J. DIFFERENTIAL GEOMETRY
21 (1985) 195–211

EXAMPLES OF COMPLETE MANIFOLDS WITH POSITIVE RICCI CURVATURE

DETLEF GROMOLL & WOLFGANG T. MEYER

Dedicated to Wilhelm Klingenberg on his sixtieth birthday

A long standing question in riemannian geometry has been: Does a complete manifold M^n with positive Ricci curvature Ric also admit a complete metric with nonnegative sectional curvature K? It is generally believed that this is not always true, but counterexamples were not known. The answer is actually affirmative for the dimension $n = 3$ (cf. [6], [16]). Note that $K > 0$ is sometimes known to be obstructed when a metric with Ric > 0 exists. Simple examples are $S^k \times \mathbf{R}^l$ in the noncompact case [5], and $\mathbf{R}P^k \times \mathbf{R}P^l$ in the nonsimply connected compact case for $k, l \geqslant 2$, as a consequence of Synge's Lemma [4].

Examples of complete manifolds with $K \geqslant 0$ remain fairly scarce. One way or another, they can all be obtained using classical spaces and quotients of isometric group actions (cf. [3] for a detailed list of references). There are several additional methods to produce complete metrics with Ric > 0. Certain fiber bundles were treated in [14] and [15], and a large class of Brieskorn varieties in [7]. Finally, by Yau's work, Kaehler metrics with Ric $\geqslant 0$ exist on any compact Kaehler manifold with first Chern class $c_1 \geqslant 0$ (cf. [17]). Interesting examples arise as complete intersections in $\mathbf{C}P^{n+r}$, notably hypersurfaces. In particular, the $K3$-surface (quartic) in $\mathbf{C}P^3$ admits a Ricci flat metric, but this is a true border line case: Since the $\hat{A}$-genus does not vanish, we have Ric $\equiv 0$ whenever Ric $\geqslant 0$ (cf. [8]). It follows that $K \geqslant 0$ would imply $K \equiv 0$, which is impossible. Therefore one can distinguish at least between the conditions Ric $\geqslant 0$ and $K \geqslant 0$, in a weak sense.

In this paper we present new classes of complete manifolds with Ric > 0. First of all we construct noncompact examples many of which cannot carry metrics with $K \geqslant 0$. This settles the above question in the noncompact case.

Received October 11, 1984. The first author was partially supported by the Alexander von Humboldt Foundation and the National Science Foundation Grant MCS 8102758A02.

Although there are no compact counterexamples as yet, we obtain series of closed manifolds with Ric > 0 and Euler number $\chi < 0$. They are either counterexamples, or contradict the global Hopf conjecture, i.e. $\chi \geqslant 0$ for $K \geqslant 0$. Spaces of the last type also arise from complete intersections. The lowest dimensional is the cubic in $\mathbf{C}P^4$ with $c_1 > 0$, $\chi = -6$. The only known invariant so far to distinguish between Ric > 0 and $K \geqslant 0$ is the homotopy type in the noncompact situation, due to the structure theory in [2].

Our starting point is somewhat reminiscent of the discussion for Brieskorn varieties given in [7]. We will consider the "stable" geometry and topology of certain real algebraic varieties with codimension 2 in euclidean space $\mathbf{R}^{m+p+q}$. Let $f(z)$ be any multihomogeneous polynomial in $\mathbf{R}^m$ for which the origin is an isolated critical point, and let $F(z, x, y) = f(z) + |x|^2 - |y|^2$, $x \in \mathbf{R}^p$, $y \in \mathbf{R}^q$. Intersecting the zero set $F = 0$ of this polynomial with a suitable ellipsoid in $\mathbf{R}^{m+p+q}$, we obtain a compact manifold V_0 with positive Ricci curvature (in the induced metric), as soon as $p + q$ is large compared to $|p - q|$. V_0 bounds the set $F \leqslant 0$ in the ellipsoid whose interior is denoted by V_-. The metric of V_- can be warped near the boundary to yield a complete metric of positive Ricci curvature, provided $p - q$ and $p + q$ are sufficiently large, depending on f.

The above warping problem is delicate in general (cf. also [9]). In §1 we discuss it to the extent needed for our asymptotic estimates. §2 deals with the class of examples for which the boundary problem can be solved. The geometric estimates are obtained in §3, topological invariants in §4, and finally, in §5, we look at some special examples.

We are indebted to B. Lawson for some valuable suggestions in connection with this work. We would also like to acknowledge helpful conversations with A. Durfee and J. H. Eschenburg.

1. The boundary problem

Let $(V, \partial V)$ be a compact riemannian manifold with boundary ∂V and metric $\langle\ ,\ \rangle$. The distance function from the boundary is denoted by t. Near the boundary we consider the unit vector field $T = -\operatorname{grad} t$. On $\operatorname{Int}(V)$ we define a complete warped metric g_ε by

$$(1.1)\qquad g_\varepsilon(X, Y) = \varphi^2(t)\langle T, X\rangle\langle T, Y\rangle + \langle X^\perp, Y^\perp\rangle,$$

where $\varphi(t) = 1 + \exp(1/t + 1/(t - \varepsilon))$ for $0 < t < \varepsilon$, $\varphi(t) = 1$ for $t \geqslant \varepsilon$, and $X^\perp$ denotes the component of X orthogonal to T. This metric is defined as soon as ε is smaller than the injectivity radius of the normal exponential map of ∂V. For our estimates, φ could be replaced by any other function satisfying the following conditions:

$$(1.2)\qquad \begin{aligned} &\varphi(t) \geqslant 1, && \\ &\varphi(t) = 1 && \text{for } t \geqslant \varepsilon, \\ &\varphi'(t) = -T\varphi < 0 && \text{for } 0 < t < \varepsilon, \\ &\int_0^\varepsilon \varphi(t)\,dt = \infty && (\text{for completeness of } g_\varepsilon), \\ &T\varphi/\varphi^3 \text{ is bounded.} && \end{aligned}$$

Assuming that the Ricci curvature of V with respect to $\langle\ ,\ \rangle$ is positive, it is an interesting problem to find conditions on $(V, \partial V)$ and φ under which $\mathrm{Int}(V)$ has positive Ricci curvature with respect to g_ε. As we shall see, ∂V must have $\mathrm{Ric}_{\partial V} \geqslant 0$ with respect to the metric induced from $\langle\ ,\ \rangle$, and the mean curvature of ∂V with respect to the outside normal must be nonnegative. In [9], Ingram gave certain sufficient conditions in the case when $\mathrm{Int}(V)$ is an open submanifold of a euclidean sphere. They are complicated and have not been verified as yet in any interesting example. We will use a different approach by studying the asymptotic geometry of a sequence of open submanifolds in suitable ellipsoids of increasing dimension.

