Michigan Math. J. 63 (2014), 41–63

Examples of Dynamical Degree Equals Arithmetic Degree

Shu Kawaguchi & Joseph H. Silverman

Introduction

Let $X/\mathbb{C}$ be a smooth projective variety, let $f : X \to X$ be a dominant rational map, and let $f^* : \mathrm{NS}(X)_\mathbb{R} \to \mathrm{NS}(X)_\mathbb{R}$ be the induced map on the Néron–Severi group $\mathrm{NS}(X)_\mathbb{R} = \mathrm{NS}(X) \otimes \mathbb{R}$. Further, let $\rho(T, V)$ denote the spectral radius of a linear transformation $T : V \to V$ of a real or complex vector space. Then the (*first*) *dynamical degree of* f is the quantity

$$\delta_f = \lim_{n\to\infty} \rho((f^n)^*, \mathrm{NS}(X)_\mathbb{R})^{1/n}.$$

Alternatively, if we let H be any ample divisor on X and $N = \dim(X)$, then δ_f is also given by the formula

$$\delta_f = \lim_{n\to\infty} ((f^n)^* H \cdot H^{N-1})^{1/n}.$$

See [13, Proposition 1.2(iii)] and [19]. Dynamical degrees have been much studied over the past couple of decades; see [19] for a partial list of references.

In two earlier papers [19; 26], the authors studied an analogous arithmetic degree, which we now describe. Assume that X and f are defined over $\bar{\mathbb{Q}}$, and write $X(\bar{\mathbb{Q}})_f$ for the set of points P whose forward f-orbit

$$\mathcal{O}_f(P) = \{P, f(P), f^2(P), \ldots\}$$

is well defined. (There are always many such points; see [1].) Further, let

$$h_X : X(\bar{\mathbb{Q}}) \to [0, \infty)$$

be a Weil height on X relative to an ample divisor, and let $h_X^+ = \max\{1, h_X\}$. The *arithmetic degree of* f *at* $P \in X(\bar{\mathbb{Q}})_f$ is the quantity

$$\alpha_f(P) = \lim_{n\to\infty} h_X^+(f^n(P))^{1/n}, \tag{1}$$

assuming that the limit exists. We also define *upper and lower arithmetic degrees* by the formulas

$$\overline{\alpha}_f(P) = \limsup_{n\to\infty} h_X^+(f^n(P))^{1/n} \quad \text{and} \quad \underline{\alpha}_f(P) = \liminf_{n\to\infty} h_X^+(f^n(P))^{1/n}.$$

Received December 18, 2012. Revision received August 27, 2013.

The first author's research is supported by KAKENHI 24740015. The second author's research is supported by NSF DMS-0854755 and Simons Foundation grant #241309.

It is proven in [19] that the values of $\overline{\alpha}_f(P)$ and $\underline{\alpha}_f(P)$ are independent of the choice of the height function h_X.

A principal result of [19] is the fundamental inequality

$$\overline{\alpha}_f(P) \le \delta_f \quad \text{for all } P \in X(\bar{\mathbb{Q}})_f. \tag{2}$$

The papers [19; 26] also contain a number of conjectures, which we recall here. The conjectures give additional properties and relations for arithmetic and dynamical degrees.

CONJECTURE 1. *Let $X/\bar{\mathbb{Q}}$ be a smooth projective variety, let $f : X \to X$ be a dominant rational map defined over $\bar{\mathbb{Q}}$, and let $P \in X(\bar{\mathbb{Q}})_f$.*

(a) *The limit (1) defining $\alpha_f(P)$ exists.*
(b) *If $\mathcal{O}_f(P)$ is Zariski dense in X, then $\alpha_f(P) = \delta_f$.*
(c) *The number $\alpha_f(P)$ is an algebraic integer.*
(d) *The collection of arithmetic degrees*

$$\{\alpha_f(Q) : \ Q \in X(\bar{\mathbb{Q}})_f\}$$

is a finite set.

In [19], we stated without proof a number of cases for which we could prove Conjecture 1, and we promised that the proofs would appear in a subsequent publication. This paper, which is that publication, contains proofs of the following results.

THEOREM 2. *Conjecture 1 is true in the following situations*:

(a) *The map f is a morphism and* $\mathrm{NS}(X)_{\mathbb{R}} = \mathbb{R}$.
(b) *The map f is the extension to $\mathbb{P}^N$ of a regular affine automorphism $\mathbb{A}^N \to \mathbb{A}^N$.* (*See Section 1 for the definition of regular affine automorphism.*)
(c) *The variety X is a smooth projective surface and f is an automorphism.*
(d) *The map $f : \mathbb{P}^N \to \mathbb{P}^N$ is a monomial map and we consider only points $P \in \mathbb{G}_m^N(\bar{\mathbb{Q}})$.*

The proofs of (a), (b), (c), and (d) are given, respectively, in Sections 1.1, 1.2, 1.3, and 1.4.

The following weaker result, which provides some additional evidence for Conjecture 1, was also stated without proof in [19]. The proof is given in this paper.

THEOREM 3. *Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be an affine morphism defined over $\bar{\mathbb{Q}}$ whose extension to $\mathbb{P}^2$, which by a slight abuse of notation we denote also by $f : \mathbb{P}^2 \to \mathbb{P}^2$, is dominant. Assume that either of the following is true*:

(a) *The map f^m is algebraically stable for some $m \ge 1$.* (*See Section 1 for the definition of algebraic stability.*)
(b) $\deg(f) = 2$, *that is, f is a quadratic map.*

Then

$$\{P \in \mathbb{A}^2(\bar{\mathbb{Q}}):\ \alpha_f(P) = \delta_f\}$$

contains a Zariski dense set of points having disjoint orbits.

The proof of Theorem 3 uses p-adic methods, weak lower canonical heights, and Guedj's classification of degree 2 planar maps [12]. The tools that we develop, specifically Proposition 16 and Lemma 21, can be used to prove Theorem 3 more generally for affine morphisms of any dimension having a periodic point in the hyperplane at infinity.

Remark 4. Jonsson and Wulcan [16] have proven a result on dynamical canonical heights that implies parts of Conjecture 1 for polynomial morphisms $\phi : \mathbb{A}^2 \to \mathbb{A}^2$ of small topological degree, that is, maps satisfying $\# f^{-1}(Q) < \delta_f$ for a general point $Q \in \mathbb{A}^2(\bar{\mathbb{Q}})$. Their proof uses a recent dynamical compactification of $\mathbb{P}^2(\mathbb{C})$ due to Favre and Jonsson [10].

Remark 5. We also mention the following related results from [20]. If $f : X \to X$ is a morphism, then for all $P \in X(\bar{\mathbb{Q}})$, the limit defining $\alpha_f(P)$ exists, and further the set $\{\alpha_f(Q):\ \ Q \in X(\bar{\mathbb{Q}})\}$ is a finite set of algebraic integers. In other words, Conjecture 1(a, c, d) is true for morphisms; but we are not able to prove Conjecture 1(b) in this general setting. However, if X is an abelian variety and $f : X \to X$ is an endomorphism, then Conjecture 1(b) is proven in [20] using results on nef canonical heights.

1. Proof of Theorem 2

In this section we prove the various parts of Theorem 2, which give cases for which Conjecture 1 is true. We begin with the definition of algebraic stability that is due to Fornaess and Sibony.

Definition. Let $f : X \to X$ be a dominant rational map. The map f is said to be *algebraically stable* (in codimension 1) if the induced maps on $\mathrm{NS}(X)_{\mathbb{R}}$ satisfy $(f^*)^n = (f^n)^*$ for all $n \geq 1$.

We remark that the maps in Theorem 2(a, b, c) are algebraically stable. This is automatic for morphisms, and it is also a standard fact that it is true for regular affine automorphisms. Further, if f is algebraically stable, then

$$\delta_f = \lim_{n\to\infty} \rho((f^n)^*)^{1/n} = \lim_{n\to\infty} \rho((f^*)^n)^{1/n} = \rho(f^*), \tag{3}$$

so δ_f is automatically an algebraic integer. Monomial maps are not, in general, algebraically stable, but their dynamical degrees are known to be algebraic integers [14]. Thus, in the proof of Theorem 2, if we prove that $\alpha_f(P) = \delta_f$, then we also know that $\alpha_f(P)$ is an algebraic integer.

1.1. Proof of Theorem 2(a)

We start with a useful lemma.

LEMMA 6. *Let $X/\bar{\mathbb{Q}}$ be a projective variety, and let $f : X \to X$ be a dominant rational map defined over $\bar{\mathbb{Q}}$. Assume further that either of the following is true:*

- *The variety X is smooth;*
- *The map $f : X \to X$ is a morphism.*

Suppose further that $\delta_f = 1$. Then, for all $P \in X(\bar{\mathbb{Q}})_f$, the arithmetic degree $\alpha_f(P)$ exists and satisfies $\alpha_f(P) = 1$.

Proof. The fundamental inequality $\overline{\alpha}_f(P) \le \delta_f$ proven in [19], combined with trivial estimates and the assumption that $\delta_f = 1$, gives a string of inequalities

$$1 \le \underline{\alpha}_f(P) \le \overline{\alpha}_f(P) \le \delta_f = 1.$$

Hence the limit $\alpha_f(P)$ exists and is equal to 1. □

We next prove a result that is somewhat more general than Theorem 2(a).

