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EXCEL BASED SETTLEMENT OF BEAMS ON ELASTIC FOUNDATIONS WITH FREE-ENDS AND ARBITRARY LOADING

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The deflection of long footings placed on homogeneous and isotropic soils involves soil-structure differential equations models whose solution may not be possible for most practical problems. The analytical solution of beams on elastic foundation problem involves soil modulus of subgrade reaction and simplifying assumptions relative to applied loading. The exact solutions are available in relatively simple cases of loading, uniform cross sectional properties of the footing and constant soil modulus of subgrade reaction. Therefore, the Finite Difference Method (FDM) or Finite Element Method (FEM), are typically used to compute the deformation of beams with variable loading and geometry resting on elastic foundations with variable modulus of subgrade reaction. The finite differences method was used to solve this problem for long beams with arbitrary loading and constant cross-sections using an Excel Workbook to compute beam deflections providing both numerical and graphical output. The foundation is modeled as a long beam with free ends and a constant modulus of subgrade reaction. The proposed solution presents an efficient method involving a complex ordinary differential equation model for beams on elastic foundations encountered in engineering practice.

Keywords: Finite differences method, Continuous beam, Modulus of subgrade reaction, Cross section, Variable loading.

1 INTRODUCTION AND BACKGROUND

The use of numerical techniques in solving complex engineering problems is widespread. Nevertheless, beams on elastic foundation are most usually analyzed based on Winkler's model in which the soil is replaced by a bed of elastic springs. The theoretical solution of the Winkler model involves several simplifying assumptions and is of limited practical use (Al-Khafaji and Andersland 1991). Disadvantages of the theoretical solution include variable loading configurations including point loads, variability of soil properties under the footing and changes in footing cross sections.

The Finite Difference Method (FDM) for solving the problem of beams resting on elastic foundations provide powerful and practical tools for the practicing geotechnical engineer (Szilard 1974). This paper presents the finite difference solution using Excel spreadsheet to calculate the settlement of continuous long footing with variable loading. The FDM Excel model used is based on dividing the beam into 20 equal segments to permit significant variations in the applied loading configuration.

2 GOVERNING EQUATIONS FOR BEAMS ON ELASTIC FOUNDATIONS

For the case of a beam on an elastic foundation shown in Figure 1, the governing differential equation is developed assuming the contact is never broken between beam and soil (Winterkorn and Fang 1975). Thus, a linear relationship between the force on the foundation (pressure q) and the deflection y is assumed using the modulus of subgrade reaction k.

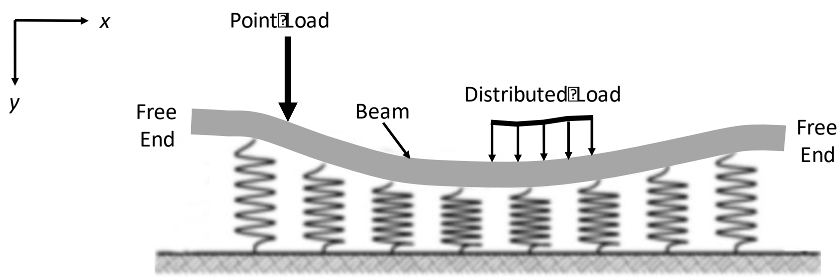

Figure 1. Beam on elastic foundation.

Note that Figure 1 shows a Winkler foundation model where the soil is replaced with elastic springs (Bowles 1982, Hetenyi 1946). The beam is assumed to have arbitrary loading q, elastically supported beam with free ends, and a reaction force ky and the curve y = y(x) is the deflected shape is given by:

$$EI\frac{d^4y}{dx^4} + ky = q \tag{1}$$

where q is the applied loading function, EI defines the beam's flexural rigidity, E is modulus of elasticity and I is the moment of inertia. Note that the Shear V and Moment M can be expressed as:

