Exceptional Field Theory. I. E[subscript 6(6)]-Covariant Form of M-Theory and Type IIB

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Exceptional field theory. I. $E_{6(6)}$-covariant form of M-theory and type IIB

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We present the details of the recently constructed $E_{6(6)}$-covariant extension of 11-dimensional supergravity. This theory requires a $5+27$-dimensional spacetime in which the "internal" coordinates transform in the $\bar{\mathbf{27}}$ of $E_{6(6)}$. All fields are $E_{6(6)}$ tensors and transform under (gauged) internal generalized diffeomorphisms. The "Kaluza-Klein" vector field acts as a gauge field for the $E_{6(6)}$-covariant "E-bracket" rather than a Lie bracket, requiring the presence of 2-forms akin to the tensor hierarchy of gauged supergravity. We construct the complete and unique action that is gauge invariant under generalized diffeomorphisms in the internal and external coordinates. The theory is subject to covariant section constraints on the derivatives, implying that only a subset of the extra 27 coordinates is physical. We give two solutions of the section constraints: the first preserves GL(6) and embeds the action of the complete (i.e. untruncated) 11-dimensional supergravity; the second preserves $\mathrm{GL}(5)\times \mathrm{SL}(2)$ and embeds complete type IIB supergravity. As a byproduct, we thus obtain an off-shell action for type IIB supergravity.

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I. INTRODUCTION

For more than three decades, since the seminal work of Cremmer and Julia [1], it has been known that the toroidal compatification of 11-dimensional supergravity [2] gives rise to the exceptional symmetries $E_{n(n)}(\mathbb{R})$, $n=6,7,8$, in dimensions $D=11-n$. Later, in the middle 1990s, the discrete subgroups $E_{n(n)}(\mathbb{Z})$ were interpreted as part of the U-duality symmetries of M-theory [3], but ever since it has remained a mystery why 11-dimensional supergravity knows about the exceptional groups and to what extent they are already present in the full theory. This fact has inspired various authors to speculate about a hidden new geometry in higher dimensions that transcends the Riemannian geometry underlying Einstein's theory [4–32], but it is fair to say that so far there is no scheme that casts the full 11-dimensional supergravity into a truly $E_{n(n)}$-covariant form. In this paper, we present in detail the construction announced recently in Ref. [33], which gives an extension of 11-dimensional supergravity that makes the exceptional group $E_{6(6)}$ manifest prior to any toroidal compactification, while also hosting the type IIB theory [34,35]. The details for the remaining finite-dimensional groups $E_{7(7)}$ and $E_{8(8)}$ will be presented in separate publications [36].

Our construction is a continuation and generalization of "double field theory" (DFT), which is an approach to make the $O(d,d)$ T-duality group of string theory manifest by introducing a generalized spacetime with doubled coordinates, subject to a "section constraint" or "strong constraint," and by reorganizing the fields into $O(d,d)$ tensors [37–41]. (For earlier results see Refs. [42–45].) Remarkably, DFT is applicable not only to (the low-energy spacetime action of) bosonic string theory, but also to the heterotic string [46], including their supersymmetric formulations [47–49], as well as massless and massive type II theories [50–53]. DFT also yields an intriguing generalization of Riemannian geometry [37,54–59], which in turn extends results in the "generalized geometry" developed in pure mathematics [60–62]. Moreover, it provides a natural framework for nongeometric fluxes [63–67]. Finally, an extension of DFT to higher-derivative α' corrections has recently been given [68]. (For a more exhaustive list of references see the recent reviews [69–71].)

In contrast to $D=10$ string theory and DFT, where the fields naturally combine into tensors under $O(10,10)$, the fields of $D=11$ supergravity do not organize directly into tensors under any of the exceptional groups. For instance, in order to realize the $E_{n(n)}$ symmetry in dimensional reduction, some field components have to be dualized into forms of lower rank. As such transformations are specific to a given dimension, it is not obvious how to build complete $E_{n(n)}$ multiplets in $D=11$ prior to any reduction. We have recently shown how to overcome these obstacles by gauge fixing the local Lorentz group and decomposing the fields and coordinates as in Kaluza-Klein compactifications, but *without* truncation [72]. The resulting formulation therefore captures all of the original 11-dimensional supergravity, at the cost of abandoning some of the Lorentz gauge freedom. The various field components, necessarily including some of their duals, can then be reorganized into $E_{n(n)}$ tensors.

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Extending the "internal" derivatives to transform in some fundamental representation of $E_{n(n)}$, subject to a generalization of the DFT section constraint proposed in Refs. [27,29], we arrive at a manifestly $E_{n(n)}$-covariant extension of 11-dimensional supergravity. The resulting theory, which we refer to in the following as "exceptional field theory" (EFT), closely resembles DFT when subjected to an analogous Kaluza-Klein-type gauge fixing of the local Lorentz group [73].

Already the early work of de Wit and Nicolai [5,6] has identified directly in 11 dimensions some of the structures found in dimensional reduction, following a Kaluza-Klein decomposition without truncation similar to the present construction. Manifest 11-dimensional covariance is abandoned, in favor of an enhanced local Lorentz symmetry in accordance with the (composite) gauge symmetries appearing in the $D = 4$ or $D = 3$ coset models. However, these constructions do not yet manifest the exceptional groups, and further work in Ref. [8] suggested that additional coordinates should be introduced in order to achieve this, an idea that also features prominently in the proposal of Ref. [14]. Later work [19,20] gave a manifestly $E_{7(7)}$-invariant action functional for a certain seven-dimensional truncation of $D = 11$ supergravity by introducing coordinates in the 56 of $E_{7(7)}$. Recently, other subsectors of $D = 11$ supergravity have been reformulated in terms of a generalized metric (see e.g., Refs. [23–25]), together with a duality-covariant formulation of part of the gauge symmetries in the form of generalized Lie derivatives. These constructions are also related to extensions of generalized geometry to the exceptional groups [16,26]. In all these truncations the match to 11-dimensional supergravity requires a Kaluza-Klein-type decomposition of the latter in which one sets to zero all off-diagonal components of the metric and the 3-form, sets to zero the external components of the 3-form and freezes the external metric to the Minkowski metric, possibly up to a warp factor. Finally, one truncates the coordinate dependence of all fields to the internal coordinates. We will explain in the Appendix the embedding of these theories into the full EFT formulation, constructed in this paper.

This formulation to be constructed requires various new mathematical tools [72], analogous to the Lorentz gauge-fixed DFT [73]. Most importantly, the off-diagonal vector field components of the Kaluza-Klein-like decomposition yield a generalization of a Yang-Mills gauge field. More precisely, these fields transform in the same way as a Yang-Mills connection, but with a bracket, in the following referred to as the "E-bracket," that does not satisfy all axioms of a Lie bracket. This, in turn, requires the introduction of forms of higher rank in order to maintain gauge covariance of the field strengths, in precise analogy to the "tensor hierarchy" of gauged supergravity [74,75]. Moreover, these higher forms play a vital role as the duals of some physical fields, which is implemented at the level of an off-shell action by means of topological Chern-Simons-like terms, as in gauged supergravity [76,77]. Finally, the "internal" field components organize into a "generalized metric" $\mathcal{M}_{MN}$ that is a covariant tensor under $E_{n(n)}$, while the "external" metric $g_{\mu\nu}$ is an $E_{n(n)}$ singlet that, however, transforms as a scalar density under the (internal) generalized Lie derivatives.

In this paper, we present in detail the construction of the $E_{6(6)}$ EFT. Dimensional reduction from 11 dimensions on a torus T^6 is known to give rise to maximal $D = 5$ supergravity with global $E_{6(6)}$ symmetry [78]. It becomes manifest in five dimensions after the proper dualization of all p-form tensors to the lowest possible degree. In particular, the 3-form descending from 11 dimensions is dualized into a scalar and joins the coordinates of the scalar target space described by the coset space $E_{6(6)}/\mathrm{USp}(8)$. The $E_{6(6)}$ EFT keeps the field and multiplet structure of the five-dimensional theory, but elevates all fields to functions of $5 + 27$ coordinates (x^μ, Y^M), where the Y^M, with dual derivatives ∂_M, live in the fundamental representation $\bar{\mathbf{27}}$ of $E_{6(6)}$. The theory is subject to covariant section constraints, which can be written in terms of the $E_{6(6)}$-invariant d-symbols d^{MNK} and d_{MNK} as follows [26,29]:

$$d^{MNK}\partial_N\partial_K A = 0, \qquad d^{MNK}\partial_N A\partial_K B = 0, \tag{1.1}$$

where A, B denote any fields or gauge parameters. This constraint is the analogue of the "strong constraint" in DFT and implies that only a subset of the 27 coordinates is physical. While in DFT the strong constraint is motivated from string theory, as implementing a strong version of the level-matching constraint, Eq. (1.1) has been postulated by analogy. However, we will discuss below that for the SO (5,5) T-duality subgroup of $E_{6(6)}$ it actually reduces to the strong constraint of DFT. The $E_{6(6)}$-covariant field content is given by

$$\{e_\mu{}^a, \mathcal{M}_{MN}, A_\mu{}^M, B_{\mu\nu M}\}, \tag{1.2}$$

where $e_\mu{}^a$ denotes the fünfbein corresponding to the external metric, while $A_\mu{}^M$ and $B_{\mu\nu\, M}$ are the tensor gauge fields relevant for the $E_{6(6)}$ EFT. The symmetric matrix $\mathcal{M}_{MN}$ parametrizes the coset space $E_{6(6)}/\mathrm{USp}(8)$ whose 42 coordinates describe the "scalar" fields of the theory. The full action is given by

$$S_{\mathrm{EFT}} = \int d^5x d^{27}Y e\Big(\hat{R} + \frac{1}{24}g^{\mu\nu}\mathcal{D}_\mu\mathcal{M}^{MN}\mathcal{D}_\nu\mathcal{M}_{MN} - \frac{1}{4}\mathcal{M}_{MN}\mathcal{F}^{\mu\nu M}\mathcal{F}_{\mu\nu}{}^N + e^{-1}\mathcal{L}_{\mathrm{top}} - V(\mathcal{M}_{MN}, g_{\mu\nu})\Big). \tag{1.3}$$

This action takes the same structural form as $D = 5$ gauged supergravity [77], with a (covariantized) Einstein-Hilbert term for $e_\mu{}^a$, a "scalar" kinetic term for $\mathcal{M}_{MN}$ and a Yang-Mills term based on the field strength $\mathcal{F}_{\mu\nu}{}^M$, the latter also depending on the 2-form $B_{\mu\nu\,M}$ in accordance with the tensor hierarchy. All fields depend on the "internal" coordinates, corresponding to the non-Abelian structure of covariant derivatives and field strengths involving the derivatives ∂_M. In addition, the "potential" $V(\mathcal{M}, g)$ is the manifestly $E_{6(6)}$-covariant expression (built using only the ∂_M derivatives) given by

$$V = -\frac{1}{24}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL} + \frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK} - \frac{1}{2}g^{-1}\partial_M g\partial_N\mathcal{M}^{MN} - \frac{1}{4}\mathcal{M}^{MN}g^{-1}\partial_M g g^{-1}\partial_N g - \frac{1}{4}\mathcal{M}^{MN}\partial_M g^{\mu\nu}\partial_N g_{\mu\nu}. \tag{1.4}$$

All terms in the action (1.3) are separately gauge invariant under the internal (generalized) diffeomorphisms of the Y^M, generated by a parameter $\Lambda^M(x, Y)$, with the $A_\mu{}^M$ taking the role of a gauge connection for this symmetry. The action is further gauge invariant under (A_μ-covariantized) "external" diffeomorphisms generated by $\xi^\mu(x, Y)$, but this symmetry is not manifest for the Y-dependent parameter ξ^μ. In fact, it is this symmetry that relates the various terms in Eq. (1.3) and fixes all relative coefficients.

Apart from the construction of the action (1.3), a central result of this paper is to show that this action after putting an appropriate solution of the section condition (1.1) reduces to full (i.e. untruncated) 11-dimensional supergravity after a rearrangement of the fields according a $5+6$ Kaluza-Klein split but keeping the dependence on all 11 coordinates. We work this out in full detail and reproduce from Eq. (1.3) the action of 11-dimensional supergravity. Moreover, it has been noted in Ref. [33] that the section condition (1.1) allows for (at least) two inequivalent solutions, the second of which reduces the theory (1.3) to the full ten-dimensional IIB theory. To this end we first break $E_{6(6)}$ under $SL(6) \times SL(2)$ such that the fundamental representation decomposes as

$$\mathbf{27} \to (15, 1) + (6, 2). \tag{1.5}$$

If we let the fields depend on six coordinates from the SL(2) doublet, the section constraints are satisfied. We are left with an unbroken GL(6) symmetry and fields depending on $5+6$ coordinates. For this choice, the action (1.3) reduces to an action that is on-shell equivalent to 11-dimensional supergravity. Alternatively, the section constraint is solved by letting fields depend on five coordinates from the 15 in Eq. (1.5), which in turn breaks the symmetry to $GL(5) \times SL(2)$. For this choice, Eq. (1.3) reduces to a ten-dimensional action with a global SL(2) symmetry and we obtain an on-shell equivalent formulation of type IIB supergravity. As a byproduct, this yields an off-shell action

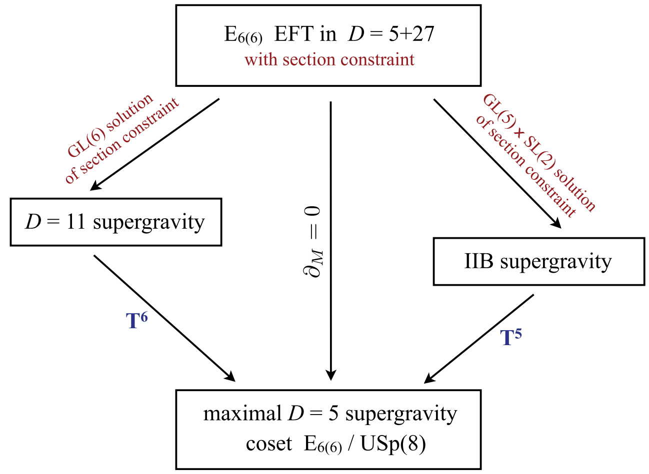

FIG. 1 (color online). $E_{6(6)}$ EFT embedding of $D = 11$ supergravity, IIB supergravity, and $D = 5$ supergravity.

for type IIB supergravity, at the cost of sacrificing manifest ten-dimensional spacetime covariance. In the sense just explained, the EFT defined by Eq. (1.3) unifies type IIB and M-theory (and thus type IIA), a feature shared with the type II DFT constructed in Refs. [50,51]. Instead, dropping all derivatives with respect to the extra internal coordinates, i.e. setting $\partial_M = 0$, the theory (1.3) directly reduces to $D = 5$ maximal supergravity in the form in which the exceptional symmetry $E_{6(6)}$ is manifest without further dualization [78]. The various links are depicted in Fig. 1, which can be thought of as a commutative diagram that explains the emergence of $E_{6(6)}$ from M-theory or type IIB.

This paper is organized as follows. In Sec. II we introduce the required $E_{6(6)}$ structures: the generalized Lie derivatives, the E-bracket, and the associated tensor hierarchy. Employing these techniques, we define in Sec. III the various terms of the $E_{6(6)}$ EFT action and discuss the (nonmanifest) gauge invariance under the external, five-dimensional diffeomorphisms. In Sec. IV we prove that 11-dimensional supergravity can be embedded in EFT, upon solving the section constraint as above and rewriting 11-dimensional supergravity appropriately for the Kaluza-Klein-inspired gauge fixing of the Lorentz group. In Sec. V we discuss the embedding and decomposition of type IIB supergravity along the same lines. We close with a summary and outlook in Sec. VI. In the Appendix we discuss truncations of our theory, in order to relate it to some of the duality-covariant truncations previously obtained in the literature.

II. $E_{6(6)}$ GENERALIZED DIFFEOMORPHISMS AND THE TENSOR HIERARCHY

We start by introducing the mathematical background needed for the definition of the theory (1.3), including the $E_{6(6)}$ generalized Lie derivatives that generate the internal (generalized) diffeomorphisms and the "E-bracket." Then we introduce the gauge fields $A_\mu{}^M$ which gauge this symmetry in the sense of making it local with respect to the "external" x space. Due to the nontrivial Jacobiator of the E-bracket, gauge covariance requires the introduction of the 2-form $B_{\mu\nu M}$ in accordance with the general tensor hierarchy of non-Abelian p-forms [74,75].

A. Generalized Lie derivatives and the E-bracket

We begin by collecting the relevant facts about the exceptional Lie group $E_{6(6)}$. Its Lie algebra is of dimension 78, with generators that we denote by t_α with the adjoint index $\alpha = 1, \ldots, 78$. In addition, $E_{6(6)}$ has two inequivalent fundamental representations of dimension 27, which we denote by $\mathbf{27}$, and $\bar{\mathbf{27}}$ for its contragredient. These representations will be indicated by lower indices $M, N = 1, \ldots, 27$ for $\mathbf{27}$ and upper indices for $\bar{\mathbf{27}}$. Note, in particular, that there is no invariant metric to raise and lower fundamental indices. In contrast, we raise and lower adjoint indices by the (rescaled) Cartan-Killing form $\kappa_{\alpha\beta} \equiv (t_\alpha)_M{}^N (t_\beta)_N{}^M$.

In the fundamental representation, there are two cubic $E_{6(6)}$-invariant tensors, the fully symmetric d-symbols d^{MNK} and d_{MNK}, which we normalize as $d_{MPQ} d^{NPQ} = \delta_M^N$. Below we will need the projector onto the adjoint representation

$$\mathbb{P}^M{}_N{}^K{}_L \equiv (t_\alpha)_N{}^M (t^\alpha)_L{}^K = \frac{1}{18}\delta_N^M \delta_L^K + \frac{1}{6}\delta_N^K \delta_L^M - \frac{5}{3} d_{NLR} d^{MKR}, \tag{2.1}$$

which satisfies

$$\mathbb{P}^M{}_N{}^N{}_M = 78. \tag{2.2}$$

We note the useful cubic relations for the d-symbols,

$$\begin{aligned} d_{S(MN} d_{PQ)T} d^{STR} &= \frac{2}{15} \delta^R{}_{(M} d_{NPQ)}, \\ d_{STR} d^{S(MN} d^{PQ)T} &= \frac{2}{15} \delta_R{}^{(M} d^{NPQ)}. \end{aligned} \tag{2.3}$$

Next, we introduce the generalized Lie derivative with respect to the vector parameter Λ^M acting on $E_{6(6)}$ tensors in the fundamental representation with an arbitrary number of upper and lower indices. Moreover, the tensors can carry an arbitrary *density weight* λ. On a vector V^M of weight λ it acts as [26,29]

$$\delta V^M = \mathbb{L}_\Lambda V^M \equiv \Lambda^K \partial_K V^M - 6\mathbb{P}^M{}_N{}^K{}_L \partial_K \Lambda^L V^N + \lambda \partial_P \Lambda^P V^M. \tag{2.4}$$

Similarly, it acts on a co-vector W_M of weight λ' as

$$\delta W_M = \mathbb{L}_\Lambda W_M \equiv \Lambda^K \partial_K W_M + 6\mathbb{P}^N{}_M{}^K{}_L \partial_K \Lambda^L W_N + \lambda' \partial_P \Lambda^P W_M, \tag{2.5}$$

and accordingly on an $E_{6(6)}$ tensor with an arbitrary number of covariant and contravariant fundamental indices. Because of the projector in Eq. (2.4), the generalized Lie derivative is compatible with the $E_{6(6)}$ algebra structure: the d-symbols are invariant tensors of weight $\lambda = 0$,

$$\mathbb{L}_\Lambda d_{MNK} = 0, \tag{2.6}$$

and its action on the $E_{6(6)}$-valued generalized metric $\mathcal{M}_{MN}$ to be introduced below (carrying weight $\lambda = 0$) preserves the group property. Moreover, the above definition is such that the $E_{6(6)}$-invariant contraction between a vector and a co-vector transforms as

$$\delta_\Lambda (V^M W_M) = \Lambda^K \partial_K (V^M W_M) + (\lambda + \lambda')\partial_P \Lambda^P V^M W_M. \tag{2.7}$$

In particular, the contraction transforms as a genuine scalar if the vectors have opposite weights, $\lambda = -\lambda'$. Writing out the projector (2.1), the Lie derivative on, say, a vector reads explicitly

$$\delta_\Lambda V^M = \Lambda^K \partial_K V^M - \partial_K \Lambda^M V^K + \left(\lambda - \frac{1}{3}\right)\partial_P \Lambda^P V^M + 10 d_{NLR} d^{MKR} \partial_K \Lambda^L V^N. \tag{2.8}$$

We observe that the projector contributes an additional density-type term, leading to an "effective weight" of $(\lambda - \frac{1}{3})$ in the action (2.8), which singles out the value $\lambda = \frac{1}{3}$. In fact, we will see below that the vector gauge parameter itself has to be thought of as a vector of weight $\lambda = \frac{1}{3}$, such that Eq. (2.8) carries no explicit weight term. We stress that by referring to the weight λ of a tensor V, sometimes denoted by $\lambda(V)$, we always denote the weight in Eq. (2.4), as opposed to the effective weight of Eq. (2.8). In the following, a careful treatment of the emerging weights will be crucial. A remarkable observation is the following: if V_M is a covariant vector of weight $\lambda(V) = \frac{2}{3}$, then the combination

$$W^M \equiv d^{MNK} \partial_K V_N \tag{2.9}$$

is a contravariant vector of weight $\lambda(W) = \frac{1}{3}$. This can be viewed as an $E_{6(6)}$ analogue of the fact that for standard diffeomorphisms the exterior derivative $\partial_{[m_0} C_{m_1 \ldots m_p]}$ of an antisymmetric p-form is a covariant tensor. [Note, however, that the tensor d^{MNK} in Eq. (2.9) is totally symmetric.] Indeed, embedding the structures of ten- and 11-dimensional spacetime diffeomorphisms, the tensor structure of Eq. (2.9) precisely encodes those exterior derivatives, as we will find from the explicit decompositions of the d-symbol in

Eqs. (4.42) and (5.5) below. The tensorial nature of Eq. (2.9) will prove crucial for the structure of the tensor hierarchy of non-Abelian p-forms. For a general study of connections and connection-free covariant derivatives in such "exceptional geometries" see Refs. [26,31,79].