We need formulas for the Ricci tensor with respect to g_ε in terms of the data from $\langle\ ,\ \rangle$. The sectional curvature, Ricci curvature, etc. of V with respect to $\langle\ ,\ \rangle$ are denoted by K, Ric, etc., and the corresponding data with respect to g_ε by $\overline{K}$, $\overline{\mathrm{Ric}}$, etc. The second fundamental tensor of the hypersurface V_t at distance t from $V_0 = \partial V$ with respect to the normal T and the metric $\langle\ ,\ \rangle$ is denoted by S_t, i.e. $S_t X = \nabla_X T$. Then it is easy to verify the following formulas:

$$(1.3)\qquad \overline{\mathrm{Ric}}(T, T) = \mathrm{Ric}(T, T) + (T\varphi/\varphi)\operatorname{tr} S_t,$$

$$(1.4)\qquad \overline{\mathrm{Ric}}(T, X) = \mathrm{Ric}(T, X) \quad \text{for } \langle X, T\rangle = 0,$$

$$(1.5)\qquad \begin{aligned} \overline{\mathrm{Ric}}(X, X) = \mathrm{Ric}(X, X) &+ \big(T\varphi/\varphi^3 + (1 - 1/\varphi^2)\operatorname{tr} S_t\big)\langle S_t X, X\rangle \\ &+ (1/\varphi^2 - 1)\big(K(X, T) + \langle S_t^2 X, X\rangle\big) \quad \text{for } \langle X, T\rangle = 0. \end{aligned}$$

Writing an arbitrary vector Z as $Z = \alpha T + \beta X$, where $\langle X, T\rangle = 0$, $\overline{\mathrm{Ric}}(Z, Z)$ becomes a quadratic form in α and β which is positive definite if and only if

$$(1.6)\qquad \overline{\mathrm{Ric}}(T, T) > 0,$$

$$(1.7)\qquad \overline{\mathrm{Ric}}(X, X) > 0,$$

$$(1.8)\qquad \overline{\mathrm{Ric}}(T, T)\,\overline{\mathrm{Ric}}(X, X) - \overline{\mathrm{Ric}}(X, T)^2 > 0.$$

Since $\lim_{t\to 0} T\varphi(t)/\varphi(t) = \infty$, (1.3) and (1.6) imply $\operatorname{tr} S_t \geqslant 0$ and in particular $\operatorname{tr} S_0 \geqslant 0$, so the mean curvature of ∂V must be nonnegative. (1.5) and (1.7) imply in the limit as $t \to 0$,

$$0 \leqslant \operatorname{Ric}(X, X) + \operatorname{tr} S_0\langle S_0 X, X\rangle - K(X, T) - \langle S_0^2 X, X\rangle = \operatorname{Ric}_{\partial V}(X, X).$$

Therefore the induced Ricci curvature of ∂V must be nonnegative.

2. The class of examples

Let F be a multihomogeneous polynomial on $\mathbf{R}^n$, i.e. F is the direct sum of homogeneous polynomials F_i on $\mathbf{R}^{n_i}$ of degree l_i,

$$2 \leqslant l_1 \leqslant l_2 \leqslant \cdots \leqslant l_k, \quad \sum_{i=1}^{k} n_i = n, \quad F(u) = \sum_{i=1}^{k} F_i(u_i).$$

Since ∇F_i is homogeneous of degree $l_i - 1$, the only singularity of F is at the origin as soon as all singularities of F are isolated. In this case, the F_i are singular exactly at the origin. For a given F as above, with an isolated singularity at the origin, we consider for any $\alpha = (\alpha_1, \cdots, \alpha_k)$, $\alpha_i > 0$ and $r > 0$, the quadratic form

$$G(u) = \sum_{i=1}^{k} \alpha_i \|u_i\|^2 - r^2.$$

The gradients ∇F and ∇G are linearly independent on $F^{-1}(0) \setminus \{0\}$. This can be seen as follows: If $a\nabla F = b\nabla G$, then $a\nabla F_i = 2b\alpha_i u_i$. Since F_i is homogeneous of degree l_i, one obtains $\langle \nabla F_i, u_i\rangle = l_i F_i(u_i)$, and therefore

$$a \sum_{i=1}^{k} F_i(u_i) = 2b \sum_{i=1}^{k} \frac{\alpha_i}{l_i} \|u_i\|^2.$$

On $F^{-1}(0)$ we have $\Sigma F_i(u_i) = F(u) = 0$, and therefore $b = 0$. The equation $a\nabla F(u) = 0$ then implies $a = 0$, since $\nabla F(u) \neq 0$ for $u \neq 0$.

As a consequence, $V_0 = F^{-1}(0) \cap G^{-1}(0)$ is a smooth hypersurface of the ellipsoid $G^{-1}(0)$, and

$$V_- = F^{-1}(-\infty, 0) \cap G^{-1}(0), \qquad V_+ = F^{-1}(0, \infty) \cap G^{-1}(0)$$

are open subsets with common boundary V_0 in the ellipsoid.

The diffeomorphism type of V_0, V_+, V_- is independent of α and r, since for different α and r the corresponding manifolds are isotopic in $\mathbf{R}^n$. To study the topology of these objects one can therefore choose $\alpha_i = 1$ and r sufficiently small. The geometry of course depends on α and r. For our geometric estimates, we will choose $r = 1$ and $\alpha_i = 2/l_i$. By this choice of α_i, we have

$\langle \nabla F, \nabla G \rangle = 4F$ which vanishes on V_0, so the gradients ∇F, ∇G are orthogonal along V_0. As a consequence, the Gauss equations for V_0 simplify considerably. But this choice of α_i is not only convenient for computations; in fact, some of the estimates definitely do not work in the sphere.

We now fix a multihomogeneous polynomial $f = \sum_{i=1}^{k} f_i$ on $\mathbf{R}^m$, degree$(f_i) = l_i$, $2 \leqslant l_1 \leqslant l_2 \leqslant \cdots \leqslant l_k$. For integers $p \geqslant 1$, $q \geqslant 1$, $n = m + p + q$, we consider the multihomogeneous polynomial

$$F(z, x, y) = f(z) + \|x\|^2 - \|y\|^2, \qquad x \in \mathbf{R}^p, y \in \mathbf{R}^q, \tag{2.1}$$

and correspondingly,

$$G(z, x, y) = \sum_{i=1}^{k} \frac{2}{l_i}\|z_i\|^2 + \|x\|^2 + \|y\|^2 - 1. \tag{2.2}$$

The manifolds V_0, V_+, V_- now depend on f, p, q. To emphasize this dependence, we shall write $V_0(f, p, q)$, $V_+(f, p, q)$, $V_-(f, p, q)$ for convenience of notation. Choosing $\varepsilon > 0$ sufficiently small, we consider the warped metrics g_ε of (1.1) on V_-. Since V_- is open in $G^{-1}(0)$, the data ∇, K, Ric, etc., are now data of this ellipsoid, with its standard metric.