$$M = -EI\frac{d^2y}{dx^2} \tag{2}$$

$$V = -EI\frac{d^3y}{dx^3} \tag{3}$$

Note that the focus of this paper is limited to the settlement of beams y for foundation placed on elastic soil. Therefore, our focus is on the model presented by Eq. (1), but Eq. (2) and Eq. (3) can be easily evaluated once the deflection values are computed. The application of FDM results in direct relationships between deflection, slope, bending moment, shear force and loading of the beam. Finite difference theory leads to approximations to these derivatives, at the discretized nodes along the beam, in terms of the unknown deflections at the nodes (Al-Khafaji and Tooley 1986).

3 THE FINITE DIFFERENCE METHOD

Finite difference method discretizes the domain into a regular grid defined by a certain number of nodes, which are separated by number of intervals as shown in Figure 2. The FDM consists of replacing the derivatives in differential equations using their equivalent difference expressions based on Taylor's series (Al-Khafaji and Tooley 1986). The resulting finite difference expression is applied repeatedly at each node to yield a system of linear algebraic equations that is solved for the deflections at each node numerically.

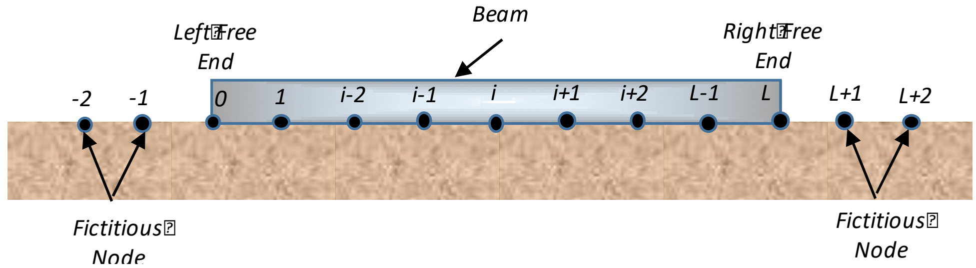

Figure 2. Finite differences grid for solving beam on elastic foundations problem.

Eq. (1) can now be solved using central difference approximations for the first through the fourth derivative as presented below in Eq. (4) to Eq. (7):

$$\left(\frac{dy}{dx}\right)_i = \frac{-y_{i-1}+y_{i+1}}{2h} \tag{4}$$

$$\left(\frac{d^2y}{dx^2}\right)_i = \frac{y_{i-1}-2y_i+y_{i+1}}{h^2} \tag{5}$$

$$\left(\frac{d^3y}{dx^3}\right)_i = \frac{-y_{i-2}+2y_{i-1}-2y_{i+1}+y_{i+2}}{2h^3} \tag{6}$$

$$\left(\frac{d^4y}{dx^4}\right)_i = \frac{y_{i-2}-4y_{i-1}+6y_i-4y_{i+1}+y_{i+2}}{h^4} \tag{7}$$

Substituting Eq. (7) into Eq. (1) at x = xi yields, Eq. (8):

$$EI\left(\frac{d^4y}{dx^4}\right)_i + ky_i = q_i \quad or \quad EI\frac{y_{i-2}-4y_{i-1}+6y_i-4y_{i+1}+y_{i+2}}{h^4} + ky_i = q_i \tag{8}$$

Simplifying gives:

$$y_{i-2} - 4y_{i-1} + \left[6 + \frac{kh^4}{EI}\right]y_i - 4y_{i+1} + y_{i+2} = \frac{q_i h^4}{EI} \tag{9}$$

Once the boundary conditions are specified, Eq. (9) can be applied at each node to form a system of linear algebraic equations that can be solved for the unknown deflections.