PROPOSITION 7. *Let $X/\bar{\mathbb{Q}}$ be a normal projective variety, let $f : X \to X$ be a morphism defined over $\bar{\mathbb{Q}}$, and suppose that there is an* ample *divisor class $D \in \mathrm{NS}(X)_{\mathbb{R}}$ satisfying*

$$f^*D \equiv \delta_f D.$$

Let $P \in X(\bar{\mathbb{Q}})$. Then

$$\alpha_f(P) = \begin{cases} 1 & \text{if } P \text{ is preperiodic,} \\ \delta_f & \text{if } P \text{ is wandering, that is, not preperiodic.} \end{cases}$$

Proof. If P is preperiodic, then directly from the definition we see that $\alpha_f(P) = 1$. Also, if $\delta_f = 1$, then Lemma 6 says that $\alpha_f(P) = 1 = \delta_f$.

We assume now that P is not preperiodic and that $\delta_f > 1$. The fact that $\delta_f > 1$ and $f^*D \equiv \delta_f D$ means that we are in the situation to apply the canonical height $\hat{h}_{D,f}$ described in [19, Theorem 5]. Since we have assumed that the divisor D is ample and that P is not preperiodic, we see from [19, Theorem 5(d)] that $\hat{h}_{D,f}(P) \ne 0$. Then [19, Theorem 5(c)] tells us that $\underline{\alpha}_f(P) \ge \delta_f$. However, (2) says that $\overline{\alpha}_f(P) \le \delta_f$, which shows that the limit defining $\alpha_f(P)$ exists and satisfies $\alpha_f(P) = \delta_f$. □

We next use Proposition 7 to prove Theorem 2(a).

Proof of Theorem 2(a). Let D be an ample divisor on X. The assumption that $\mathrm{NS}(X)_{\mathbb{R}} = \mathbb{R}$ implies that $f^*D \equiv dD + T$ for some $d \in \mathbb{R}$ and $T \in \mathrm{NS}(X)_{\mathrm{tors}}$. Replacing D by a multiple, we may assume that $T = 0$, so $f^*D \equiv dD$. Since f is a morphism, we have $(f^n)^*D \equiv (f^*)^n D \equiv d^n D$, so $d = \delta_f$. We are thus in exactly the situation to apply Proposition 7. We conclude that $\alpha_f(P) = 1$ or δ_f, respectively, depending on whether P is or is not preperiodic. □

REMARK 8. We mention that for a variety such as $\mathbb{P}^N$, which has $\mathrm{Pic}(X)_{\mathbb{R}} = \mathrm{NS}(X)_{\mathbb{R}} = \mathbb{R}$, Proposition 7 is an immediate consequence of the classical theory of canonical heights for polarized dynamical systems; see for example [8; 25]. Thus f^*D is linearly equivalent to dD with $d = \delta_f$, and the associated canonical height $\hat{h}_{D,f}$ satisfies

$$h_D(f^n(P)) = \hat{h}_{D,f}(f^n(P)) + O(1) = d^n \hat{h}_{D,f}(P) + O(1).$$

Hence

$$\alpha_f(P) = \lim_{n\to\infty} h^+(f^n(P))^{1/n} = \begin{cases} d & \text{if } \hat{h}_{D,f}(P) > 0, \\ 1 & \text{if } \hat{h}_{D,f}(P) = 0. \end{cases}$$

This completes the proof, since $\hat{h}_{D,f}(P) > 0$ if P is wandering and $\hat{h}_{D,f}(P) = 0$ if P is preperiodic. We note that the proof of Proposition 7 is similar, but since Proposition 7 assumes only an algebraic equivalence $f^*D \equiv \delta_f D$, it requires an expanded theory of canonical heights [19] in which the $O(1)$ error is replaced by the weaker $O(\sqrt{h_D})$ one.

1.2. *Proof of Theorem 2(b)*

We first recall two definitions, the latter due to Sibony.

DEFINITION. Let $f : X \to X$ be a rational map. The *indeterminacy locus of f*, which we denote by I_f, is the subvariety of X on which f is not well defined.

DEFINITION. Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be an automorphism. By abuse of notation, we write f and f^{-1} also for the extensions of f and f^{-1} to rational maps $\mathbb{P}^N \to \mathbb{P}^N$, and we write I_f and $I_{f^{-1}}$ for their indeterminacy loci in $\mathbb{P}^N$. The map f is a *regular affine automorphism* if $I_f \cap I_{f^{-1}} = \emptyset$.

The next result on regular affine automorphisms will be used in the proof of Proposition 11.

PROPOSITION 9. *Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a regular affine automorphism of degree $d \geq 2$, let $X = I_{f^{-1}} \subset \mathbb{P}^N \setminus \mathbb{A}^N$, and let $\ell = \dim(X)$.*

(a) *There exists a surjective morphism $\pi : \mathbb{P}^\ell \to X$, so in particular, X is irreducible.*
(b) *We have $f(\mathbb{P}^N \setminus (\mathbb{A}^N \cup I_f)) = X$.*
(c) *The restriction of f to X gives a surjective morphism*

$$g = f|_X : X \to X.$$

Proof. See [23, Proposition 2.5.4] for (a) and [23, Proposition 2.5.3] for (b) and (c). □

We will also need a basic lemma on the injectivity of Néron–Severi maps.

LEMMA 10. *Let $f : Y \to X$ be a surjective morphism of normal projective varieties. Then $f^* : \mathrm{NS}(X) \to \mathrm{NS}(Y)$ is injective.*

Proof. Suppose that $D \in \mathrm{NS}(X)$ satisfies $f^*D = 0$ in $\mathrm{NS}(Y)$. Let C be a curve in X, and choose a curve $\tilde{C} \subset Y$ such that $f(\tilde{C}) = C$ as sets. Also, let d be the degree of the restriction $f|_{\tilde{C}} : \tilde{C} \to C$, so in particular $d \ge 1$. Then the projection formula gives

$$0 = f_*(f^*D \cdot \tilde{C}) = D \cdot f_*\tilde{C} = d(D \cdot C).$$

This is true for every curve $C \subset X$, and hence $D \equiv 0$ in $\mathrm{NS}(X)$. □

We are now ready to prove a result on regular affine automorphisms that implies Theorem 2(b).

PROPOSITION 11. *Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a regular affine automorphism of degree $d \ge 2$ defined over $\bar{\mathbb{Q}}$, and let g denote the restriction of f to $\mathbb{P}^N \setminus \mathbb{A}^N$. Then*

$$\alpha_f(P) = \begin{cases} 1 & \text{if } P \text{ is periodic,} \\ \delta_f & \text{if } P \in \mathbb{A}^N(\bar{\mathbb{Q}}) \text{ is wandering,} \\ \delta_g & \text{if } P \in (\mathbb{P}^N \setminus \mathbb{A}^N)(\bar{\mathbb{Q}})_f \text{ is wandering.} \end{cases}$$

Proof. If P is periodic, it is clear from the definition that $\alpha_f(P) = 1$. We assume henceforth that P is wandering.

If $P \in \mathbb{A}^N(\bar{\mathbb{Q}})$, the proof is similar to the proof sketched in Remark 8, using the theory of canonical heights for regular affine automorphisms developed by the first author. It is proven in [18] that for all $Q \in \mathbb{A}^N(\bar{\mathbb{Q}})$, the limit

$$\hat{h}^+(P) = \lim_{n\to\infty} \frac{1}{d^n} h(f^n(P))$$

exists and satisfies

$$\hat{h}^+(Q) = 0 \quad \Longleftrightarrow \quad Q \text{ is periodic.}$$

Since P is assumed wandering, we have $\hat{h}^+(P) > 0$. Choose an n_0 such that $h(f^n(P)) \ge (d^n/2)\hat{h}^+(P)$ for all $n \ge n_0$. Then

$$\underline{\alpha}_f(P) = \liminf_{n\to\infty} h^+(f^n(P))^{1/n} \ge \liminf_{n\to\infty} \left(\frac{d^n}{2}\hat{h}^+(P)\right)^{1/n} = d.$$

Hence $\underline{\alpha}_f(P) \ge \delta_f$, and combined with (2), we deduce as usual that $\alpha_f(P)$ exists and equals δ_f.

It remains to deal with wandering points in $\mathbb{P}^N(\bar{\mathbb{Q}})_f \setminus \mathbb{A}^N(\bar{\mathbb{Q}})$, that is, points P lying on the hyperplane at infinity. Let $X = I_{f^{-1}}$ and $g = f|_X : X \to X$ be as in Proposition 9. We claim that for $P \in X(\bar{\mathbb{Q}})$, the map g satisfies

$$\alpha_g(P) = \begin{cases} \delta_g & \text{if } P \text{ is wandering,} \\ 1 & \text{if } P \text{ is preperiodic.} \end{cases} \tag{4}$$

We will verify this claim by using Theorem 2(a).

Let $\tilde{X}$ be the normalization of X, and let $p : \tilde{X} \to X$ denote the normalization homomorphism. Since $\mathbb{P}^\ell$ is normal, the map $\pi : \mathbb{P}^\ell \to X$ from Proposition 9(a) lifts to a surjective morphism $\tilde{\pi} : \mathbb{P}^\ell \to \tilde{X}$. Lemma 10 then tells us that $\tilde{\pi}^* : \mathrm{NS}(\tilde{X}) \to \mathrm{NS}(\mathbb{P}^\ell)$ is injective. Since $\mathrm{NS}(\mathbb{P}^\ell) = \mathbb{Z}$, we conclude that $\mathrm{NS}(\tilde{X}) = \mathbb{Z}$.

