m_s > m_{s+1} \geq \cdots \geq m_{s'}, m_1 + \cdots + m_{s'} = m$. By Example 5.1, $M = \mathbf{R}^2 \ltimes_\varphi \mathbf{R}^{2m}/\mathbf{Z}^2 \ltimes_\varphi \mathbf{Z}^{2m}$ has symplectic structures. Moreover, we obtain that $M = \mathbf{R}^2 \ltimes_\varphi \mathbf{R}^{2m}/\mathbf{Z}^2 \ltimes_\varphi \mathbf{Z}^{2m}$ doesn't have the Hard Lefschetz property for any symplectic form.

In fact,

$$[\alpha \wedge \beta \wedge \omega_1 \wedge \cdots \wedge \hat{\omega}_{2k_1-1} \wedge \hat{\omega}_{2k_1} \wedge \cdots \wedge \hat{\omega}_{2k_s-1} \wedge \hat{\omega}_{2k_s} \wedge \cdots \wedge \omega_{2m}] \in H^{2(m+1-s)}(\mathfrak{g})$$

isn't contained in $L^{(m+1)-2s}(H^{2s}(\mathfrak{g}))$.

Moreover, consider $\mathbf{R}^{2m}$ as a vector space $V = \mathrm{span}\{e_1, \cdots, e_{2m}\}$. Then for each $\varphi(t,x) \in Aut(\mathbf{R}^{2m})$, if necessary, take an other basis of V we can write

$$\varphi(t,x) = \varphi(t,x,m_1) \oplus \cdots \oplus \varphi(t,x,m_{s'}),$$

where $m_1 = \cdots = m_s > m_{s+1} \geq \cdots \geq m_{s'}, m_1 + \cdots + m_{s'} = m$. In fact, note that $\varphi(0,1) \underset{\{e_1,e_3,\cdots,e_{2m-1},e_2,\cdots,e_{2m}\}}{\longrightarrow} \begin{pmatrix} (b^{kh}) & 0 \\ 0 & (b^{kh}) \end{pmatrix}$, where (b^{kh}) is an (m,m)-type nilpotent matrix, then consider a Jordan normal form of the (m,m)-type nilpotent matrix.

EXAMPLE 5.3. Let $B \in SL(m,\mathbf{Z})$ be a unimodular matrix with distinct positive eigenvalues $\lambda_1, \cdots, \lambda_m$. Then we define $\varphi : \mathbf{R}^2 \to Aut(\mathbf{R}^{4m})$ by the following:

$$(t,x) \mapsto \varphi(t,x) = \begin{pmatrix} A(t) & 0 & xA(t) & 0 \\ 0 & A(-t) & 0 & xA(-t) \\ 0 & 0 & A(t) & 0 \\ 0 & 0 & 0 & A(-t) \end{pmatrix},$$

where

$$A(t) = \begin{pmatrix} e^{\mathrm{tlog}\lambda_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & e^{\mathrm{tlog}\lambda_m} \end{pmatrix}.$$

$\mathbf{R}^2 \ltimes_\varphi \mathbf{R}^{4m}$ admits the following lattice:

$$\mathbf{Z}^2 \ltimes_\varphi \underbrace{\left\{ P \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix} \middle| \mu_1, \cdots, \mu_m \in \mathbf{Z} \right\} \times \cdots \times \left\{ P \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix} \middle| \mu_1, \cdots, \mu_m \in \mathbf{Z} \right\}}_{4 \quad \text{times}},$$

where $P \in GL(m, \mathbf{R})$ which admits

$$PBP^{-1} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_m \end{pmatrix}.$$

By a straightforward computation, we see

$$\begin{aligned} d\omega_k &= -\log\lambda_k \alpha \wedge \omega_k - \beta \wedge \omega_{2m+k}\,, \\ d\omega_{m+k} &= +\log\lambda_k \alpha \wedge \omega_{m+k} - \beta \wedge \omega_{3m+k}\,, \\ d\omega_{2m+k} &= -\log\lambda_k \alpha \wedge \omega_{2m+k}\,, \\ d\omega_{3m+k} &= +\log\lambda_k \alpha \wedge \omega_{3m+k}\,. \end{aligned}$$

Then

$$\omega = \alpha \wedge \beta + \sum_{k=1}^{m} \omega_k \wedge \omega_{3m+k} + \sum_{k=1}^{m} \omega_{m+k} \wedge \omega_{2m+k}$$

is a symplectic form of $\mathbf{R}^2 \ltimes_\varphi \mathbf{R}^{4m}/\mathbf{Z}^2 \ltimes_\varphi \mathbf{Z}^{4m}$.