4 FREE BOUNDARY CONDITION

In dealing with boundary conditions, the end nodes and the interior nodes next to them at each free end require special consideration before the finite difference analysis is completed. Consider the case when node 0 is at the left-hand end of the beam (Figure 2) where the bending moment and the shear at node 0 are both zero (Al-Khafaji and Tooley 1986). This allows us to remove the fictitious displacements at nodes -1 and -2 as follows:

At the free end at the left-hand side, we have Eq. (10) and Eq. (11):

$$y_{-1} = 2y_o - 2y_1 \tag{10}$$

$$y_{-2} = 4y_o - 4y_1 + y_2 \tag{11}$$

At the free end at the right-hand side, we have Eq. (12) and Eq. (13):

$$y_{L+1} = 2y_L - 2y_{L-1} \tag{12}$$

$$y_{L+2} = 4y_{L-2} - 4y_{L-1} + 4y_l \tag{13}$$

The fictitious deflections at nodes -2 and -1 at the free end on the left-hand side of the beam and those at nodes L+1 and L+2 at the free end on the right-hand side of the beam can now be replaced using the four equations listed above. Thus,

At the free end located at node 0:

$$\left(2+\frac{kh^4}{EI}\right)y_o - 4y_1 + 2y_2 = \frac{h^4}{EI}\,q_o \qquad (14)$$

At the first node, node 1:

$$-2y_o + \left(5+\frac{kh^4}{EI}\right)y_1 - 4y_2 + 2y_3 = \frac{h^4}{EI}q_1 \qquad (15)$$

At the free end located at node L:

$$2y_{L-2} - 4y_{L-1} + \left(2+\frac{kh^4}{EI}\right)y_L = \frac{h^4}{EI}q_L \qquad (16)$$

At node $\mathrm{L}-1$:

$$y_{L-3} - 4y_{L-2} + \left(5+\frac{kh^4}{EI}\right)y_{L-1} - 2y_l = \frac{h^4}{EI}q_{L-1} \qquad (17)$$

Eq. (9) can be used for all interior nodes of the beam and Eq. (14) and Eq. (15) should be used at nodes 0 and 1. Eq. (16) and Eq. (17) should be used at nodes L-1 and L.

5 NUMERICAL EXAMPLE

Consider the concrete beam shown in Figure 3 with free ends resting on elastic foundation and compute the deflections along its length assuming 20 length increments, L = 20 m, E = 3x107 kN/m^2, k = 24,000 kN/m^3. The beam cross section is h = 0.3 m high and b = 1.0 m wide.

6 SOLUTION

Note that the Excel program developed is based on dividing any beam into 20 equal length increments along the beam. This is to maximize the accuracy of the computed deflections and permit significant variations in the loading on the beam. In this case, the length increment h = 20/20 = 1.0 m. The beam characteristics are calculated as:

$$\mathrm{I} = \frac{1}{12}\mathrm{bh}^3 = \frac{1}{12}1.0(0.3)^3 = 0.0105833\ \mathrm{m}^4 \qquad (18)$$

$$\mathrm{EI} = 0.0105833\ (3\mathrm{x}107) = 317{,}500\ \mathrm{kN}\cdot\mathrm{m}^2 \qquad (19)$$

Thus, using the equations developed earlier and the loading presented above, the deflected shape of the beam is shown below in Figure 4.

Note that the deflection values are given in meters. Furthermore, the deflection on the right-hand side of the beam is actually above the ground! This is referred to as the corner effect. Clearly, this example illustrates the power of the Excel spreadsheet to analyze geotechnical problems that are impossible to solve using analytical methods.