Let $\tilde{g} : \tilde{X} \to \tilde{X}$ be the normalization morphism. Choose any ample divisor D on X. Since p is a finite morphism, we see that p^*D is ample on $\tilde{X}$. Let $P \in X(\bar{\mathbb{Q}})$ and choose a point $Q \in p^{-1}(P) \subset \tilde{X}(\bar{\mathbb{Q}})$. Then, by functoriality of height functions [15, Theorem B.3.2(b)],

$$h_{p^*D}(\tilde{g}^n(Q)) = h_D(p \circ \tilde{g}^n(Q)) + O(1) = h_D(g^n(P)) + O(1),$$

where the $O(1)$ is independent of n, P, and Q. It follows directly from the definition that $\alpha_{\tilde{g}}(Q)$ exists if and only if $\alpha_g(P)$ exists, and if they exist, then they are equal. Since the dynamical degree is a birational invariant, we have $\delta_g = \delta_{\tilde{g}}$, and since $\tilde{g}$ is a morphism, its dynamical degree is equal to $\rho(\tilde{g}^*)$, which is an algebraic integer.

Since p is finite, the inverse image $p^{-1}(P)$ is a finite set, so P is preperiodic if and only if Q is preperiodic. Assertion (4) now follows from Theorem 2(a).

We now resume the proof of Proposition 11, where we recall that we are reduced to the case that $P \in \mathbb{P}^N_f(\bar{\mathbb{Q}}) \setminus \mathbb{A}^N$. Proposition 9(b) and the fact that $I_{f^{-1}} \cap I_f = \emptyset$ (which is the definition of regularity) imply that $Q \notin I_f$, so the entire forward orbit of Q is well defined, that is,

$$\mathbb{P}^N(\bar{\mathbb{Q}})_f = \mathbb{P}^N(\bar{\mathbb{Q}}) \setminus I_f.$$

In any case, from Proposition 9(b) we see that our wandering point P satisfies $f(P) \in I_{f^{-1}} = X$, so the assertion (4) gives

$$\alpha_g(f(P)) = \delta_g.$$

Since $g = f|_X$ and since we can compute the arithmetic degree using the height associated to any ample divisor, we can compute α_g using a very ample height on X that is the restriction of a very ample height on $\mathbb{P}^N$. Thus

$$\alpha_g(f(P)) = \alpha_f(f(P)) = \alpha_f(P),$$

where the last equality is [19, Lemma 12]. Hence $\alpha_f(P) = \delta_g$. □

1.3. Proof of Theorem 2(c)

In order to prove Theorem 2(c), which deals with automorphisms of smooth projective surfaces, we use the following result of the first author [17], which generalized the second author's construction on K3 surfaces [24]. We also refer the reader to [16], which gives results for surface maps of small topological degree.

PROPOSITION 12 [17]. *Let X be a smooth projective surface defined over $\bar{\mathbb{Q}}$, and let $f : X \to X$ be an automorphism with $\delta_f > 1$.*

(a) *There are only finitely many f-periodic irreducible curves in X. Let E_f be the union of these curves.*
(b) *There are divisors D^+ and D^- in* $\mathrm{Div}(X)_{\mathbb{R}}$ *and associated canonical height functions $\hat{h}^+$ and $\hat{h}^-$ satisfying*

$$\hat{h}^{\pm} = h_{D^{\pm}} + O(1) \quad \textit{and} \quad \hat{h}^{\pm} \circ f^{\pm 1} = \delta_f \hat{h}^{\pm}.$$

(c) *The function* $\hat{h}^+ + \hat{h}^-$ *is a Weil height for a divisor in* $\operatorname{Div}(X)_{\mathbb{R}}$ *that is nef and big.*
(d) *We have* $\hat{h}^+(P) \geq 0$ *and* $\hat{h}^-(P) \geq 0$ *for all* $P \in X(\bar{\mathbb{Q}})$.
(e) *Let* $P \in (X \setminus E_f)(\bar{\mathbb{Q}})$. *Then*

$$\hat{h}^+(P) = 0 \iff \hat{h}^-(P) = 0 \iff P \text{ is periodic.}$$

Proof. (a) is [17, Proposition 3.1]. The rest of Proposition 12 is [17, Theorem 5.2] (including the proof) and [17, Proposition 5.5]. □

Proof of Theorem 2(c). Let X be a smooth projective surface defined over $\bar{\mathbb{Q}}$, let $f : X \to X$ be an automorphism, let E_f be the union of the f-periodic irreducible curves in X as in Proposition 12, and let $P \in X(\bar{\mathbb{Q}})$. We are going to prove that

$$\alpha_f(P) = \begin{cases} 1 & \text{if } P \text{ is periodic or } P \in E_f, \\ \delta_f & \text{if } P \text{ is wandering and } P \notin E_f. \end{cases}$$

If $\delta_f = 1$, then Lemma 6 gives $\alpha_f(P) = 1 = \delta_f$, while if P is periodic, then directly from the definition we have $\alpha_f(P) = 1$, so we assume henceforth that $\delta_f > 1$ and that P is not periodic.

If E_f is nonempty, let $\phi : E_f \to E_f$ denote the restriction of f to E_f. Writing $E = \bigcup C_i$ as a finite union of irreducible curves, there is an iterate ϕ^m such that $\phi^m \in \operatorname{Aut}(C_i)$ for all i. Considering the three cases of genus 0, 1, and greater than 1, we see that automorphisms of curves have dynamical degree 1, so $\delta_{f^m}(C_i) = 1$. It follows as above that $\alpha_{f^m}(P) = 1$ since we can restrict an ample height on X to each C_i. Replacing P by $f^i(P)$ for $0 \leq i < m$, we deduce that $\alpha_f(P) = 1$.

We are now reduced to the case that $\delta_f > 1$, $P \notin E_f$, and P is not periodic. Let $\hat{h}^\pm$ be the canonical heights associated to $D^\pm$ as described in Proposition 12. In particular, we have

$$\hat{h}^+(P) > 0, \qquad \hat{h}^+(f^n(P)) = \delta_f^n \hat{h}^+(P), \quad \text{and} \quad \hat{h}^-(f^n(P)) = \delta_f^{-n} \hat{h}^-(P).$$

We set $h_X = \hat{h}^+ + \hat{h}^-$, which is a Weil height associated to a divisor that is big and nef. This allows us to compute

$$\begin{aligned} \underline{\alpha}_f(P) &= \liminf_{n\to\infty} h_X^+(f^n(P))^{1/n} \\ &= \liminf_{n\to\infty} (\hat{h}^+(f^n(P)) + \hat{h}^-(f^n(P)))^{1/n} \\ &= \liminf_{n\to\infty} (\delta_f^n \hat{h}^+(P) + \delta_f^{-n} \hat{h}^-(P))^{1/n} \\ &= \delta_f, \end{aligned}$$

where to deduce the final equality, we are using the fact that Proposition 12 tells us that $\hat{h}^+(P) > 0$.

On the other hand, we know from (2) that the upper arithmetic degree satisfies $\delta_f \geq \overline{\alpha}_f(P)$, so we have proven that

$$\underline{\alpha}_f(P) \geq \delta_f \geq \overline{\alpha}_f(P) \geq \underline{\alpha}_f(P). \tag{5}$$

Hence all of these quantities are equal, which proves that the limit $\alpha_f(P)$ exists and is equal to δ_f. □

REMARK 13. Interesting cases to which Theorem 2(c) applies are compositions of noncommuting involutions of K3 surfaces in $\mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The height theory of these maps was studied in [3; 4; 24; 27], and Theorem 2(c) for K3 surfaces in $\mathbb{P}^2 \times \mathbb{P}^2$ was already proven by a similar argument in [26, Section 12]. There are also higher dimensional versions of these constructions in which the associated involutions are rational maps, not morphisms. It would be interesting to study $\alpha_f(P)$ for these reversible dynamical systems on Calabi–Yau manifolds. We note that Oguiso and Truong [22] have recently constructed an explicit Calabi–Yau threefold having a primitive automorphism of positive entropy.

1.4. Proof of Theorem 2(d)

The case of monomial maps described in Theorem 2(d) is an immediate consequence of results in [26].

PROPOSITION 14. *Let $A = (a_{ij})$ be an N-by-N matrix with integer coefficients and* $\det(A) \neq 0$, *and let $f_A : \mathbb{P}^N \to \mathbb{P}^N$ be the associated monomial map extending the endomorphism of $\mathbb{G}_m^N \to \mathbb{G}_m^N$ defined by A, that is, extending the map*

$$(t_1, \dots, t_N) \longmapsto (t_1^{a_{11}} t_2^{a_{12}} \cdots t_N^{a_{1N}}, \dots, t_1^{a_{N1}} t_2^{a_{N2}} \cdots t_N^{a_{NN}}).$$

(a) *The set of arithmetic degrees of f_A for points in $\mathbb{G}_m^N(\bar{\mathbb{Q}})$ satisfies*

$$\{\alpha_{f_A}(P) : \ P \in \mathbb{G}_m^N(\bar{\mathbb{Q}})\} \subset \{\textit{eigenvalues of } A\}.$$

In particular, $\alpha_{f_A}(P)$ is an algebraic integer.