3. Geometric estimates

Our aim is to adjust ε, p, q so that the warped metric g_ε on $V_-(f, p, q)$ has positive Ricci curvature. According to §1 we have to establish (1.6)–(1.8). From (1.3)–(1.5) it is obvious that we need estimates for $\|S_t\|$, $\operatorname{tr} S_t$, and $T\varphi/\varphi^3$. Since an upper bound for $T\varphi/\varphi^3$ depends on ε, it is necessary to find a lower bound for the injectivity radius of the normal exponential map of $V_0(f, p, q)$, independent of p and q.

Let us start with estimates on the ellipsoid $G^{-1}(0)$ defined by the function G of (2.2). An elementary calculation gives the extremal values of the sectional curvature,

$$K_{\max} = \frac{l_k}{2}, \qquad K_{\min} = \frac{4}{l_k l_{k-1}}, \tag{3.1}$$

independent of p, q. The Ricci curvature therefore satisfies

$$(n-2)K_{\min} \leqslant \operatorname{Ric}(Z, Z) \leqslant (n-2)K_{\max} \tag{3.2}$$

for any unit tangent vector Z of $G^{-1}(0)$. Since Ric is positive definite symmetric, the following estimate holds for any pair of orthonormal tangent vectors X, T:

$$\mu_{\min}^2 \leqslant \operatorname{Ric}(X, X)\operatorname{Ric}(T, T) - \operatorname{Ric}(X, T)^2 \leqslant \mu_{\max}^2,$$

where $\mu_{\min}$ and $\mu_{\max}$ are the minimum and maximum eigenvalues of Ric. This inequality and (3.2) imply

$$(3.3)\quad (n-2)^2K_{\min}^2 \leqslant \operatorname{Ric}(X,X)\operatorname{Ric}(T,T)-\operatorname{Ric}(X,T)^2 \leqslant (n-2)^2K_{\max}^2.$$

Next we obtain estimates for curvature quantities of V_0 in the ellipsoid. The tangent space of V_0 at $p \in V_0$ is the orthogonal complement of the span of the gradients ∇F, ∇G at p. Since $\langle \nabla F_p, \nabla G_p\rangle = 0$, the second fundamental tensor S_0 of V_0 is given by

$$(3.4)\qquad S_0X = \|\nabla F\|^{-1}(\nabla_X\nabla F)^{\mathscr{T}} = \|\nabla F\|^{-1}(H_FX)^{\mathscr{T}},$$

where $(\ )^{\mathscr{T}}$ denotes the projection to the tangent space of V_0, and ∇ is the derivative of $\mathbf{R}^n$. Note that $\|\nabla F\|$ is bounded, and bounded away from zero on $V_0(f,p,q)$, independent of p, q. The eigenvalues of the hessian H_F of F are the eigenvalues of H_f and the values 2, -2. Therefore, there is a constant C_0 such that

$$(3.5)\qquad \|S_0\| < C_0, \quad \text{independent of } p \text{ and } q.$$

The mean curvature $\operatorname{tr} S_0$ of V_0 is given by

$$(3.6)\qquad \begin{aligned}\operatorname{tr} S_0 &= \|\nabla F\|^{-1}\operatorname{tr}(H_F)^{\mathscr{T}}\\ &= \|\nabla F\|^{-1}\left[\operatorname{tr} H_f + 2(p-q) - \frac{\langle H_F\nabla F, \nabla F\rangle}{\|\nabla F\|^2} - \frac{\langle H_F\nabla G, \nabla G\rangle}{\|\nabla G\|^2}\right],\end{aligned}$$

from which we get estimates

$$(3.7)\qquad \begin{aligned}&\|\nabla F\|^{-1}\left(\operatorname{tr} H_f + 2(p-q) - 2\|H_F\|\right)\\ &\qquad\leqslant \operatorname{tr} S_0 \leqslant \|\nabla F\|^{-1}\left(\operatorname{tr} H_F + 2(p-q) + 2\|H_F\|\right).\end{aligned}$$

As an immediate consequence we have:

Lemma 1. *There is an integer s such that for p, q which $p-q \geqslant s$, the mean curvature $\operatorname{tr} S_0$ of $V_0(f,p,q)$ is positive. For a fixed $s_0 \geqslant s$, $\operatorname{tr} S_0$ is bounded for all p, q in terms of $p-q=s$.*

This was observed already in [9].

Next we give an estimate for the Ricci curvature of V_0. By the Gauss equation for V_0 in the ellipsoid $G^{-1}(0)$, the Ricci curvature of V_0 in direction X is found to be

$$(3.8)\qquad \begin{aligned}\operatorname{Ric}_{V_0}(X,X) &= \operatorname{Ric}(X,X) - K(X, \nabla F/\|\nabla F\|)\\ &\qquad + \operatorname{tr} S_0\langle S_0X, X\rangle - \|S_0X\|^2.\end{aligned}$$

Using (3.2) and (3.6) we obtain

$$\mathrm{Ric}_{V_0}(X, X) \geqslant (n-3)K_{\min} - \|H_F\|/\|\nabla F\|^2\big(|\mathrm{tr}\, H_f| + 2|p-q| + 3\|H_F\|\big).$$

Since $n = m + p + q$, this proves:

Theorem 1. *For any integer s there is an integer r such that for any p, q satisfying $p - q = s$, $p + q \geqslant r$, the Ricci curvature of $V_0(f, p, q)$ is strictly positive.*

This result is also contained in [9].

We finally turn to the estimates for $\|S_t\|$, $\mathrm{tr}\, S_t$, and the injectivity radius of the normal exponential map of V_0. For these estimates, a basic differential equation for the second fundamental tensor S_t is useful. Consider for any $p \in V_0$ the geodesic $\gamma(\tau) = \exp(-\tau T_p)$, where $T = -\mathrm{grad}\, t$ as before, and the tensor field $\tau \to S_\tau \circ \gamma(\tau)$ with covariant derivative S_τ' on the normal bundle along γ. One has

$$S_t' = R_T + S_t^2, \tag{3.9}$$

where $R_T X = R(X, T)T$. This can be checked by taking the second derivative of variational Jacobi fields X along γ, satisfying

$$\langle X, \dot\gamma\rangle = 0 \quad \text{and} \quad X_t' = -S_t X_t. \tag{3.10}$$

The following proposition contains the estimates needed.