Let us now discuss a few properties of the generalized Lie derivatives, which all require the section constraints (1.1). First, we note that there are "trivial" gauge parameters, i.e., gauge parameters that do not generate a gauge transformation via Eq. (2.4). These are of the form

$$\Lambda^M = d^{MNK}\partial_N\chi_K, \tag{2.10}$$

for an arbitrary covariant vector χ_K. To prove this claim we compute from Eq. (2.8)

$$\delta_\Lambda V^M = (-d^{MPQ}\partial_N\partial_P\chi_Q + 10 d_{NLR}d^{MKR}d^{LPQ}\partial_K\partial_P\chi_Q)V^N. \tag{2.11}$$

Here we have set to zero the transport term and the density term, since for the above parameter they vanish by the section constraints (1.1). Next we apply the cubic identity (2.3), noticing that

$$\begin{aligned} &d_{RLN}d^{R(MK}d^{PQ)L}\partial_K\partial_P\chi_Q \\ &\quad = \frac{1}{6}d_{RLN}(2d^{RMK}d^{PQL} + 2d^{RQK}d^{PML})\partial_K\partial_P\chi_Q \\ &\quad = \frac{2}{3}d_{RLN}d^{RMK}d^{PQL}\partial_K\partial_P\chi_Q, \end{aligned} \tag{2.12}$$

where we used the symmetry in K,P and the section constraint. The cubic identity thus implies

$$\begin{aligned} &10 d_{RLN}d^{RMK}d^{PQL}\partial_K\partial_P\chi_Q \\ &\quad = 2\delta_N{}^{(M}d^{KPQ)}\partial_K\partial_P\chi_Q = d^{PMQ}\partial_N\partial_P\chi_Q, \end{aligned} \tag{2.13}$$

where, in the last equality, we used again the section constraint. Inserting this in Eq. (2.11) we observe that this cancels the first term, thus proving $\delta_\Lambda V^M = 0$ and so the triviality of the action of this gauge parameter. In the above proof we have given the detailed steps that will recur in similar form in many of the computations below, making repeated use of the section constraints (1.1) and the cubic identity (2.3). As such, in the following derivations we will not repeat all intermediate steps in similar detail.

Next, we turn to the gauge algebra. A direct computation as above shows that, modulo the section constraints (1.1), the gauge transformations close

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}] = \delta_{[\Lambda_2,\Lambda_1]_{\rm E}}, \tag{2.14}$$

according to the "E-bracket"

$$[\Lambda_2, \Lambda_1]_{\rm E}^M = 2\Lambda_{[2}^K\partial_K\Lambda_{1]}^M - 10 d^{MNP}d_{KLP}\Lambda_{[2}^K\partial_N\Lambda_{1]}^L. \tag{2.15}$$

Put differently, the generalized Lie derivatives satisfy the algebra [26,29][1]

$$[\mathbb{L}_{\Lambda_1}, \mathbb{L}_{\Lambda_2}] = \mathbb{L}_{[\Lambda_1,\Lambda_2]_{\rm E}}. \tag{2.16}$$

The E-bracket is the M-theory or EFT analogue of the C-bracket in DFT. Like the C-bracket, the E-bracket does not define a Lie algebra in that it has a nontrivial "Jacobiator,"

$$\begin{aligned} J(U,V,W) \equiv{}& [[U,V]_{\rm E}, W]_{\rm E} + [[V,W]_{\rm E}, U]_{\rm E} \\ &+ [[W,U]_{\rm E}, V]_{\rm E}. \end{aligned} \tag{2.17}$$

As in DFT, however, the Jacobiator takes the form of a trivial parameter [Eq. (2.10)] and is therefore consistent with the Jacobi identity for the symmetry variations, $[[\delta_{\Lambda_1}, \delta_{\Lambda_2}], \delta_{\Lambda_3}] + \text{cycl} = 0$. The proof is formally identical to that for the Courant bracket in generalized geometry [61] or for the C-bracket in DFT [39] and proceeds as follows.[2] First, we define the Dorfman-type product (or bracket) between vectors of weight $\frac{1}{3}$,

$$\begin{aligned} (V\circ W)^M \equiv (\mathbb{L}_V W)^M ={}& V^N\partial_N W^M - W^N\partial_N V^M \\ &+ 10 d^{MKR}d_{PLR}\partial_K V^L W^P. \end{aligned} \tag{2.18}$$

A comparison with Eq. (2.15) then shows that the product differs from the E-bracket by a term symmetric in the two arguments,

$$(V\circ W)^M = [V,W]_{\rm E}^M + 5 d^{MKR}\partial_K(d_{RPL}V^P W^L). \tag{2.19}$$

Note that the symmetric contribution takes the trivial form (2.10) and so $(V\circ W)$ and $[V,W]_{\rm E}$ generate the same generalized Lie derivative. Using this and the algebra (2.16) it is straightforward to verify that the product satisfies the Jacobi-like identity

$$U\circ(V\circ W) - V\circ(U\circ W) - (U\circ V)\circ W = 0. \tag{2.20}$$

In fact, with Eq. (2.18) we compute

$$\begin{aligned} U\circ(V\circ W) - V\circ(U\circ W) &= U\circ(\mathbb{L}_V W) - V\circ(\mathbb{L}_U W) \\ &= \mathbb{L}_U\mathbb{L}_V W - \mathbb{L}_V\mathbb{L}_U W \\ &= \mathbb{L}_{[U,V]_{\rm E}} W \\ &= \mathbb{L}_{(U\circ V)} W = (U\circ V)\circ W, \end{aligned} \tag{2.21}$$

[1] Note that the seeming sign difference between Eqs. (2.14) and (2.16) originates from the difference between a field variation, acting on fields first, and an abstract operator like the Lie derivative.

[2] See also the analysis in the context of exceptional generalized geometry [26], to which our discussion reduces for one solution of the section constraint.

thus proving Eq. (2.20). Next we use Eq. (2.19) to compute

$$[[U,V]_{\rm E},W]_{\rm E} = ([U,V]_{\rm E}\circ W)^M - 5d^{MKR}\partial_K(d_{RPL}[U,V]_E^P W^L) = ((U\circ V)\circ W)^M - 5d^{MKR}\partial_K(d_{RPL}[U,V]_E^P W^L). \quad (2.22)$$

Using the fact that as a consequence of Eq. (2.19) the E-bracket Jacobiator is proportional to the "Jacobiator" for the Dorfman product, one computes with the identity (2.20)

$$J^M(U,V,W) = \frac{5}{3}d^{MKR}\partial_K(d_{RPL}([U,V]_E^P W^L + [W,U]_{\rm E}^P V^L + [V,W]_{\rm E}^P U^L)). \quad (2.23)$$

This completes the proof that the Jacobiator is of the trivial form Eq. (2.11).

B. $E_{6(6)}$ tensor hierarchy

We now turn to a discussion of external covariant derivatives, gauge connections, and covariant curvatures. These are necessary because in the above gauge transformations we will take the gauge parameters Λ^M to be functions of the (internal) $E_{6(6)}$ coordinates Y^M but also of the (external) five-dimensional coordinates x^μ. Thus, the gauge transformations are local with respect to the x space and the corresponding partial derivatives ∂_μ need to be covariantized. We thus introduce a gauge connection $A_\mu{}^M$ and define the covariant derivative

$$\mathcal{D}_\mu \equiv \partial_\mu - \mathbb{L}_{A_\mu}. \quad (2.24)$$

For instance, the covariant derivative of a vector (of weight λ) is given by

$$\mathcal{D}_\mu V^M = \partial_\mu V^M - A_\mu{}^K\partial_K V^M + 6\mathbb{P}^M{}_N{}^K{}_L\partial_K A_\mu{}^L V^N - \lambda\partial_P A_\mu{}^P V^M. \quad (2.25)$$

Sometimes, we will explicitly split off the density term and write

$$\mathcal{D}_\mu V^M = D_\mu V^M - \lambda\partial_P A_\mu{}^P V^M \quad (2.26)$$

for a vector V^M of weight λ. The transformation of the gauge connection is obtained by requiring gauge covariance of the covariant derivatives. An explicit computation shows that with

$$\delta A_\mu{}^M = \partial_\mu\Lambda^M - A_\mu{}^K\partial_K\Lambda^M + \Lambda^K\partial_K A_\mu{}^M - 10d^{MNP}d_{KLP}\Lambda^L\partial_N A_\mu{}^K = D_\mu\Lambda^M - \frac{1}{3}(\partial_K A_\mu{}^K)\Lambda^M \equiv \mathcal{D}_\mu\Lambda^M, \quad (2.27)$$

the covariant derivatives are indeed covariant. This confirms that the gauge parameter Λ^M is a contravariant tensor of weight $\lambda = \frac{1}{3}$.

Next, we introduce a non-Abelian field strength for the above gauge connection. The naive non-Abelian Yang-Mills field strength reads

$$F_{\mu\nu}{}^M = 2\partial_{[\mu}A_{\nu]}{}^M - [A_\mu, A_\nu]_{\rm E}^M = 2\partial_{[\mu}A_{\nu]}{}^M - 2A_{[\mu}{}^K\partial_K A_{\nu]}{}^M + 10d^{MKR}d_{NLR}A_{[\mu}{}^N\partial_K A_{\nu]}{}^L. \quad (2.28)$$

Since the E-bracket does not satisfy the Jacobi identity, however, this field strength does not transform fully covariantly. We first compute its variation with respect to an arbitrary $\delta A_\mu{}^M$, which is a contravariant vector of weight $\lambda = \frac{1}{3}$,

$$\delta F_{\mu\nu}{}^M = 2\mathcal{D}_{[\mu}\delta A_{\nu]}{}^M + 10d^{MKR}d_{NLR}\partial_K(A_{[\mu}{}^N\delta A_{\nu]}{}^L). \quad (2.29)$$

The final term here is noncovariant, but of the "trivial" form (2.10). In the spirit of the tensor hierarchy [74,75], this suggests introducing 2-form potentials $B_{\mu\nu M}$ and defining the full covariant field strength by

$$\mathcal{F}_{\mu\nu}{}^M \equiv F_{\mu\nu}{}^M + 10d^{MNK}\partial_K B_{\mu\nu N}, \quad (2.30)$$

such that its general variation is given by

$$\delta\mathcal{F}_{\mu\nu}{}^M = 2\mathcal{D}_{[\mu}\delta A_{\nu]}{}^M + 10d^{MNK}\partial_K\Delta B_{\mu\nu\, N}, \quad (2.31)$$

with

$$\Delta B_{\mu\nu N} \equiv \delta B_{\mu\nu N} + d_{NKL}A_{[\mu}{}^K\delta A_{\nu]}{}^L. \quad (2.32)$$

The covariant field strength also appears in the commutator of covariant derivatives,

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = -\mathbb{L}_{F_{\mu\nu}} = -\mathbb{L}_{\mathcal{F}_{\mu\nu}}, \quad (2.33)$$

where the last equality uses the triviality of Eq. (2.10). With these results at hand we can now verify the gauge covariance of the curvature. In addition to the gauge symmetry parametrized by Λ^M, the newly introduced gauge potential $B_{\mu\nu\, M}$ comes with its own tensor gauge symmetry, whose parameter we denote by $\Xi_{\mu M}$. Explicitly, the complete gauge variations are given by

$$\delta A_\mu{}^M = D_\mu\Lambda^M - \frac{1}{3}(\partial_K A_\mu{}^K)\Lambda^M - 10d^{MNK}\partial_K\Xi_{\mu N}, \quad \Delta B_{\mu\nu M} = 2D_{[\mu}\Xi_{\nu]M} - \frac{4}{3}(\partial_K A_{[\mu}{}^K)\Xi_{\nu]M} + d_{MKL}\Lambda^K\mathcal{H}_{\mu\nu}{}^L + \mathcal{O}_{\mu\nu M}, \quad (2.34)$$

up to yet unspecified terms $\mathcal{O}_{\mu\nu M}$ satisfying

$$d^{MNK}\partial_K\mathcal{O}_{\mu\nu N} = 0, \tag{2.35}$$

which do not contribute to Eq. (2.31). It is a straightforward calculation to show that under Eq. (2.34), the field strength (2.30) transforms as a contravariant vector [Eq. (2.8)] of weight $\lambda = \frac{1}{3}$. Moreover, the form of Eq. (2.34) shows that the 2-form gauge parameter $\Xi_{\mu M}$ is a covariant vector of weight $\lambda = \frac{2}{3}$.

After having introduced a gauge-covariant field strength, we will now discuss the Bianchi identities, which is also a convenient trick in order to define the covariant field strength of the 2-form $B_{\mu\nu\, M}$. To this end we note the following useful relation, which follows from the observation in Eq. (2.9):

$$\mathcal{D}_\mu(d^{MNK}\partial_K V_N) = d^{MNK}\partial_K\mathcal{D}_\mu V_N, \tag{2.36}$$

which is valid for any covariant vector V_N of weight $\lambda = \frac{2}{3}$. An explicit computation shows that the field strength (2.30) satisfies the Bianchi identities

$$3\mathcal{D}_{[\mu}\mathcal{F}_{\nu\rho]}{}^M = 10 d^{MNK}\partial_K\mathcal{H}_{\mu\nu\rho N}, \tag{2.37}$$

with the 3-form field strength $\mathcal{H}_{\mu\nu\rho M}$ defined by the following equation (up to terms that vanish under the projection with $d^{MNK}\partial_K$):

$$\begin{aligned}\mathcal{H}_{\mu\nu\rho\, M} = {} & 3\mathcal{D}_{[\mu}B_{\nu\rho]M} - 3 d_{MKL}A_{[\mu}{}^K\partial_\nu A_{\rho]}{}^L \\ & + 2 d_{MKL}A_{[\mu}{}^K A_\nu{}^P\partial_P A_{\rho]}{}^L \\ & - 10 d_{MKL}d^{LPR}d_{RNQ}A_{[\mu}{}^K A_\nu{}^N\partial_P A_{\rho]}{}^Q + \cdots .\end{aligned} \tag{2.38}$$

With respect to the generalized Lie derivative, this is a covariant vector of weight $\lambda = \frac{2}{3}$. Next, we determine the Bianchi identity for $\mathcal{H}_M$. From the derivative of Eq. (2.37),

$$\begin{aligned} 20 d^{MNK}\partial_K\mathcal{D}_{[\mu}\mathcal{H}_{\nu\rho\sigma]N} &= 6\mathcal{D}_{[\mu}\mathcal{D}_\nu\mathcal{F}_{\rho\sigma]}{}^M \\ &= -15 d^{MNK}\partial_K(d_{NPQ}\mathcal{F}_{[\mu\nu}{}^P\mathcal{F}_{\rho\sigma]}{}^Q),\end{aligned} \tag{2.39}$$

we conclude the Bianchi identity

$$4\mathcal{D}_{[\mu}\mathcal{H}_{\nu\rho\sigma]M} = -3 d_{MPQ}\mathcal{F}_{[\mu\nu}{}^P\mathcal{F}_{\rho\sigma]}{}^Q + \cdots, \tag{2.40}$$

again up to terms annihilated by the projection with $d^{MNK}\partial_K$.

III. COVARIANT $E_{6(6)}$ THEORY

We are now in the position to define all terms in the $E_{6(6)}$ EFT action (1.3), specifically the kinetic terms for the propagating fields $e_\mu{}^a$, $\mathcal{M}_{MN}$ and $A_\mu{}^M$. The dynamics of the 2-form tensors $B_{\mu\nu M}$ is governed by a topological Chern-Simons-type term that implies the required duality relations between $A_\mu{}^M$ and $B_{\mu\nu M}$. We define the "potential" term as a function of the generalized metric $\mathcal{M}_{MN}$ and the external metric $g_{\mu\nu}$, and prove its gauge invariance under the internal generalized diffeomorphisms. Finally, we discuss the nonmanifest invariance of the action under the (covariantized) five-dimensional external diffeomorphisms, which in turn fixes all relative coefficients of the action.

A. Kinetic and topological terms

Let us start by recalling the field content as given in Eq. (1.2) above,

$$\{e_\mu{}^a, \mathcal{M}_{MN}, A_\mu{}^M, B_{\mu\nu M}\}. \tag{3.1}$$

In the following we define the kinetic terms for the first three fields. The five-dimensional vielbein ("fünfbein") $e_\mu{}^a$ is a scalar density under Λ^M gauge transformations, with weight $\lambda = \frac{1}{3}$. In order to write a gauge-invariant action we thus have to employ the covariant derivatives

$$\mathcal{D}_\mu e_\nu{}^a \equiv \partial_\mu e_\nu{}^a - A_\mu{}^M\partial_M e_\nu{}^a - \frac{1}{3}\partial_M A_\mu{}^M e_\nu{}^a \tag{3.2}$$

in the usual definition of the spin connection coefficients $\omega_\mu{}^{ab}$, which then become Λ^M scalars (i.e. carry weight $\lambda = 0$). The correspondingly covariantized Riemann tensor $R_{\mu\nu}{}^{ab}$ defined in the usual fashion then also transforms as a Λ^M scalar. However, because of the noncommutativity of the covariant derivatives $\mathcal{D}_\mu$, the covariantized Riemann tensor does not transform tensorially under local Lorentz transformations $\delta_\lambda\omega_\mu{}^{ab} = -\mathcal{D}_\mu\lambda^{ab}$. This can be repaired by defining the improved Riemann tensor [73]

$$\hat{R}_{\mu\nu}{}^{ab} \equiv R_{\mu\nu}{}^{ab} + \mathcal{F}_{\mu\nu}{}^M e^{a\rho}\partial_M e_\rho{}^b, \tag{3.3}$$

which transforms covariantly under internal generalized diffeomorphisms and local Lorentz transformations.[3] The covariantized Einstein-Hilbert term

$$S_{\rm EH} = \int d^5x d^{27}Y e\hat{R} = \int d^5x d^{27}Y e e_a{}^\mu e_b{}^\nu\hat{R}_{\mu\nu}{}^{ab} \tag{3.4}$$

then is gauge invariant under these symmetries. In particular, the weight $\lambda = \frac{5}{3}$ carried by the fünfbein determinant e according to Eq. (3.2), combines with the weights of the inverse fünfbeins to a total weight of 1, as required in order for the Lagrangian to vary under Λ^M transformations into a total derivative.

[3]One could also write an A-covariantized Einstein-Hilbert term in terms of the metric $g_{\mu\nu}$, in which case there is no such extra term present, with Lorentz symmetry being already manifest.

Next, we turn to the kinetic term for $\mathcal{M}_{MN}$. This matrix parametrizes the scalar coset space $E_{6(6)}/USp(8)$ whose 42 coordinates describe the scalar fields of the theory. Under the generalized diffeomorphisms (2.5) it transforms as a symmetric 2-tensor of weight $\lambda' = 0$. Note in particular, that this transformation is compatible with the group property $\det \mathcal{M} = 1$. Introducing its covariant derivative according to Eq. (2.24), we can define the gauge-invariant kinetic term

$$\mathcal{L}_{\rm sc} = \frac{1}{24} e g^{\mu\nu} \mathcal{D}_\mu \mathcal{M}_{MN} \mathcal{D}_\nu \mathcal{M}^{MN}, \tag{3.5}$$

with the inverse matrix $\mathcal{M}^{MN}$. In particular, with the inverse metric $g^{\mu\nu}$ carrying weight $\lambda = -\frac{2}{3}$ and the fünfbein determinant the carrying weight $\lambda = \frac{5}{3}$, the total weight of this term in the Lagrangian is 1, as required for Λ^M gauge invariance. Similarly, the Yang-Mills kinetic term $-\frac{1}{4} e \mathcal{M}_{MN} \mathcal{F}^{\mu\nu M} \mathcal{F}_{\mu\nu}{}^N$ in Eq. (1.3) carries the correct weight of 1 and is hence gauge invariant. Indeed, we saw above that the field strengths $\mathcal{F}_{\mu\nu}{}^M$ are gauge covariant and carry a weight of $\lambda = \frac{1}{3}$, which is precisely the correct weight given the presence of two inverse metrics $g^{\mu\nu}$.