By Theorem 4.3, we see $\mathbf{R}^2 \ltimes_\varphi \mathbf{R}^{4m}/\mathbf{Z}^2 \ltimes_\varphi \mathbf{Z}^{4m}$ is a compact Lefschetz manifold.

EXAMPLE 5.4 (cf. [AFLM]). Let consider the following matrices:

$$\begin{aligned} A &= \sum_{k=1}^{m} a_k (E_{2k-1,2k-1} - E_{2k,2k}) \\ B &= 0\,. \end{aligned}$$

Then the automorphism $\varphi(t, x) = \varphi(t)$ induced by A and B can be written as the following matrix:

$$\varphi = \begin{pmatrix} e^{a_1 t} & 0 & & & & 0 \\ 0 & e^{-a_1 t} & & & & \\ & & \ddots & & & \\ & & & & e^{a_m t} & 0 \\ 0 & & & & 0 & e^{-a_m t} \end{pmatrix}.$$

Then $M(a_1, \cdots, a_m) = \mathbf{R}^2 \ltimes_\varphi \mathbf{R}^{2m}/\mathbf{Z}^2 \ltimes_\varphi \mathbf{Z}^{2m}$ has the following non-degenerate close 2-form:

$$\omega = \alpha \wedge \beta + \omega_1 \wedge \omega_2 + \cdots + \omega_{2m-1} \wedge \omega_{2m}\,.$$

By Proposition 4.4, for each symplectic form, we have

$$L^k : H^{(m+1)-q}(\mathfrak{g}) \to H^{(m+1)+q}(\mathfrak{g}) \qquad \text{for } q = 0, \cdots, m+1$$

is an isomorphism. Hence $M(a_1, \cdots, a_m)$ has the Hard Lefschetz property.

EXAMPLE 5.5. Let define an automorphism φ of $\mathbf{R}^8$ as follows.

$$\varphi(t,x,y)=\begin{pmatrix} e^t & 0 & xe^t & 0 & ye^t & 0 & xye^t & 0\\ 0 & e^{-t} & 0 & xe^{-t} & 0 & ye^{-t} & 0 & xye^{-t}\\ 0 & 0 & e^t & 0 & 0 & 0 & ye^t & 0\\ 0 & 0 & 0 & e^{-t} & 0 & 0 & 0 & ye^{-t}\\ 0 & 0 & 0 & 0 & e^t & 0 & xe^t & 0\\ 0 & 0 & 0 & 0 & 0 & e^{-t} & 0 & xe^{-t}\\ 0 & 0 & 0 & 0 & 0 & 0 & e^t & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-t} \end{pmatrix}.$$

Thus we have

$$\begin{aligned} &d\omega_1=-\alpha\wedge\omega_1-\beta\wedge\omega_3-\gamma\wedge\omega_5\,,\quad d\omega_2=\alpha\wedge\omega_2-\beta\wedge\omega_4-\gamma\wedge\omega_6\,,\\ &d\omega_3=-\alpha\wedge\omega_3-\gamma\wedge\omega_7\,,\quad d\omega_4=\alpha\wedge\omega_4-\gamma\wedge\omega_8\,,\\ &d\omega_5=-\alpha\wedge\omega_5-\beta\wedge\omega_7\,,\quad d\omega_6=\alpha\wedge\omega_6-\beta\wedge\omega_8\,,\\ &d\omega_7=-\alpha\wedge\omega_7\,,\quad d\omega_8=\alpha\wedge\omega_8\,, \end{aligned}$$

