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EXAMPLES OF UNGRADABLE ALGEBRAS

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ABSTRACT. Examples are presented of finite-dimensional algebras that admit no positive grading (that is, a nontrivial grading indexed by the natural numbers). Some of these examples have finite global dimension (they are even quasihereditary), and yield a negative answer to a question of Anick and Green.

A finite-dimensional algebra A over a field K has a *positive semisimple grading* in case there is a K-decomposition

$$A = \bigoplus_{n \geq 0} A_n$$

such that $A_m A_n \subseteq A_{m+n}$ for all nonnegative integers m and n, and the radical of A is

$$J = \bigoplus_{n > 0} A_n.$$

In this case the initial subring $A_0 \cong A/J$. (In [1] such an algebra is called *nontrivially* $\mathbf{N}$*-gradable.*)

It is easy to see that any split finite-dimensional K-algebra A with radical $J = J(A)$ such that $J^3 = 0$ has a positive semisimple grading. Indeed, then A contains a subalgebra A_0 (isomorphic to a direct sum of matrix rings over K) such that ${}_K A = A_0 \oplus J$ and $A_0 \otimes_K A_0^{op}$ is semisimple, so that $J = A_1 \oplus J^2$ as an $A_0 - A_0$ bimodule; the grading is $A = A_0 \oplus A_1 \oplus A_2$ with $A_2 = J^2$. In particular, any split algebra of dimension four over K has a positive semisimple grading because it must have $J^3 = 0$ or be uniserial.

With these observations in mind we present a split K-algebra with

$$\dim({}_K A) = 5 \quad \text{and} \quad J^4 = 0$$

that admits no positive semisimple grading.

1. **Example.** In the free associative algebra $K\langle X, Y\rangle$ over a field K, let I be the ideal generated by the elements X^3, XY, YX^2, $X^2 - Y^3$, and $YX - Y^3$,

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and let $A = K\langle X, Y\rangle/I$. Then A is a local QF algebra that admits no positive semisimple grading.

Proof. Let x and y denote the residue classes of X and Y, respectively. Then the elements $1, x, y, y^2$, and y^3 form a K-basis of A, and we have $x^2 = yx = y^3$, $xy = 0$, and $y^4 = 0$. Letting $J = J(A)$, direct calculations show that the left and right lower Loewy series of A coincide (we denote them by $S = \mathrm{Soc}(R)$, $S_2 = \mathrm{Soc}_2(R)$, etc.) and that

$$J = S_3 = Kx \oplus Ky \oplus Ky^2 \oplus Ky^3; \tag{1}$$

$$S_2 = Kx \oplus Ky^2 \oplus Ky^3 = \{a \in A\colon a^3 = 0\}; \tag{2}$$

$$J^2 = Ky^2 \oplus Ky^3 = \{a \in A\colon a^2 = 0\}; \tag{3}$$

$$J^3 = S = Ky^3. \tag{4}$$

It follows from (4) that A is QF. Considering the K-dimension and the Loewy length of A, we see that a positive semisimple grading of A must have either four or five nonzero terms. We shall show that neither is possible.

Suppose that there is a grading of the form

$$A = A_0 \oplus A_i \oplus A_j \oplus A_k \oplus A_l$$

with each $\dim A_n = 1$ and $0 < i < j < k < l$. Then clearly $A_l = J^3 \neq 0$, so one of A_i^2, A_iA_j, A_jA_i, or A_j^2 is not contained in A_l. If $A_i^2 \not\subseteq A_l$, then $A_i^2 = A_j$ or $A_i^2 = A_k$. But if $A_i^2 = A_j$ then $J^2 = A_j \oplus A_l$ and by (3), $0 \neq A_k^2 = A_l$ and by (2), $0 \neq A_i^3 = A_l$. This leads to the contradiction