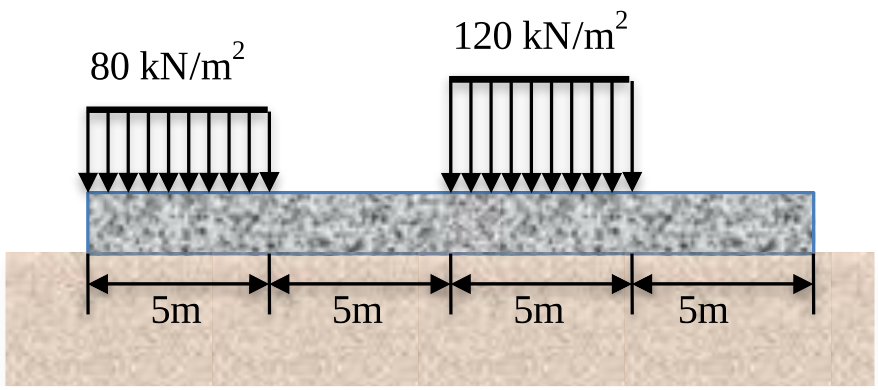

Figure 3. Schematic for numerical example for beam on elastic foundation.

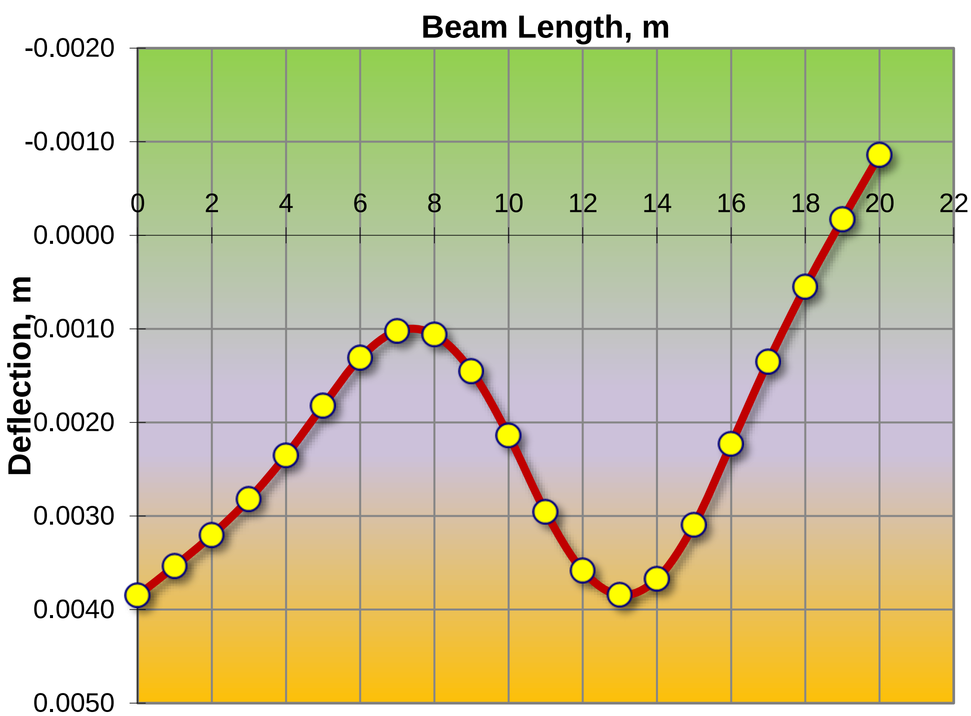

Figure 4. Excel generated solution using the developed finite differences grid.

7 SUMMARY AND CONCLUSIONS

The majority of analytical solutions for beam on elastic foundations are of limited practical use because of the many simplifying assumptions used and their failure to deal with load discontinuities and other complexities. In most cases, the variability in the imposed loading on a beam resting on elastic foundation makes it impossible to compute the deflections of beams analytically. The finite differences method was used to solve this problem for long beams with free ends, arbitrary loading and constant cross-sections using an Excel Workbook providing both numerical and graphical output. The Excel workbook was developed using 20 length increments to best describe arbitrary loadings on the beam. That is, if a concentrated load is applied at any location it can be transformed into a uniform load by dividing its value by the length increment h which is 1/20 of the beam's length. The solution of this problem can be extended to beams with other end conditions, variable cross sections and modulus of subgrade reaction. The finite difference method is readily applicable to other soil-structure interaction problems that are important in geotechnical engineering.

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