where α, β and γ are left invariant forms such that

$$\alpha_e=(dt)_e\,,\quad (\beta)_e=(dx)_e\,,\quad (\gamma)_e=(dy)_e\,.$$

Therefore, $(\mathbf{R}^3\ltimes_\varphi\mathbf{R}^8/\mathbf{Z}^3\ltimes_\varphi\mathbf{Z}^8)\times S^1$ has a symplectic structure, for example

$$\alpha\wedge\beta+\gamma\wedge\theta+\omega_1\wedge\omega_8-\omega_3\wedge\omega_6+\omega_4\wedge\omega_5-\omega_2\wedge\omega_7\,,$$

where θ is a left invariant form of $\mathbf{R}$. By Theorem 4.3, $(\mathbf{R}^3\ltimes_\varphi\mathbf{R}^8/\mathbf{Z}^3\ltimes_\varphi\mathbf{Z}^8)\times S^1$ is a Lefschetz manifold.

6. Harmonic cohomology groups on certain compact symplectic solvmanifolds.

In this section, we study certain completely solvable Lie groups G which satisfy the conditions in Proposition 3.1 and have symplectic structures. We use the same notations introduced in sections 3, 4.

DEFINITION 6.1. Let V be a vector space and $x^*\in\bigwedge^p V^*$. Then we say $y^*\in\bigwedge^q V^*$ is divisible by x^* if y^* can be written as

$$y^*=x^*\wedge z^*\,,$$

where $z^*\in\bigwedge^{q-p}V^*$.

Let A, P be the matrices given by

$$\begin{aligned} A&=\sum_{k=1}^{m}(E_{2k-1,2k-1}-E_{2k,2k})\\ P&=\sum_{k=1}^{m-1}(E_{2k-1,2k+1}+E_{2k,2k+2})\,. \end{aligned}$$

For each l ($l = 1, \cdots, m-1$), we define $(\varphi_l)_* : \mathbf{R}^2 \to \mathrm{End}(\mathbf{R}^{2m})$ by

$$(\varphi_l)_*(t, x_l) = tA + x_l P^l \quad \text{for } t, x_l \in \mathbf{R}\,.$$

Now we consider closed 2-forms on $\mathbf{R}^2 \ltimes_{\varphi_l} \mathbf{R}^{2m}$. By definition, we see

$$\begin{cases} d\omega_{2k-1} = -\alpha \wedge \omega_{2k-1} - \beta_l \wedge \omega_{2(k+l)-1} \\ d\omega_{2k} = \alpha \wedge \omega_{2k} - \beta_l \wedge \omega_{2(k+l)}\,. \end{cases}$$

Now we define $\gamma_{i,l} = \gamma_i$ by

$$\gamma_{i,l} = \gamma_i = \sum_{k=0}^{K} (-1)^k \omega_{2(lk+i)+1} \wedge \omega_{2m-2(lk+i)} \quad \text{for } i = 0, \cdots, l-1\,,$$

where $K = K(i, l)$ is the integer such that

$$lK \leq m - i - 1\,, \quad l(K+1) > m - i - 1\,.$$

Then we have

PROPOSITION 6.2. *$\gamma_{i,l} = \gamma_i$ is a closed form on $\mathbf{R}^2 \ltimes_{\varphi_l} \mathbf{R}^{2m}$.*

PROOF. By a straightforward computation, we see

$$\begin{aligned} &\sum_{k=0}^{K} (-1)^k \omega_{2(lk+i)+1} \wedge \omega_{2m-2(lk+i)} \\ \xrightarrow{d}\ & d(\omega_{2i+1} \wedge \omega_{2m-2i}) + \cdots + (-1)^K d(\omega_{2(lK+i)+1} \wedge \omega_{2m-2(lK+i)}) \\ =& -\beta_l \wedge \omega_{2(l+i)+1} \wedge \omega_{2m-2i} \\ & + \beta_l \wedge \omega_{2(2l+i)+1} \wedge \omega_{2m-2(l+i)} + \beta_l \wedge \omega_{2(l+i)+1} \wedge \omega_{2m-2i} \\ & \qquad \vdots \\ & + (-1)^{K-1}(-\beta_l \wedge \omega_{2(lK+i)+1} \wedge \omega_{2m-2(l(K-1)+i)} \\ & \qquad - \beta_l \wedge \omega_{2(l(K-1)+i)+1} \wedge \omega_{2m-2(l(K-2)+i)}) \\ & + (-1)^K(-\beta_l \wedge \omega_{2(lK+i)+1} \wedge \omega_{2m-2(l(K-1)+i)}) \\ & = 0\,. \quad \square \end{aligned}$$