After having discussed the kinetic terms, we now turn to the topological Chern-Simons-like term. By this we mean a term that is written without use of the metric (i.e., only through exterior products of forms) and that contains bare gauge potentials such that it is only gauge invariant up to boundary terms. Its structure is analogous to the topological term in general $D = 5$ gauged supergravity [77], such that its field equations yield the desired first-order duality equations relating $A_\mu{}^M$ and $B_{\mu\nu\, M}$. Such a term may be written more conveniently as a total derivative in one higher dimension, which has the advantage of making the gauge invariance manifest. Using form notation for the invariant curvatures introduced in Eqs. (2.30) and (2.38),

$$\mathcal{F}^M \equiv \frac{1}{2} \mathcal{F}_{\mu\nu}{}^M dx^\mu \wedge dx^\nu, \quad \mathcal{H}_M \equiv \frac{1}{3!} \mathcal{H}_{\mu\nu\rho M} dx^\mu \wedge dx^\nu \wedge dx^\rho, \tag{3.6}$$

the topological term can be written as an integral of an exact 6-form over a six-dimensional space $\mathcal{M}_6$,

$$\begin{aligned} S_{\rm top} &= \int d^5x d^{27}Y \mathcal{L}_{\rm top} \\ &= \kappa \int d^{27}Y \int_{\mathcal{M}_6} (d_{MNK} \mathcal{F}^M \wedge \mathcal{F}^N \wedge \mathcal{F}^K - 40 d^{MNK} \mathcal{H}_M \wedge \partial_N \mathcal{H}_K), \end{aligned} \tag{3.7}$$

whose overall constant κ will be determined below. From this we may determine the nonmanifestly gauge-invariant five-dimensional form, but it is not very illuminating and will also not be needed in the following. What will be needed in the following is the general variation of the topological term, which is derived from Eq. (3.7) and takes the form

$$\delta \mathcal{L}_{\rm top} = \kappa \varepsilon^{\mu\nu\rho\sigma\tau} \left(\frac{3}{4} d_{MNK} \mathcal{F}_{\mu\nu}{}^M \mathcal{F}_{\rho\sigma}{}^N \delta A_\tau{}^K + 5 d^{MNK} d_{KPQ} \partial_N \mathcal{H}_{\mu\nu\rho\, M} A_\sigma{}^P \delta A_\tau{}^Q + 5 d^{MNK} \partial_N \mathcal{H}_{\mu\nu\rho\, M} \delta B_{\sigma\tau K} \right). \tag{3.8}$$

In terms of the covariant variation (2.32) it takes the even simpler form

$$\delta \mathcal{L}_{\rm top} = \kappa \varepsilon^{\mu\nu\rho\sigma\tau} \left(\frac{3}{4} d_{MNK} \mathcal{F}_{\mu\nu}{}^M \mathcal{F}_{\rho\sigma}{}^N \delta A_\tau{}^K + 5 d^{MNK} \partial_N \mathcal{H}_{\mu\nu\rho\, M} \Delta B_{\sigma\tau\, K} \right). \tag{3.9}$$

With this form it is straightforward to explicitly verify gauge invariance under the Λ and Ξ transformations (2.34), integrating by parts and using the Bianchi identities (2.37) and (2.40). Note that due to Eq. (2.36) in this computation we can exchange the relevant ∂_M and $\mathcal{D}_\mu$ derivatives.

We close this subsection by giving the field equations of the topological fields $B_{\mu\nu M}$, which enter the topological term and the Yang-Mills term via the covariant field strength $\mathcal{F}_{\mu\nu}{}^M$. The field equations obtained by varying $B_{\mu\nu P}$ in these terms read

$$d^{PML} \partial_L (e \mathcal{M}_{MN} \mathcal{F}^{\mu\nu N} + \kappa \varepsilon^{\mu\nu\rho\sigma\tau} \mathcal{H}_{\rho\sigma\tau M}) = 0. \tag{3.10}$$

We will see in the following sections that upon taking appropriate solutions of the constraints (1.1), these relations reduce to the required first-order duality relations of either 11-dimensional supergravity or type IIB supergravity.

B. The potential

We now discuss the final term in the EFT action, namely the potential, which is a function of $g_{\mu\nu}$ and $\mathcal{M}_{MN}$ given by

$$V = -\frac{1}{24}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL} + \frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK} - \frac{1}{2}g^{-1}\partial_M g\partial_N\mathcal{M}^{MN} - \frac{1}{4}\mathcal{M}^{MN}g^{-1}\partial_M g g^{-1}\partial_N g - \frac{1}{4}\mathcal{M}^{MN}\partial_M g^{\mu\nu}\partial_N g_{\mu\nu}. \qquad (3.11)$$

The relative coefficients in here are determined by Λ^M gauge invariance, and in the following we will verify this gauge symmetry. As the potential is an $E_{6(6)}$ singlet, with all indices being properly contracted, it is sufficient to verify the cancellation of all terms that are "noncovariant" in the following sense. For a generic object with an arbitrary number of upper and lower $E_{6(6)}$ fundamental indices, we define

$$\Delta_\Lambda \equiv \delta_\Lambda - \mathbb{L}_\Lambda. \qquad (3.12)$$

Put differently, by Δ we denote all terms in its variation that differ from the covariant ones (in turn given by the generalized Lie derivative). As the covariant generalized Lie derivative terms automatically combine into the Lie derivative of a scalar, it is sufficient to verify the cancellation of the noncovariant terms. The only terms that lead to a nontrivial Δ are those involving a partial derivative, so we have to compute those terms for $\partial\mathcal{M}$ and ∂g. First, we compare

$$\delta_\Lambda(\partial_M\mathcal{M}^{KL}) = \partial_M(\Lambda^P\partial_P\mathcal{M}^{KL} - 12\mathbb{P}^{(K}{}_R{}^{|P}{}_Q\partial_P\Lambda^{Q|}\mathcal{M}^{L)R}), \qquad (3.13)$$

with the covariant

$$\mathbb{L}_\Lambda(\partial_M\mathcal{M}^{KL}) = \Lambda^P\partial_P(\partial_M\mathcal{M}^{KL}) - 12\mathbb{P}^{(K}{}_R{}^{|P}{}_Q\partial_P\Lambda^{Q|}\partial_M\mathcal{M}^{L)R} + 6\mathbb{P}^R{}_M{}^P{}_Q\partial_P\Lambda^Q\partial_R\mathcal{M}^{KL} + \lambda\partial_P\Lambda^P\partial_M\mathcal{M}^{KL}. \qquad (3.14)$$

Here we introduced λ in order to allow for a possible weight of $\partial\mathcal{M}$. In fact, we will show momentarily that although $\mathcal{M}$ has weight zero, its derivative has a nontrivial weight. To see this we note that the first term in the second line of Eq. (3.14) simplifies by the section constraint, so that by writing out the projector according to Eq. (2.1) we obtain

$$\mathbb{L}_\Lambda(\partial_M\mathcal{M}^{KL}) = \Lambda^P\partial_P(\partial_M\mathcal{M}^{KL}) - 12\mathbb{P}^{(K}{}_R{}^{|P}{}_Q\partial_P\Lambda^{Q|}\partial_M\mathcal{M}^{L)R} + \frac{1}{3}\partial_P\Lambda^P\partial_M\mathcal{M}^{KL} + \partial_M\Lambda^P\partial_P\mathcal{M}^{KL} + \lambda\partial_P\Lambda^P\partial_M\mathcal{M}^{KL}. \qquad (3.15)$$

In Eq. (3.13) there are no density-type terms, so in order to match this as closely as possible with Eq. (3.15) we have to cancel the density term by setting $\lambda = -\frac{1}{3}$. We then infer that Eq. (3.13) agrees with Eq. (3.15), up to terms that involve second derivatives of the gauge parameter. In total, we have shown that $\partial\mathcal{M}$ comes with weight $\lambda = -\frac{1}{3}$ while its noncovariant variation is given by

$$\Delta_\Lambda(\partial_M\mathcal{M}^{KL}) = -12\mathbb{P}^{(K}{}_R{}^{|P}{}_Q\partial_M\partial_P\Lambda^{Q|}\mathcal{M}^{L)R}. \qquad (3.16)$$

Similarly, we have

$$\Delta_\Lambda(\partial_M\mathcal{M}_{KL}) = +12\mathbb{P}^R{}_{(K}{}^P{}_{|Q}\partial_M\partial_{P|}\Lambda^Q\mathcal{M}_{L)R}, \qquad (3.17)$$

again taking $\partial\mathcal{M}$ to have weight $\lambda = -\frac{1}{3}$. Taking the trace of Eq. (3.16) we obtain in particular

$$\Delta_\Lambda(\partial_N\mathcal{M}^{MN}) = -\frac{5}{3}\partial_N\partial_P\Lambda^P\mathcal{M}^{MN} - \partial_N\partial_P\Lambda^M\mathcal{M}^{PN} + \cdots, \qquad (3.18)$$

up to terms that vanish upon contraction with ∂_M by the section constraint. Finally we need to determine Δ_Λ for ∂g. By an exactly analogous computation we find that $g^{-1}\partial g$ has weight $\lambda = -\frac{1}{3}$. Moreover, derivatives ∂_M acting on $g^{\mu\nu}$ and $g_{\mu\nu}$ induce additional weights of $-\frac{1}{3}$, such that we find the total weights to be

$$\lambda(g^{-1}\partial_M g) = -\frac{1}{3}, \quad \lambda(\partial_M g^{\mu\nu}) = -1, \quad \lambda(\partial_M g_{\mu\nu}) = \frac{1}{3}, \qquad (3.19)$$

with the noncovariant gauge variations given by

$$\Delta_\Lambda(g^{-1}\partial_M g) = \frac{10}{3}\partial_M\partial_P\Lambda^P, \quad \Delta_\Lambda(\partial_M g^{\mu\nu}) = -\frac{2}{3}\partial_M\partial_P\Lambda^P g^{\mu\nu}, \quad \Delta_\Lambda(\partial_M g_{\mu\nu}) = \frac{2}{3}\partial_M\partial_P\Lambda^P g_{\mu\nu}. \qquad (3.20)$$

Let us now verify the gauge invariance of the potential. First, we note that the weights of the partial derivatives of the fields are as required in order to combine to a total weight of 1 with the weight $\lambda = \frac{5}{3}$ of the fünfbein determinant e multiplying the potential term in the action. Thus, the complete Λ invariance of the action is proven once we check that all Δ_Λ variations above cancel, which we will now show. We compute for the first term of Eq. (3.11)

$$\delta_\Lambda\left(-\frac{1}{24}e\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL}\right)$$
$$=\frac{1}{6}e\partial_M\partial_P\Lambda^K\mathcal{M}^{MN}\mathcal{M}^{PL}\partial_N\mathcal{M}_{KL}$$
$$-\frac{5}{3}ed_{RQS}d^{KPS}\partial_M\partial_P\Lambda^Q\mathcal{M}^{MN}\mathcal{M}^{RL}\partial_N\mathcal{M}_{KL}$$
$$=e\partial_M\partial_P\Lambda^K\mathcal{M}^{MN}\mathcal{M}^{PL}\partial_N\mathcal{M}_{KL}. \tag{3.21}$$

Here, in the second equality, we used the fact that $\mathcal{M}$ is $E_{6(6)}$ valued with determinant 1, which allows for simplifications. In order to explain this we first note that the current

$$(J_N)^K{}_L\equiv\mathcal{M}^{KP}\partial_N\mathcal{M}_{PL} \tag{3.22}$$

lives in the adjoint representation and is traceless. Therefore it satisfies

$$\mathbb{P}^M{}_N{}^K{}_L(J_P)^L{}_K=(J_P)^M{}_N. \tag{3.23}$$

Spelling out the projector with Eq. (2.1), this condition implies

$$d_{NLS}d^{MKS}J^L{}_K=-\frac{1}{2}J^M{}_N. \tag{3.24}$$

Using this in the second term on the right-hand side of the first equality in Eq. (3.21) then reproduces the final equality. For the second term in the potential (3.11) we compute

$$\delta_\Lambda\left(\frac{1}{2}e\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK}\right)$$
$$=\frac{2}{3}e\partial_M\partial_P\Lambda^P\partial_N\mathcal{M}^{MN}-e\partial_M\partial_P\Lambda^K\mathcal{M}^{MN}\mathcal{M}^{PL}\partial_N\mathcal{M}_{LK}$$
$$+e\partial_M\partial_P\Lambda^L\partial_L\mathcal{M}^{MP}. \tag{3.25}$$

Here we used again that the current J is Lie algebra valued, so that the invariance of the d symbol implies

$$0=3d^{K(SP}J^{M)}{}_K=d^{KPM}J^S{}_K+2d^{SK(P}J^{M)}{}_K. \tag{3.26}$$

The last term in here appears in the above variation, and by this relation it has been rewritten in terms of the first term, which then in turn gives zero by the section constraint. We observe that the cubic term in $\mathcal{M}$ in Eq. (3.25) precisely cancels the same term in Eq. (3.21), which in turn determined the relative coefficient between these terms. By using Eq. (3.20) it is straightforward to verify that the remaining terms linear in $\partial\mathcal{M}$ are cancelled by the Δ_Λ variation of the terms in the second line of Eq. (3.11). This proves the full Λ^M gauge invariance of the potential.

C. $(4+1)$-dimensional diffeomorphisms

In the previous subsections, we have determined the various terms of the EFT action (1.3) by invariance under generalized internal Λ^M diffeomorphisms. While this has uniquely fixed the form of the five different terms in Eq. (1.3), they could in principle have appeared with arbitrary relative coefficients. In this section we show that all relative factors are determined by invariance of the full action under the remaining gauge symmetries, which are a covariantized version of the $(4+1)$-dimensional diffeomorphisms with parameters $\xi^\mu(x,Y)$. If ξ^μ is independent of Y these are manifest symmetries for each term in the action separately. For general ξ^μ, however, this gauge invariance is far from manifest and in particular it relates all terms in the action. As a result, the action (1.3) is the unique action (with no free parameter left up to an overall rescaling) that is not only invariant under generalized internal diffeomorphisms $\Lambda^M(x,Y)$ but also under the appropriate version of the external diffeomorphisms $\xi^\mu(x,Y)$. The actions of these diffeomorphisms on the various fields are given by

$$\delta e_\mu{}^a=\xi^\nu\mathcal{D}_\nu e_\mu{}^a+\mathcal{D}_\mu\xi^\nu e_\nu{}^a,$$
$$\delta\mathcal{M}_{MN}=\xi^\mu\mathcal{D}_\mu\mathcal{M}_{MN},$$
$$\delta A_\mu{}^M=\xi^\nu\mathcal{F}_{\nu\mu}{}^M+\mathcal{M}^{MN}g_{\mu\nu}\partial_N\xi^\nu,$$
$$\Delta B_{\mu\nu M}=\frac{1}{16\kappa}\xi^\rho e\varepsilon_{\mu\nu\rho\sigma\tau}\mathcal{F}^{\sigma\tau N}\mathcal{M}_{MN}, \tag{3.27}$$

written for $B_{\mu\nu M}$ in terms of the covariant variation (2.32). They take the form of conventional diffeomorphisms, but they are "covariantized" with respect to the connection A of the separate Λ gauge symmetry, except for an additional $\mathcal{M}$-dependent term in $\delta A_\mu{}^M$ and an on-shell modification in $\Delta B_{\mu\nu\,M}$. More precisely, the naive covariant variation of $B_{\mu\nu\,M}$ would take the form $\Delta_\xi B_{\mu\nu M}=\xi^\rho\mathcal{H}_{\mu\nu\rho\,M}$, with the covariant field strength defined in Eq. (2.38), but it turns out that off-shell gauge invariance of the action requires one to replace this field strength according to the duality relation (3.10). Thus, the gauge variations (3.27) are only on-shell equivalent to the conventional form of the (covariantized) diffeomorphisms.

Next, we discuss the gauge invariance of the action under Eq. (3.27) in some detail. The explicit verification of this gauge invariance is quite tedious and so we focus on a subset of terms that provide a very strong consistency check and that are sufficient in order to determine all relative coefficients in the action. Specifically, for various structures the cancellation proceeds completely parallel to the calculation that ensures standard diffeomorphism invariance in 11-dimensional supergravity in a $5+6$ splitting of fields and coordinates. They can therefore be omitted. In particular, as explained in Ref. [72], terms linear in $\mathcal{M}$ that are of the structural form $\mathcal{M}^{MN}\partial_M(\cdots)\partial_N(\ldots)$ have to cancel separately, and this computation is formally identical to the corresponding one for standard diffeomorphisms. In the following we focus on those terms for which cancellation involves the novel features of the EFT action.

We start by computing the variation of the sum of Yang-Mills and the topological term, denoted in the following by $\mathcal{L}_{\rm VT}$,

$$\mathcal{L}_{\rm VT} \equiv -\frac{1}{4} e \mathcal{F}_{\mu\nu}{}^{M} \mathcal{F}^{\mu\nu\,N} \mathcal{M}_{MN} + \kappa \mathcal{L}_{\rm CS}. \tag{3.28}$$

Using Eq. (3.9) one easily sees that its general variation is given by

$$\delta\mathcal{L}_{\rm VT} = (\kappa \varepsilon^{\mu\nu\rho\sigma\tau} d_{MNK} \mathcal{F}_{\nu\rho}{}^{K} \mathcal{F}_{\sigma\tau}{}^{N} - \mathcal{D}_\nu (e \mathcal{M}_{MN} \mathcal{F}^{\mu\nu\,N})) \delta A_\mu{}^M + 5 d^{MKN} \partial_K \left(e \mathcal{F}^{\mu\nu\,N} \mathcal{M}_{MN} + \frac{4\kappa}{3} \varepsilon^{\mu\nu\rho\sigma\tau} \mathcal{H}_{\rho\sigma\tau M} \right) \Delta B_{\mu\nu\,N} \\ + \mathcal{O}(\delta g_{\mu\nu}) + \mathcal{O}(\delta \mathcal{M}_{MN}). \tag{3.29}$$

Next, we insert the gauge variations (3.27) and first focus on the $\mathcal{F}\wedge\mathcal{F}$ terms in the variation,

$$\begin{aligned} \delta\mathcal{L}_{\rm VT}|_{\mathcal{F}\wedge\mathcal{F}} &= \kappa \varepsilon^{\mu\nu\rho\sigma\tau} d_{MNK} \mathcal{F}_{\nu\rho}{}^{K} \mathcal{F}_{\sigma\tau}{}^{N} \mathcal{M}^{ML} g_{\mu\lambda} \partial_L \xi^\lambda + \frac{5}{16\kappa} d^{MKN} \partial_K (e \mathcal{F}^{\mu\nu Q} \mathcal{M}_{MQ}) \xi^\rho e \varepsilon_{\mu\nu\rho\sigma\tau} \mathcal{F}^{\sigma\tau P} \mathcal{M}_{NP} \\ &= \kappa \varepsilon^{\mu\nu\rho\sigma\tau} d_{MNK} \mathcal{M}^{ML} \mathcal{F}_{\nu\rho}{}^{K} \mathcal{F}_{\sigma\tau}{}^{N} g_{\mu\lambda} \partial_L \xi^\lambda - \frac{5}{32\kappa} \varepsilon^{\mu\nu\sigma\tau\rho} d^{MKN} \mathcal{M}_{MQ} \mathcal{M}_{NP} \mathcal{F}_{\mu\nu}{}^{P} \mathcal{F}_{\sigma\tau}{}^{Q} g_{\rho\lambda} \partial_K \xi^\lambda. \end{aligned} \tag{3.30}$$

We can simplify this variation by using that $\mathcal{M}$ is $E_{6(6)}$ valued, so that the invariance of the d symbol implies $d^{MKN} \mathcal{M}_{MQ} \mathcal{M}_{NP} = d_{PQM} \mathcal{M}^{MN}$. Using this in Eq. (3.30) we infer that this variation vanishes for

$$\kappa^2 = \frac{5}{32}. \tag{3.31}$$

Let us now return to Eq. (3.29) and focus on the variation coming from the second term in the first line, restricted to the covariant, $\mathcal{M}$-independent term of $\delta A_\mu{}^M$ in Eq. (3.27). Integrating by parts we compute

$$e \mathcal{F}^{\mu\nu\,N} \mathcal{M}_{MN} \mathcal{D}_\nu (\xi^\rho \mathcal{F}_{\rho\mu}{}^M) = e \mathcal{F}^{\mu\nu\,N} \mathcal{M}_{MN} \mathcal{D}_\nu \xi^\rho \mathcal{F}_{\rho\mu}{}^M - \frac{1}{2} e \mathcal{F}^{\mu\nu\,N} \mathcal{M}_{MN} \xi^\rho \mathcal{D}_\rho \mathcal{F}_{\mu\nu}{}^M + 5 e d^{MPQ} \xi^\rho \mathcal{F}^{\mu\nu\,N} \mathcal{M}_{MN} \partial_P \mathcal{H}_{\mu\nu\rho\,Q}, \tag{3.32}$$

where we rewrote the $\mathcal{D}\mathcal{F}$ term as a total curl and then used the Bianchi identity (2.37) in the last term in the second line. Let us note that the first two terms of Eq. (3.32) occur already in a completely analogous form in the usual diffeomorphism variation, and so their cancellation against the variation of $g_{\mu\nu}$ and $\mathcal{M}_{MN}$ from Eq. (3.29) is standard. The term in the last line originating from the novel Bianchi identity, however, needs to be cancelled separately. This is achieved by the variation originating from the second term in the second line of Eq. (3.29). In fact, by inserting ΔB from Eq. (3.27) we compute for this term

$$\begin{aligned} &\frac{5}{12} e \varepsilon_{\mu\nu\lambda\sigma'\tau'} \varepsilon^{\mu\nu\rho\sigma\tau} d^{MKN} \partial_K \mathcal{H}_{\rho\sigma\tau\,M} \xi^\lambda \mathcal{F}^{\sigma'\tau'\,Q} \mathcal{M}_{NQ} \\ &\quad = -5 e d^{MKN} \partial_K \mathcal{H}_{\rho\sigma\tau\,M} \xi^\rho \mathcal{F}^{\sigma\tau\,Q} \mathcal{M}_{NQ}, \end{aligned}$$

which cancels precisely the final term in Eq. (3.32).