Proposition 1. *Let s be as in Lemma* 1 *and $p - q \geqslant s$, C_0 as in* (3.5). *Then, independent of p and q,*

(a) *A lower bound for the injectivity radius of the normal exponential map of $V_0(f, p, q)$ is given by*

$$\rho = K_{\max}^{-1/2}\tan^{-1}\big(K_{\max}^{1/2}/C_0\big),$$

(b) $\|S_t\|$ *is bounded for* $0 \leqslant t \leqslant \rho/2$,

(c) $\mathrm{tr}\, S_t > 0$ *for* $0 \leqslant t \leqslant \rho$, *and there is a constant C_1 such that* $\mathrm{tr}\, S_t \leqslant \mathrm{tr}\, S_0 + (n-2)C_1 t$ *for* $0 \leqslant t \leqslant \rho/2$.

Proof. (a) the injectivity radius ρ_0 and the focal radius of the normal exponential map coincide. Otherwise, the boundary of the tubular neighborhood of radius ρ_0 about V_0 will intersect itself tangentially somewhere in V_-, thus giving rise to a geodesic of length $2\rho_0$, locally minimizing the distance between points on V_0. But since the Ricci curvature of the ellipsoid is positive and $\mathrm{tr}\, S_0 > 0$, standard variational techniques [12] lead to a contradiction. This is completely analogous to Klingenberg's basic argument to estimate the injectivity radius from a point (cf. [11]).

A lower bound for the focal radius is now easily obtained by basic comparison: Since $K \leqslant K_{\max}$ and $S_0 \leqslant C_0$ (meaning $C_0 \cdot I - S_0$ is a nonnegative operator), the first focal point of V_0 along any normal unit speed geodesic in V_- cannot come before ρ (cf. [18]). Part (b) contains another explicit argument.

(b) Consider the solution of thè differential equation

$$h' = K_{\max} + h^2, \qquad h(0) = h_0 = C_0, \tag{3.11}$$

given by $h_t = K_{\max}^{1/2}(C_0 + K_{\max}^{1/2} \tan tK_{\max}^{1/2})(K_{\max}^{1/2} - C_0 \tan tK_{\max}^{1/2})^{-1}$. Note that

$$h_0 \pm S_0 > 0. \tag{3.12}$$

By (3.9) and (3.11),

$$(h + S)_t' > 0 \tag{3.13}$$

whenever S_t is defined. Similarly,

$$(h - S)' = (h + S)(h - S) + (K_{\max} - R_T) > 0, \tag{3.14}$$

certainly if $h \pm S > 0$. Now let $[0, t_0)$ be the largest interval on which h, S are defined and $h - S$ is positive. Then by (3.12)–(3.14),

$$(h \pm S)_t \geqslant (h \pm S)_0 > 0 \tag{3.15}$$

and

$$\|S_t\| \leqslant h_t, \tag{3.16}$$

$0 \leqslant t < t_0$. Then $t_0 = \rho$, the first singular point of h: If $t_0 < \rho$, we conclude from (3.16) and (3.10) that t_0 is smaller than the focal radius of V_0. Therefore, S_{t_0} is defined, and by (3.15), $(h - S)_{t_0}$ is positive, contradicting the choice of t_0. Now (b) is an immediate consequence of (3.16). The last argument simplifies only slightly if we assume the estimate in (a).

(c) The equation (3.9) implies $0 \leqslant (\operatorname{tr} S_t)' \leqslant (n - 2)(K_{\max} + \|S_t\|^2)$. Since $\operatorname{tr} S_0 > 0$ and $\|S_t\|$ is bounded by (b), one can choose C_1 as an upper bound for $K_{\max} + \|S_t\|^2$.

We are now in a position to prove our main result concerning the estimates for Ricci curvature.

Theorem 2. *Let f be a multihomogeneous polynomial. Fix s so that the mean curvature $\operatorname{tr} S_0$ of $V_0(f, p, q)$ is positive for $p - q \geqslant s$. Then there is a number $\varepsilon > 0$ and an integer r, such that $V_-(f, p, q)$ has positive Ricci curvature with respect to the warped metric g_ε for any p, q satisfying $p - q = s$ and $p + q \geqslant r$.*

Proof. Let ρ and C_1 be as in Proposition 1, and $\delta = \frac{1}{2}K_{\min}^2 \leqslant \frac{1}{2}K_{\min}$, compare (3.1). By (b) of Proposition 1 we can choose $0 < \varepsilon < \rho/2$, so that $\delta \leqslant K_{\min}^2 - \varepsilon C_1\|S_t\|K_{\max}$ and $\delta \leqslant K_{\min} - \varepsilon C_1\|S_t\|$ for $0 \leqslant t \leqslant \varepsilon$.

Using (c) we obtain for any X with $\|X\| = 1$, $\langle X, T\rangle = 0$:

$$\left|\left(\frac{T\varphi}{\varphi^3} + \left(1 - \frac{1}{\varphi^2}\right)\operatorname{tr} S_t\right)\langle S_t X, X\rangle + \left(\frac{1}{\varphi^2} - 1\right)\left(K(X, T) + \langle S_t^2 X, X\rangle\right)\right|$$
$$\leqslant (n-2)\varepsilon C_1\|S_t\| + A,$$

where $A = T\varphi/\varphi^3\|S_t\| + \operatorname{tr} S_0\|S_t\| + K_{\max} + \|S_t\|^2$ is uniformly bounded, independent of $p + q$, $0 \leqslant t \leqslant \varepsilon$.

The last estimate, (1.5), and (3.2) yield

$$\overline{\operatorname{Ric}}(X, X) \geqslant (n-2)\left(K_{\min} - \varepsilon C_1\|S_t\|\right) - A \geqslant (n-2)\delta - A.$$

Similarly, (1.3)–(1.5), (3.2), and (3.3) give us

$$\overline{\operatorname{Ric}}(X, X)\,\overline{\operatorname{Ric}}(T, T) - \overline{\operatorname{Ric}}(X, T)^2$$
$$\geqslant \overline{\operatorname{Ric}}(X, X)\frac{T\varphi}{\varphi}\operatorname{tr} S_t + (n-2)^2\left(K_{\min}^2 - \varepsilon C_1\|S_t\|K_{\max}\right) - (n-2)K_{\max}A$$
$$\geqslant \overline{\operatorname{Ric}}(X, X)\frac{T\varphi}{\varphi}\operatorname{tr} S_t + (n-2)^2\delta - (n-2)K_{\max}A.$$

By choosing $p + q$ and hence $n = m + p + q$ large enough we see first that $\overline{\operatorname{Ric}}(X, X)$ and in turn $\overline{\operatorname{Ric}}(X, X)\overline{\operatorname{Ric}}(T, T) - \overline{\operatorname{Ric}}(X, T)^2$ become positive. For this one should note $(T\varphi/\varphi)\operatorname{tr} S_t > 0$ by (1.2) and (c) of the proposition.