THEOREM 6.3. *We define a map $\varphi_* : \mathbf{R}^{n+1} \to End(\mathbf{R}^{2m})$ by*

$$\varphi_*(t, x_1, \cdots, x_n) = tA + \sum_{j=1}^{n} x_j B_{2j-1} \quad \textit{for } t, x_1, \cdots, x_n \in \mathbf{R}\,,$$

where $B_{2j-1} = P^{2j-1}$ ($2j - 1 \leq m - 1$). Then $\mathbf{R}^{n+1} \ltimes_\varphi \mathbf{R}^{2m}$ or $\mathbf{R}^{n+1} \ltimes_\varphi \mathbf{R}^{2m} \times \mathbf{R}^1$ has a symplectic structure.

PROOF. Note that

$$\begin{cases} d\omega_{2k-1} = -\alpha \wedge \omega_{2k-1} - \sum_{j=1}^{n} \beta_j \wedge \omega_{2(k+J)-1} \\ d\omega_{2k} = \alpha \wedge \omega_{2k} - \sum_{j=1}^{n} \beta_j \wedge \omega_{2(k+J)}\,, \end{cases}$$

where $J = 2j - 1$. We show that

$$\omega_{0,2} = \sum_{k=0}^{m} (-1)^k \omega_{2k+1} \wedge \omega_{2m-2k}$$

is a closed 2-form. Fixing an odd number $L = 2l - 1 \leq m - 1$, we see

$$\begin{aligned} \omega_{0,2} &= \sum_{k=0}^{m-1} (-1)^k \omega_{2k+1} \wedge \omega_{2m-2k} \\ &= \sum_{\nu=0}^{L-1} \sum_{h=0} (-1)^{Lh+\nu} \omega_{2(Lh+\mu)+1} \wedge \omega_{2m-2(Lh+\nu)} \\ &= \sum_{\nu=0}^{L-1} (-1)^{\nu} \sum_{h=0} (-1)^h \omega_{2(Lh+\nu)+1} \wedge \omega_{2m-2(Lh+\nu)}\,. \end{aligned}$$

By Proposition 6.2, this implies that

$$d\omega_{0,2} = \sum_{j \neq l} c_{jkh} \beta_j \wedge \omega_{2k+1} \wedge \omega_{2h}\,.$$

Since $L = 2l - 1$ is any odd number, we see that $d\omega_{0,2} = 0$. Thus if $n + 1$ is even,

$$\omega = \alpha \wedge \beta_1 + \beta_2 \wedge \beta_3 + \cdots + \beta_{n-1} \wedge \beta_n + \sum_{k=0}^{m-1} (-1)^k \omega_{2k+1} \wedge \omega_{2m-2k}$$

is a symplectic structure. Similarly, if $n + 1$ is odd, we see that $(\mathbf{R}^{n+1} \ltimes \mathbf{R}^{2m}) \times \mathbf{R}^1$ has a symplectic structure. $\square$

For simplicity, we assume that $n + 1$ is even unless explicitly stated to the contrary.

Now let ω be a closed 2-form on $\mathbf{R}^{n+1} \ltimes_{\varphi} \mathbf{R}^{2m}$ given by

$$\sum p_i \alpha \wedge \beta_i + \sum q_{ij} \beta_i \wedge \beta_j + \sum_{i=0}^{m-1} r_i \left(\sum_{k=0} (-1)^k \omega_{2(k+i)+1} \wedge \omega_{2m-2k} \right),$$

where $p_i, q_{ij}, r_i \in \mathbf{R}$. Then ω is non-degenerate if and only if

$$\det \begin{pmatrix} 0 & p_1 & \cdots & \cdots & p_n \\ -p_1 & 0 & q_{12} & \cdots & q_{1n} \\ \vdots & -q_{12} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & q_{n-1n} \\ -p_n & -q_{1n} & \cdots & -q_{n-1n} & 0 \end{pmatrix} \neq 0 \quad \text{and} \quad r_0 \neq 0 \,.$$