(b) *If $P \in \mathbb{G}_m^N(\bar{\mathbb{Q}})$ has a Zariski dense orbit, then $\alpha_{f_A}(P) = \delta_{f_A}$.*

(c) *If $P \in \mathbb{G}_m^N(\bar{\mathbb{Q}})$ satisfies $\alpha_{f_A}(P) < \delta_{f_A}$, then the orbit of P lies in a proper f_A-invariant algebraic subgroup of $\mathbb{G}_m^N$.*

Proof. This (and more) is proven in [26]. In particular, (a) follows from [26, Corollary 32], and (b) and (c) follow by combining [26, Proposition 19(d) and Corollary 29]. □

2. Large Sets of Points Satisfying $\alpha_f(P) = \delta_f$

In this section we describe our main results concerning large sets for which we can prove that $\alpha_f(P) = \delta_f$. The proofs are given in subsequent sections. We recall from Section 1 that f is algebraically stable if $(f^n)^* = (f^*)^n$, and that one consequence of algebraic stability is that δ_f is simply the largest eigenvalue of f^*; see (3). In particular, if $X = \mathbb{P}^N$ and f is algebraically stable, then $\delta_f = \deg(f)$.

We recall that Theorem 3 states that for certain affine morphisms $f : \mathbb{A}^2 \to \mathbb{A}^2$, there is a large set of points P such that $\alpha_f(P) = \delta_f$. Our proof of Theorem 3 actually yields a stronger result, which we now describe.

DEFINITION. Let $f : X \to X$ be a rational map defined over $\bar{\mathbb{Q}}$ with dynamical degree $\delta_f > 1$, and let $D \in \mathrm{Div}(X)_{\mathbb{R}}$. The *weak lower canonical height* associated to f and D is the function

$$\underline{\hat{h}}^{\circ}_{f,D} : X(\bar{\mathbb{Q}})_f \longrightarrow \mathbb{R} \cup \{\infty\}, \qquad \underline{\hat{h}}^{\circ}_{f,D}(P) = \liminf_{n\to\infty} \frac{h_D(f^n(P))}{\delta_f^n}.$$

Here h_D is any Weil height associated to D. Since any two such heights differ by $O(1)$, we see that the value of $\underline{\hat{h}}^{\circ}_{f,D}(P)$ is independent of the choice of h_D. We also note that ∞ is an allowable value of $\underline{\hat{h}}^{\circ}_{f}$.

REMARK 15. The canonical height associated to eigendivisors of morphisms was defined in [8]. A more general definition for rational maps $f : \mathbb{P}^N \to \mathbb{P}^N$ would be

$$\hat{h}_f(P) = \limsup_{n\to\infty} \frac{h(f^n(P))}{n^{\ell}\delta_f^n},$$

where $\ell \geq 0$ is determined by the conjectural estimate $\deg(f^n) \approx n^{\ell}\delta_f^n$ as $n \to \infty$; see [26]. The function $\underline{\hat{h}}^{\circ}_{f}$ differs from $\hat{h}_f$ in two ways. First, it is defined using the liminf, rather than the limsup. Second, the denominator includes only δ_f^n, it has no n^{ℓ} correction factor. The utility of the weak lower canonical height in studying arithmetic degrees is explained in Proposition 16. See also [16] for additional material on canonical heights attached to dominant rational maps.

For self-maps of $\mathbb{P}^N$, it is proven in [26] that

$$\hat{h}_f(P) > 0 \quad \Longrightarrow \quad \overline{\alpha}_f(P) = \delta_f, \tag{6}$$

so the positivity of $\hat{h}_f(P)$ is at least as strong as the equality of the dynamical degree and the upper arithmetic degree. The proof works, mutatis mutandis, to show that if $\underline{\hat{h}}^{\circ}_{f}(P) > 0$, then $\underline{\alpha}_f(P) \geq \delta_f$, and combined with (2), this implies that $\alpha_f(P) = \delta_f$, as in the following useful result.

PROPOSITION 16. *Let $f : X \to X$ be a dominant rational map defined over $\bar{\mathbb{Q}}$ with dynamical degree satisfying $\delta_f > 1$, let $D \in \mathrm{Div}(X)_{\mathbb{R}}$ be any divisor, and let $P \in X(\bar{\mathbb{Q}})_f$. Then*

$$\underline{\hat{h}}^{\circ}_{f,D}(P) > 0 \quad \Longrightarrow \quad \alpha_f(P) = \delta_f.$$

In particular, if $\underline{\hat{h}}^{\circ}_{f,D} > 0$, then the limit (1) defining $\alpha_f(P)$ converges.

Proof. The assumption that $\underline{\hat{h}}^{\circ}_{f,D}(P) > 0$ implies in particular that P is a wandering point. Further, since by definition the height $\underline{\hat{h}}^{\circ}_{f,D}(P)$ is the liminf of $\delta_f^{-n} h_D(f^n(P))$, we can find an integer n_0 such that

$$\delta_f^{-n} h_D(f^n(P)) \geq \frac{1}{2}\underline{\hat{h}}^{\circ}_{f,D}(P) > 0 \quad \text{for all } n \geq n_0.$$

It follows that

$$\liminf_{n\to\infty} h_D(f^n(P))^{1/n} \geq \liminf_{n\to\infty} \delta_f \left(\frac{1}{2}\underline{\hat{h}}^{\circ}_{f,D}(P)\right)^{1/n} = \delta_f.$$

Let $H \in \mathrm{Div}(X)$ be an ample divisor such that $H - D$ is also ample. Then $h_H \geq h_D - C$ for a constant C, so

$$\begin{aligned}\underline{\alpha}_f(P) &= \liminf_{n\to\infty} h_H^+(f^n(P))^{1/n} \\ &\geq \liminf_{n\to\infty} (h_D(f^n(P)) - C)^{1/n} \\ &\geq \delta_f.\end{aligned}$$

This lower bound, combined with the upper bound $\overline{\alpha}_f(P) \leq \delta_f$ from (2), implies that $\alpha_f(P)$ exists and equals δ_f; compare with the final step (5) in the proof of Theorem 2(c). □

QUESTION 17. In the context of Proposition 16, if $\alpha_f(P) = \delta_f$, is it true that there exists a divisor $D \in \mathrm{Div}(X)_{\mathbb{R}}$ such that $\underline{\hat{h}}^{\circ}_{f,D}(P) > 0$?

Using Proposition 16, we see that Theorem 3 is an immediate consequence of the following result.

PROPOSITION 18. *Let $K/\mathbb{Q}$ be a number field, and let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be an affine morphism defined over K whose extension to $\mathbb{P}^2$, which by a slight abuse of notation we denote also by $f : \mathbb{P}^2 \to \mathbb{P}^2$, is dominant and satisfies $\delta_f > 1$. Assume that one of the following is true*:

(a) *The map f^m is algebraically stable for some $m \geq 1$.*
(b) $\deg(f) = 2$, *that is*, f *is a quadratic map.*

Then there is a finite extension K' of K, a prime $\mathfrak{p}$ of K' and a $\mathfrak{p}$-adic open set $U \subset \mathbb{P}^2(K'_{\mathfrak{p}})$ such that

$$\underline{\hat{h}}^{\circ}_f(P) > 0 \quad \textit{for all } P \in U \cap \mathbb{A}^2(K').$$

The next two lemmas will be used in the proof of Proposition 18. The first characterizes algebraic stability in the case of affine morphisms, and the second describes how the dynamical degree and the canonical height change when f is replaced by an iterate.

LEMMA 19. *Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be an affine morphism, and, by abuse of notation, let $f : \mathbb{P}^N \to \mathbb{P}^N$ also denote the rational map obtained by extending f to $\mathbb{P}^N$. Define inductively a sequence of subvarieties of $\mathbb{P}^N$ by*

$$V_0 = \mathbb{P}^N \setminus \mathbb{A}^N \quad \textit{and} \quad V_{n+1} = \overline{f(V_n \setminus I_f)},$$

where the overline indicates taking the Zariski closure. Then

$$f \text{ is algebraically stable} \quad \Longleftrightarrow \quad V_n \neq \emptyset \quad \text{for all } n \geq 0.$$

Proof. This follows from [23, Proposition 1.4.3], although it is not written there in quite this form. □

LEMMA 20. *Let $f : X \to X$ be a dominant rational map, and let $m \geq 1$.*

(a) *The dynamical degree satisfies*

$$\delta_{f^m} = \delta_f^m.$$

(b) *Assume that X and f are defined over $\bar{\mathbb{Q}}$ and that $\delta_f > 1$, and let $P \in X(\bar{\mathbb{Q}})_f$. Then*

$$\underline{\hat{h}}^{\circ}_f(P) = \min_{0 \leq i < m} \delta_f^{-i} \underline{\hat{h}}^{\circ}_{f^m}(f^i(P)).$$

Proof. The assertions follow easily from the definitions of the quantities involved. □

3. Canonical Heights, $\mathfrak{p}$-adic Neighborhoods, and Periodic Points

The following result provides our primary tool for proving Proposition 18.

LEMMA 21. *Let $K/\mathbb{Q}$ be a number field, and let $f : \mathbb{A}^N \to \mathbb{A}^N$ be an affine morphism defined over K whose extension $f : \mathbb{P}^N \to \mathbb{P}^N$ is dominant and satisfies $\delta_f > 1$. Suppose that there exists an integer $m \geq 1$ and a point $Q_0 \in \mathbb{P}^N(K)$ lying on the hyperplane at infinity such that f^m is defined at Q_0 and such that $f^m(Q_0) = Q_0$. Then there is a prime $\mathfrak{p}$ of K, that is, a prime ideal of the ring of integers of K, and a $\mathfrak{p}$-adic neighborhood $U \subset \mathbb{P}^N(K_{\mathfrak{p}})$ of Q_0 such that*

$$\underline{\hat{h}}^{\circ}_f(P) > 0 \quad \text{for all } P \in U \cap \mathbb{A}^N(K).$$

REMARK 22. The study of algebraic points on varieties via $\mathfrak{p}$-adic neighborhoods that are mapped into themselves by algebraic maps, as in Lemma 21, has a long history. One might start by citing the Skolem–Lech–Mahler theorem on linear recurrences and Chaubauty's result on rational points on curves [9], as well as more recent results in arithmetic dynamics, including for example papers on the dynamical Mordell–Lang conjecture and applications to potential density [1; 2; 6; 7; 11; 21]. One might also compare Lemma 21 to the argument in [16], where a key point is the construction of an f-invariant p-adic open set on which the associated local canonical height is positive.