$$4i = 2j < 2k = l = 3i.$$

Thus all products A_mA_n must be contained in $A_k \oplus A_l$, and so

$$A_k \oplus A_l = J^2.$$

Now, since $J^3 \neq 0$, it follows that $A_iA_k = A_l$ or $A_jA_k = A_l$. The latter implies $A_iA_k = 0$, so $A_iJ^2 = 0$ and hence $A_i \subseteq S_2$ but $A_i \not\subseteq J^2$. Then by (3) $A_i^2 = A_l$, which yields the contradiction $j + k = l = 2i$. Finally, suppose that $A_iA_k = A_l$, then $A_jA_k = 0$ so $A_jJ^2 = 0$ and $S_2 = A_j \oplus A_k \oplus A_l$. Now by (2) and (3) there are K-generators a for A_i and b for A_j that are K-linear combinations of the form

$$a = \kappa_1 x + y + \kappa_3 y^2 + \kappa_4 y^3$$

and

$$b = x + \lambda_3 y^2 + \lambda_4 y^3.$$

But since $A_iA_j \subseteq JS_2 \subseteq A_l$ and $i + k = l$, $A_iA_j = 0 = A_jA_i$, so this implies

$$(\kappa_1 + 1 + \lambda_3)y^3 = ab = 0 = ba = (\kappa_1 + \lambda_3)y^3.$$

Thus a positive semisimple grading with five nonzero terms cannot exist.

The remaining possibility is a positive semisimple grading of the form

$$A = A_0 \oplus A_i \oplus A_j \oplus A_k$$

with each $A_n \neq 0$. Here we must have $0 \neq J^3 = A_i^3$, which implies

$$j = 2i \quad \text{and} \quad k = 3i;$$

and consequently $A_j^2 = 0$. Thus by (3), $A_j \oplus A_k = J^2$ and if

$$x = a_i + a_j + a_k$$

with the $a_n \in A_n$ then a_i must have no y-coordinate. But then by (2), $a_i J^2 = 0 = J^2 a_i$, so

$$y^3 = x^2 = a_i^2 \in A_k \cap A_j = 0.$$

This contradiction completes the proof. □

In [2, Theorem 10.3] Auslander showed that if A is a finite-dimensional K-algebra with radical J and Loewy length l then the algebra $\mathrm{End}({}_A M)$, with $M = A/J \oplus \cdots \oplus A/J^{l-1} \oplus A$ has global dimension $\leq l$. This construction, together with the example above, yields a split finite-dimensional K-algebra of finite global dimension which admits no positive semisimple grading.

2. **Proposition.** *Let A be a finite-dimensional K-algebra, ${}_A M$ be a finitely generated generator, and $B = \mathrm{End}({}_A M)$. If A admits no positive semisimple grading, then neither does B.*

Proof. It follows from [7, Proposition 2.4] that the existence of a positive semisimple grading is a Morita invariant, so we may assume that ${}_A M = N \oplus A$. Suppose that $B = \mathrm{End}({}_A M)$ and there is such a grading

$$B = \bigoplus_{n \geq 0} B_n.$$

Then according to [3, Proposition 9] B_0 contains a complete set of primitive idempotents for B, so there is an idempotent $e \in B_0$ such that