4. The topology of the examples

In this section we study the topology of the manifolds $V_0(f, p, q)$, $V_-(f, p, q)$, $V_+(f, p, q)$ introduced in §2. In the special case when $p = 1$ or $q = 1$, some of our results follow from Proposition 4 in [10]. For topological conclusions it is not essential that f is a multihomogeneous polynomial on $\mathbf{R}^m$, f may be any real analytic function with an isolated singularity at the origin, $f(0) = 0$. However, the function G of (2.2) will be replaced by

$$G(z, x, y) = \|z\|^2 + \|x\|^2 + \|z\|^2 - \varepsilon^2, \tag{4.1}$$

with ε sufficiently small. For a multihomogeneous f, the topology of V_0, V_-, V_+ is not changed, as pointed out in §2. For an analytic f, the topology of V_0, V_-, V_+ is independent of ε, as soon as ε is small enough.

It will be more convenient here to work with the closures $\overline{V}_\pm$ of $V_\pm$. The sets V_0, $\overline{V}_-$, $\overline{V}_+$ consist of all $(z, x, y) \in \mathbf{R}^{m+p+q}$ satisfying $f(z) + \|x\|^2 - \|y\|^2 = 0, \leqslant 0, \geqslant 0$ respectively, and $\|z\|^2 + \|x\|^2 + \|y\|^2 = \varepsilon^2$. Furthermore, we consider W_0, W_-, W_+ given by all $z \in \mathbf{R}^m$ with $f(z) = 0, \leqslant 0, \geqslant 0$ respectively, and $\|z\|^2 = \varepsilon^2$. We also need U_0, U_-, U_+ given by all $(z, x) \in \mathbf{R}^{m+p}$ satisfying $f(z) + \|x\|^2 = 0, \leqslant 0, \geqslant 0$ respectively, and $\|z\|^2 + \|x\|^2 = \varepsilon^2$.

Clearly, the topology of V_0, V_-, V_+ only depends on f, p, q. However, it is difficult in general to determine the invariants of an arbitrary f which will enter the computations. They are reflected in the topology of W_0, W_-, W_+. We will show that the topology of V_0, V_-, V_+ is determined by the topology of W_0, W_-, W_+. For some functions f, the topology of W_0, W_-, W_+ and hence of V_0, V_-, V_+ can be computed.

For technical reasons we introduce the set $\tilde{W}_-$ given by all $z \in \mathbf{R}^m$ satisfying $f(z) + \varepsilon^2 - \|z\|^2 = 0$ and $\|z\|^2 \leqslant \varepsilon^2$, as well as the sets C_-, C_+ consisting of all $z \in \mathbf{R}^m$ with $f(z) + \varepsilon^2 - \|z\|^2 \leqslant 0, \geqslant 0$ respectively, and $\|z\|^2 \leqslant \varepsilon^2$. They will be needed in the following proposition.

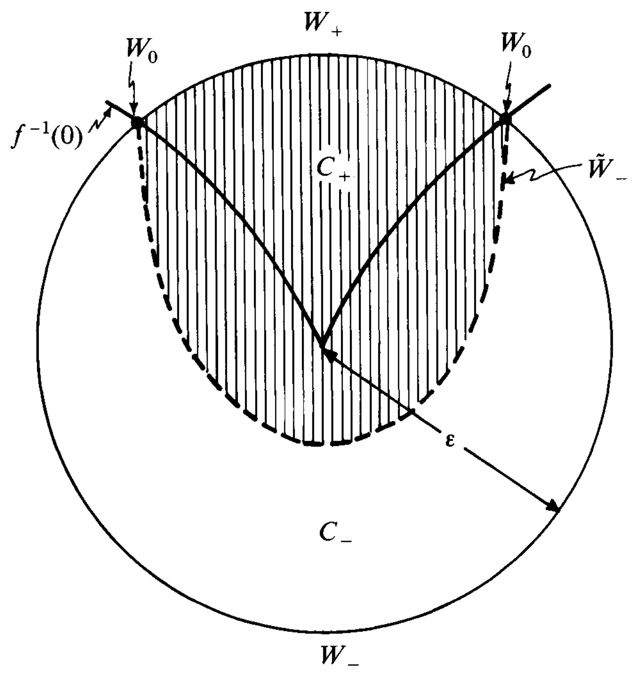

Figure 1 indicates the location of the last sets in $\mathbf{R}^m$, when $m = 2$. $f^{-1}(0)$ is the curve having a singularity at the origin. W_0 consists of the two points where $f^{-1}(0)$ intersects the circle of radius ε. W_+, W_- are arcs of this circle. $\tilde{W}_-$ is the dashed curve dividing the disc of radius ε into the two regions C_-, C_+.

Proposition 2. *For sufficiently small* $\varepsilon > 0$, *there is a continuous function* τ: $W_- \to (0, 1]$ *so that*

(a) $\tau_z \cdot z \in \tilde{W}_-$,

(b) $\tau_z = 1$ *if and only if* $z \in W_0$,

(c) $C_- = \bigcup_{z \in W_-} \{tz | \tau_z \leqslant t \leqslant 1\}$,

(d) C_+ *is homeomorphic to the m-cell* D_ε^m *of radius* ε.

In particular, W_- *and* $\tilde{W}_-$ *are homeomorphic and* (W_-, W_0), $(\tilde{W}_-, W_0)$ *are strong deformation retracts of* (C_-, W_0).

Remark. In fact, τ is differentiable and $\tilde{W}_-$ is a smooth hypersurface with boundary W_0, diffeomorphic to W_-. C_- is a topological manifold, whose boundary $W_- \cup \tilde{W}_-$ is the double of W_-.