Let us next inspect the variation of the second term in the first line of Eq. (3.29), but now under the non-covariant, $\mathcal{M}$-dependent term of $\delta A_\mu{}^M$ in Eq. (3.27). Upon integration by parts we obtain

$$\begin{aligned} &- \mathcal{D}_\nu (e \mathcal{M}_{MN} \mathcal{F}^{\mu\nu\,N}) \mathcal{M}^{MK} g_{\mu\rho} \partial_K \xi^\rho \\ &\quad = e \mathcal{F}_{\mu\nu}{}^M \mathcal{M}_{MN} \mathcal{D}^\nu \mathcal{M}^{NK} \partial_K \xi^\mu - \mathcal{F}^{\mu\nu\,M} \mathcal{D}_\mu (g_{\nu\rho} \partial_M \xi^\rho). \end{aligned} \tag{3.33}$$

The second term precisely cancels against the main contribution from the variation of the Einstein-Hilbert term. This computation is formally identical to that presented in Ref. [72], cf. Eq. (4.16) in that paper. The first term in Eq. (3.33) will cancel against the variation of the scalar kinetic term. In order to show this, let us first compute the variation of the "scalar current,"

$$
\begin{aligned}
\delta_\xi(\mathcal{D}_\mu\mathcal{M}_{MN}) &= \mathcal{D}_\mu(\xi^\nu\mathcal{D}_\nu\mathcal{M}_{MN}) - \mathbb{L}_{\delta A_\mu}\mathcal{M}_{MN} \\
&= \mathcal{L}_\xi(\mathcal{D}_\mu\mathcal{M}_{MN}) - \xi^\nu\mathbb{L}_{\mathcal{F}_{\mu\nu}}\mathcal{M}_{MN} + \mathbb{L}_{\xi^\nu\mathcal{F}_{\mu\nu}}\mathcal{M}_{MN} - \mathbb{L}_{\mathcal{M}^{\cdot K}g_{\mu\nu}\partial_K\xi^\nu}\mathcal{M}_{MN} \\
&= \mathcal{L}_\xi(\mathcal{D}_\mu\mathcal{M}_{MN}) + 12\mathbb{P}^P{}_Q{}^K{}_{(M}\mathcal{M}_{N)K}\mathcal{F}_{\mu\nu}{}^Q\partial_P\xi^\nu - \mathcal{M}^{KL}\partial_L\mathcal{M}_{MN}g_{\mu\nu}\partial_K\xi^\nu + 12\mathbb{P}^P{}_Q{}^K{}_{(M}\mathcal{M}_{N)K}\partial_P(\mathcal{M}^{LQ}g_{\mu\nu}\partial_L\xi^\nu) \\
&= \mathcal{L}_\xi(\mathcal{D}_\mu\mathcal{M}_{MN}) + \frac{2}{3}\mathcal{M}_{MN}\mathcal{F}_{\mu\nu}{}^P\partial_P\xi^\nu + 2\mathcal{F}_{\mu\nu}{}^K\mathcal{M}_{K(M}\partial_{N)}\xi^\nu - 20d^{PKL}d_{QL(M}\mathcal{M}_{N)K}\mathcal{F}_{\mu\nu}{}^Q\partial_P\xi^\nu \\
&\quad - \mathcal{M}^{KL}\partial_L\mathcal{M}_{MN}g_{\mu\nu}\partial_K\xi^\nu + \frac{2}{3}\mathcal{M}_{MN}\partial_P(\mathcal{M}^{LP}g_{\mu\nu}\partial_L\xi^\nu) + 2\mathcal{M}_{K(M}\partial_{N)}(\mathcal{M}^{KL}g_{\mu\nu}\partial_L\xi^\nu) \\
&\quad - 20d^{PKL}d_{QL(M}\mathcal{M}_{N)K}\partial_P(\mathcal{M}^{RQ}g_{\mu\nu}\partial_R\xi^\nu).
\end{aligned} \tag{3.34}
$$

After some tedious algebra, using in particular the fact that $(\mathcal{D}^\mu\mathcal{M}^{-1}\mathcal{M})^M{}_N$ is an $\mathfrak{e}_{6(6)}$ algebra-valued matrix on which the projector $\mathbb{P}^P{}_Q{}^N{}_M$ acts as the identity, one then computes for the variation of the scalar kinetic term

$$
\begin{aligned}
\delta\mathcal{L}_{\rm kin} &= \mathcal{D}^\mu\mathcal{M}^{MN}\mathcal{M}_{NK}\mathcal{F}_{\mu\nu}{}^K\partial_M\xi^\nu + \mathcal{D}^\mu\mathcal{M}^{MN}\partial_M(g_{\mu\nu}\partial_N\xi^\nu) \\
&\quad + \left(\mathcal{M}_{NL}\partial_M\mathcal{M}^{LK} - \frac{1}{12}\mathcal{M}^{KL}\partial_L\mathcal{M}_{MN}\right) \\
&\quad \times \mathcal{D}_\mu\mathcal{M}^{MN}\partial_K\xi^\mu.
\end{aligned} \tag{3.35}
$$

The first term in here precisely cancels the first term in Eq. (3.33). The second term is of the form $\mathcal{M}^{MN}\partial_M\partial_N$, which we consistently omitted, cf. the discussion above and Ref. [72]. Finally, the last line will be cancelled against part of the variation of the potential (thereby determining the overall coefficient of the potential). In fact, it is not difficult to see, using the analogue of the first of the equations in Eq. (4.22) in Ref. [72], that the variation of the leading terms in the potential read

$$
\begin{aligned}
\delta V &= \delta\left(\frac{1}{2}\mathcal{M}_{NL}\partial_M\mathcal{M}^{LK} - \frac{1}{24}\mathcal{M}^{KL}\partial_L\mathcal{M}_{MN}\right)\partial_K\mathcal{M}^{MN} + \cdots \\
&= \left(\mathcal{M}_{NL}\partial_M\mathcal{M}^{LK} - \frac{1}{12}\mathcal{M}^{KL}\partial_L\mathcal{M}_{MN}\right) \\
&\quad \times \mathcal{D}_\mu\mathcal{M}^{MN}\partial_K\xi^\mu + \cdots.
\end{aligned} \tag{3.36}
$$

As claimed, in the combination $\mathcal{L}_{\rm kin} - V$ they cancel the terms in Eq. (3.35). We have thus succeeded in determining all relative coefficients in the action (1.3) from ξ^μ gauge invariance and have shown how the nonstandard diffeomorphism symmetry is realized in the EFT action. This concludes our discussion of the $(4+1)$-dimensional diffeomorphisms.

IV. EMBEDDING OF $D = 11$ SUPERGRAVITY

In this section we show explicitly how to embed 11-dimensional supergravity into the EFT constructed above. To this end, in the first subsection we rewrite $D = 11$ supergravity in a Lorentz gauge-fixed form that would be appropriate for Kaluza-Klein compactification to $D = 5$, but keeping the dependence on all 11 coordinates. In the second subsection we reduce the EFT (1.3) by choosing a specific solution for the section constraint (1.1) that breaks $E_{6(6)}$ to GL(6), with all fields depending on $5+6$ coordinates. After the explicit dualization of some fields, we establish complete equivalence with $D = 11$ supergravity.

A. Decomposition of $D = 11$ supergravity

We start by briefly recalling the bosonic sector of $D = 11$ supergravity [2], whose fields consist of the elfbein $E_{\hat\mu}{}^{\hat a}$ and the 3-form potential $C_{\hat\mu\hat\nu\hat\rho}$, where $\hat\mu, \hat\nu = 0, \ldots, 10$, and $\hat a, \hat b = 0, \ldots, 10$, denote $D = 11$ curved and flat indices, respectively. The action reads

$$
\begin{aligned}
S_{11} = \int d^{11}x E\bigg(& R - \frac{1}{12}F^{\hat\mu\hat\nu\hat\rho\hat\sigma}F_{\hat\mu\hat\nu\hat\rho\hat\sigma} \\
& + \frac{1}{12\cdot 216}E^{-1}\epsilon^{\hat\mu_1\cdots\hat\mu_{11}}F_{\hat\mu_1\cdots\hat\mu_4}F_{\hat\mu_5\cdots\hat\mu_8}C_{\hat\mu_9\hat\mu_{10}\hat\mu_{11}}\bigg),
\end{aligned} \tag{4.1}
$$

with the Abelian field strength

$$
F_{\hat\mu\hat\nu\hat\rho\hat\sigma} = 4\partial_{[\hat\mu}C_{\hat\nu\hat\rho\hat\sigma]}. \tag{4.2}
$$

This theory is invariant under 3-form gauge transformations $\delta C_{\hat\mu\hat\nu\hat\rho} = 3\partial_{[\hat\mu}\Lambda_{\hat\nu\hat\rho]}$ and under 11-dimensional diffeomorphisms as well as local Lorentz transformations. Next we reduce the Lorentz gauge symmetry from SO(1,10) to $\mathrm{SO}(1,4)\times\mathrm{SO}(6)$, choosing an upper-triangular gauge for the elfbein, and accordingly split the indices and field components in the above three terms of the action.

1. Einstein-Hilbert term

First we consider the decomposition of the Einstein-Hilbert term, following Refs. [80,81]. For future application it is convenient to keep the decomposition general, so for the moment we consider a D-dimensional Einstein-Hilbert term and split the indices as $D = n + d$,

$$
\hat\mu = (\mu, m), \qquad \hat a = (a, \alpha), \tag{4.3}
$$

where $\mu = 1, \ldots n$, and $m = 1, \ldots, d$, and similarly for the flat indices. The Lorentz gauge symmetry is partially fixed by choosing the upper-triangular form of the

D-dimensional vielbein as follows:

$$E_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} \phi^{\gamma} e_{\mu}{}^{a} & A_{\mu}{}^{m}\phi_{m}{}^{\alpha} \\ 0 & \phi_{m}{}^{\alpha} \end{pmatrix}, \tag{4.4}$$

where $\phi = \det(\phi_m{}^{\alpha})$. The inverse is then given by

$$E_{\hat{a}}{}^{\hat{\mu}} = \begin{pmatrix} \phi^{-\gamma} e_{a}{}^{\mu} & -\phi^{-\gamma} e_{a}{}^{\nu} A_{\nu}{}^{m} \\ 0 & \phi_{\alpha}{}^{m} \end{pmatrix}. \tag{4.5}$$

The constant parameter γ depends on the "external" dimension n and is determined as

$$\gamma = -\frac{1}{n-2} \tag{4.6}$$

by requiring an Einstein-frame metric in the n-dimensional theory.

Before we compute the form of the Einstein-Hilbert term in the gauge (4.4) it is convenient to investigate the form of the gauge symmetries after this splitting. The original Einstein-Hilbert term is invariant under D-dimensional diffeomorphisms $x^{\hat{\mu}} \to x^{\hat{\mu}} - \xi^{\hat{\mu}}$ and local Lorentz transformations parametrized by $\lambda^{\hat{a}}{}_{\hat{b}}$, which act on the elfbein as

$$\delta E_{\hat{\mu}}{}^{\hat{a}} = \xi^{\hat{\nu}}\partial_{\hat{\nu}} E_{\hat{\mu}}{}^{\hat{a}} + \partial_{\hat{\mu}}\xi^{\hat{\nu}} E_{\hat{\nu}}{}^{\hat{a}} + \lambda^{\hat{a}}{}_{\hat{b}} E_{\hat{\mu}}{}^{\hat{b}}. \tag{4.7}$$

After the splitting of indices, the diffeomorphisms give rise to two types of gauge symmetries according to

$$\xi^{\hat{\mu}} = (\xi^{\mu}, \Lambda^{m}). \tag{4.8}$$

We will refer to the gauge transformations parametrized by Λ^m as "internal" diffeomorphisms. From Eq. (4.7) we compute

$$\begin{aligned} \delta_\Lambda e_\mu{}^a &= \Lambda^m \partial_m e_\mu{}^a - \gamma \partial_m \Lambda^m e_\mu{}^a, \\ \delta_\Lambda \phi_m{}^\alpha &= \Lambda^n \partial_n \phi_m{}^\alpha + \partial_m \Lambda^n \phi_n{}^\alpha, \\ \delta_\Lambda \phi &= \Lambda^n \partial_n \phi + \partial_n \Lambda^n \phi, \\ \delta_\Lambda A_\mu{}^m &= \partial_\mu \Lambda^m - A_\mu{}^n \partial_n \Lambda^m + \Lambda^n \partial_n A_\mu{}^m. \end{aligned} \tag{4.9}$$

We infer that e and ϕ transform as tensors (or tensor densities) under the symmetry of Λ^m transformations, for which $A_\mu{}^m$ provides a gauge connection. In fact, we can define covariant derivatives and field strengths as follows:

$$\begin{aligned} D_\mu e_\nu{}^a &= \partial_\mu e_\nu{}^a - A_\mu{}^m \partial_m e_\nu{}^a + \gamma \partial_n A_\mu{}^n e_\nu{}^a, \\ D_\mu \phi_m{}^\alpha &= \partial_\mu \phi_m{}^\alpha - A_\mu{}^n \partial_n \phi_m{}^\alpha - \partial_m A_\mu{}^n \phi_n{}^\alpha, \\ F_{\mu\nu}{}^m &= \partial_\mu A_\nu{}^m - \partial_\nu A_\mu{}^m - A_\mu{}^n \partial_n A_\nu{}^m + A_\nu{}^n \partial_n A_\mu{}^m, \end{aligned} \tag{4.10}$$

and it is straightforward to verify that they transform covariantly under Eq. (4.9). In order to compute the form of the gauge transformations parametrized by ξ^μ, which we refer to as "external" diffeomorphisms in the following, we have to add a compensating local Lorentz transformation in order to preserve the gauge choice in Eq. (4.4). The Lorentz parameter is found to be

$$\lambda^a{}_\beta = -\phi^\gamma \phi_\beta{}^m \partial_m \xi^\nu e_\nu{}^a. \tag{4.11}$$

Moreover, it turns out to be convenient to present these "external" diffeomorphisms in the form of covariant or "improved" diffeomorphisms, for which we add a field-dependent gauge transformation with parameter $\Lambda^m = -\xi^\nu A_\nu{}^m$. The full transformation rules can then be written directly in terms of the covariant objects from Eq. (4.10),

$$\begin{aligned} \delta_\xi e_\mu{}^a &= \xi^\nu D_\nu e_\mu{}^a + D_\mu \xi^\nu e_\nu{}^a, \\ \delta_\xi \phi_m{}^\alpha &= \xi^\nu D_\nu \phi_m{}^\alpha, \\ \delta_\xi A_\mu{}^m &= \xi^\nu F_{\nu\mu}{}^m + \phi^{2\gamma}\phi^{mn} g_{\mu\nu}\partial_n \xi^\nu, \end{aligned} \tag{4.12}$$

with $\phi^{mn} = \phi_\alpha{}^m \phi^{\alpha n}$.

After having discussed the form of the gauge symmetries, we are now ready to decompose the Einstein-Hilbert term. To this end it is convenient to use the following formula:

$$\begin{aligned} S_{\rm EH} &= \int d^D x E E_{\hat{a}}{}^{\hat{\mu}} E_{\hat{b}}{}^{\hat{\nu}} R_{\hat{\mu}\hat{\nu}}{}^{\hat{a}\hat{b}} \\ &= \int d^n x d^d y E\Big(-\frac{1}{4}\hat{\Omega}^{\hat{a}\hat{b}\hat{c}}\hat{\Omega}_{\hat{a}\hat{b}\hat{c}} + \frac{1}{2}\hat{\Omega}^{\hat{a}\hat{b}\hat{c}}\hat{\Omega}_{\hat{b}\hat{c}\hat{a}} \\ &\quad + \hat{\Omega}_{\hat{c}\hat{b}}{}^{\hat{b}}\hat{\Omega}^{\hat{c}}{}_{\hat{a}}{}^{\hat{a}}\Big), \end{aligned} \tag{4.13}$$

where we introduced the coefficients of anholonomy,

$$\hat{\Omega}_{\hat{a}\hat{b}\hat{c}} = E_{\hat{a}}{}^{\hat{\mu}} E_{\hat{b}}{}^{\hat{\nu}} (\partial_{\hat{\mu}} E_{\hat{\nu}\hat{c}} - \partial_{\hat{\nu}} E_{\hat{\mu}\hat{c}}). \tag{4.14}$$

Inserting the elfbein (4.4) and its inverse in here we find for the various components

$$\begin{aligned} \hat{\Omega}_{abc} &= \phi^{-\gamma}\Omega_{abc} + 2\gamma\phi^{-\gamma-1} e_{[a}{}^{\mu}\eta_{b]c} D_\mu \phi, \\ \hat{\Omega}_{ab\gamma} &= \phi^{-2\gamma} e_a{}^\mu e_b{}^\nu F_{\mu\nu}{}^m \phi_{m\gamma}, \\ \hat{\Omega}_{a\beta\gamma} &= \phi^{-\gamma}\phi_\beta{}^m e_a{}^\mu D_\mu \phi_{m\gamma}, \\ \hat{\Omega}_{\alpha bc} &= e_b{}^\nu \phi_\alpha{}^m D_m e_{\nu c}, \\ \hat{\Omega}_{\alpha\beta c} &= 0, \\ \hat{\Omega}_{\alpha\beta\gamma} &= \Omega_{\alpha\beta\gamma}, \end{aligned} \tag{4.15}$$

where we introduced the "external" and "internal" coefficients of anholonomy,

$$\Omega_{abc} = 2e_{[a}{}^{\mu}e_{b]}{}^{\nu}D_{\mu}e_{\nu c},$$
$$\Omega_{\alpha\beta\gamma} = 2\phi_{[\alpha}{}^{m}\phi_{\beta]}{}^{n}\partial_{m}\phi_{n\gamma}, \tag{4.16}$$

and defined

$$D_m e_{\nu c} \equiv \partial_m e_{\nu c} + \gamma\phi^{-1}\partial_m\phi e_{\nu c}. \tag{4.17}$$

This latter derivative is covariant under the internal diffeomorphisms (4.9) in that $D_m e_{\nu c}$ transforms as a vector density (with the same weight $-\gamma$ as $e_{\nu c}$). Moreover, we see that in Eq. (4.15) all the components are organized already into the covariant objects (4.10), so that the Λ gauge invariance of the action will be manifest.

Next we determine the form of the Einstein-Hilbert term by inserting the components (4.15) into Eq. (4.13) and using

$$E \equiv \det E_{\hat{\mu}}{}^{\hat{a}} = \phi^{n\gamma+1}e. \tag{4.18}$$

We find

$$\begin{aligned} S_{\rm EH} = \int d^n x d^d y e\Big[&-\frac{1}{4}\Omega^{abc}\Omega_{abc} + \frac{1}{2}\Omega^{abc}\Omega_{bca} + \Omega^a\Omega_a - e^{a\mu}e^{b\nu}F_{\mu\nu}{}^m(e_b{}^\rho\partial_m e_{\rho a}) \\ &-\frac{1}{2}\phi^{mn}g^{\mu\nu}D_\mu\phi_m{}^\alpha D_\nu\phi_{n\alpha} - \gamma^2(n-2)\phi^{-2}g^{\mu\nu}D_\mu\phi D_\nu\phi - \frac{1}{2}g^{\mu\nu}(\phi^{\alpha m}D_\mu\phi_m{}^\gamma)(\phi_\gamma{}^n D_\nu\phi_{n\alpha}) - \frac{1}{4}\phi^{-2\gamma}\phi_{mn}F^{\mu\nu m}F_{\mu\nu}{}^n \\ &+\phi^{2\gamma}\Big(-\frac{1}{2}\phi^{mn}g^{\mu\nu}D_m e_\mu{}^a D_n e_{\nu a} - \frac{1}{2}\phi^{mn}(e^{b\mu}D_m e_\mu{}^c)(e_c{}^\nu D_n e_{\nu b}) + \phi^{mn}(e^{-1}D_m e)(e^{-1}D_n e) - \frac{1}{4}\Omega^{\alpha\beta\gamma}\Omega_{\alpha\beta\gamma} \\ &+\frac{1}{2}\Omega^{\alpha\beta\gamma}\Omega_{\beta\gamma\alpha} + \Omega^\alpha\Omega_\alpha + 2\phi^{\alpha m}\Omega_\alpha e^{-1}D_m e\Big)\Big]. \end{aligned} \tag{4.19}$$

Let us now write the various terms more geometrically. The terms in the first line combine into the n-dimensional Einstein-Hilbert term for $e_\mu{}^a$, but with the additional covariantization that all derivatives are covariant according to Eq. (4.10) and the Ricci scalar is based on the "improved" Riemann tensor

$$\hat{R}_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} + F_{\mu\nu}{}^m e^{a\rho}\partial_m e_\rho{}^b, \tag{4.20}$$

which is necessary in order to preserve local SO(1,4) Lorentz invariance, as discussed above for the full EFT. Next, the terms in the last line in the potential can also be written more geometrically, using

$$\begin{aligned} e\phi^{2\gamma}R(\phi_m{}^\alpha) = e\phi^{2\gamma}\Big(&-\frac{1}{4}\Omega^{\alpha\beta\gamma}\Omega_{\alpha\beta\gamma} + \frac{1}{2}\Omega^{\alpha\beta\gamma}\Omega_{\beta\gamma\alpha} \\ &+\Omega^\alpha\Omega_\alpha + 2\phi_\alpha{}^m e^{-1}\partial_m e\Omega^\alpha \\ &+2(2\gamma-1)\phi_\alpha{}^m\phi^{-1}\partial_m\phi\Omega^\alpha\Big) + \text{total der.}, \end{aligned} \tag{4.21}$$

which for γ as determined in Eq. (4.6) reproduces the last line of Eq. (4.19). Finally, we can reorganize the De terms into Dg terms in order to make the local Lorentz invariance manifest. In total we obtain

$$\begin{aligned} S_{\rm EH} = \int d^n x d^d y e\Big[&\hat{R} - \frac{1}{4}\phi^{-2\gamma}\phi_{mn}F^{\mu\nu m}F_{\mu\nu}{}^n \\ &-\frac{1}{2}\phi^{mn}g^{\mu\nu}D_\mu\phi_m{}^\alpha D_\nu\phi_{n\alpha} - \gamma^2(n-2)\phi^{-2}g^{\mu\nu}D_\mu\phi D_\nu\phi \\ &-\frac{1}{2}g^{\mu\nu}(\phi^{\alpha m}D_\mu\phi_m{}^\gamma)(\phi_\gamma{}^n D_\nu\phi_{n\alpha}) - V(\phi,e)\Big], \end{aligned} \tag{4.22}$$

with the "Einstein-Hilbert potential"

$$\begin{aligned} V_{\rm EH}(\phi,e) \\ = -\phi^{2\gamma}\Big(R(\phi) + \frac{1}{4}\phi^{mn}(D_m g^{\mu\nu}D_n g_{\mu\nu} + g^{-1}D_m g g^{-1}D_n g)\Big). \end{aligned} \tag{4.23}$$

Below we will also need the form of the potential in terms of the symmetric tensor $\phi_{mn} = \phi_m{}^\alpha\phi_{n\alpha}$, as opposed to the vielbein. Integrating by parts, and setting $\gamma = -\frac{1}{3}$, the term involving the internal Ricci scalar can be written as

$$\begin{aligned} e\phi^{-\frac{2}{3}}R(\phi) = e\phi^{-\frac{2}{3}}\Big[&\frac{1}{2}\phi^{mn}\phi^{kl}\phi^{pq}\partial_k\phi_{mq}\partial_p\phi_{nl} \\ &-\frac{1}{4}\phi^{mn}\phi^{kl}\phi^{pq}\partial_p\phi_{mk}\partial_q\phi_{nl} \\ &-\frac{2}{3}\partial_m\phi^{mn}\phi^{-1}\partial_n\phi - \frac{21}{9}\phi^{mn}(\phi^{-1}\partial_m\phi)(\phi^{-1}\partial_n\phi) \\ &+\partial_m\phi^{mn}e^{-1}\partial_n e + 2\phi^{mn}(e^{-1}\partial_m e)(\phi^{-1}\partial_n\phi)\Big], \end{aligned} \tag{4.24}$$

which is the form that is convenient for the comparison with the $E_{6(6)}$-covariant theory.