REMARK. If $B_1 = P^1, B_2 = P^2$, then $(\mathbf{R}^3 \ltimes \mathbf{R}^{2m}) \times \mathbf{R}^1$ $(m \geq 3)$ does not have a symplectic structure. In fact, we set

$$\omega_{0,2} = \sum_{k=0} (-1)^k \omega_{2k+1} \wedge \omega_{2m-2k} \,.$$

By Proposition 6.2, $d\omega_{0,2} = \sum_{k,h} c_{2kh} \beta_2 \wedge \omega_{2k+1} \wedge \omega_{2h}$. Note that it is necessary that $\omega_{0,2}$ is a closed 2-form to exist a non-degenerate closed 2-form. On the other hand, by Proposition 6.2, if ζ is a 2-form such that $d\zeta = \sum_{k,h} c_{1kh} \beta_1 \wedge \omega_{2k+1} \wedge \omega_{2h}$, then ζ is a sum of the following terms:

$$\sum_{k=0} (-1)^k \omega_{2(2k+i)+1} \wedge \omega_{2m-2(2k+i)} \,.$$

In particular, let $i = 0$, we get

$$\sum_{k=0} (-1)^k \omega_{4k+1} \wedge \omega_{2m-4k} \,.$$

However, we see that

$$\begin{aligned} \omega_{0,2} &= \sum_{k=0} (-1)^k \omega_{2k+1} \wedge \omega_{2m-2k} \\ &= \sum_{h=0} (-1)^{2h} \omega_{4h+1} \wedge \omega_{2m-4h} + \sum_{h=0} (-1)^{2h+1} \omega_{4h+3} \wedge \omega_{2m-4h-2} \\ &= \sum_{h=0} \omega_{4h+1} \wedge \omega_{2m-4h} - \sum_{h=0} \omega_{4h+3} \wedge \omega_{2m-4h-2} \,. \end{aligned}$$

Then $\omega_{0,2}$ is not a closed 2-form which implies that $(\mathbf{R}^3 \ltimes_\varphi \mathbf{R}^{2m}) \times \mathbf{R}^1$ $(m \geq 3)$ does not have a symplectic structure.

The matrix form of $\mathbf{R}^{n+1} \ltimes_\varphi \mathbf{R}^{2m}$ constructed in Theorem 6.3 has the following linear transformation part:

$$\begin{aligned} \varphi(t, \mathbf{x}) &= \exp(\varphi_*(t, \mathbf{x})) \\ &= \sum_{k \leq h} f_{kh}(x_1, \cdots, x_n)(e^t E_{2k-1,2h-1} + e^{-t} E_{2k,2h}) \,, \end{aligned}$$

where $t \in \mathbf{R}$, $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbf{R}^n$ and $f_{kh}(x_1, \cdots, x_n)$ are the following polynomials :

$$f_{kh}(x_1, \cdots, x_n) = \sum_{\substack{k_1 + 3k_2 + \cdots + (2n-1)k_n = h-k \\ k_1, \cdots, k_n > 0}} \frac{1}{k_1! \cdots k_n!} {x_1}^{k_1} \cdots {x_n}^{k_n} \,.$$

Now we set

$$\Phi(x_1, \cdots, x_n; m) = \sum_{1\le k\le h\le m} f_{kh}(x_1, \cdots, x_n) E_{kh} \,.$$