Proof. ε will be determined so that

$$\text{(4.2)} \qquad \langle \nabla F, I \rangle < 2\|I\|^2 \text{ on } C_-, \text{ and } \nabla f, I \text{ independent on } W_0,$$

where I denotes the position vector field. Assuming this estimate we proceed as follows: As a consequence of (4.2), we conclude that $\tilde{W}_-$ is a differentiable manifold with boundary W_0. Furthermore, the position field I is transversal to $\tilde{W}_-$, and the function $g(z) = f(z) + \varepsilon^2 - \|z\|^2$ decreases radially in C_-. For $z \in W_-$ we define $\tau(z) = \max\{t \,|\, 0 < t \leqslant 1,\ tz \in \tilde{W}_-\}$. Note $g(z) \leqslant 0$ and $g(0) = \varepsilon^2 > 0$. By transversality, τ is continuous, also $\{tz \,|\, \tau_z \leqslant t \leqslant 1\} \subset C_-$, $\tau_z z \in \tilde{W}_-$, and (b) holds. To check (c), take any $z \in C_-$, so $g(z) \leqslant 0$. Since g decreases along the radial ray through z, the ray stays in C_- until it meets W_- at the point $\varepsilon z/\|z\|$. To prove (d), we give a homeomorphism h of the ε-ball D_ε^m with C_+. Let $h(0) = 0$. For $z \neq 0$, let $h(z) = z$ if $\varepsilon z/\|z\| \in W_+$ and $h(z) = \tau(z/\|z\|)z$ if $\varepsilon z/\|z\| \in W_-$. h is a homeomorphism which carries W_- to $\tilde{W}_-$.

To establish the first statement of (4.2), it suffices to find ε so that

$$\text{(4.3)} \qquad \langle \nabla f, I \rangle < 2f(z) + 2\varepsilon^2 \quad \text{for } z \in C_-,$$

since $f(z) + \varepsilon^2 \leqslant \|z\|^2$ on C_-. Write $f = f_d + (f - f_d)$, where f_d is the lowest order term of the Taylor expansion about the origin, of degree d. Note that $d \geqslant 2$, since the origin is a singular point of f. Then (4.3) becomes

$$\text{(4.4)} \qquad (d-2)f_d - 2(f - f_d) + \langle \nabla(f - f_d), I \rangle < 2\varepsilon^2.$$

The left-hand side of (4.4) is of order $\geqslant 3$, therefore if $\varepsilon > 0$ is sufficiently small, the inequality holds for all $z \in C_-$.

The second statement of (4.2) follows immediately from §2, when f is multihomogeneous. The general algebraic case is contained in Corollary 2.9 of [13]. For analytic functions, one can use the "Curve Selection Lemma" (cf. Theorem 1 of [1], for example).

The following lemma contains all the topological information we need.

Lemma 2. (a) *U_- is homotopy equivalent to W_-,*
(b) *U_0 is homeomorphic to*

$$(W_- \times S^{p-1}) \cup (W_0 \times D^p) = \partial(W_- \times D^p),$$

(c) *U_+ is homeomorphic to $(D_\varepsilon^m \times S^{p-1}) \cup (W_+ \times D^p)$,*
where D_ε^m is the closed m-cell of radius ε and $W_+ \subset \partial D_\varepsilon^m = S_\varepsilon^{m-1}$.
(a′) *$\bar{V}_+$ is homotopy equivalent to U_+,*
(b′) *V_0 is homeomorphic to $\partial(U_+ \times D^q)$,*
(c′) *$\bar{V}_-$ is homeomorphic to $(D_\varepsilon^{m+p} \times S^{q-1}) \cup (U_- \times D^q)$.*

Proof. We will prove (a), (b) and (c). The corresponding statements (a′), (b′), (c′) can be obtained similarly by observing that the roles of U_- and U_+ in the proof of Proposition 2 are interchanged if $f(z) + \|x\|^2$ is replaced by $f(z) - \|x\|^2$.

(a) $U_- = \{(x, z) \mid z \in C_-,\ \|x\|^2 = \varepsilon^2 - \|z\|^2\} = \bigcup\{z\} \times S_z^{p-1},\ z \in C_-$, where S_z^{p-1} denotes the sphere of radius $(\varepsilon^2 - \|z\|^2)^{1/2}$ in $\mathbf{R}^p$, i.e. U_- is a singular sphere bundle over C_-. The spheres S_z^{p-1} degenerate to points on W_-. For $z \in W_-$, the union of the sets $\{tz\} \times S_{tz}^{p-1}$, $\tau(z) \leqslant t \leqslant 1$, is homeomorphic to $\{z\} \times D_z^p$, where D_z^p denotes the p-cell of radius $\varepsilon(1 - \tau_z)$. Using (c) of Proposition 2, one can see that U_- is homeomorphic to the singular disc bundle $\bigcup\{z\} \times D_z^p$, $z \in W_-$, over W_-. The cells degenerate to points on $W_0 = \partial W_-$. Clearly this bundle is homotopy equivalent to W_-.

For the proof of (b), note that $U_0 = \{(z, x) \mid z \in \tilde{W}_-, \|x\|^2 = \varepsilon^2 - \|z\|^2\} = \bigcup\{z\} \times S_z^{p-1}, z \in \tilde{W}_-$, which is homeomorphic to $(W_- \times S^{p-1}) \cup (W_0 \times D^p)$. This can be seen by using a collar neighborhood of W_0 in W_-. Observe again that the spheres S_z^{p-1} degenerate to points on $W_0 = \partial W_-$. D^p is the union of the spheres along a normal geodesic in such a collar, starting orthogonal from W_0.

For (c), we first describe U_+ as a singular sphere bundle over the cell C_+: $U_+ = \{(z, x) \mid f(z) + \varepsilon^2 - \|z\|^2 \geqslant 0,\ \|x\|^2 = \varepsilon^2 - \|z\|^2\} = \bigcup\{z\} \times S_z^{p-1},\ z \in C_+$. The spheres over points of $W_+ \subset C_+$ degenerate to points. Since $\tilde{W}_-$ intersects the ε-sphere transversally in W_0, there is a nonvanishing vector field in a neighborhood of W_+ in C_+, which is transversal to W_+ and tangent along $\tilde{W}_-$. Using the flow of this vector field, a neighborhood N_δ of W_+ in C_+ is seen to be homeomorphic to $W_+ \times [0, \delta]$, such that $N_\delta \cap \tilde{W}_- \approx W_0 \times [0, \delta]$. The part of the degenerate sphere bundle over N_δ is homeomorphic to $\bigcup(w, t) \times S_t^{p-1}$, $(w, t) \in W_+ \times [0, \delta]$. S_t^{p-1} is the sphere of radius t. The homeomorphism carries $\{z\} \times S_z^{p-1}$ to $(w, t) \times S_t^{p-1}$. The closure of the complement of N_δ in C_+ is still homeomorphic to C_+, and $\bigcup\{t\} \times S_t^{p-1}$, $t \in [0, \delta]$, is homeomorphic to D^p. From this we obtain that U_+ is homeomorphic to $(C_+ \times S^{p-1}) \cup (W_+ \times D^p)$. By (d) of Proposition 2, C_+ is homeomorphic to D_ε^m. The homeomorphism $D_\varepsilon^m \to C_+$ given there leaves W_+ fixed. This completes the proof of (c).