2. 3-form kinetic and topological terms

We now turn to the decomposition of the kinetic term for the 3-form. First, we have to perform field redefinitions of the various components of $C_{\hat\mu\hat\nu\hat\rho}$ in terms of the Kaluza-Klein vector in order to obtain forms that transform covariantly under the gauge symmetries. The general prescription for Kaluza-Klein reductions is to "flatten" all $D=11$ curved indices with $E_a{}^{\hat\mu}$ and then to "unflatten" with the external n-bein components $E_\mu{}^a$. For instance, the vectors originating from the 3-form are redefined according to

$$A_{\mu\, mn} \equiv E_\mu{}^a E_a{}^{\hat\nu} C_{\hat\nu mn}. \tag{4.25}$$

Performing the analogous field redefinition for the other components we obtain the following field variables originating from the 3-form $C_{\hat\mu\hat\nu\hat\rho}$, denoted by A:

$$\begin{aligned}
&A_{mnk} = C_{mnk}, \qquad A_{\mu mn} = C_{\mu mn} - A_\mu{}^k C_{kmn},\\
&A_{\mu\nu m} = C_{\mu\nu m} - 2A_{[\mu}{}^n C_{\nu]mn} + A_\mu{}^n A_\nu{}^k C_{mnk},\\
&A_{\mu\nu\rho} = C_{\mu\nu\rho} - 3A_{[\mu}{}^m C_{\nu\rho]m} + 3A_{[\mu}{}^m A_\nu{}^n C_{\rho]mn}\\
&\qquad - A_\mu{}^m A_\nu{}^n A_\rho{}^k C_{mnk}.
\end{aligned} \tag{4.26}$$

This definition is such that the fields transform covariantly under internal diffeomorphisms, i.e., simply according to their "internal" index structure. In order to display the transformation under the components of the 3-form gauge parameter $\Lambda_{\hat\mu\hat\nu}$, we also have to perform redefinitions of the parameters with the Kaluza-Klein vector, following exactly the same prescription as for the fields. Thus, we define the new parameters

$$\Lambda'_{\mu m} = \Lambda_{\mu m} - A_\mu{}^n \Lambda_{nm}, \quad \text{etc.} \tag{4.27}$$

Dropping the prime on the parameters in the following, we obtain the gauge transformations under $(\Lambda_{mn}, \Lambda_{\mu m}, \Lambda_{\mu\nu})$ which act on the fields as

$$\begin{aligned}
\delta A_{mnk} &= 3\partial_{[m}\Lambda_{nk]},\\
\delta A_{\mu mn} &= D_\mu \Lambda_{mn} - 2\partial_{[m}\Lambda_{|\mu|n]},\\
\delta A_{\mu\nu m} &= 2D_{[\mu}\Lambda_{\nu]m} - F_{\mu\nu}{}^n \Lambda_{mn} + \partial_m \Lambda_{\mu\nu},\\
\delta A_{\mu\nu\rho} &= 3D_{[\mu}\Lambda_{\nu\rho]} - 3F_{[\mu\nu}{}^m \Lambda_{\rho]m}.
\end{aligned} \tag{4.28}$$

As usual, all derivatives are covariant with respect to the internal diffeomorphisms. We observe that after the decomposition the formerly Abelian 3-form gauge transformations of $D=11$ supergravity take a nontrivial form with noncommuting covariant derivatives and extra Stückelberg-type transformations, reminiscent of the tensor hierarchy introduced above. Moreover, the Kaluza-Klein Yang-Mills field strength $F_{\mu\nu}{}^n$ explicitly appears in the transformation rules.

Let us now turn to the form of the field strength components. As for the fields, redefinitions are required, in order to arrive at field strengths that are covariant under internal diffeomorphisms and invariant under Eq. (4.28). We define

$$F'_{\mu mnk} \equiv E_\mu{}^a E_a{}^{\hat\nu} F_{\hat\nu mnk}, \quad \text{etc.}, \tag{4.29}$$

which are manifestly invariant under the 3-form gauge transformations as a consequence of the invariance of the original field strength $F_{\hat\mu\hat\nu\hat\rho\hat\sigma}$. Dropping the primes in the following, one finds for the redefined field strength in terms of the redefined fields

$$\begin{aligned}
F_{mnkl} &= 4\partial_{[m}A_{nkl]},\\
F_{\mu\, nkl} &= D_\mu A_{nkl} - 3\partial_{[n}A_{|\mu|kl]},\\
F_{\mu\nu\, mn} &= 2D_{[\mu}A_{\nu]mn} + F_{\mu\nu}{}^k A_{kmn} + 2\partial_{[m}A_{|\mu\nu|n]},\\
F_{\mu\nu\rho m} &= 3D_{[\mu}A_{\nu\rho]m} + 3F_{[\mu\nu}{}^n A_{\rho]mn} - \partial_m A_{\mu\nu\rho},\\
F_{\mu\nu\rho\sigma} &= 4D_{[\mu}A_{\nu\rho\sigma]} + 6F_{[\mu\nu}{}^m A_{\rho\sigma]m}.
\end{aligned} \tag{4.30}$$

These field strengths are manifestly covariant with respect to internal diffeomorphisms. Moreover, one may verify by an explicit computation that the field strengths are gauge invariant under Eq. (4.28). Due to the non-Abelian gauge connections entering the fields strengths, the latter satisfy nonstandard Bianchi identities,

$$\begin{aligned}
D_\mu F_{mnkl} &= 4\partial_{[m}F_{|\mu|nkl]},\\
2D_{[\mu}F_{\nu]nkl} &= -3\partial_{[n}F_{|\mu\nu|kl]} - F_{\mu\nu}{}^m F_{mnkl},\\
3D_{[\mu}F_{\nu\rho]mn} &= 2\partial_{[m}F_{|\mu\nu\rho|n]} + 3F_{[\mu\nu}{}^k F_{\rho]kmn},\\
4D_{[\mu}F_{\nu\rho\sigma]m} &= -\partial_m F_{\mu\nu\rho\sigma} + 6F_{[\mu\nu}{}^n F_{\rho\sigma]mn},\\
5D_{[\mu}F_{\nu\rho\sigma\lambda]} &= 10F_{[\mu\nu}{}^m F_{\rho\sigma\lambda]m}.
\end{aligned} \tag{4.31}$$

As for the tensor hierarchy, the Bianchi identities relate the exterior derivatives of a field strength to the "next higher" field strength in the hierarchy.

We are now in a position to give the decomposition of the kinetic term for the 3-form. Due to the form of the redefinition (4.29) of the field strengths, it is straightforward to rewrite the F^2 term, by simply going to flattened indices,

$$\begin{aligned}\mathcal{L}_{\text{3-form}} &= -\frac{1}{12}EF^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}F_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} = -\frac{1}{12}EF^{\hat{a}\hat{b}\hat{c}\hat{d}}F_{\hat{a}\hat{b}\hat{c}\hat{d}}\\ &= -\frac{1}{12}\phi^{n\gamma+1}e(\phi^{-8\gamma}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}+4\phi^{-6\gamma}\phi^{mn}F^{\mu\nu\rho}{}_{m}F_{\mu\nu\rho n}+6\phi^{-4\gamma}\phi^{mn}\phi^{kl}F^{\mu\nu}{}_{mk}F_{\mu\nu nl}\\ &\quad+4\phi^{-2\gamma}\phi^{mn}\phi^{kl}\phi^{pq}F^{\mu}{}_{mkp}F_{\mu nlq}+\phi^{mn}\phi^{kl}\phi^{pq}\phi^{rs}F_{mkpr}F_{nlqs})\\ &= -\frac{1}{12}e(\phi^{2}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}+4\phi^{\frac{4}{3}}\phi^{mn}F^{\mu\nu\rho}{}_{m}F_{\mu\nu\rho n}+6\phi^{\frac{2}{3}}\phi^{mn}\phi^{kl}F^{\mu\nu}{}_{mk}F_{\mu\nu nl}\\ &\quad+4\phi^{mn}\phi^{kl}\phi^{pq}F^{\mu}{}_{mkp}F_{\mu nlq}+\phi^{-\frac{2}{3}}\phi^{mn}\phi^{kl}\phi^{pq}\phi^{rs}F_{mkpr}F_{nlqs}).\end{aligned} \tag{4.32}$$

Here we left the raising of spacetime indices with $g^{\mu\nu}$ implicit, and we inserted the value for γ [see Eq. (4.6)] for $n=5$.

Next we have to decompose the topological Chern-Simons-like term in Eq. (4.1) and write it in terms of the invariant field strengths defined in Eq. (4.30). One finds

$$\begin{aligned}\mathcal{L}_{\text{top}} = -\frac{1}{108}\epsilon^{\mu\nu\rho\sigma\lambda}\epsilon^{mnklpq}\Big(&A_{\mu\nu m}F_{\rho\sigma\lambda n}F_{klpq}\\ &+6A_{\mu\nu m}F_{\rho\sigma nk}F_{\lambda lpq}-\frac{1}{2}A_{\mu\nu\rho}F_{\sigma\lambda mn}F_{klpq}\\ &+\frac{2}{3}A_{\mu\nu\rho}F_{\sigma mnk}F_{\lambda lpq}-\frac{1}{4}A_{\mu mn}F_{klpq}F_{\nu\rho\sigma\lambda}\\ &+4A_{\mu mn}F_{\nu klp}F_{\rho\sigma\lambda q}-\frac{9}{2}A_{\mu mn}F_{\nu\rho kl}F_{\sigma\lambda pq}\\ &+\frac{1}{3}A_{mnk}F_{\mu lpq}F_{\nu\rho\sigma\lambda}+2A_{mnk}F_{\mu\nu lp}F_{\rho\sigma\lambda q}\Big).\end{aligned} \tag{4.33}$$

The validity of this expression can be checked explicitly by verifying gauge invariance under Eq. (4.28). As the field strengths are already gauge invariant by construction, we only have to vary the bare gauge potentials A. After this we may integrate by parts and show the cancellation by use of the Bianchi identities (4.31). This computation requires repeated use of Schouten identities according to which terms with total antisymmetrization over seven internal indices $m,n,\ldots$ vanish identically. Let us note that up to total derivatives, the form of Eq. (4.33) is uniquely determined by gauge invariance under Eq. (4.28), up to the overall coefficient that is determined by $D=11$ supergravity.

Finally, we can give the complete action of $D=11$ supergravity under the $5+6$ decomposition and the corresponding gauge fixing of the local Lorentz group,

$$\begin{aligned}S_{11} = \int d^5x d^6y e\Big[&\hat{R}-\frac{1}{4}\tilde{\mathcal{M}}_{\mathfrak{m}\mathfrak{n}}\mathcal{F}^{\mu\nu\mathfrak{m}}\mathcal{F}_{\mu\nu}{}^{\mathfrak{n}}-\frac{1}{12}\phi^{2}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}\\ &-\frac{1}{3}\phi^{\frac{4}{3}}\phi^{mn}F^{\mu\nu\rho}{}_{m}F_{\mu\nu\rho n}-\frac{1}{2}\phi^{mn}D^{\mu}\phi_{m}{}^{\alpha}D_{\mu}\phi_{n\alpha}\\ &-\frac{1}{3}\phi^{-2}D^{\mu}\phi D_{\mu}\phi-\frac{1}{2}(\phi^{\alpha m}D^{\mu}\phi_{m}{}^{\gamma})(\phi_{\gamma}{}^{n}D_{\mu}\phi_{n\alpha})\\ &-\frac{1}{3}\phi^{mn}\phi^{kl}\phi^{pq}F^{\mu}{}_{mkp}F_{\mu nlq}-V(e,\phi)+e^{-1}\mathcal{L}_{\text{top}}\Big].\end{aligned} \tag{4.34}$$

Here we fixed $\gamma=-\frac{1}{3}$ according to Eq. (4.6). Moreover, we combined the 2-form field strengths of the Kaluza-Klein gauge vector and the vector originating from the 3-form,

$$\mathcal{F}_{\mu\nu}{}^{\mathfrak{m}} = (\mathcal{F}_{\mu\nu}{}^{m},\mathcal{F}_{\mu\nu mn}) \equiv (F_{\mu\nu}{}^{m},F_{\mu\nu mn}-F_{\mu\nu}{}^{k}A_{kmn}), \tag{4.35}$$

by introducing the scalar-dependent kinetic metric

$$\tilde{\mathcal{M}}_{m,n} = \phi^{\frac{2}{3}}(\phi_{mn}+2\phi^{kl}\phi^{pq}A_{mkp}A_{nlq}),\qquad \tilde{\mathcal{M}}_{m,}{}^{kl} = 2\phi^{\frac{2}{3}}\phi^{kp}\phi^{lq}A_{mpq},\qquad \tilde{\mathcal{M}}^{mn,kl} = 2\phi^{\frac{2}{3}}\phi^{m[k}\phi^{l]n}, \tag{4.36}$$

with the index ${}_{\mathfrak{m}}=({}_{m},{}^{[mn]})$. The topological term is given by Eq. (4.33) and the full potential reads

$$\begin{aligned}eV = -e\phi^{-\frac{2}{3}}\Big[&\frac{1}{2}\phi^{mn}\phi^{kl}\phi^{pq}\partial_k\phi_{mq}\partial_p\phi_{nl}-\frac{1}{4}\phi^{mn}\phi^{kl}\phi^{pq}\partial_p\phi_{mk}\partial_q\phi_{nl}-\frac{2}{3}\partial_m\phi^{mn}\phi^{-1}\partial_n\phi-\frac{1}{9}\phi^{mn}(\phi^{-1}\partial_m\phi)(\phi^{-1}\partial_n\phi)\\ &+\partial_m\phi^{mn}e^{-1}\partial_n e-\frac{2}{3}\phi^{mn}(e^{-1}\partial_m e)(\phi^{-1}\partial_n\phi)+\frac{1}{4}\phi^{mn}(\partial_m g^{\mu\nu}\partial_n g_{\mu\nu}+g^{-1}\partial_m g g^{-1}\partial_n g)-\frac{1}{12}\phi^{mn}\phi^{kl}\phi^{pq}\phi^{rs}F_{mkpr}F_{nlqs}\Big].\end{aligned} \tag{4.37}$$

It is obtained by combining Eq. (4.23) with the purely internal F^2 term from Eq. (4.32). Moreover, we used Eq. (4.24) and expanded the Dg terms according to Eq. (4.17). This is the final form of the action, and it is still equivalent to the full $D=11$ supergravity. In the following, we will compare and match this result with the action obtained by evaluating the EFT (1.3) for a particular solution of the section constraints.

B. GL(6)-invariant reduction of EFT

In this subsection, we will consider the $E_{6(6)}$-covariant EFT (1.3) upon specifying an explicit solution of the section condition, that breaks $E_{6(6)}$ down to GL(6). We will show that the resulting theory upon further dualization precisely coincides with 11-dimensional supergravity in the form presented in the previous subsection.

1. GL(6)-invariant solution of the section condition

The relevant embedding of GL(6) into $E_{6(6)}$ is given by

$$\mathrm{GL}(6) = \mathrm{SL}(6)\times \mathrm{GL}(1) \subset \mathrm{SL}(6)\times \mathrm{SL}(2) \subset E_{6(6)}, \qquad (4.38)$$

with the fundamental representation of $E_{6(6)}$ breaking as

$$\bar{\mathbf{27}} \to 6_{+1} + 15'_0 + 6_{-1}, \qquad (4.39)$$

and the adjoint breaking into

$$\mathbf{78} \to 1_{-2} + 20_{-1} + (1+35)_0 + 20_{+1} + 1_{+2}, \qquad (4.40)$$

with the subscripts referring to the GL(1) charges. An explicit solution to the section condition (1.1) is given by restricting the Y^M dependence of all fields to the six coordinates in the 6_{+1}. Explicitly, by splitting the coordinates Y^M according to Eq. (4.39) into

$$\{Y^M\} \to \{y^m, y_{mn}, y^{\bar{m}}\}, \qquad (4.41)$$

with indices $m, n = 1, \ldots, 6$, the nonvanishing components of the d symbol are given by[4]

$$d^{MNK}:\ d^{m\bar{n}}{}_{kl} = \frac{1}{\sqrt{5}}\delta^m_{[k}\delta^n_{l]}, \quad d_{mnklpq} = \frac{1}{4\sqrt{5}}\varepsilon_{mnklpq},$$
$$d_{MNK}:\ d_{m\bar{n}}{}^{kl} = \frac{1}{\sqrt{5}}\delta^k_{[m}\delta^l_{n]}, \quad d^{mnklpq} = \frac{1}{4\sqrt{5}}\varepsilon^{mnklpq}, \qquad (4.42)$$

and all those related by symmetry, $d^{MNK} = d^{(MNK)}$. In particular, the GL(1) grading guarantees that all components d^{mnk} vanish, such that the section condition (1.1) indeed is solved by restricting the coordinate dependence of all fields according to

$$\{\partial_{\bar{m}}A = 0, \partial^{mn}A = 0\} \Leftrightarrow A(x^\mu, Y^M) \to A(x^\mu, y^m). \qquad (4.43)$$

Let us first revisit the resulting field content of the model. The $E_{6(6)}$-covariant formulation presented above carries all 27 vector fields $A_\mu{}^M$, now breaking according to Eq. (4.39), whereas the 2-forms appear only under the projection $d^{MNK}\partial_N B_{\mu\nu K}$. With Eq. (4.42) we find that only the components $B_{\mu\nu\bar{n}}$ and $B_{\mu\nu}{}^{mn}$ enter the Lagrangian; moreover, they enter under ∂_m derivatives according to

$$\partial_m B_{\mu\nu\bar{n}} - \partial_n B_{\mu\nu\bar{m}}, \quad \text{and} \quad \partial_m B_{\mu\nu}{}^{mn}. \qquad (4.44)$$

In other words, with this parametrization the Lagrangian comes with an additional local shift symmetry

$$\delta B_{\mu\nu\bar{n}} = \partial_n \Omega_{\mu\nu}, \qquad \delta B_{\mu\nu}{}^{mn} = \partial_k \Omega_{\mu\nu}{}^{[kmn]}, \qquad (4.45)$$

for arbitrary $\Omega_{\mu\nu}$, $\Omega_{\mu\nu}{}^{[kmn]}$. In total, the full p-form field content of the $E_{6(6)}$ Lagrangian in this basis is thus given by

$$\{A_\mu{}^m, A_{\mu mn}, A_\mu{}^{\bar{m}}\}, \qquad \{B_{\mu\nu\bar{m}}, B_{\mu\nu}{}^{mn}\}, \qquad (4.46)$$

modulo Eq. (4.45). Comparing Eq. (4.46) to the field content of the Kaluza-Klein reduction of $D = 11$ supergravity in the split of Sec. IV A suggests identifying the $A_\mu{}^m$ with the Kaluza-Klein vector fields sitting in the 11-dimensional vielbein (4.4), and to relate the fields $\{A_{\mu mn}, B_{\mu\nu\bar{m}}\}$ to the different components of the 11-dimensional 3-form (4.26). The index structure of the remaining fields $\{B_{\mu\nu}{}^{mn}, A_\mu{}^{\bar{m}}\}$ suggests relating them to the corresponding components of the 11-dimensional 6-form, i.e. to describe degrees of freedom on-shell dual to $\{A_{\mu mn}, B_{\mu\nu\bar{m}}\}$. Finally the six 2-form tensors $B_{\mu\nu m}$ that are absent in Eq. (4.46) represent the degrees of freedom that are on-shell dual to the Kaluza-Klein vector fields, i.e. descending from the 11-dimensional dual graviton. They do not figure in the action (1.3) and we comment on their role in the conclusions. We recall that in the EFT formulation, all vector fields appear with a Yang-Mills kinetic term whereas the 2-forms couple via a topological term. The latter do not represent additional degrees of freedom but are on-shell dual to the vector fields. In order to match the structure of $D = 11$ supergravity, we will thus have to trade the Yang-Mills vector field $A_\mu{}^{\bar{m}}$ for a propagating 2-form $B_{\mu\nu\bar{m}}$ as we shall describe in detail in Sec. IV B 3 below.