For example, we see

$$\Phi(x;6) = \begin{pmatrix} 1 & x & \frac{1}{2!}x^2 & \frac{1}{3!}x^3 & \frac{1}{4!}x^4 & \frac{1}{5!}x^5 \\ 0 & 1 & x & \frac{1}{2!}x^2 & \frac{1}{3!}x^3 & \frac{1}{4!}x^4 \\ 0 & 0 & 1 & x & \frac{1}{2!}x^2 & \frac{1}{3!}x^3 \\ 0 & 0 & 0 & 1 & x & \frac{1}{2!}x^2 \\ 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Phi(x,y,z;6) = \begin{pmatrix} 1 & x & \frac{1}{2!}x^2 & \frac{1}{3!}x^3+y & \frac{1}{4!}x^4+xy & \frac{1}{5!}x^5+\frac{1}{2!}x^2y+z \\ 0 & 1 & x & \frac{1}{2!}x^2 & \frac{1}{3!}x^3+y & \frac{1}{4!}x^4+xy \\ 0 & 0 & 1 & x & \frac{1}{2!}x^2 & \frac{1}{3!}x^3+y \\ 0 & 0 & 0 & 1 & x & \frac{1}{2!}x^2 \\ 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

By Proposition 3.1, we see that $\mathbf{R}^{n+1} \ltimes_\varphi \mathbf{R}^{2m}$ admits a lattice and by Theorem 4.3, $\mathbf{R}^{n+1} \ltimes_\varphi \mathbf{R}^{2m}/\mathbf{Z}^{n+1} \ltimes_\varphi \mathbf{Z}^{2m}(\times S^1)$ is a compact Lefschetz manifold.

COROLLARY 6.5. *Let*

$$Q = \sum_{k<h} q_{kh}(E_{2k-1,2h-1} + E_{2k,2h}) \,, \quad q_{kh} \in \mathbf{Q}$$

and set

$$B_j = \sum_{k=1}^{m-1} c_k^j Q^{2k-1} \,, \quad j = 1, \cdots, n \,,$$

where $c_k^j \in \mathbf{Q}$. *We define* $\varphi_* : \mathbf{R}^{n+1} \to End(\mathbf{R}^{2m})$ *by*

$$\varphi_*(t, x_1, \cdots, x_n) = tA + \sum_{j=1}^{n} x_j B_j \quad \text{for} \quad t, x_1, \cdots, x_n \in \mathbf{R} \,.$$

Then $\mathbf{R}^{n+1} \ltimes_\varphi \mathbf{R}^{2m}(\times \mathbf{R}^1)$ *is a symplectic solvable Lie group which admits a lattice.*

PROOF. Consider the Jordan normal form of Q and applying Theorem 6.3 (cf. Example 5.2). □

Moreover, a completely solvable Lie group which is constructed in Theorem 6.3 has the following exact forms.

LEMMA 6.6. *For $m - k > h$*

$$\alpha \wedge \beta_1 \wedge \cdots \wedge \beta_n \wedge \omega_1 \wedge \omega_3 \wedge \cdots \wedge \hat{\omega}_{2k+1} \wedge \cdots \wedge \omega_{2m-1} \wedge \omega_2 \wedge \cdots \wedge \hat{\omega}_{2h} \wedge \cdots \wedge \omega_{2m}$$

is an exact form on $\mathbf{R}^{n+1} \ltimes_\varphi \mathbf{R}^{2m}$. (It is not necessary that $n+1$ is even.)

PROOF. Let $\beta = \beta_1 \wedge \cdots \wedge \beta_n$. By a straightforward computation, we see