As an immediate consequence we have

Corollary 1. *Let* $k_0 = \min\{p-1, q-1\}$.

(a) U_+ *and* V_+ *are* $(p-1)$*-connected,*

(b) V_- *is* $(q-1)$*-connected,*

(c) V_0 *is* k_0*-connected.*

Proof. For $0 \leqslant k < p$, a map $S^k \to U^+ \approx (D_\varepsilon^m \times S^{p-1}) \cup (W_+ \times D^p)$ is homotopic to a map $S^k \to D_\varepsilon^m \times S^{p-1}$, since it can be approximated by a map with an image not intersecting $W_+ \times \{0\}$, for dimension reasons. But then it is homotopic to a constant, since it can be first deformed into $\{w\} \times S^{p-1}$, where $w \in W_+$, and then to a point in $\{w\} \times D^p$. The argument for V_- is analogous.

For $0 \leqslant k < k_0$, a map $S^k \to V_0 \approx (U_+ \times S^{q-1}) \cup (U_0 \times D^q)$ is homotopic to a map $\psi: S^k \to U_+ \times S^{q-1}$, as above. The map $\pi \circ \psi$, where $\pi: U_+ \times S^{q-1} \to U_+$ denotes the projection, is homotopic to a constant mapping of S^k to some point $u_0 \in U_0$, since U_+ is $(p-1)$-connected. Hence ψ is homotopic to a map $S^k \to \{u_0\} \times S^{q-1}$, which in turn is homotopic to a constant in $\{u_0\} \times D^q$.

Let A denote any of the three W, U, $\overline{V}$. A_0, A_+, A_- are submanifolds of a sphere S^ν, $A_+ \cup A_- = S^\nu$, $A_+ \cap A_- = A_0$, where $\nu = m-1$ for $A = W$, $\nu = m + p - 1$ for $A = U$ and $\nu = m + p + q - 1$ for $A = \overline{V}$. In the following H means homology, $\tilde{H}$ reduced homology, with coefficients in a field of characteristic zero. Now we have

Lemma 3. $\tilde{H}_k(A_+) \simeq \tilde{H}_{\nu-k-1}(A_-)$, $H_k(A_0) \simeq H_k(A_-) \oplus H_{\nu-k-1}(A_-)$.

Proof. By duality, for any closed $A \subset S^\nu$, $\tilde{H}_k(A) \simeq \tilde{H}_{\nu-k-1}(S^\nu - A)$. Hence

$$\tilde{H}_k(A_+) \simeq \tilde{H}_{\nu-k-1}(S^\nu - A_+) \simeq \tilde{H}_{\nu-k-1}(A_-).$$

The last isomorphism holds, since $A_0 = \partial A_+ = \partial A_-$ has a collar neighborhood in S^ν, and $S^\nu = A_+ \cup A_-$. Similarly

$$\tilde{H}_k(A_0) \simeq \tilde{H}_{\nu-k-1}(S^\nu - A_0) \simeq \tilde{H}_{\nu-k-1}(A_+) \oplus H_{\nu-k-1}(A_-)$$
$$\simeq \tilde{H}_k(A_-) \oplus H_{\nu-k-1}(A_-),$$

and thus $H_k(A_0) \simeq H_k(A_-) \oplus H_{\nu-k-1}(A_-)$.

Corollary 2. *The Euler characteristics satisfy*

$$\chi(A_+) - 1 = (-1)^{\nu-1}(\chi(A_-) - 1),$$
$$\chi(A_0) = \left(1 + (-1)^{\nu-1}\right)\chi(A_-),$$
$$\chi(A_+) = \chi(A_-), \quad \chi(A_0) = 2\chi(A_-) \quad \textit{for } \nu \textit{ odd}.$$

Lemma 3 and (a), (a′) of Lemma 2 show that the homology of all the nine spaces W_0, W_+, W_-, U_0, U_+, U_-, V_0, V_+, V_- is determined by the homology of any one of these spaces. Since we are mainly interested in V_0 and V_-, we only note:

Theorem 3.

$$\tilde{H}_k(V_-) \simeq \tilde{H}_{k-q}(W_-),$$
$$H_k(V_0) \simeq H_k(V_-) \oplus H_{m+p+q-k-2}(V_-).$$
$$\chi(V_-) = (-1)^q(\chi(W_-) - 1) + 1,$$
$$\chi(V_0) = \left(1 + (-1)^{m+p+q}\right)\chi(V_-).$$

If m is even and p, q are odd, then

$$\chi(V_0) = 4 - 2\chi(W_-) = 4 - \chi(W_0).$$

Proof. The proof is obvious from (a), (a′) in Lemma 2, Lemma 3, and its Corollary 2.

5. The special examples

From the geometric point of view, interesting examples arise when V_0 has negative Euler characteristic, and V_- is not a vector bundle over a closed manifold. According to Theorem 3, $\chi(V_0)$ is negative as soon as m is even, p, q are odd, and $\chi(W_0) > 4$. The simplest examples of this type occur when f is a function on $\mathbf{C} \simeq \mathbf{R}^2$ so that W_0 consists of more than 4 points on the unit circle.

For an integer $l \geqslant 2$, we consider the function f: $\mathbf{C} \to \mathbf{R}$, $f(z) = \mathrm{Re}(z^l)$. Then W_0 consists of $2l$ points on the circle. Both W_- and W_+ are unions of l disjoint arcs on the circle. In this context we also write $V(l, p, q)$ instead of $V(f, p, q)$.

Theorem 4. (a) $\chi(V_0) = 4 - 2l$ *for p, q odd.*

(b) *For $q \geqslant 2$, $V_-(l, p, q)$ is a simply connected manifold, which is not of the homotopy type of any closed manifold, if $l \geqslant 3$. In particular, V_- does not admit any complete metric of nonnegative sectional curvature.*

(c) *For any integer* l, *there is an integer* s_0 *such that for any fixed* $s \geqslant s_0$, *there exist* $\varepsilon > 0$ *and an integer* r *so that whenever* $p - q = s$ *and* $p + q \geqslant r$, *the following holds*:

(i) $V_0(l, p, q)$ *has positive Ricci curvature with respect to the natural metric of* §2;

(ii) $V_-(l, p, q)$ *has positive Ricci curvature with respect to the warped metric* g_ε *introduced in* §1.

Proof. (a) is obvious from Theorem 3.