Let us now work out the details of this identification by evaluating the general EFT formulas in the basis (4.39) and imposing the explicit solution of the section condition (4.43) on all fields. We first consider the six vector fields $A_\mu{}^m$ transforming in the same representation as the surviving coordinates (4.43). Under the general gauge transformations (2.27) they transform according to

$$\delta_\Lambda A_\mu{}^m = \partial_\mu \Lambda^m - A_\mu{}^n \partial_n \Lambda^m + \Lambda^n \partial_n A_\mu{}^m, \qquad (4.47)$$

while they remain invariant under all higher tensor gauge transformations from Eq. (2.34). The associated gauge transformations close into the Lie algebra

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}] = \delta_{\Lambda_{12}}, \qquad \Lambda_{12}^m \equiv \Lambda_2^k \partial_k \Lambda_1^m - \Lambda_1^k \partial_k \Lambda_2^m \qquad (4.48)$$

of standard six-dimensional diffeomorphisms, embedded into the E-bracket (2.15). The six vector fields $A_\mu{}^m$ thus

[4] We use the summation conventions $X^M Y_M = X^m Y_m + X_{mn} Y^{mn} + X^{\bar{m}} Y_{\bar{m}}$.

ensure that the theory is invariant under internal diffeomorphisms with the parameters Λ^m. As anticipated above, we will identify them with the Kaluza-Klein vector fields from the 11-dimensional vielbein (4.4). For the following and just as in the previous section, cf. Eq. (4.10), we thus define the covariant derivatives

$$D_\mu = \partial_\mu - \mathcal{L}_{A_\mu}, \tag{4.49}$$

corresponding to the action of six-dimensional internal diffeomorphisms. Accordingly, the covariant field strength as evaluated from the corresponding components of the $E_{6(6)}$ object $\mathcal{F}_{\mu\nu}{}^M$ coincides with the non-Abelian field strength for the Kaluza-Klein vector field in Eq. (4.10),

$$\mathcal{F}_{\mu\nu}{}^m = 2\partial_{[\mu}A_{\nu]}{}^m - A_\mu{}^n\partial_n A_\nu{}^m + A_\nu{}^n\partial_n A_\mu{}^m = F_{\mu\nu}{}^m. \tag{4.50}$$

Evaluating the remaining components of the covariant field strengths (2.30) yields the field strengths for the other gauge fields as

$$\begin{aligned}\mathcal{F}_{\mu\nu mn} &= 2D_{[\mu}A_{\nu]mn} + \partial_m\tilde{B}_{\mu\nu\bar{n}} - \partial_n\tilde{B}_{\mu\nu\bar{m}},\\ \mathcal{F}_{\mu\nu}{}^{\bar{m}} &= 2D_{[\mu}A_{\nu]}{}^{\bar{m}} - 2(\partial_k A_{[\mu}{}^k)A_{\nu]}{}^{\bar{m}} - \frac{1}{2}\epsilon^{mnrskl}A_{[\mu|rs}\partial_{n|}A_{\nu]kl}\\ &\quad + 2\partial_n\tilde{B}_{\mu\nu}{}^{nm},\end{aligned} \tag{4.51}$$

where we have redefined the 2-form tensors as

$$\begin{aligned}\tilde{B}_{\mu\nu\bar{m}} &= \sqrt{5}B_{\mu\nu\bar{m}} + A_{[\mu}{}^n A_{\nu]nm},\\ \tilde{B}_{\mu\nu}{}^{mn} &= \sqrt{5}B_{\mu\nu}{}^{mn} + \frac{1}{2}(A_{[\mu}{}^m A_{\nu]}{}^{\bar{n}} - A_{[\mu}{}^n A_{\nu]}{}^{\bar{m}}).\end{aligned} \tag{4.52}$$

In turn, we obtain the field strengths for these 2-form tensors by evaluating the corresponding components of the $E_{6(6)}$ object $\mathcal{H}_{\mu\nu\rho M}$,

$$\begin{aligned}\tilde{\mathcal{H}}_{\mu\nu\rho\bar{m}} &\equiv \sqrt{5}\mathcal{H}_{\mu\nu\rho\bar{m}} - \partial_m\mathcal{O}_{\mu\nu\rho} = 3D_{[\mu}\tilde{B}_{\nu\rho]\bar{m}} + 3A_{[\mu|mn|}F_{\nu\rho]}{}^n,\\ \tilde{\mathcal{H}}_{\mu\nu\rho}{}^{mn} &\equiv \sqrt{5}\mathcal{H}_{\mu\nu\rho}{}^{mn} - \partial_k\mathcal{O}_{\mu\nu\rho}{}^{[kmn]}\\ &= 3D_{[\mu}\tilde{B}_{\nu\rho]}{}^{mn} - 3\partial_k A_{[\mu}{}^k\tilde{B}_{\nu\rho]}{}^{mn}\\ &\quad + \frac{3}{2}(A_{[\mu}{}^{\bar{m}}F_{\nu\rho]}{}^n - A_{[\mu}{}^{\bar{n}}F_{\nu\rho]}{}^m)\\ &\quad - \frac{3}{4}\epsilon^{mnklpq}(A_{[\mu|kl|}D_\nu A_{\rho]pq} + 2A_{[\mu|kl}\partial_{p|}\tilde{B}_{\nu\rho]\bar{q}}),\end{aligned} \tag{4.53}$$

where we have split off the additional contributions

$$\begin{aligned}\mathcal{O}_{\mu\nu\rho} &\equiv -A_{[\mu}{}^k A_\nu{}^l A_{\rho]kl},\\ \mathcal{O}_{\mu\nu\rho}{}^{[kmn]} &\equiv A_{[\mu}{}^k A_\nu{}^m A_{\rho]}{}^{\bar{n}} + A_{[\mu}{}^n A_\nu{}^k A_{\rho]}{}^{\bar{m}} + A_{[\mu}{}^m A_\nu{}^n A_{\rho]}{}^{\bar{k}}\\ &\quad + \frac{1}{2}\epsilon^{kmnlpq}(3A_{[\mu|lp|}\tilde{B}_{\nu\rho]\bar{q}} - 2A_{[\mu|lp|}A_\nu{}^r A_{\rho]rq})\end{aligned} \tag{4.54}$$

that are projected out from the Lagrangian, since—just as with the tensor fields—their field strengths also appear only under the projection $d^{MNK}\partial_N\mathcal{H}_{\mu\nu\rho K}$, cf. Eq. (4.44).

For completeness, let us also give the vector and tensor gauge transformations of the various components as obtained from evaluating the general formulas (2.34),

$$\begin{aligned}\delta A_{\mu mn} &= D_\mu\Lambda_{mn} + \mathcal{L}_\Lambda A_{\mu mn} - 2\partial_{[m}\tilde{\Xi}_{|\mu|n]},\\ \delta A_\mu{}^{\bar{m}} &= D_\mu\Lambda^{\bar{m}} - \partial_n A_\mu{}^n\Lambda^{\bar{m}} + \mathcal{L}_\Lambda A_\mu{}^{\bar{m}} - 2\partial_n\tilde{\Xi}_\mu{}^{nm},\\ \delta\tilde{B}_{\mu\nu\bar{m}} &= 2D_{[\mu}\tilde{\Xi}_{\nu]m} + \mathcal{L}_\Lambda\tilde{B}_{\mu\nu m} + \Lambda_{km}F_{\mu\nu}{}^k - \partial_m(\Lambda^k\tilde{B}_{\mu\nu\bar{k}}),\end{aligned} \tag{4.55}$$

with the tensor gauge parameters redefined in accordance with Eq. (4.52),

$$\begin{aligned}\tilde{\Xi}_{\mu m} &\equiv \sqrt{5}\Xi_{\mu m} + \Lambda^n A_{\mu nm},\\ \tilde{\Xi}_\mu{}^{mn} &= \sqrt{5}\Xi_\mu{}^{mn} + \frac{1}{2}(\Lambda^m A_\mu{}^{\bar{n}} - \Lambda^n A_\mu{}^{\bar{m}}).\end{aligned} \tag{4.56}$$

2. Scalar sector

Let us now discuss the scalar field content of the theory. In the $E_{6(6)}$-covariant formulation they parametrize the coset space $E_{6(6)}/USp(8)$ in terms of the symmetric matrix $\mathcal{M}_{MN}$. To relate to $D = 11$ supergravity, we need to choose a parametrization of this matrix in accordance with the decomposition (4.40). Following Ref. [82], we build the matrix as $\mathcal{M} = \mathcal{V}\mathcal{V}^T$ from a "vielbein" $\mathcal{V}$ in triangular gauge,

$$\mathcal{V} \equiv \exp[\Phi t_{(0)}]\mathcal{V}_6\exp[c_{kmn}t^{kmn}_{(+1)}]\exp[\varphi t_{(+2)}]. \tag{4.57}$$

Here, $t_{(0)}$ is the $E_{6(6)}$ generator associated to the GL(1) grading, $\mathcal{V}_6$ denotes a general matrix in the SL(6) subgroup, whereas the $t_{(+n)}$ refer to the $E_{6(6)}$ generators of positive grading in Eq. (4.40). All generators are evaluated in the fundamental 27 representation (4.39), such that the symmetric matrix $\mathcal{M}_{MN}$ takes the block form

$$\mathcal{M}_{KM} = \begin{pmatrix} \mathcal{M}_{km} & \mathcal{M}_k{}^{mn} & \mathcal{M}_{k\bar{m}} \\ \mathcal{M}^{kl}{}_m & \mathcal{M}^{kl,mn} & \mathcal{M}^{kl}{}_{\bar{m}} \\ \mathcal{M}_{\bar{k}m} & \mathcal{M}_{\bar{k}}{}^{mn} & \mathcal{M}_{\bar{k}\bar{m}} \end{pmatrix}. \qquad (4.58)$$

An explicit evaluation of Eq. (4.57) determines the various blocks in Eq. (4.58). E.g. its last line is given by

$$\mathcal{M}_{\bar{m}n} = \frac{1}{24} e^{\Phi} m_{mk} \epsilon^{klpqrs} c_{nlp} c_{qrs} - e^{\Phi} m_{mn} \varphi,$$
$$\mathcal{M}_{\bar{m}}{}^{kl} = -\frac{1}{6\sqrt{2}} m_{mn} \epsilon^{nklpqr} e^{\Phi} c_{pqr}, \qquad \mathcal{M}_{\bar{m}\bar{n}} = e^{\Phi} m_{mn}, \qquad (4.59)$$

parametrized by Φ, φ, c_{kmn}. The symmetric matrix $m_{mn} \equiv (\nu\nu^T)_{mn}$ is built from the SL(6) vielbein ν that parametrizes the standard embedding of this subgroup via $\mathcal{V}_6$ in Eq. (4.57) as

$$(\mathcal{V}_6)_M{}^A = \begin{pmatrix} \nu_m{}^a & 0 & 0 \\ 0 & (\nu^{-1})^{[m}{}_a (\nu^{-1})^{n]}{}_b & 0 \\ 0 & 0 & \nu_{\bar{m}}{}^{\bar{a}} \end{pmatrix}. \qquad (4.60)$$

The remaining blocks of Eq. (4.58) yield more lengthy expressions, but can be expressed in compact form via the corresponding blocks of the matrix

$$\tilde{\mathcal{M}}_{MN} \equiv \mathcal{M}_{MN} - \mathcal{M}_{M\bar{m}} (\mathcal{M}_{\bar{m}\bar{n}})^{-1} \mathcal{M}_{\bar{n}N}, \qquad (4.61)$$

which take the form

$$\tilde{\mathcal{M}}_{mn} = e^{-\Phi} m_{mn} + \frac{1}{2} c_{mkp} c_{nlq} m^{kl} m^{pq},$$
$$\tilde{\mathcal{M}}_m{}^{kl} = -\frac{1}{\sqrt{2}} c_{mpq} m^{pk} m^{ql}, \qquad \tilde{\mathcal{M}}^{kl,mn} = m^{m[k} m^{l]n}. \qquad (4.62)$$

The matrix (4.61) will play a central role in the following after redualizing some of the vector fields. From the inverse matrix $\mathcal{M}^{MN}$ we will need only the particular block

$$\mathcal{M}^{mn} = e^{\Phi} m^{mn}. \qquad (4.63)$$

Now, that we have specified the field content according to the explicit solution (4.43), we can work out the $E_{6(6)}$-covariant Lagrangian in this parametrization. Let us start with the scalar kinetic term. First, we should evaluate the covariant derivatives $\mathcal{D}_\mu \mathcal{M}_{MN}$ in the split (4.39). With Eq. (4.42) we find for the covariant derivatives of the components of a general vector V^M

$$\mathcal{D}_\mu V^m = D_\mu V^m + \frac{1}{3} (\partial_k A_\mu{}^k) V^m,$$
$$\mathcal{D}_\mu V_{mn} = D_\mu V_{mn} + \frac{1}{3} (\partial_k A_\mu{}^k) V_{mn} + V^k \partial_k A_{\mu mn} + V^k \partial_m A_{\mu nk} + V^k \partial_n A_{\mu km},$$
$$\mathcal{D}_\mu V^{\bar{m}} = D_\mu V^{\bar{m}} - \frac{2}{3} (\partial_k \Lambda^k) V^{\bar{m}} + \frac{1}{2} \epsilon^{mnklpq} \partial_n A_{\mu kl} V_{pq} + (\partial_k A_\mu{}^{\bar{k}}) V^m, \qquad (4.64)$$

where as above the derivatives D_μ are only covariantized with respect to the Kaluza-Klein gauge transformations, i.e. $D_\mu \equiv \partial_\mu - \mathcal{L}_{A_\mu}$. Comparing this to the parametrization (4.59) of the matrix $\mathcal{M}_{MN}$, we derive the covariant derivatives on the parameters of this matrix as

$$\mathcal{D}_\mu m_{mn} = D_\mu m_{mn} + \frac{1}{3} (\partial_k A_\mu{}^k) m_{mn},$$
$$\mathcal{D}_\mu \Phi = D_\mu \Phi + (\partial_n A_\mu{}^n),$$
$$\mathcal{D}_\mu c_{klm} = D_\mu c_{klm} + 3\sqrt{2} \partial_{[k} A_{|\mu| lm]},$$
$$\mathcal{D}_\mu \varphi = D_\mu \varphi - (\partial_n A_\mu{}^n) \varphi + \partial_n A_\mu{}^{\bar{n}} + \frac{\sqrt{2}}{24} \epsilon^{klmnpq} c_{klm} \partial_n A_{\mu pq}. \qquad (4.65)$$

From the first two lines we infer that the combination

$$\phi_{mn} \equiv e^{-\Phi/3} m_{mn} \qquad (4.66)$$

transforms as a genuine tensor (of vanishing weight) under six-dimensional diffeomorphisms. As anticipated by the notation, we will identify it with the internal part $\phi_{mn} = \phi_{m\alpha} \phi_n{}^\alpha$ of the metric of 11-dimensional supergravity (4.4).

Putting all this together, we obtain after some calculation the explicit form of the scalar kinetic term from Eq. (1.3),

$$\begin{aligned} e^{-1} \mathcal{L}_{\text{kin},0} &\equiv \frac{1}{24} \mathcal{D}_\mu \mathcal{M}_{MN} \mathcal{D}^\mu \mathcal{M}^{MN} \\ &= \frac{1}{4} D_\mu \phi_{mn} D^\mu \phi^{mn} - \frac{1}{3} \phi^{-2} D_\mu \phi D^\mu \phi \\ &\quad - \frac{1}{12} \phi^{-2/3} \phi^{kn} \phi^{lp} \phi^{mq} \mathcal{D}_\mu c_{klm} \mathcal{D}^\mu c_{npq} \\ &\quad - \frac{1}{2} \phi^{-2} \left(\mathcal{D}_\mu \varphi + \frac{1}{72} \epsilon^{klmnpq} c_{klm} \mathcal{D}_\mu c_{npq} \right)^2, \end{aligned} \qquad (4.67)$$

with $\phi \equiv e^{-\Phi} = (\det \phi_{mn})^{1/2}$ as above. Next, we can evaluate the $E_{6(6)}$-covariant potential (3.11) in the parametrization (4.59) and (4.62) and obtain

$$
\begin{aligned}
V = & -\frac{1}{3}\phi^{-2/3}\partial_m\phi_{nk}\partial_l\phi_{pq}\phi^{mn}\phi^{kl}\phi^{pq} + \frac{1}{36}\phi^{-2/3}\partial_m\phi_{nk}\partial_l\phi_{pq}\phi^{ml}\phi^{nk}\phi^{pq} \\
& + \frac{1}{4}\phi^{-2/3}\partial_m\phi_{nk}\partial_l\phi_{pq}\phi^{ml}\phi^{np}\phi^{kq} - \frac{1}{2}\phi^{-2/3}\partial_m\phi_{nk}\partial_l\phi_{pq}\phi^{mp}\phi^{nq}\phi^{kl} \\
& + \frac{2}{3}\phi^{-5/3}\phi^{mn}e^{-1}\partial_m e\partial_n\phi - e^{-1}\phi^{-2/3}\partial_m e\partial_n\phi^{mn} + \frac{1}{3}\phi^{-2/3}\partial_{[k}c_{lmn]}\partial_{[p}c_{qrs]}\phi^{kp}\phi^{lq}\phi^{mr}\phi^{ns} \\
& - \phi^{-2/3}\phi^{mn}e^{-1}\partial_m e e^{-1}\partial_n e - \frac{1}{4}\phi^{-2/3}\phi^{mn}\partial_m g^{\mu\nu}\partial_n g_{\mu\nu}.
\end{aligned} \tag{4.68}
$$

In particular, the second line of the potential (3.11) is straightforwardly evaluated with Eq. (4.63).

3. Dualization

Before explicitly evaluating the remaining parts of the $E_{6(6)}$-covariant Lagrangian, let us recall the field content. From Eq. (4.46) and the subsequent discussion, we have vectors and 2-forms given by

$$\{A_\mu{}^m, A_{\mu mn}, A_\mu{}^{\bar m}\}, \qquad \{\tilde B_{\mu\nu\bar m}, \tilde B_{\mu\nu}{}^{mn}\}, \tag{4.69}$$

of which only the vectors represent propagating degrees of freedom. In the previous subsection we introduced the parametrization of the scalar fields of the model as

$$\{\phi_{mn}, c_{kmn}, \varphi\}. \tag{4.70}$$

Comparing this to the form of 11-dimensional supergravity in the $5+6$ split presented in Sec. IV A, we see that we will have to dualize the singlet scalar field φ into a 3-form tensor field and eliminate the fields $A_\mu{}^{\bar m}$ and $\tilde B_{\mu\nu}{}^{mn}$. In particular, the latter step should introduce a kinetic term for the 2-form tensor fields $\tilde B_{\mu\nu\bar m}$, promoting these fields to propagating degrees of freedom.

For the dimensionally reduced theory this is precisely the pattern of dualizations of p-forms into $(3-p)$-forms that is required to make the $E_{6(6)}$ symmetry apparent [82]. In the following, we give a version of that dualization which applies even for the fully y-dependent fields despite the non-Abelian structure of the internal diffeomorphisms that may pose an obstacle for the possibility of dualization. It is rather similar to the mechanisms of non-Abelian dualizations appearing in gauged supergravity [83,84] empowered by the compensating fields of the tensor hierarchy. As a result, we will show in this section that upon this dualization, the Lagrangian evaluated from Eq. (1.3) precisely coincides with $D=11$ supergravity.

We start by dualizing the singlet scalar field φ into a 3-form. To this end, we first note that the Lagrangian (1.3) after the resolution of the section condition according to Eq. (4.43) has a global symmetry that acts by a shift on φ. Its origin is the $E_{6(6)}$ generator $t_{(+2)}$ in the basis of Eq. (4.57) with action

$$\delta_\lambda\varphi = \lambda, \qquad \delta_\lambda A_\mu{}^{\bar m} = \lambda A_\mu{}^m \tag{4.71}$$

on scalar and vector fields. This symmetry is compatible with the solution of the section constraint (4.43) due to

$$\delta_\lambda\partial_{\bar m} = 0, \qquad \delta_\lambda\partial^{mn} = 0, \tag{4.72}$$

as an immediate consequence of the grading (4.39) and (4.40). As a result, this symmetry survives after imposing the explicit solution of the section constraint. Moreover, due to our field redefinitions (4.52), the same generator has a nontrivial action on the 2-forms as

$$\delta_\lambda\tilde B_{\mu\nu}{}^{mn} = \lambda A_{[\mu}{}^m A_{\nu]}{}^n. \tag{4.73}$$

For dualizing the scalar fields φ we will now follow a standard routine: we gauge the shift symmetry (4.71) by the introduction of an auxiliary vector field and eliminate the latter by its field equations. Specifically, in the scalar sector we introduce covariant derivatives

$$\mathcal{D}_\mu \to \hat{\mathcal{D}}_\mu \equiv \mathcal{D}_\mu - a_\mu t_{(+2)}, \tag{4.74}$$

such that the kinetic term (4.67) remains invariant under the local form of Eq. (4.71) provided the auxiliary vector a_μ transforms as

$$\delta_\lambda a_\mu = \partial_\mu\lambda, \qquad \delta_\Lambda a_\mu = \mathcal{L}_\Lambda a_\mu + (\partial_k\Lambda^k)a_\mu. \tag{4.75}$$

In the vector sector, the gauging of Eq. (4.71) is more intricate, since the new gauge symmetry interferes with the existing non-Abelian structure (4.55) of the vector fields. As a result, this further deformation necessitates the introduction of additional Stückelberg-type couplings on the level of the field strengths according to

$$
\begin{aligned}
\mathcal{F}_{\mu\nu}{}^{\bar m} \to \hat{\mathcal{F}}_{\mu\nu}{}^{\bar m} \equiv\ & 2\hat D_{[\mu}A_{\nu]}{}^{\bar m} - 2(\partial_k A_{[\mu}{}^k)A_{\nu]}{}^{\bar m} \\
& - \frac{1}{2}\epsilon^{mnrskl}A_{[\mu|rs}\partial_{n|}A_{\nu]kl} \\
& + 2\partial_n\tilde B_{\mu\nu}{}^{nm} + b_{\mu\nu}{}^m,
\end{aligned} \tag{4.76}
$$

with the new auxiliary 2-form $b_{\mu\nu}$ transforming as

$$
\begin{aligned}
& \delta_\lambda b_{\mu\nu}{}^m = 0, \\
& \delta_\Lambda b_{\mu\nu}{}^m = \mathcal{L}_\Lambda b_{\mu\nu}{}^m + (\partial_k\Lambda^k)b_{\mu\nu}{}^m + 2a_{[\mu}\partial_{\nu]}\Lambda^m,
\end{aligned} \tag{4.77}
$$

in order to guarantee the covariant transformation behavior of the field strength. With these extra fields and modified transformations, the kinetic part of the Lagrangian is thus invariant under λ and Λ^M transformations. Moreover, the auxiliary 2-form $b_{\mu\nu}{}^m$ comes with its own tensor gauge invariance,

$$\delta_\xi b_{\mu\nu}{}^m = 2\partial_{[\mu}\xi_{\nu]}{}^{\bar{m}}, \qquad \delta_\xi a_\mu = -\partial_n \xi_\mu{}^{\bar{n}},$$
$$\delta_\xi A_\mu{}^{\bar{m}} = -\xi_\mu{}^{\bar{m}}, \qquad \delta_\xi \tilde{B}_{\mu\nu}{}^{mn} = -A_{[\mu}{}^m \xi_{\nu]}{}^{\bar{n}} + A_{[\mu}{}^n \xi_{\nu]}{}^{\bar{m}}, \tag{4.78}$$

which separately leaves the kinetic part of the Lagrangian invariant.