$$\begin{aligned}
&\sum_{i=1}^{h} \alpha \wedge \beta_2 \wedge \cdots \wedge \beta_n \wedge \omega_1 \wedge \cdots \wedge \hat{\omega}_{2(k+i)+1} \wedge \cdots \wedge \omega_{2m-1} \\
&\qquad\qquad\qquad\qquad \wedge \omega_2 \wedge \cdots \wedge \hat{\omega}_{2(h-i+1)} \wedge \cdots \wedge \omega_{2m} \\
\xrightarrow{d}\; & -\alpha \wedge \beta \wedge \omega_1 \wedge \cdots \wedge \hat{\omega}_{2k+1} \wedge \cdots \wedge \omega_{2m-1} \wedge \omega_2 \wedge \cdots \wedge \hat{\omega}_{2h} \wedge \cdots \wedge \omega_{2m} \\
& +\alpha \wedge \beta \wedge \omega_1 \wedge \cdots \wedge \hat{\omega}_{2k+3} \wedge \cdots \wedge \omega_{2m-1} \wedge \omega_2 \wedge \cdots \wedge \hat{\omega}_{2h-2} \wedge \cdots \wedge \omega_{2m} \\
& -\alpha \wedge \beta \wedge \omega_1 \wedge \cdots \wedge \hat{\omega}_{2k+3} \wedge \cdots \wedge \omega_{2m-1} \wedge \omega_2 \wedge \cdots \wedge \hat{\omega}_{2h-2} \wedge \cdots \wedge \omega_{2m} \\
& +\alpha \wedge \beta \wedge \omega_1 \wedge \cdots \wedge \hat{\omega}_{2k+5} \wedge \cdots \wedge \omega_{2m-1} \wedge \omega_2 \wedge \cdots \wedge \hat{\omega}_{2h-4} \wedge \cdots \wedge \omega_{2m} \\
& \quad \vdots \\
& -\alpha \wedge \beta \wedge \omega_1 \wedge \cdots \wedge \hat{\omega}_{2(k+h-2)+1} \wedge \cdots \wedge \omega_{2m-1} \wedge \omega_2 \wedge \hat{\omega}_4 \wedge \cdots \wedge \omega_{2m} \\
& +\alpha \wedge \beta \wedge \omega_1 \wedge \cdots \wedge \hat{\omega}_{2(k+h)-1} \wedge \cdots \wedge \omega_{2m-1} \wedge \hat{\omega}_2 \wedge \cdots \wedge \omega_{2m} \\
& -\alpha \wedge \beta \wedge \omega_1 \wedge \cdots \wedge \hat{\omega}_{2(k+h)-1} \wedge \cdots \wedge \omega_{2m-1} \wedge \hat{\omega}_2 \wedge \cdots \wedge \omega_{2m} \\
=\; & -\alpha \wedge \beta \wedge \omega_1 \wedge \cdots \wedge \hat{\omega}_{2k+1} \wedge \cdots \wedge \omega_{2m-1} \wedge \omega_2 \wedge \cdots \wedge \hat{\omega}_{2h} \wedge \cdots \wedge \omega_{2m}. \quad \square
\end{aligned}$$

THEOREM 6.7. *For any symplectic structure, we have*

$$\dim H_{DR}^{2m+(n+1)-2}(\mathbf{R}^{n+1} \ltimes_\varphi \mathbf{R}^{2m}) - \dim H_{hr}^{2m+(n+1)-2}(\mathbf{R}^{n+1} \ltimes_\varphi \mathbf{R}^{2m}) = m - 1\,.$$

Then we see that $\mathbf{R}^{n+1} \ltimes_\varphi \mathbf{R}^{2m}$ does not have the Hard Lefschetz property.

PROOF. Note that

$$H^2(\mathfrak{g}) = \mathrm{span}\left\{ \begin{array}{l} \alpha \wedge \beta_i\,, \quad \beta_i \wedge \beta_j \quad (i < j)\,, \\ \displaystyle\sum_{k=0}^{m-i-1} \omega_{2(k+i)+1} \wedge \omega_{2m-2k} \ (i = 0, \cdots, m-1) \end{array} \right\}.$$

By a straightforward computation, we see that a decomposable $(2m-2)$-form δ which is generated by the wedge product of $\omega_{2(k+i)+1} \wedge \omega_{2m-2k}$ $(i = 0, \cdots, m-1, k = 0, \cdots, m-i-1)$ is as follows.

$$\omega_1 \wedge \omega_3 \wedge \cdots \wedge \hat{\omega}_{2k+1} \wedge \cdots \wedge \omega_{2m-1} \wedge \omega_2 \wedge \cdots \wedge \hat{\omega}_{2h} \wedge \cdots \wedge \omega_{2m} \quad m - k \le h\,.$$

Since $\sum_{k=0}^{m-i-1} \omega_{2(k+i)+1} \wedge \omega_{2m-2k}$ is a sum of $\omega_1 \wedge \omega_{2m}$, $\omega_3 \wedge \omega_{2m}$, $\omega_3 \wedge \omega_{2m-2}$, $\omega_5 \wedge \omega_{2m}, \cdots$, if δ is divisible by ω_1, then it is divisible by ω_{2m}. Moreover, if it is divisible by $\omega_1 \wedge \omega_3$, then it is divisible by $\omega_{2m} \wedge \omega_{2m-2}$. Thus if δ is divisible by $\omega_1 \wedge \omega_3 \wedge \cdots \wedge \omega_{2k-1}$, then it is divisible by $\omega_{2m} \wedge \omega_{2m-2} \wedge \cdots \wedge \omega_{2m-2k+2}$.