EXAMPLE 23. We note that it is possible for f^m to be defined at Q_0 even if some lower iterate of f is not defined at Q_0. For example, the map

$$f : \mathbb{P}^2 \longrightarrow \mathbb{P}^2, \qquad f([X, Y, Z]) = [Y^2, XZ, Z^2]$$

is not defined at $[1, 0, 0]$, but $f^2 = [X^2, Y^2, Z^2]$ is a morphism. In particular, we have $\delta_{f^2} = \deg(f^2) = 2$, so $\delta_f = \delta_{f^2}^{1/2} = \sqrt{2}$ from Lemma 20(a).

REMARK 24. Under the assumptions of Lemma 21, the map f^m is algebraically stable. Indeed, with notation as in Lemma 19, the assumption that $f^m(Q_0) = Q_0$ implies that $Q_0 \in V_n$ for all n. Then Lemma 19 tells us that f^m is algebraically stable.

EXAMPLE 25. It is easy to construct examples of affine morphisms that do not have any periodic points. For example, consider the map

$$f : \mathbb{A}^2 \longrightarrow \mathbb{A}^2, \qquad f(x,y) = (xy, y+1).$$

Then

$$f^n(x,y) = (xy(y+1)\cdots(y+n-1), y+n),$$

so f has no periodic points in $\mathbb{A}^2$. The extension of f to $\mathbb{P}^2$ satisfies

$$f^n([X,Y,Z]) = [XY(Y+Z)\cdots(Y+(n-1)Z), YZ^n + nZ^{n+1}, Z^{n+1}],$$

so the only possible periodic point of f in $\mathbb{P}^2 \setminus \mathbb{A}^2$ is the point $[1,0,0]$. But f is not defined at $[1,0,0]$, and hence f has no periodic points in $\mathbb{P}^2$. Of course, the map f is not algebraically stable, since $\deg(f^n) = n+1$, so we already know from Remark 24 that Lemma 21 does not apply to f.

EXAMPLE 26. Let $a \in \bar{\mathbb{Q}}^*$ be a number that is not a root of unity. The map

$$f : \mathbb{P}^2 \longrightarrow \mathbb{P}^2, \qquad f([X,Y,Z]) = [aX^2Y, XY^2, Z^3],$$

which extends the affine morphism

$$\mathbb{A}^2 \longrightarrow \mathbb{A}^2, \qquad (x,y) \longmapsto (ax^2y, xy^2),$$

gives an example of an algebraically stable affine morphism having no periodic points on the line $\{Z=0\}$ at infinity. Indeed, if we let

$$e(n) = \frac{1}{2}(3^n+1) \quad \text{and} \quad u(n) = \frac{1}{4}(3^n - 1 + 2n),$$

then one easily checks that

$$f^n([X,Y,Z]) = [a^{u(n)}X^{e(n)}Y^{e(n)-1}, a^{u(n)-n}X^{e(n)-1}Y^{e(n)}, Z^{3^n}],$$

so $\deg(f^n) = 3^n = \deg(f)^n$, which shows that f is algebraically stable. The indeterminacy locus of f is

$$I_f = \{[1,0,0],[0,1,0]\}.$$

Suppose that $[\alpha,\beta,0] \notin I_f$ is a periodic point lying on the line at infinity. Then $\alpha\beta \neq 0$, so using the formula for f^n, we must have

$$\frac{a^{u(n)}\alpha^{e(n)}\beta^{e(n)-1}}{a^{u(n)-n}\alpha^{e(n)-1}\beta^{e(n)}} = \frac{\alpha}{\beta}.$$

This implies that $a^n = 1$, contradicting our choice of a as a nonroot of unity. Hence f has no periodic points on the line at infinity. Of course, it does have periodic points in $\mathbb{A}^2$. More precisely, it has exactly one periodic point, namely the fixed point $(0,0)$.

Proof of Lemma 21. Lemma 20(b) says that

$$\underline{\hat{h}}^{\circ}_f(P) = \min_{0\le i<m} \delta_f^{-i}\,\underline{\hat{h}}^{\circ}_{f^m}(f^i(P)).$$

So if we can prove the theorem for f^m, then we can apply the theorem to the map f^m and each of the points $P, f(P), f^2(P), \dots, f^{m-1}(P)$ to deduce that the theorem is true for the map f and the point P. We may thus replace f with f^m, which reduces us to the case that Q_0 is a fixed point of f. Then, as noted in Remark 24, the map f is automatically algebraically stable, that is, $\delta_f = d = \deg(f)$.

We let $X_1, \dots, X_N, W$ be projective coordinates on $\mathbb{P}^N$, with the hyperplane at infinity being the set $\{W = 0\}$. Making a change of coordinates, we move Q_0 to the point

$$Q_0 = [1, 0, 0, \dots, 0] \in \mathbb{P}^N.$$

Then the assumptions that $f : \mathbb{A}^N \to \mathbb{A}^N$ and $f(Q_0) = Q_0$ imply that f can be written in the form

$$f = [aX_1^d + G_1, G_2, \dots, G_N, W^d] \tag{7}$$

with $a \in K^*$ and $G_1, \dots, G_N \in K[X_1, \dots, X_N, W]$ homogeneous polynomials of degree d that vanish at Q_0, that is, $G_1, \dots, G_N$ are in the ideal generated by $X_2, \dots, X_N, W$.

For any prime $\mathfrak{p}$ of K, we let

$$R_\mathfrak{p} = \{x \in K_\mathfrak{p} : |x|_\mathfrak{p} \le 1\} \quad \text{and} \quad M_\mathfrak{p} = \{x \in K_\mathfrak{p} : |x|_\mathfrak{p} < 1\}$$

denote, respectively, the ring of integers of $K_\mathfrak{p}$ and the maximal ideal of $R_\mathfrak{p}$. We choose a prime $\mathfrak{p}$ of K such that

$$a \in R_\mathfrak{p}^* \quad \text{and} \quad G_1, \dots, G_N \in R_\mathfrak{p}[X, Y],$$

and we consider the $\mathfrak{p}$-adic neighborhood of Q_0 defined by

$$U = \{[x_1, x_2, \dots, x_N, w] : x_1 \in R_\mathfrak{p}^* \text{ and } x_2, \dots, x_N, w \in M_\mathfrak{p}\}.$$

Using (7) and the facts that $a \in R_\mathfrak{p}^*$ and $G_1, \dots, G_N$ are in the ideal generated by $X_2, \dots, X_N$, it is clear that $f(U) \subset U$. More precisely, if we choose a point

$$P = [\alpha_1, \dots, \alpha_N, \beta] \in U \quad \text{with } \alpha_1 \in R_\mathfrak{p}^* \text{ and } \alpha_2, \dots, \alpha_N, \beta \in M_\mathfrak{p},$$

then we can write $f^n(P)$ as

$$f^n(P) = [\alpha_1^{(n)}, \dots, \alpha_N^{(n)}, \beta^{d^n}] \in U \quad \text{with } \alpha_1^{(n)} \in R_\mathfrak{p}^* \text{ and } \alpha_2^{(n)}, \dots, \alpha_N^{(n)}, \beta \in M_\mathfrak{p}.$$

The key point to note here is that we cannot cancel any factors of $\mathfrak{p}$ from these homogeneous coordinates of $f^n(P)$, because the first coordinate is a unit.

We now compute

$$\begin{aligned} h(f^n(P)) &= h([\alpha_1^{(n)}, \dots, \alpha_N^{(n)}, \beta^{d^n}]) \\ &= h\left(\left[\frac{\alpha_1^{(n)}}{\beta^{d^n}}, \dots, \frac{\alpha_N^{(n)}}{\beta^{d^n}}, 1\right]\right) \\ &= \sum_{v \in M_K} \log \max\left\{\left\|\frac{\alpha_1^{(n)}}{\beta^{d^n}}\right\|_v, \dots, \left\|\frac{\alpha_N^{(n)}}{\beta^{d^n}}\right\|_v, 1\right\} \\ &\ge \log \max\left\{\left\|\frac{\alpha_1^{(n)}}{\beta^{d^n}}\right\|_\mathfrak{p}, \dots, \left\|\frac{\alpha_N^{(n)}}{\beta^{d^n}}\right\|_\mathfrak{p}, 1\right\} \end{aligned}$$

$$= \log \|\beta\|_{\mathfrak{p}}^{-d^n} \quad \text{since } \|\alpha_1^{(n)}\|_{\mathfrak{p}} = 1 \text{ and } \|\alpha_i^{(n)}\|_{\mathfrak{p}} \le 1 \text{ for all } i,$$
$$= d^n \log \|\beta\|_{\mathfrak{p}}^{-1}.$$

Note that this inequality holds for all $n \ge 0$. We are given that $\beta \in M_{\mathfrak{p}}$, so $\log \|\beta\|_{\mathfrak{p}}^{-1} > 0$. Hence

$$\underline{\hat{h}}^{\circ}_f(P) = \liminf_{n\to\infty} \frac{h(f^n(P))}{\delta_f^n} = \liminf_{n\to\infty} \frac{h(f^n(P))}{d^n} \ge \log \|\beta\|_{\mathfrak{p}}^{-1} > 0.$$

This concludes the proof of Lemma 21. □

4. Algebraically Stable Affine Maps on $\mathbb{A}^2$

In this section we use Lemma 21 to prove Proposition 18(a). We use the assumed algebraic stability of f and a case-by-case analysis to find the required periodic point lying on the line at infinity.