Let us now turn to the topological term (3.7) in order to render it invariant under the new gauge symmetries (4.71), (4.73), and (4.78). After evaluating this term with the solution of the section condition (4.43), it is invariant under the global symmetry (4.71) and (4.73) but acquires a nontrivial variation for a local gauge parameter λ according to

$$\delta_\lambda \mathcal{L}_{\text{top},0} = -\frac{1}{\sqrt{2}}\varepsilon^{\mu\nu\rho\sigma\tau}\partial_\mu\lambda(F_{\nu\rho}{}^m A_{\sigma mn} A_\tau{}^n + \partial_m \tilde{B}_{\nu\rho\bar{n}} A_\sigma{}^m A_\tau{}^n). \tag{4.79}$$

In view of Eq. (4.75), this variation can be cancelled by adding the additional topological term

$$\mathcal{L}_{\text{top},1} \equiv \frac{1}{\sqrt{2}}\varepsilon^{\mu\nu\rho\sigma\tau} a_\mu (F_{\nu\rho}{}^m A_{\sigma mn} A_\tau{}^n + \partial_m \tilde{B}_{\nu\rho\bar{n}} A_\sigma{}^m A_\tau{}^n), \tag{4.80}$$

such that the sum $\mathcal{L}_{\text{top},0} + \mathcal{L}_{\text{top},1}$ is invariant under local λ transformations. In turn, the variation of this combined topological term under the local tensor gauge symmetry (4.78) is given by

$$\begin{aligned}\delta_\xi \mathcal{L}_{\text{top},0+1} &= -\frac{1}{\sqrt{2}}\epsilon^{\mu\nu\rho\sigma\tau}(2\partial_\mu A_\nu{}^k \partial_{[k}\tilde{B}_{\rho\sigma\bar{m}]} - 2A_\mu{}^k \partial_\nu \partial_{[k}\tilde{B}_{|\rho\sigma|\bar{m}]} \\ &\quad - \partial_\mu(A_{\nu mn} F^n_{\rho\sigma}))\xi_\tau{}^{\bar{m}} \\ &= \frac{1}{3\sqrt{2}}\epsilon^{\mu\nu\rho\sigma\tau}(\tilde{\mathcal{H}}_{\mu\nu\rho\bar{m}} + 3\partial_m(A_\mu{}^n \tilde{B}_{\nu\rho\bar{n}}))\partial_\sigma \xi_\tau{}^{\bar{m}},\end{aligned} \tag{4.81}$$

and thus it can be cancelled by the introduction of a second addition to the topological term,

$$\mathcal{L}_{\text{top},2} = -\frac{1}{6\sqrt{2}}\epsilon^{\mu\nu\rho\sigma\tau}(\tilde{\mathcal{H}}_{\mu\nu\rho\bar{m}} + 3\partial_m(A_\mu{}^n \tilde{B}_{\nu\rho\bar{n}}))b_{\sigma\tau}{}^m. \tag{4.82}$$

Finally, we have to ensure that the combined topological term $\mathcal{L}_{\text{top},0+1+2}$ remains invariant under the original Λ^M and $\Xi_{\mu M}$ gauge transformations of Eq. (2.34). After some lengthy but straightforward calculations, we find for this variation

$$\begin{aligned}\delta\mathcal{L}_{\text{top},0+1+2} &= \frac{1}{2\sqrt{2}}\epsilon^{\mu\nu\rho\sigma\tau}(2A_\mu{}^k A_\nu{}^n \partial_k \tilde{\Xi}_{\rho n} - A_\mu{}^k F_{\nu\rho}{}^n \Lambda_{kn} \\ &\quad - \Lambda^n \partial_\mu \tilde{B}_{\nu\rho\bar{n}})(2\partial_\sigma a_\tau + \partial_m b_{\sigma\tau}{}^m) \\ &\quad - \frac{1}{2\sqrt{2}}\epsilon^{\mu\nu\rho\sigma\tau}\partial_m(2A_\mu{}^k \tilde{\Xi}_{\nu k} - \Lambda^n \tilde{B}_{\mu\nu\bar{n}})\partial_\rho b_{\sigma\tau}{}^m.\end{aligned} \tag{4.83}$$

This variation is cancelled by adding to the topological Lagrangian the final contribution

$$\mathcal{L}_{\text{top},3} = \frac{1}{4\sqrt{2}}\epsilon^{\mu\nu\rho\sigma\tau}(2a_\mu \partial_\nu \mathcal{A}_{\rho\sigma\tau} + \partial_m b_{\mu\nu}{}^m \mathcal{A}_{\rho\sigma\tau}), \tag{4.84}$$

with the new field $\mathcal{A}_{\rho\sigma\tau}$, transforming as

$$\begin{aligned}\delta\mathcal{A}_{\mu\nu\rho} &= \mathcal{L}_\Lambda \mathcal{A}_{\mu\nu\rho} + 2\Lambda^n \partial_{[\mu}\tilde{B}_{\nu\rho]n} + 2A_{[\mu}{}^m F_{\nu\rho]}{}^n \Lambda_{mn} \\ &\quad - 4\partial_m \tilde{\Xi}_{[\mu|\bar{n}|} A_\nu{}^m A_{\rho]}{}^n + 2\partial_{[\mu}(2A_\nu{}^k \tilde{\Xi}_{\rho]k} - \Lambda^n \tilde{B}_{\nu\rho]\bar{n}}).\end{aligned} \tag{4.85}$$

A short calculation also shows that the terms in the variation of Eq. (4.84) proportional to $\mathcal{A}_{\rho\sigma\tau}$ cancel. Moreover, the term (4.84) is separately invariant under the new gauge symmetries (4.71) and (4.78), so no further compensation is required. To clean up the construction, we may eventually combine all new contributions with the topological term, which can be put into the more compact form

$$\begin{aligned}\mathcal{L}_{\text{top},1+2+3} &= \frac{1}{4\sqrt{2}}\varepsilon^{\mu\nu\rho\sigma\tau}\bigg(2a_\mu(D_\nu \tilde{\mathcal{A}}_{\rho\sigma\tau} - \tilde{B}_{\nu\rho\bar{m}} F_{\sigma\tau}{}^m) \\ &\quad - \frac{1}{3}\tilde{b}_{\mu\nu}{}^m(2\tilde{\mathcal{H}}_{\rho\sigma\tau\bar{m}} + 3\partial_m \tilde{\mathcal{A}}_{\rho\sigma\tau})\bigg),\end{aligned} \tag{4.86}$$

with the auxiliary fields redefined as

$$\begin{aligned}\tilde{b}_{\mu\nu}{}^m &\equiv b_{\mu\nu}{}^m - 2a_{[\mu}A_{\nu]}{}^m, \\ \tilde{\mathcal{A}}_{\mu\nu\rho} &\equiv \mathcal{A}_{\mu\nu\rho} + 2A_\mu{}^n \tilde{B}_{\nu\rho\bar{n}}.\end{aligned} \tag{4.87}$$

After these redefinitions, the gauge transformations of $\mathcal{A}_{\mu\nu\rho}$ in Eq. (4.85) take the fully covariant and more compact form

$$\delta\tilde{\mathcal{A}}_{\mu\nu\rho} = \mathcal{L}_\Lambda \tilde{\mathcal{A}}_{\mu\nu\rho} + 2F_{[\mu\nu}{}^n \tilde{\Xi}_{\rho]n}. \tag{4.88}$$

In the course of our construction, something interesting has happened. We recall that the original Lagrangian carried the 2-form $\tilde{B}_{\mu\nu\bar{n}}$ exclusively under ∂_m derivative *à la* Eq. (4.44). This is still true for its variation (4.81) (although not manifest in the final expression), but no longer for the compensating term (4.82). Consequently, the new topological term (4.86) carries the longitudinal part of $\tilde{B}_{\mu\nu\bar{n}}$ as a new field. Nevertheless, the shift symmetry (4.45) of the original Lagrangian can be preserved, if the field $\tilde{\mathcal{A}}_{\mu\nu\rho}$ simultaneously transforms as

$$\delta\tilde{\mathcal{A}}_{\mu\nu\rho} = -2D_{[\mu}\Omega_{\nu\rho]}, \qquad \delta\tilde{B}_{\mu\nu\bar{m}} = \partial_m\Omega_{\mu\nu}. \tag{4.89}$$

I.e. this symmetry is identified with the tensor gauge symmetry of the new 3-form $\tilde{\mathcal{A}}_{\mu\nu\rho}$.

Let us pause and summarize what we have achieved. Upon introducing the new covariant derivatives and field strengths (4.74) and (4.76) in the Lagrangian, as well as extending its topological term $\mathcal{L}_{\rm top,0}$ to $\mathcal{L}_{\rm top,0+1+2+3}$ from Eq. (4.86) we have modified the original Lagrangian such that in addition to the former gauge symmetries it is also invariant under the new local gauge symmetries (4.71), (4.78), and (4.89). The modification has introduced the auxiliary vector and tensor gauge fields a_μ, $b_{\mu\nu}{}^m$, and $\mathcal{A}_{\mu\nu\rho}$. The resulting Lagrangian provides an efficient tool to perform the dualization of the original theory. We can show that depending on how we treat the auxiliary fields, the Lagrangian either reduces to the original one or takes a different form, in which the former fields φ and $A_\mu{}^{\bar{m}}$ disappear. Thereby we arrive at the dual version of the original Lagrangian.

Let us first show that the new Lagrangian is equivalent to the original theory obtained from the $E_{6(6)}$-covariant EFT after solving the section condition. We recall that the only term in which $\tilde{B}_{\mu\nu\bar{m}}$ appears without derivative, is Eq. (4.82). It thus gives separate equations of motion [by a variation of the type (4.45) under which all other terms are invariant] implying that

$$\partial_m\partial_{[\mu}b_{\nu\rho]}{}^m = 0. \tag{4.90}$$

With the local gauge symmetry (4.78) we can thus set

$$\partial_m b_{\mu\nu}{}^m = 0 \Rightarrow b_{\mu\nu}{}^m = \partial_n\Upsilon_{\mu\nu}{}^{[mn]}, \tag{4.91}$$

for some locally defined $\Upsilon_{\mu\nu}{}^{[mn]}$. Upon making use of yet another local symmetry of the full Lagrangian,[5]

$$\delta\tilde{B}_{\mu\nu}{}^{mn} = \frac{1}{2}\Upsilon_{\mu\nu}{}^{[mn]}, \qquad \delta b_{\mu\nu}{}^m = -\partial_n\Upsilon_{\mu\nu}{}^{[nm]}, \tag{4.92}$$

we can then completely eliminate the field $b_{\mu\nu}{}^m$. The field equations following from the variation of $\mathcal{A}_{\mu\nu\rho}$ in Eq. (4.84) imply that

$$2\partial_{[\mu}a_{\nu]} = -\partial_m b_{\mu\nu}{}^m = 0. \tag{4.93}$$

[5]This is not a novel gauge symmetry but simply illustrates some redundancy in the introduction of the auxiliary field $b_{\mu\nu}$ in Eq. (4.76).

Thus, a_μ is also pure gauge and can be set to zero with the local symmetry (4.75). As a result, all auxiliary fields a_μ, $b_{\mu\nu}{}^m$, and $\mathcal{A}_{\mu\nu\rho}$ disappear from the equations of motion and we are back to the theory obtained from the $E_{6(6)}$-covariant formulation.

Alternatively, we may integrate out the auxiliary gauge fields a_μ, $b_{\mu\nu}$ upon using their algebraic field equations. The local symmetries (4.71), (4.78), and (4.92) which formally remain present in this procedure, show that after integrating out a_μ and $b_{\mu\nu}$, the resulting Lagrangian no longer depends on the fields φ, $A_\mu{}^{\bar{m}}$, and $\tilde{B}_{\mu\nu}{}^{mn}$. Instead, the fields $\mathcal{A}_{\mu\nu\rho}$ and $\tilde{B}_{\mu\nu}{}^{\bar{m}}$ are promoted to propagating fields with proper kinetic terms. We thus obtain a dual version of the original Lagrangian with precisely the field content of $D = 11$ supergravity. To conclude this discussion, we will now show in detail that the result indeed coincides with the $D = 11$ supergravity Lagrangian after Kaluza-Klein decomposition.

With the kinetic terms from Eq. (1.3) evaluated according to Eqs. (4.58) and (4.67), and covariantized according to Eqs. (4.74) and (4.76), the equations of motion for the auxiliary fields a_μ, $\tilde{b}_{\mu\nu}{}^m$ read

$$\begin{aligned} a_\mu &= \mathcal{D}_\mu\varphi + \frac{1}{72}\epsilon^{klmnpq}c_{klm}\mathcal{D}_\mu c_{npq} \\ &\quad + 2\varepsilon^{\mu\nu\rho\sigma\tau}\phi^2(D_\nu\tilde{\mathcal{A}}_{\rho\sigma\tau} - \tilde{B}_{\nu\rho\bar{m}}F_{\sigma\tau}{}^m), \\ \tilde{b}_{\mu\nu}{}^m &= -(\mathcal{M}_{\bar{m}\bar{n}})^{-1}\mathcal{M}_{\bar{n}M}\mathcal{F}^{\mu\nu M} \\ &\quad - \frac{2}{3}\varepsilon^{\mu\nu\rho\sigma\tau}(\mathcal{M}_{\bar{m}\bar{n}})^{-1}(2\tilde{\mathcal{H}}_{\rho\sigma\tau\bar{n}} + 3\partial_n\tilde{\mathcal{A}}_{\rho\sigma\tau}). \end{aligned} \tag{4.94}$$

Inserting this into the Lagrangian produces the new kinetic terms

$$\begin{aligned} e^{-1}\mathcal{L}_{\rm kin,2+3} = &-\frac{1}{24}\phi^{4/3}\phi^{mn}(2\tilde{\mathcal{H}}_{\mu\nu\rho\bar{m}} + 3\partial_m\tilde{\mathcal{A}}_{\mu\nu\rho}) \\ &\times(2\tilde{\mathcal{H}}^{\mu\nu\rho}{}_{\bar{n}} + 3\partial_n\tilde{\mathcal{A}}^{\mu\nu\rho}) \\ &-\frac{3}{2}\phi^2(D_{[\mu}\tilde{\mathcal{A}}_{\nu\rho\sigma]} - \tilde{B}_{[\mu\nu|\bar{m}|}F_{\rho\sigma]}{}^m) \\ &\times(D^\mu\tilde{\mathcal{A}}^{\nu\rho\sigma} - \tilde{B}^{\mu\nu}{}_{\bar{n}}F^{\rho\sigma n}), \end{aligned} \tag{4.95}$$

for the 2-forms $\tilde{B}_{\mu\nu}{}^{\bar{m}}$ and 3-form $\tilde{\mathcal{A}}_{\mu\nu\rho}$, while the vector kinetic term turns into

$$e^{-1}\mathcal{L}_{\rm kin,1} = -\frac{1}{4}\mathcal{F}_{\mu\nu}{}^M\mathcal{F}^{\mu\nu N}\tilde{\mathcal{M}}_{MN}, \tag{4.96}$$

with the matrix $\tilde{\mathcal{M}}_{MN}$ from Eqs. (4.61) and (4.62). In particular, the form of this matrix shows that the vector fields $A_\mu{}^{\bar{m}}$ have disappeared from the kinetic term (4.96) as expected. In order to calculate the topological term after the elimination of the auxiliary fields, let us first consider the original topological term (3.7). After explicitly solving the section condition (4.43) we can give a fairly compact expression for this term upon integrating up Eq. (3.8) as

$$\mathcal{L}_{\text{top},0} = \frac{1}{4\sqrt{2}}\varepsilon^{\mu\nu\rho\sigma\tau}\varepsilon^{mnklpq}\bigg(\frac{1}{2}A_{\mu mn}\mathcal{F}_{\nu\rho kl}\partial_p\tilde{B}_{\sigma\tau q} + \frac{1}{3}D_\mu A_{\nu mn}D_\rho A_{\sigma kl}A_{\tau pq} + \frac{1}{3}\partial_m A_{\mu pq}A_{\nu kl}A_{\rho nr}F_{\sigma\tau}{}^r + \mathcal{O}(A_\mu{}^{\bar{m}}) + \mathcal{O}(B_{\mu\nu}{}^{mn})\bigg). \tag{4.97}$$

Eventually, we are only interested in this term at vanishing $A_\mu{}^{\bar{m}}$, $B_{\mu\nu}{}^{mn}$, since we know from the general symmetry argument above that these fields will no longer enter the Lagrangian after the elimination of the auxiliary fields. Moreover, plugging Eq. (4.94) into the original Lagrangian gives the following additional contributions to the topological term:

$$\mathcal{L}_{\text{top, dual}} = \frac{1}{4\sqrt{2}}\varepsilon^{\mu\nu\rho\sigma\tau}\varepsilon^{mnklpq}\bigg[\frac{1}{12}c_{mnk}\bigg(\sqrt{2}\partial_l A_{\mu pq} + \frac{1}{3}\mathcal{D}_\mu c_{lpq}\bigg)(D_\nu\tilde{\mathcal{A}}_{\rho\sigma\tau} - \tilde{B}_{\nu\rho\bar{r}}F_{\sigma\tau}{}^r) + \frac{1}{72}(F_{\mu\nu}{}^r c_{rlp}c_{qmn} - 12A_{\mu kl}\partial_n A_{\nu pq})(2\tilde{\mathcal{H}}_{\rho\sigma\tau\bar{k}} + 3\partial_k\tilde{\mathcal{A}}_{\rho\sigma\tau}) - \frac{1}{18\sqrt{2}}c_{pqn}\mathcal{F}_{\mu\nu kl}(2\tilde{\mathcal{H}}_{\rho\sigma\tau\bar{m}} + 3\partial_m\tilde{\mathcal{A}}_{\rho\sigma\tau})\bigg]. \tag{4.98}$$

Comparing the resulting parts of the Lagrangian (4.95)–(4.98) to the Kaluza-Klein decomposition of 11-dimensional supergravity presented in Sec. IV A, we are led to the following redefinition of fields:

$$\tilde{\mathcal{A}}_{\mu\nu\rho} \to \frac{2\sqrt{2}}{3}A_{\mu\nu\rho}, \qquad \tilde{B}_{\mu\nu\,\bar{m}} \to \sqrt{2}A_{\mu\nu\,m},$$
$$A_{\mu\,mn} \to \sqrt{2}A_{\mu\,mn}, \qquad c_{mnk} \to -2A_{mnk}. \tag{4.99}$$

With this translation, the above combinations of field strengths become

$$2\tilde{\mathcal{H}}_{\mu\nu\rho\bar{m}} + 3\partial_m\tilde{\mathcal{A}}_{\mu\nu\rho} \to 2\sqrt{2}F_{\mu\nu\rho m},$$
$$D_{[\mu}\tilde{\mathcal{A}}_{\nu\rho\sigma]} - \tilde{B}_{[\mu\nu|\bar{m}|}F_{\rho\sigma]}{}^m \to -\frac{\sqrt{2}}{6}F_{\mu\nu\rho\sigma},$$
$$\mathcal{F}_{\mu\nu\,mn} \to \sqrt{2}\mathcal{F}_{\mu\nu\,mn},$$
$$\mathcal{D}_\mu c_{klm} \to -2F_{\mu klm}, \tag{4.100}$$

i.e. translated directly into the field strengths (4.30) and (4.35) introduced in the discussion of the Kaluza-Klein decomposition of 11-dimensional supergravity. It is then straightforward to verify that the combination of kinetic terms (4.67), (4.95), and (4.96), indeed precisely coincides with the corresponding terms of Eq. (4.34), from 11-dimensional supergravity. Likewise, the combination of the topological terms (4.86), (4.97), and (4.98), and using the dictionary (4.99) reproduces the 11-dimensional result (4.33) up to total derivatives. Although this comparison is not straightforward since there is no canonical form in which to give these nonmanifestly gauge covariant terms, they can be systematically matched by comparing their general variation with respect to the various gauge fields. Similarly, agreement is found between the potential terms (4.68) and (4.37). Finally, the Einstein-Hilbert terms from 11 dimensions and from EFT are based on the improved Riemann tensors (3.3) and (4.20), that are readily identified since

$$\mathcal{F}_{\mu\nu}{}^M\partial_M \to F_{\mu\nu}{}^m\partial_m, \tag{4.101}$$

on the solution of the section constraint (4.43). Thus we have shown total agreement between the EFT evaluated for Eq. (4.43) and the full 11-dimensional supergravity cast into the $(5+6)$-dimensional Kaluza-Klein form.