By the same argument, it is easy to verify that

$$\omega_1 \wedge \omega_3 \wedge \cdots \wedge \hat{\omega}_{2k+1} \wedge \cdots \wedge \omega_{2m-1} \wedge \omega_2 \wedge \cdots \wedge \hat{\omega}_{2m-2k} \wedge \cdots \wedge \omega_{2m}$$

and a $2m$-form

$$\omega_1 \wedge \omega_3 \wedge \cdots \wedge \omega_{2m-1} \wedge \omega_2 \wedge \omega_4 \wedge \cdots \wedge \omega_{2m}$$

are only generated by the wedge product of $\omega_1 \wedge \omega_{2m}, \omega_3 \wedge \omega_{2m-2}, \cdots, \omega_{2m-1} \wedge \omega_2$. By Lemma 6.6, the above argument implies that the image of

$$\sum_{k=0}^{m-i-1} (-1)^k \omega_{2(k+i)+1} \wedge \omega_{2m-2k} \quad (i = 1, \cdots, m-1)$$

by $L^{m+(n+1)/2-2}$ is exact. Hence,

$$\dim H_{DR}^{2m+(n+1)-2}(\mathbf{R}^{n+1} \ltimes_\varphi \mathbf{R}^{2m}) - \dim H_{hr}^{2m+(n+1)-2}(\mathbf{R}^{n+1} \ltimes_\varphi \mathbf{R}^{2m}) \geq m-1\,.$$

Let

$$\tau_1 = \sum p_i \alpha \wedge \beta_i + \sum q_{ij} \beta_i \wedge \beta_j$$
$$\tau_2 = \sum_{k=0}^{m-1} (-1)^k \omega_{2k+1} \wedge \omega_{2m-2k}\,,$$

where $p_i, q_{ij} \in \mathbf{R}$. We set $\tau = \tau_1 + r\tau_2$, where $r \in \mathbf{R}$. We shall show $L^{m+(n+1)/2-2}\tau$ is not exact. Assume that $L^{m+(n+1)/2-2}\tau = d\theta$. On the other hand,

$$\begin{aligned}\tau = \tau_1 + r\tau_2 \xrightarrow{L^{m+(n+1)/2-2}} & \sum_i P_i \hat{\alpha} \wedge \beta_1 \wedge \cdots \wedge \hat{\beta}_i \wedge \cdots \wedge \beta_n \wedge \Omega \\ & + \sum\nolimits_{i<j} Q_{ij} \alpha \wedge \beta_1 \wedge \cdots \wedge \hat{\beta}_i \wedge \cdots \wedge \hat{\beta}_j \wedge \cdots \wedge \beta_n \wedge \Omega \\ & + R\alpha \wedge \beta_1 \wedge \cdots \wedge \beta_n \wedge \tau_2^{m-1}\,,\end{aligned}$$

where $P_i, Q_{ij}, R \in \mathbf{R}$ and $\Omega = \omega_1 \wedge \cdots \wedge \omega_{2m-1} \wedge \omega_2 \wedge \cdots \wedge \omega_{2m}$. Since $L^{m+(n+1)/2-2} : \mathcal{H}^2(\mathfrak{g}) \to \mathcal{H}^{2m+(n+1)-2}(\mathfrak{g})$ is an isomorphism, there exists a non-zero coefficient. For example, if $P_i \neq 0$, then we have

$$\begin{aligned}\alpha \wedge \beta_i \wedge L^{m+(n+1)/2-2}\omega &= d(\alpha \wedge \beta_i \wedge \theta) \\ &= (-1)^i a \cdot \alpha \wedge \beta_1 \wedge \cdots \wedge \beta_n \wedge \Omega\,.\end{aligned}$$

It is a contradiction. □

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