Proof of Theorem 18(a). If $\delta_f = 1$, then Lemma 6 says that $\alpha_f(P) = \delta_f = 1$, so we may assume that $\delta_f > 1$. Let $d = \deg(f)$. We write f in homogeneous form as

$$f(X,Y,Z) = [F(X,Y) + ZF_1(X,Y,Z), G(X,Y) + ZG_1(X,Y,Z), Z^d].$$

Since f has degree d, we see that at least one of F and G is nonzero. Changing coordinates, we may assume that $F \ne 0$.

Let $H = \gcd(F,G) \in K[X,Y]$, and write

$$F = HF_0 \quad \text{and} \quad G = HG_0 \quad \text{with } \gcd(F_0,G_0) = 1,$$

so the map f has the form

$$\begin{aligned} f(X,Y,Z) = [&H(X,Y)F_0(X,Y) + ZF_1(X,Y,Z), \\ &H(X,Y)G_0(X,Y) + ZG_1(X,Y,Z), Z^d]. \end{aligned}$$

Since

$$\deg G_0 = \deg G - \deg H = d - \deg H = \deg F - \deg H = \deg F_0, \tag{8}$$

we have a well-defined map

$$\phi = [F_0, G_0] : \mathbb{P}^1 \longrightarrow \mathbb{P}^1. \tag{9}$$

We consider three subcases, depending on the degree of F_0.

Case 1. $\deg(F_0) \ge 2$.

In Case 1, the map ϕ in (9) has degree at least 2. Such a map ϕ has infinitely many distinct periodic orbits in $\mathbb{P}^1(\bar{\mathbb{Q}})$ [5], while there are only finitely many points in $\mathbb{P}^1$ satisfying $H(X,Y) = 0$. (If H is constant, there will be no such points.) So, after replacing K by a finite extension, we can find a ϕ-periodic point $Q_0 = [x_0, y_0] \in \mathbb{P}^1(K)$, say of period m, such that

$$H(\phi^i(Q_0)) \ne 0 \quad \text{for all } 0 \le i < m. \tag{10}$$

By abuse of notation, we also write $Q_0 = [x_0, y_0, 0] \in \mathbb{P}^2$, using the natural identification of $\mathbb{P}^1$ with the line $Z = 0$ in $\mathbb{P}^2$. We note that (10) implies that

$$\begin{aligned} f(Q_0) &= f([x_0, y_0, 0]) \\ &= [H(x_0, y_0)F_0(x_0, y_0), H(x_0, y_0)G_0(x_0, y_0), 0] \\ &= [F_0(x_0, y_0), G_0(x_0, y_0), 0] \end{aligned}$$

is well defined, since F_0 and G_0 have no nontrivial common roots, and more generally (10) ensures that $f^i(Q_0)$ is well defined for all $i \geq 0$. With the identification $\mathbb{P}^1 = \{Z = 0\} \subset \mathbb{P}^2$, we have

$$f^i(Q_0) = \phi^i(Q_0) \quad \text{for all } i \geq 0,$$

and hence the point $Q_0 \in \mathbb{P}^2(K)$ is an m-periodic point for f. It follows from Lemma 21 that there is a prime $\mathfrak{p}$ and a $\mathfrak{p}$-adic neighborhood $Q_0 \in U \subset \mathbb{P}^2(K_\mathfrak{p})$ such that $\underline{\hat{h}}^\circ_f(P) > 0$ for all $P \in U \cap \mathbb{A}^2(K)$.

Case 2. $\deg(F_0) = 0$.

From (8) we see that G_0 is also constant, so f has the form

$$f = [\alpha H(X, Y) + ZF_1(X, Y, Z), \beta H(X, Y) + ZG_1(X, Y, Z), Z^d]$$

for some $[\alpha, \beta] \in \mathbb{P}^1$. If $\beta \neq 0$, we conjugate f by the map

$$\psi(X, Y, Z) = [Y, \beta X - \alpha Y, Z],$$

to obtain

$$f^\psi = \psi \circ f \circ \psi^{-1} = [\beta H \circ \psi^{-1} + ZG_1 \circ \psi^{-1}, Z(\beta F_1 \circ \psi^{-1} - \alpha G_1 \circ \psi^{-1}), Z^d].$$

So in all cases, after possibly changing coordinates and relabeling, we are reduced to studying maps of the form

$$f(X, Y, Z) = [H(X, Y) + ZF_1(X, Y, Z), ZG_1(X, Y, Z), Z^d].$$

Case 2.a. $H(1, 0) = 0$.

Then f is not defined at $[1, 0, 0]$, that is, $[1, 0, 0] \in I_f$, and hence

$$\overline{f(\{Z = 0\} \setminus I_f)} = \{[1, 0, 0]\} \subset I_f.$$

It follows from Lemma 19 that f is not algebraically stable. In fact, already at the second iterate we have $\deg(f^2) < \deg(f)^2$.

Case 2.b. $H(1, 0) \neq 0$.

Then f is defined at $[1, 0, 0]$, and $[1, 0, 0]$ is a fixed point of f, so we can take $Q_0 = [1, 0, 0]$ and $m = 1$ in Lemma 21 to obtain the desired conclusion.

Case 3. $\deg(F_0) = 1$.

In this case the map

$$\phi = [F_0, G_0] : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

has degree 1. Thus ϕ is a linear fractional transformation, so after a change of coordinates of $\mathbb{P}^2$ mapping the line at infinity to itself, the map ϕ may be put into one of the following two forms:

$$\phi(X, Y) = [aX, Y] \quad \text{or} \quad \phi(X, Y) = [X + bY, Y] \quad \text{with } a, b \in \bar{\mathbb{Q}}^*.$$

We note that for any $\gamma \in \mathrm{PGL}_3(\bar{\mathbb{Q}})$, we have

$$\delta_{\gamma\circ f\circ\gamma^{-1}} = \delta_f \quad \text{and} \quad \overline{\alpha}_{\gamma\circ f\circ\gamma^{-1}}(P) = \overline{\alpha}_f(\gamma P),$$

so it is permissible to make this change of coordinates. We write

$$H(X, Y) = c_k X^k Y^{d-1-k} + \cdots \quad \text{with } k \geq 0 \text{ and } c_k \neq 0.$$

We note that unless $a = 1$, the map ϕ has either one or two fixed points, and no other periodic points. If one of those fixed points is not in I_f, then we can apply Lemma 21 to conclude the proof. However, if the fixed points are in I_f, that is, if $H(X, Y)$ vanishes at the fixed points, then f has no periodic points on the line at infinity, so we cannot use Lemma 21. We give an alternative argument that works in all cases.

Let K be a number field containing the coefficients of the polynomials defining f, and let $\mathfrak{p}$ be a nonarchimedean place such that the nonzero coefficients of H, F_0, F_1, G_0, G_1 have $\mathfrak{p}$-adic absolute value 1. We consider the $\mathfrak{p}$-adic open set

$$U = \{P = [x, y, z] \in \mathbb{P}^2(K_\mathfrak{p}) : \ |x|_\mathfrak{p} > |y|_\mathfrak{p} > |z|_\mathfrak{p} \text{ and } |y|_\mathfrak{p}^d > |x|_\mathfrak{p}^{d-1}|z|_\mathfrak{p}\}.$$

We note that a point $[\alpha, \beta, 1] \in \mathbb{A}^2(K_\mathfrak{p})$ is in U if and only if

$$|\alpha|_\mathfrak{p} > |\beta|_\mathfrak{p} > |\alpha|_\mathfrak{p}^{1-1/d} > 1. \tag{11}$$

We are going to prove that

$$\overline{\alpha}_f(P) = \delta_f \quad \text{for all } P \in U \cap \mathbb{A}^2(K).$$

Let $P \in U \cap \mathbb{A}^2(K_\mathfrak{p})$ and write P and $f(P)$ as

$$P = [\alpha, \beta, 1] \quad \text{and} \quad f(P) = [\alpha', \beta', 1].$$

We claim that

$$f(P) \in U \quad \text{and} \quad |\beta'|_\mathfrak{p} \geq |\beta|_\mathfrak{p}^d. \tag{12}$$

The assumption that $P \in U$ tells us that we can write

$$|\beta|_\mathfrak{p} = R \quad \text{and} \quad |\alpha|_\mathfrak{p} = RS \quad \text{with } R > 1, S > 1, \text{ and } R > S^{d-1}. \tag{13}$$

We now verify (12) for the two cases for ϕ.

Case 3.a. $\deg(F_0) = 1$ and $\phi = [aX, Y]$.