(b) V_- is simply connected for $q \geqslant 2$ by Corollary 1. The Betti numbers of V_- can be computed as follows: $b_0(W_-) = l$, $b_k(W_-) = 0$ for $k \geqslant 1$, since W_- consists of l arcs on the circle. From Theorem 3 we obtain $b_0(V_-) = 1$, $b_q(V_-) = l - 1$, and $b_k(V_-) = 0$ for $k \neq 0, q$. Since V_- is simply connected, any closed manifold of the same homotopy type must be orientable and hence satisfy Poincaré duality. This is excluded by the Betti numbers, as soon as $l \geqslant 3$.

(c) is the statement of Theorem 2 in the case of the special examples.

Remarks. (i) It is a curious fact that some $V_0(l, p, q)$ contain an exotic Brieskorn sphere of codimension $p - q + 1$ with positive Ricci curvature, given by the equations

$$z_0^l + z_1^2 + \cdots + z_q^2 = 0, \qquad \frac{2}{l}|z_0|^2 + |z_1|^2 + \cdots + |z_q|^2 = 1,$$

where $z_0, \cdots, z_q$ are complex variables (cf. [7]). In this context z_0 corresponds to the variable z of $f(z)$, and $z_k = x_k + iy_k$, where x_k, y_k are the variables in our equations, $1 \leqslant k \leqslant q$.

(ii) Lemma 2 can be used to show that V_0 is homeomorphic to a manifold obtained from $S^{p+1} \times S^{q-1}$ by surgery. Take l disjoint $(p+1)$-cells D^{p+1} in S^{p+1}, remove $\operatorname{int}(D^{p+1} \times S^{q-1})$ from $S^{p+1} \times S^{q-1}$, and attach l disjoint copies of $S^p \times D^q$ along the common boundary:

$$V_0 \simeq \{S^{p+1} \times S^{q-1} - l(D^{p+1} \times S^{q-1})\} \cup l(S^p \times D^q).$$

This can be seen as follows. According to Lemma 2, $V_0 = \partial\{[(D^m \times S^{p-1}) \cup (W_+ \times D^p)] \times D^q\}$. Since $W_+ = S^{m-1}\text{-int}(W_-)$, we have

$$\begin{aligned} V_0 &= \partial\{[D^m \times S^{p-1} \cup S^{m-1} \times D^q - \operatorname{int}(W_- \times D^p)] \times D^q\} \\ &= \{D^m \times S^{p-1} \cup S^{m-1} \times D^p - \operatorname{int}(W_- \times D^p)\} \\ &\quad \times S^{q-1} \cup (\partial(W_- \times D^p)) \times D^q. \end{aligned}$$

Now $D^m \times S^{p-1} \cup S^{m-1} \times D^p = \partial(D^m \times D^p) \simeq S^{m+p+1}$.

In our examples, $m = 2$ and W_- consists of l disjoint 1-cells, so $W_- \times D^p$ is a union of l disjoint $(p+1)$-cells.

(iii) The number of diffeomorphism types of compact manifolds $V_0(f, p, q)$ with positive Ricci curvature is increasing rapidly with the dimension n, although still finite for any fixed n. The possible homotopy types seem quite general, in a stable sense (cf. §3.) Similar considerations apply to the complete examples $V_-(f, p, q)$. They are of finite type by construction. We finally point out that the V_0 arise metrically as submanifolds of euclidean spaces, with the smallest interesting codimension 2. A positively Ricci curved hypersurface in $\mathbf{R}^n$ has necessarily positive sectional curvature, and is therefore the boundary of a strictly convex body. Furthermore, it is easy to see that V_- with the metric g_ε arises isometrically with optimal codimension 2 in $\mathbf{R}^{n+1}$ as the graph of a function on V_- in the ellipsoid.

References

[1] F. Bruhat & H. Cartan, *Sur la structure des sous-ensembles analytiques rèels*, C. R. Sci. Paris Sér. A **244** (1957) 988–990.

[2] J. Cheeger & D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. (2) **96** (1972) 413–443.

[3] J. H. Eschenburg, *Freie isometrische Aktionen auf kompakten Lie-Gruppen mit positiv gekrümmten Orbiträumen*, Schriftenreihe des Math. Instituts der Universität Münster.

[4] D. Gromoll, W. Klingenberg & W. Meyer, *Riemannsche Geometrie im Grossen*, Lecture Notes in Math., Vol. 55, Springer, Berlin, 1975.

[5] D. Gromoll & W. Meyer, *On complete open manifolds of positive curvature*, Ann. of Math. (2) **90** (1969) 75–90.

[6] R. S. Hamilton, *3-manifolds with positive Ricci curvature*, J. Differential Geometry **17** (1982) 255–306.

[7] H. Hernandez-Andrade, *A class of compact manifolds with positive Ricci curvature*, Proc. Sympos. Pure Math., Vol. 27, Part 1, Amer. Math. Soc., Providence, RI, 1975, 73–88.

[8] N. J. Hitchin, *Harmonic spinors*, Advances in Math. **14** (1974) 1–55.

[9] P. M. Ingram, *Extension aspects and new examples of positively Ricci curved manifolds*, Thesis, Stony Brook, 1981.

[10] H. King, *The number of critical points in Morse approximations*, Compositio Math. **34** (1977) 285–288.

[11] W. Klingenberg, *Contributions to riemannian geometry in the large*, Ann. of Math. (2) **69** (1959) 654–666.

[12] H. B. Lawson, *The unknottedness of minimal embeddings*, Invent. Math. **11** (1970).

[13] J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies No. 61, Princeton University Presss, Princeton, NJ, 1968.

[14] J. Nash. *Positive Ricci curvature on fibre bundles and exotic spheres*, J. Differential Geometry **14** (1979) 241–254.

[15] W. A. Poor, *Some exotic spheres with positive Ricci curvature*, Math. Ann. **216** (1975) 245–252.

[16] R. Schoen & S. T. Yau, *Complete 3-dimensional manifolds with positive Ricci curvature and scalar curvature*, Seminar on Differential Geometry, Ann. of Math. Studies No. 102, Princeton University Press, Princeton, NJ, 1982, 209–238.

[17] Seminaire Palaiseau 1978, *Première classe de Chern et courbure de Ricci: preuve de la conjecture de Calabi*, Société Mathématique de France, Astérisque 58 (1978).

[18] F. Warner, *Extensions of the Rauch comparison theorem to submanifolds*, Trans. Amer. Math. Soc. **122** (1966) 341–356.

STATE UNIVERSITY OF NEW YORK AT STONY BROOK
UNIVERSITÄT MÜNSTER