$$eBe \cong \mathrm{End}({}_A A) \cong A.$$

This yields a positive semisimple grading

$$A \cong eBe = \bigoplus_{n \geq 0} eB_n e. \quad \square$$

3. *Remarks.* (a) The fact is that the algebra A of Example 1 does not admit *any* nontrivial integral grading. Indeed, if ${}_K A = \bigoplus_{n \in \mathbf{Z}} A_n$ with $A_j A_k \subseteq A_{j+k}$ and $J_0 = J(A_0)$, then $J = J_0 \oplus (\bigoplus_{n \neq 0} A_n)$ and there are at most four nonzero terms with $n \neq 0$. If there were only three of these whose indices were not all of the same sign, or if there were two with positive indices and two with negative indices, then the cube of each term in $J = J_0 \oplus (\bigoplus_{n \neq 0} A_n)$ would be zero, contrary to (2). A grading with nonzero terms $A = A_i \oplus A_0 \oplus A_j \oplus A_k \oplus A_l$ and $i < 0 < j < k < l$ would have $J^2 = A_i \oplus A_l$ by (3), and $J_0 = 0$, leading to $A_i = J^3 = A_l$. Thus we see that in an integral grading of A the indices of the nonzero terms must all be nonnegative (or what is the same, nonpositive). Thus, in view of Example 1, we are left with the possibility of $J_0 \neq 0$ and one, two, or three nonzero terms with positive indices; and then since A_0 must be a proper factor ring of A, $J_0^3 = 0$. The cases of one or two nonzero terms can be eliminated using (2) as before. Finally, a grading $A = A_0 \oplus A_i \oplus A_j \oplus A_k$ with

$0 < i < j < k$ and $J_0 \neq 0$ would have $J_0^2 = A_j^3 = A_k^2 = 0$, so by (3) $A_i^2 = A_j$ and $A_i^3 = A_k$. This implies $j = 2i$, $k = 3i$, and so $A_j^2 \subseteq A_{4i} = 0$. Thus by (3), we arrive at the contradiction $J_0 \oplus A_j \oplus A_k \subseteq J^2$.

(b) Exercise 1 in [8, §11.6] presents a four-dimensional commutative algebra R over a field F which is uniserial of composition length 2 such that $R/J(R)$ is not isomorphic to any subring of R (and, so, R is not split). If $R = \bigoplus_{n \in \mathbf{Z}} R_n$ were a nontrivial grading then each R_i, $i \neq 0$, being nilpotent, is in $J(R)$. But $\dim J(R) = 2$ and $R_0 \ncong R/J(R)$, so $R = R_0 \oplus R_i$, $i \neq 0$ and $\dim R_i = 1$. But then R_i is an ideal and R has no ideals of dimension one.

If $C = \operatorname{End}({}_R(R/J(R) \oplus R))$, then C is of global dimension 2 and admits no nontrivial positive grading $C = C_0 \oplus \cdots \oplus C_m$ with $C_m \neq 0$. Indeed, such a grading would yield $R \cong eC_0e \oplus eC_me$ where e is the projection on ${}_RR$ in C, but $eC_me \neq 0$ since C_m is an ideal and eC and Ce are C-faithful.

(c) It was shown in [6] that the endomorphism algebra $\operatorname{End}({}_AM)$, with $M = A/J \oplus \cdots \oplus A/J^{l-1} \oplus A$, is in fact quasihereditary (see [4] or [5] for the definition). Hence there are even quasihereditary algebras with no positive semisimple gradings.

(d) A curious example of a split commutative algebra is obtained when

$$A = K[X, Y]/(X^4 - X^2Y + XY^2, X^4 - X^3, X^5, Y^4).$$

When K is of characteristic not equal to 2, A can be graded. For example, $A_0 = K\cdot 1$, $A_3 = K\cdot(x - \frac{1}{2}y)$, $A_4 = K\cdot(-\frac{5}{4}x^2 + y)$, $A_6 = K\cdot(x^2 + \frac{1}{4}y^2 - xy)$, $A_7 = K\cdot(-\frac{5}{4}x^3 - \frac{1}{2}y^2 + xy + \frac{5}{8}x^2y)$, $A_8 = K\cdot(y^2 - \frac{5}{2}x^2y)$, $A_9 = K\cdot(x^3 - \frac{7}{8}x^4 - \frac{3}{4}x^2y)$, $A_{11} = K\cdot(-\frac{3}{2}x^4 + xy^2)$, and $A_{12} = K\cdot x^4$, where x and y are the images, respectively, of X and Y in A and the K-basis $\{1, x, x^2, x^3, x^4, y, y^2, xy, x^2y\}$ is chosen. On the other hand it can be shown that if $K = \mathbf{Z}/2\mathbf{Z}$, A has no nontrivial integral grading.

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