We estimate the size of the two terms in the first coordinate of $f(P)$ as

$$|F_0(\alpha,\beta)H(\alpha,\beta)|_{\mathfrak{p}} = |a\alpha|_{\mathfrak{p}} |c_k \alpha^k \beta^{d-1-k}|_{\mathfrak{p}} = |\alpha|_{\mathfrak{p}}^{k+1} |\beta|_{\mathfrak{p}}^{d-1-k} = R^d S^{k+1}$$

and

$$|F_1(\alpha,\beta,1)|_{\mathfrak{p}} \le \max_{0\le i\le d-1} |\alpha|_{\mathfrak{p}}^i |\beta|_{\mathfrak{p}}^{d-1-i} = \max_{0\le i\le d-1} R^{d-1} S^i = (RS)^{d-1}.$$

We know from (13) that $R^d > (RS)^{d-1}$, so the ultrametric inequality gives

$$\begin{aligned} |\alpha'|_{\mathfrak{p}} &= |F_0(\alpha,\beta)H(\alpha,\beta) + F_1(\alpha,\beta,1)|_{\mathfrak{p}} = |F_0(\alpha,\beta)H(\alpha,\beta)|_{\mathfrak{p}} \\ &= R^d S^{k+1}. \end{aligned} \tag{14}$$

Similarly, the second coordinate of $f(P)$ has the two terms

$$|G_0(\alpha,\beta)H(\alpha,\beta)|_{\mathfrak{p}} = |\beta|_{\mathfrak{p}} |c_k \alpha^k \beta^{d-1-k}|_{\mathfrak{p}} = |\alpha|_{\mathfrak{p}}^{k} |\beta|_{\mathfrak{p}}^{d-k} = R^d S^{k} \ge R^d$$

and

$$|G_1(\alpha,\beta,1)|_{\mathfrak{p}} \le \max_{0\le i\le d-1} |\alpha|_{\mathfrak{p}}^i |\beta|_{\mathfrak{p}}^{d-1-i} = \max_{0\le i\le d-1} R^{d-1} S^i = (RS)^{d-1}.$$

Again using $R^d > (RS)^{d-1}$ from (13), we have

$$\begin{aligned} |\beta'|_{\mathfrak{p}} &= |G_0(\alpha,\beta)H(\alpha,\beta) + G_1(\alpha,\beta,1)|_{\mathfrak{p}} = |G_0(\alpha,\beta)H(\alpha,\beta)|_{\mathfrak{p}} \\ &= R^d S^{k}. \end{aligned} \tag{15}$$

Using the formulas $|\alpha'|_{\mathfrak{p}} = R^d S^{k+1}$ and $|\beta'|_{\mathfrak{p}} = R^d S^k$ from (14) and (15), it is now easy to verify the claims in (12). First, we check that $f(P) \in U$. We have

$$|\beta'|_{\mathfrak{p}} = R^d S^k \ge R^d > 1 \quad \text{and} \quad \frac{|\alpha'|_{\mathfrak{p}}}{|\beta'|_{\mathfrak{p}}} = S > 1,$$

and further

$$\frac{|\beta'|_{\mathfrak{p}}^d}{|\alpha'|_{\mathfrak{p}}^{d-1}} = \frac{R^d}{S^{d-k-1}} \ge \frac{R^d}{S^{d-1}} > R^{d-1} > 1.$$

(We have used the inequality $R > S^{d-1}$ from (13).) This shows that $f(P)$ satisfies (11), so $f(P) \in U$. Finally, we have

$$|\beta'|_{\mathfrak{p}} = R^d S^k \ge R^d = |\beta|_{\mathfrak{p}}^d,$$

which completes the proof of (12) in Case 3a.

Case 3.b. $\deg(F_0) = 1$ and $\phi = [X + b, Y]$.

The proof is similar, so we just quickly sketch. We have

$$|F_0(\alpha,\beta)H(\alpha,\beta)|_{\mathfrak{p}} = |\alpha + b|_{\mathfrak{p}} |c_k \alpha^k \beta^{d-1-k}|_{\mathfrak{p}} = |\alpha|_{\mathfrak{p}}^{k+1} |\beta|_{\mathfrak{p}}^{d-1-k} = R^d S^{k+1},$$

$$|F_1(\alpha,\beta,1)|_{\mathfrak{p}} \le \max_{0\le i\le d-1} |\alpha|_{\mathfrak{p}}^i |\beta|_{\mathfrak{p}}^{d-1-i} = \max_{0\le i\le d-1} R^{d-1} S^i = (RS)^{d-1},$$

$$|G_0(\alpha,\beta)H(\alpha,\beta)|_{\mathfrak{p}} = |\beta|_{\mathfrak{p}}|c_k\alpha^k\beta^{d-1-k}|_{\mathfrak{p}} = |\alpha|_{\mathfrak{p}}^k|\beta|_{\mathfrak{p}}^{d-k} = R^dS^k \geq R^d,$$
$$|G_1(\alpha,\beta,1)|_{\mathfrak{p}} \leq \max_{0\leq i\leq d-1} |\alpha|_{\mathfrak{p}}^i|\beta|_{\mathfrak{p}}^{d-1-i} = \max_{0\leq i\leq d-1} R^{d-1}S^i = (RS)^{d-1}.$$

These are the same estimates that we proved in Case 3a, so the rest of the proof of Case 3a carries over verbatim.

We now resume the proof of Case 3. We let $P = [\alpha_0, \beta_0, 1] \in U \cap \mathbb{A}^2(K)$, and for $n \geq 0$ we write

$$f^n(P) = [\alpha_n, \beta_n, 1].$$

Using (12), we see by induction that for all $n \geq 0$ we have

$$f^n(P) \in U \quad \text{and} \quad |\beta_n|_{\mathfrak{p}} = |\beta_0|_{\mathfrak{p}}^{d^n}.$$

We use this to compute

$$\begin{aligned} h(f^n(P)) &= \frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K} \log\max\{|\alpha_n|_v, |\beta_n|_v, 1\} \\ &\geq \frac{1}{[K:\mathbb{Q}]}\log\max\{|\alpha_n|_{\mathfrak{p}}, |\beta_n|_{\mathfrak{p}}, 1\} \\ &\geq \frac{1}{[K:\mathbb{Q}]}\log|\beta_0|_{\mathfrak{p}}^{d^n} \\ &= d^n\frac{\log|\beta_0|_{\mathfrak{p}}}{[K:\mathbb{Q}]}. \end{aligned}$$

Hence

$$\overline{\alpha}_f(P) = \limsup h_X^+(f^n(P))^{1/n} \geq d.$$

On the other hand, from (2) we have $\overline{\alpha}_f(P) \leq \delta_f \leq d$, which completes the proof in Case 3 that $\overline{\alpha}_f(P) = \delta_f$ for all $P \in U \cap \mathbb{A}^2(K)$. (We remark that in Case 3 it is easy to check that f is algebraically stable, so $\delta_f = d$; but in any case, one always has the inequality $\delta_f \leq d$, which is all that we need here.)

Cases 1, 2, and 3 cover all of the possibilities for the map ϕ, which completes the proof of Proposition 18(a). □

5. Degree 2 Affine Maps on $\mathbb{A}^2$

In this section we give the proof of Proposition 18(b), which deals with degree 2 affine morphism of $\mathbb{A}^2$. The proof is a case-by-case analysis, using the classification of dominant quadratic polynomial maps of $\mathbb{A}^2(\mathbb{C})$ due to Guedj. We note that Guedj's proof works over any algebraically closed field of characteristic 0, so in particular it is valid over $\bar{\mathbb{Q}}$. We also note that it suffices to prove Proposition 18(b) for maps that are not algebraically stable, since algebraically stable maps of $\mathbb{A}^2$ are covered by Proposition 18(a). It is worth noting that some nonalgebraically stable maps actually have no periodic points on the line at infinity, so we cannot directly apply Lemma 21 to these maps. For example, taking $f(x, y) = (y, xy)$, it is easy to check that for all $m \geq 1$, the map f^m has no fixed points on the line at infinity. However, for maps of this sort, we are able to directly prove the desired growth of $h_X^+(f^n(P))$ in an appropriate $\mathfrak{p}$-adic neighborhood, leading to the desired conclusion.

PROPOSITION 27 (Guedj [12]). *Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a dominant quadratic polynomial map defined over $\bar{\mathbb{Q}}$. Suppose that f is not algebraically stable and that $\delta_f > 1$. Then f is conjugated by a $\bar{\mathbb{Q}}$-linear automorphism of $\mathbb{A}^2$ to one of the following maps* (*the numbering is from* [12]):

Case	$f(x,y)$	*Conditions*	δ_f
1.1	$(y+c_1, xy+c_2)$	$c_1, c_2 \in \bar{\mathbb{Q}}$	$(1+\sqrt{5})/2$
3.1	(y, x^2+ay+c)	$a, c \in \bar{\mathbb{Q}}$	$\sqrt{2}$
3.2	$(ay+c_1, x(x-y)+c_2)$	$a, c_1, c_2 \in \bar{\mathbb{Q}}$, $a \neq 0$	$(1+\sqrt{5})/2$

REMARK 28. We note that in [12], Guedj gives normal forms for all dominant quadratic polynomial maps of $\mathbb{A}^2$ up to conjugation by $\bar{\mathbb{Q}}$-linear automorphisms.

Proof of Theorem 18(b). For a given point $P = (x_0, y_0) \in \mathbb{A}^2(K)$, we write

$$f^n(P) = (x_n, y_n).$$

As in the proof of Lemma 21, our aim is to find a prime $\mathfrak{p}$ of K and a point P such that

$$\liminf_{n\to\infty} \frac{\log \max\{|x_n|_{\mathfrak{p}}, |y_n|_{\mathfrak{p}}, 1\}}{\delta_f^n} > 0.$$

For such P we have

$$\begin{aligned}
\underline{\hat{h}}_f^{\circ}(P) &= \liminf_{n\to\infty} \frac{h(f^n(P))}{\delta_f^n} \\
&= \liminf_{n\to\infty} \frac{1}{\delta_f^n[K:\mathbb{Q}]} \sum_{w\in M_K} \log\max\{|x_n|_{\mathfrak{p}}, |y_n|_{\mathfrak{p}}, 1\} \\
&\geq \liminf_{n\to\infty} \frac{\log\max\{|x_n|_{\mathfrak{p}}, |y_n|_{\mathfrak{p}}, 1\}}{\delta_f^n[K:\mathbb{Q}]} \\
&> 0,
\end{aligned}$$

which suffices to prove the theorem. We fix a prime $\mathfrak{p}$ such that every nonzero coefficient of f has $\mathfrak{p}$-adic norm equal to 1.

Proposition 18(a) covers all maps that are algebraically stable, that is, maps satisfying $\delta_f = 2$, while maps with $\delta_f = 1$ always have $\alpha_f(P) = 1$ by Lemma 6. We are thus reduced to studying maps satisfying $1 < \delta_f < 2$, which are Cases 1.1, 3.1, and 3.2 in Guedj's classification as described in Proposition 27. We consider each in turn.

CASE 1.1. Take $P = (x_0, y_0) \in \mathbb{A}^2(K)$ with $|x_0|_{\mathfrak{p}} = |y_0|_{\mathfrak{p}} > 1$, and, to ease notation, let $R = |x_0|_{\mathfrak{p}} = |y_0|_{\mathfrak{p}}$. For $n \geq 1$, an easy induction shows that

$$|x_n|_{\mathfrak{p}} = R^{F_{n+1}} \quad \text{and} \quad |y_n|_{\mathfrak{p}} = R^{F_{n+2}},$$

where F_n is the nth Fibonacci number. Hence

$$\liminf_{n\to\infty} \frac{\log\max\{|x_n|_{\mathfrak{p}}, |y_n|_{\mathfrak{p}}, 1\}}{\delta_f^n} = \liminf_{n\to\infty} \frac{F_{n+2}\log R}{\delta_f^n} = \frac{\delta_f^2 \log R}{\sqrt{5}} > 0,$$

since $\delta_f = \frac{1}{2}(1+\sqrt{5})$ and $F_n = (\delta_f^n - \delta_f^{-n})/\sqrt{5}$.

CASE 3.1. Although f is not a morphism of $\mathbb{P}^2$, its second iterate

$$f^2([X, Y, Z]) = [X^2 + aYZ + cZ^2, aX^2 + Y^2 + a^2YZ + (a+1)cZ^2, Z^2]$$

extends to a morphism of $\mathbb{P}^2$. Let ξ be a root of $\xi^2 - \xi + a = 0$ and replace K with $K(\xi)$. Then the point $[1, \xi, 0]$ is a fixed point of f^2 lying on the line at infinity, so we can apply Lemma 21 to obtain the desired result.

CASE 3.2. Take $P = (x_0, y_0) \in \mathbb{A}^2(K)$ with $1 < |x_0|_{\mathfrak{p}} < |y_0|_{\mathfrak{p}}$, and let $R = |y_0|_{\mathfrak{p}}$. For $n \geq 1$, we claim that $|x_n|_{\mathfrak{p}} = R^{F_{n+1}}$ and $|y_n| = R^{F_{n+2}}$, where F_n is the nth Fibonacci number. Indeed, by induction we find that

$$|x_{n+1}|_{\mathfrak{p}} = |ay_n + c|_{\mathfrak{p}} = |y_n|_{\mathfrak{p}} = R^{F_{n+2}}$$

and

$$|y_{n+1}|_{\mathfrak{p}} = |x_n(x_n - y_n) + c_2|_{\mathfrak{p}} = |x_n y_n|_{\mathfrak{p}} = R^{F_{n+1}+F_{n+2}} = R^{F_{n+3}}.$$

Hence just as in Case 1.1 we have

$$\liminf_{n\to\infty} \frac{\log\max\{|x_n|_{\mathfrak{p}}, |y_n|_{\mathfrak{p}}, 1\}}{\delta_f^n} = \liminf_{n\to\infty} \frac{F_{n+2}\log R}{\delta_f^n} = \frac{\delta_f^2 \log R}{\sqrt{5}} > 0.$$

This completes the proof of Proposition 18(b). □

ACKNOWLEDGMENTS. The authors would like to thank ICERM for providing a stimulating research environment during their spring 2012 visits, and the organizers of conferences on Automorphisms (Shirahama 2011), Algebraic Dynamics (Berkeley 2012), and the SzpiroFest (CUNY 2012), during which some of this research was done. The authors would also like to thank Mattias Jonsson for his helpful comments on the initial draft, and the referees for their extensive comments that greatly improved the exposition.

References

[1] E. Amerik, *Existence of non-preperiodic algebraic points for a rational self-map of infinite order,* Math. Res. Lett. 18 (2011), 251–256.

[2] E. Amerik, F. Bogomolov, and M. Rovinsky, *Remarks on endomorphisms and rational points,* Compos. Math. 147 (2011), 1819–1842.

[3] A. Baragar, *Rational points on K3 surfaces in* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, Math. Ann. 305 (1996), 541–558.

[4] A. Baragar and R. van Luijk, *K3 surfaces with Picard number three and canonical vector heights,* Math. Comput. 76 (2007), 1493–1498 (electronic).

[5] A. F. Beardon, *Iteration of rational functions,* Graduate Texts in Mathematics, 132, Springer, New York, 1991.

[6] J. P. Bell, D. Ghioca, and T. J. Tucker, *The dynamical Mordell–Lang problem for étale maps,* Amer. J. Math. 132 (2010), 1655–1675.

[7] R. L. Benedetto, D. Ghioca, P. Kurlberg, and T. J. Tucker, *A case of the dynamical Mordell–Lang conjecture,* Math. Ann. 352 (2012), 1–26.

[8] G. S. Call and J. H. Silverman, *Canonical heights on varieties with morphisms,* Compositio Math. 89 (1993), 163–205.

[9] C. Chabauty, *Sur les points rationnels des courbes algébriques de genre supérieur à l'unité,* C. R. Acad. Sci. Paris 212 (1941), 882–885.

[10] C. Favre and M. Jonsson, *Dynamical compactifications of* $\mathbf{C}^2$*,* Ann. of Math. (2) 173 (2011), 211–248.

[11] D. Ghioca and T. J. Tucker, *Periodic points, linearizing maps, and the dynamical Mordell–Lang problem,* J. Number Theory 129 (2009), 1392–1403.

[12] V. Guedj, *Dynamics of quadratic polynomial mappings of* $\mathbb{C}^2$*,* Michigan Math. J. 52 (2004), 627–648.

[13] ———, *Ergodic properties of rational mappings with large topological degree,* Ann. of Math. (2) 161 (2005), 1589–1607.

[14] B. Hasselblatt and J. Propp, *Degree-growth of monomial maps,* Ergodic Theory Dynam. Systems 27 (2007), 1375–1397. Corrigendum: 27 (2007), 1999.

[15] M. Hindry and J. H. Silverman, *Diophantine geometry: an introduction,* Graduate Texts in Mathematics, 201, Springer, New York, 2000.

[16] M. Jonsson and E. Wulcan, *Canonical heights for plane polynomial maps of small topological degree,* Math. Res. Lett. 19 (2012), 1207–1217.

[17] S. Kawaguchi, *Projective surface automorphisms of positive topological entropy from an arithmetic viewpoint,* Amer. J. Math. 130 (2008), 159–186.

[18] ———, *Local and global canonical height functions for affine space regular automorphisms,* Algebra and Number Theory 7 (2013), 1225–1252, doi: 10.2140/ant.2013.7.1225.

[19] S. Kawaguchi and J. H. Silverman, *On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties,* J. Reine Angew. Math. (2012, to appear), arXiv:1208.0815.

[20] ———, *Dynamical canonical heights for Jordan blocks, arithmetic degrees of orbits, and nef canonical heights on abelian varieties,* 2013, arXiv:1301.4964.

[21] B. Poonen, *p-adic interpolation of iterates,* 2013, arXiv:1307.5887.

[22] K. Oguiso and T. Truong, *Explicit examples of rational and Calabi–Yau threefolds with primitive automorphisms of positive entropy,* 2013, arXiv:1306.1590.

[23] N. Sibony, *Dynamique des applications rationnelles de* $\mathbb{P}^k$*,* Dynamique et géométrie complexes (Lyon, 1997), Panor. Synthèses, 8, pp. ix–x, xi–xii, 97–185, Soc. Math. France, Paris, 1999.

[24] J. H. Silverman, *Rational points on K3 surfaces: a new canonical height,* Invent. Math. 105 (1991), 347–373.

[25] ———, *The arithmetic of dynamical systems,* Graduate Texts in Mathematics, 241, Springer, New York, 2007.

[26] ———, *Dynamical degree, arithmetic entropy, and canonical heights for dominant rational self-maps of projective space,* Ergodic Theory Dynam. Systems, doi:http://dx.doi.org/10.1017/etds.2012.144. Published online: 29 November 2012. arXiv:1111.5664.

[27] L. Wang, *Rational points and canonical heights on K3-surfaces in* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$*,* Recent developments in the inverse Galois problem (Seattle, WA, 1993), Contemp. Math., 186, pp. 273–289, Amer. Math. Soc., Providence, RI, 1995.

S. Kawaguchi
Department of Mathematics
Faculty of Science
Kyoto University
Kyoto, 606-8502
Japan

kawaguch@math.kyoto-u.ac.jp

J. H. Silverman
Mathematics Department
Brown University
Box 1917
Providence, RI 02912
USA

jhs@math.brown.edu