V. EMBEDDING OF TYPE IIB SUPERGRAVITY

In the previous section, we have shown that upon imposing the explicit GL(6)-invariant solution (4.43) of the section condition and subsequent dualization of some of the fields, the $E_{6(6)}$-covariant EFT precisely reproduces the full 11-dimensional supergravity in the $5+6$ Kaluza-Klein split. In this section, we discuss an inequivalent solution [33] to the section condition upon which the EFT reproduces the full ten-dimensional IIB theory [34,35].[6]

A. $\mathrm{GL}(\mathbf{5})\times\mathrm{SL}(\mathbf{2})$-invariant solution of the section condition

The corresponding solution of the section condition preserves the group $\mathrm{GL}(5)\times\mathrm{SL}(2)$ embedded according to

$$\mathrm{GL}(5)\times\mathrm{SL}(2) \subset \mathrm{SL}(6)\times\mathrm{SL}(2) \subset E_{6(6)} \tag{5.1}$$

into $E_{6(6)}$. In this case, the fundamental and the adjoint representation of $E_{6(6)}$ break as

$$\overline{\mathbf{27}} \to (5,1)_{+4} + (5',2)_{+1} + (10,1)_{-2} + (1,2)_{-5}, \tag{5.2}$$

$$\mathbf{78} \to (5,1)_{-6} + (10',2)_{-3} + (1+15+20)_0 + (10,2)_{+3} + (5',1)_{+6}, \tag{5.3}$$

[6]An analogous IIB solution of the SL(5)-covariant section condition, corresponding to some three-dimensional truncation of type IIB supergravity, has been studied recently [85] in the truncation of the theory to its potential term.

with the subscripts referring to the charges under $\mathrm{GL}(1) \subset \mathrm{GL}(5)$. An explicit solution to the section condition (1.1) is given by restricting the Y^M dependence of all fields to the five coordinates in the $(5,1)_{+4}$. Explicitly, by splitting the coordinates Y^M and the fundamental indices according to Eq. (5.2) into

$$\{Y^M\} \to \{y^m, y_{m\alpha}, y^{mn}, y_\alpha\}, \qquad (5.4)$$

with internal indices $m, n = 1, \ldots, 5$ and SL(2) indices $\alpha = 1, 2$, the nonvanishing components of the d symbol are given by

$$\begin{aligned} d^{MNK}\colon\; d^m{}_{n\alpha,\beta} &= \frac{1}{\sqrt{10}} \delta^m_n \epsilon_{\alpha\beta}, \qquad d^{mn}{}_{k\alpha,l\beta} = \frac{1}{\sqrt{5}} \delta^{mn}_{kl} \epsilon_{\alpha\beta}, \\ d^{mn,kl,p} &= \frac{1}{\sqrt{40}} \epsilon^{mnklp}, \\ d_{MNK}\colon\; d_m{}^{n\alpha,\beta} &= \frac{1}{\sqrt{10}} \delta^n_m \epsilon^{\alpha\beta}, \qquad d_{mn}{}^{k\alpha,l\beta} = \frac{1}{\sqrt{5}} \delta^{kl}_{mn} \epsilon^{\alpha\beta}, \\ d_{mn,kl,p} &= \frac{1}{\sqrt{40}} \epsilon_{mnklp}, \end{aligned} \qquad (5.5)$$

and all those related by symmetry, $d^{MNK} = d^{(MNK)}$. In particular, the GL(1) grading guarantees that all components d^{mnk} vanish, such that the section condition (1.1) indeed is solved by restricting the coordinate dependence of all fields according to

$$\{\partial^{m\alpha} A = 0, \partial_{mn} A = 0, \partial^\alpha A = 0\} \Leftrightarrow A(x^\mu, Y^M) \to A(x^\mu, y^m). \qquad (5.6)$$

Moreover, the form of the d symbol (5.5) shows that any further coordinate dependence of a field A on combinations of the remaining coordinates violates the section condition. This explicitly shows that Eq. (5.6) is not a subcase of Eq. (4.43), but rather a different inequivalent solution.

B. $\mathrm{GL}(5) \times \mathrm{SL}(2)$-invariant reduction of EFT

In this subsection, we evaluate the EFT Lagrangian (1.3) upon splitting fields and tensors according to Eqs. (5.2)–(5.5) and assuming the explicit solution (5.6) of the section condition. Having gone through this analysis in great detail for the case of $D = 11$ supergravity in Sec. IV, we will keep the discussion much shorter here, and restrict it to the essential new ingredients. In particular, in this case, due to the presence of the self-dual 4-form in IIB, there is no known ten-dimensional Lagrangian to which the result can immediately be compared. Rather, the procedure will produce an action, in which only an $\mathrm{SO}(1,4) \times \mathrm{SO}(5)$ subgroup of the ten-dimensional Lorentz group is realized, much in the spirit of Refs. [86,87] in which Lorentz symmetry appears broken to SO(9) but is recovered on the level of the equations of motion.[7]

In analogy to the discussion in Sec. IV B above, let us first revisit the resulting field content of the model. With the split (5.2) and (5.3), the full p-form field content of the $\mathrm{E}_{6(6)}$ Lagrangian in this basis is thus given by

$$\{A_\mu{}^m, A_{\mu m\alpha}, A_{\mu kmn}, A_{\mu\alpha}\}, \qquad \{B_{\mu\nu}{}^\alpha, B_{\mu\nu mn}, B_{\mu\nu}{}^{m\alpha}\}, \qquad (5.7)$$

where we have defined $A_{\mu kmn} = \frac{1}{2}\epsilon_{kmnpq} A_\mu{}^{pq}$. More precisely, the Lagrangian depends on the 2-forms only under derivatives,

$$\{\partial_m B_{\mu\nu}{}^\alpha, \partial_{[k} B_{|\mu\nu|mn]}, \partial_m B_{\mu\nu}{}^{m\alpha}\}. \qquad (5.8)$$

Similar to the case of $D = 11$ supergravity, the vector fields $A_\mu{}^m$ will be identified with the IIB Kaluza-Klein vector fields. Indeed, they transform under the general gauge transformations (2.27) according to

$$\delta_\Lambda A_\mu{}^m = \partial_\mu \Lambda^m - A_\mu{}^n \partial_n \Lambda^m + \Lambda^n \partial_n A_\mu{}^m, \qquad (5.9)$$

with the associated gauge transformations closing into the algebra

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}] = \delta_{\Lambda_{12}}, \qquad \Lambda^m_{12} \equiv \Lambda^k_2 \partial_k \Lambda^m_1 - \Lambda^k_1 \partial_k \Lambda^m_2 \qquad (5.10)$$

of five-dimensional diffeomorphisms, embedded into the E-bracket (2.15). Comparing the remaining fields of Eq. (5.7) to the field content of the Kaluza-Klein reduction of IIB supergravity suggests relating the fields $\{A_{\mu m\alpha}, B_{\mu\nu}{}^\alpha\}$ in Eq. (5.7) to the different components of the doublet of ten-dimensional 2-forms, and the fields $A_{\mu kmn}, B_{\mu\nu mn}$ with the components of the (self-dual) IIB f4-form. The remaining fields $A_{\mu\alpha}, B_{\mu\nu}{}^{m\alpha}$ descend from components of the doublet of dual 6-forms. Again, the 2-form tensors $B_{\mu\nu m}$ that do not figure in the $\mathrm{E}_{6(6)}$-covariant Lagrangian represent the degrees of freedom on-shell dual to the Kaluza-Klein vector fields, i.e. descending from the ten-dimensional dual graviton. We recall that in the EFT formulation, all vector fields appear with a Yang-Mills kinetic term whereas the 2-forms couple via a topological term and are on-shell dual to the vector fields. In order to match the structure of IIB supergravity, we will thus have to trade the Yang-Mills vector fields $A_{\mu\alpha}$ for a propagating 2-form $B_{\mu\nu}{}^\alpha$.

The details of this identification can be worked out by evaluating the general formulas of the $\mathrm{E}_{6(6)}$-covariant formulation with Eq. (5.5) and imposing the explicit solution of the section condition (5.6) on all fields. Without repeating the details of the derivation which goes in close analogy to the analysis of Sec. IV B, we summarize the covariant field strengths for the different vector fields from Eq. (5.7),

[7]Covariant Pasti-Sorokin-Tonin-type formulations of IIB supergravity have been constructed in Refs. [88,89].

$$
\begin{aligned}
\mathcal{F}_{\mu\nu}{}^{m} &= 2\partial_{[\mu}A_{\nu]}{}^{m} - A_{\mu}{}^{n}\partial_{n}A_{\nu}{}^{m} + A_{\nu}{}^{n}\partial_{n}A_{\mu}{}^{m},\\
\mathcal{F}_{\mu\nu m\alpha} &= 2D_{[\mu}A_{\nu]m\alpha} + \epsilon_{\alpha\beta}\partial_{m}\tilde{B}_{\mu\nu}{}^{\beta},\\
\mathcal{F}_{\mu\nu kmn} &= 2D_{[\mu}A_{\nu]kmn} - 3\sqrt{2}\epsilon^{\alpha\beta}A_{[\mu[k|\alpha|}\partial_{m}A_{\nu]n]\beta}\\
&\quad + 3\partial_{[k}\tilde{B}_{|\mu\nu|mn]},\\
\mathcal{F}_{\mu\nu\alpha} &= 2D_{[\mu}A_{\nu]\alpha} - 2(\partial_{k}A_{[\mu}{}^{k})A_{\nu]\alpha}\\
&\quad - \sqrt{2}A_{[\mu}{}^{mn}\partial_{n}A_{\nu]m\alpha} - \sqrt{2}A_{[\mu|m\alpha|}\partial_{n}A_{\nu]}{}^{mn}\\
&\quad - \epsilon_{\alpha\beta}\partial_{k}\tilde{B}_{\mu\nu}{}^{k\beta},
\end{aligned}
\tag{5.11}
$$

with the modified 2-forms

$$
\begin{aligned}
\tilde{B}_{\mu\nu}{}^{\alpha} &\equiv \sqrt{10}B_{\mu\nu}{}^{\alpha} - \epsilon^{\alpha\beta}A_{[\mu}{}^{n}A_{\nu]n\beta},\\
\tilde{B}_{\mu\nu mn} &\equiv \sqrt{10}B_{\mu\nu mn} + A_{[\mu}{}^{k}A_{\nu]kmn},\\
\tilde{B}_{\mu\nu}{}^{k\alpha} &\equiv \sqrt{10}B_{\mu\nu}{}^{k\alpha} + \epsilon^{\alpha\beta}A_{[\mu}{}^{k}A_{\nu]\beta}.
\end{aligned}
\tag{5.12}
$$

All covariant derivatives $D_{\mu} \equiv \partial_{\mu} - \mathcal{L}_{A_{\mu}}$ correspond to the action of five-dimensional internal diffeomorphisms. The corresponding vector gauge transformations are given by

$$
\begin{aligned}
\delta A_{\mu}{}^{m} &= D_{\mu}\Lambda^{m},\\
\delta A_{\mu m\alpha} &= D_{\mu}\Lambda_{m\alpha} + \mathcal{L}_{\Lambda}A_{\mu m\alpha} - \epsilon_{\alpha\beta}\partial_{m}\tilde{\Xi}_{\mu}{}^{\beta},\\
\delta A_{\mu kmn} &= D_{\mu}\Lambda_{kmn} + \mathcal{L}_{\Lambda}A_{\mu kmn} - 3\sqrt{2}\epsilon^{\alpha\beta}\partial_{[k}A_{|\mu|m|\alpha|}\Lambda_{n]\beta}\\
&\quad - 3\partial_{[k}\tilde{\Xi}_{|\mu|mn]},
\end{aligned}
\tag{5.13}
$$

with

$$
\tilde{\Xi}_{\mu}{}^{\alpha} \equiv \sqrt{10}\Xi_{\mu}{}^{\alpha} - \epsilon^{\alpha\beta}\Lambda^{n}A_{\mu n\beta}, \quad \tilde{\Xi}_{\mu mn} \equiv \sqrt{10}\Xi_{\mu mn} + \Lambda^{k}A_{\mu kmn}.
\tag{5.14}
$$

As for the vector fields $A_{\mu\alpha}$, it will be sufficient to observe that its gauge variation is given by

$$
\delta A_{\mu\alpha} = \cdots + \epsilon_{\alpha\beta}\partial_{k}\tilde{\Xi}_{\mu}{}^{k\beta},
\tag{5.15}
$$

implying that it can entirely be gauged away by the tensor gauge symmetry associated with the 2-forms $B_{\mu\nu}{}^{k\beta}$. Consequently, it will automatically disappear from the Lagrangian upon integrating out $\partial_{k}B_{\mu\nu}{}^{k\beta}$. The remaining 2-form field strengths in turn come with the gauge transformations

$$
\begin{aligned}
\delta\tilde{B}_{\mu\nu}{}^{\alpha} &= 2D_{[\mu}\tilde{\Xi}_{\nu]}{}^{\alpha} + \mathcal{L}_{\Lambda}\tilde{B}_{\mu\nu}{}^{\alpha} - \epsilon^{\alpha\beta}\Lambda_{n\beta}F_{\mu\nu}{}^{n},\\
\delta\tilde{B}_{\mu\nu mn} &= 2D_{\mu}\left(\tilde{\Xi}_{\nu mn} + \frac{1}{\sqrt{2}}\epsilon^{\alpha\beta}A_{\nu m\alpha}\Lambda_{n\beta}\right) + \sqrt{2}\partial_{m}A_{\mu n\alpha}\tilde{\Xi}_{\nu}{}^{\alpha}\\
&\quad + \mathcal{L}_{\Lambda}\tilde{B}_{\mu\nu mn} - \frac{1}{\sqrt{2}}\Lambda_{[m|\alpha|}\partial_{n]}\tilde{B}_{\mu\nu}{}^{\alpha} + \Lambda_{mnk}F_{\mu\nu}{}^{k}
\end{aligned}
\tag{5.16}
$$

and field strengths

$$
\begin{aligned}
\tilde{\mathcal{H}}_{\mu\nu\rho}{}^{\alpha} &\equiv \sqrt{10}\mathcal{H}_{\mu\nu\rho}{}^{\alpha} = 3D_{[\mu}\tilde{B}_{\nu\rho]}{}^{\alpha} + 3\epsilon^{\alpha\beta}F_{[\mu\nu}{}^{n}A_{\rho]n\beta},\\
\tilde{\mathcal{H}}_{\mu\nu\rho mn} &\equiv \sqrt{10}\mathcal{H}_{\mu\nu\rho mn}\\
&= 3D_{\mu}\tilde{B}_{\nu\rho mn} - 3F_{\mu\nu}{}^{k}A_{\rho kmn} - 3\sqrt{2}\epsilon^{\alpha\beta}A_{\mu m\alpha}D_{\nu}A_{\rho n\beta}\\
&\quad + 3\sqrt{2}A_{\mu m\alpha}\partial_{n}\tilde{B}_{\nu\rho}{}^{\alpha},
\end{aligned}
\tag{5.17}
$$

up to terms that are projected out from the Lagrangian under y derivatives. The expressions on the rhs in Eqs. (5.16) and (5.17) are understood to be projected onto the corresponding antisymmetrizations in their parameters, i.e. $[mn]$, $[\mu\nu]$, $[\mu\nu\rho]$, etc.

Finally, we note that the topological term (3.7) in this parametrization is given by

$$
\begin{aligned}
\mathcal{L}_{\rm top} = \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma\tau}\epsilon^{klmnp}\bigg(&\frac{\sqrt{2}}{6}\epsilon^{\alpha\beta}\mathcal{F}_{\mu\nu m\alpha}\mathcal{F}_{\rho\sigma n\beta}A_{\tau pkl} + \frac{1}{6}\mathcal{F}_{\mu\nu mnq}F_{\rho\sigma}{}^{q}A_{\tau klp} - \frac{\sqrt{2}}{2}\epsilon^{\alpha\beta}A_{\mu m\alpha}\partial_{n}A_{\nu p\beta}F_{\rho\sigma}{}^{q}A_{\tau klq} + \frac{1}{2}\partial_{p}\tilde{B}_{\mu\nu mn}F_{\rho\sigma}{}^{q}A_{\tau klq}\\
&+ \sqrt{2}\epsilon^{\alpha\beta}A_{\mu m\alpha}D_{\nu}A_{\rho n\beta}\partial_{p}\tilde{B}_{\sigma\tau kl} - \sqrt{2}A_{\mu m\alpha}\partial_{n}\tilde{B}_{\nu\rho}{}^{\alpha}\partial_{p}\tilde{B}_{\sigma\tau kl} + \frac{2}{3}\epsilon^{\alpha\beta}A_{\mu m\alpha}\partial_{n}A_{\nu k\beta}A_{\rho l\gamma}\partial_{p}\tilde{B}_{\sigma\tau}{}^{\gamma} - \epsilon^{\alpha\beta}\epsilon^{\gamma\delta}A_{\mu m\alpha}\partial_{n}A_{\nu k\beta}A_{\rho l\gamma}D_{\sigma}A_{\tau p\delta}\\
&+ \frac{\sqrt{2}}{9}\partial_{m}\tilde{\mathcal{H}}_{\mu\nu\rho}{}^{\alpha}A_{\sigma n\alpha}A_{\tau klp} - D_{\mu}\tilde{B}_{\nu\rho mn}\partial_{p}\tilde{B}_{\sigma\tau kl} - \frac{2}{3}\epsilon_{\alpha\beta}\tilde{\mathcal{H}}_{\mu\nu\rho}{}^{\beta}\partial_{k}\tilde{B}_{\sigma\tau}{}^{k\alpha} + \mathcal{O}(A_{\mu\alpha})\bigg).
\end{aligned}
\tag{5.18}
$$

Let us now move to the scalar field content of the theory. In the EFT formulation, they parametrize the symmetric matrix $\mathcal{M}_{MN}$. To relate to IIB supergravity, we need to choose a parametrization of this matrix in accordance with the decomposition (5.3). In standard fashion, we build the matrix as $\mathcal{M} = \mathcal{V}\mathcal{V}^{T}$ from a "vielbein" $\mathcal{V}$ in triangular gauge,

$$
\mathcal{V} \equiv \exp[\Phi t_{(0)}]\mathcal{V}_{2}\mathcal{V}_{5}\exp[b_{mn}{}^{\alpha}t_{(+3)}{}_{\alpha}^{mn}]\exp[\epsilon^{klmnp}c_{klmn}t_{(+6)p}].
\tag{5.19}
$$

Here, $t_{(0)}$ is the $E_{6(6)}$ generator associated to the GL(1) grading of Eq. (5.3), $\mathcal{V}_{2}$, $\mathcal{V}_{5}$ denote matrices in the SL(2) and SL(5) subgroup, respectively, parametrized by the vielbeins ν_{2}, ν_{5} in analogy to Eq. (4.60). The $t_{(+n)}$ refer to the $E_{6(6)}$ generators of positive grading in Eq. (5.3), with the nontrivial commutator

$$
[t_{(+3)}{}_{\alpha}^{kl}, t_{(+3)}{}_{\beta}^{mn}] = \epsilon_{\alpha\beta}\epsilon^{klmnp}t_{(+6)p}.
\tag{5.20}
$$

All generators are evaluated in the fundamental 27 representation (5.2), such that the symmetric matrix $\mathcal{M}_{MN}$ takes the block form

$$\mathcal{M}_{KM} = \begin{pmatrix} \mathcal{M}_{km} & \mathcal{M}_k{}^{m\beta} & \mathcal{M}_{k,mn} & \mathcal{M}_k{}^{\beta} \\ \mathcal{M}^{k\alpha}{}_m & \mathcal{M}^{k\alpha,m\beta} & \mathcal{M}^{k\alpha}{}_{mn} & \mathcal{M}^{k\alpha,\beta} \\ \mathcal{M}_{kl,m} & \mathcal{M}_{kl}{}^{m\beta} & \mathcal{M}_{kl,mn} & \mathcal{M}_{kl}{}^{\beta} \\ \mathcal{M}^{\alpha}{}_m & \mathcal{M}^{\alpha,m\beta} & \mathcal{M}^{\alpha}{}_{mn} & \mathcal{M}^{\alpha\beta} \end{pmatrix}. \quad (5.21)$$

An explicit evaluation of Eq. (5.19) determines the various blocks in Eq. (5.21). For instance, its last line is given by

$$\begin{aligned} \mathcal{M}^{\alpha\beta} &= e^{5\Phi/3} m^{\alpha\beta}, \qquad \mathcal{M}^{\alpha}{}_{mn} = \sqrt{2} e^{5\Phi/3} b_{mn}{}^{\alpha}, \\ \mathcal{M}^{\alpha,m\beta} &= \frac{1}{2} e^{5\Phi/3} m^{\alpha\gamma} \epsilon_{\gamma\delta} \epsilon^{mklpq} b_{kl}{}^{\beta} b_{pq}{}^{\delta} \\ &\quad - e^{5\Phi/3} m^{\alpha\beta} \epsilon^{mklpq} c_{klpq}, \\ \mathcal{M}^{\alpha}{}_m &= \frac{1}{3} e^{5\Phi/3} m^{\alpha\beta} \epsilon_{\beta\gamma} \epsilon^{kpqrs} \\ &\quad \times \left(\epsilon_{\delta_1\delta_2} b_{mk}{}^{\delta_1} b_{pq}{}^{\delta_2} b_{rs}{}^{\gamma} + \frac{1}{4} b_{mk}{}^{\gamma} c_{pqrs} \right), \end{aligned} \quad (5.22)$$

with the symmetric matrix $m^{\alpha\beta} = (\nu_2)^{\alpha}{}_u (\nu_2)^{\beta u}$ build from the SL(2) vielbein from Eq. (5.19). Later, after integrating out some of the fields, we will need the components of (cf. the discussion in the previous section)

$$\tilde{\mathcal{M}}_{MN} \equiv \mathcal{M}_{MN} - \mathcal{M}_M{}^{\alpha} (\mathcal{M}^{\alpha\beta})^{-1} \mathcal{M}_N{}^{\beta}, \quad (5.23)$$

for which we find

$$\begin{aligned} \tilde{\mathcal{M}}^{mn,kl} &= e^{2\Phi/3} m^{m[k} m^{l]n}, \\ \tilde{\mathcal{M}}^{mn}{}_{k\alpha} &= \frac{1}{\sqrt{2}} e^{2\Phi/3} \epsilon^{mnpqr} m_{kp} m_{\alpha\beta} b_{qr}{}^{\beta}, \\ \tilde{\mathcal{M}}^{mn,k} &= \frac{1}{\sqrt{2}} e^{2\Phi/3} \epsilon^{mnpqr} \epsilon_{\alpha\beta} m^{kl} b_{lp}{}^{\alpha} b_{qr}{}^{\beta} \\ &\quad - \frac{1}{6\sqrt{2}} e^{2\Phi/3} \epsilon^{mnpqr} m^{kl} c_{lpqr}, \\ \tilde{\mathcal{M}}_{m\alpha,n\beta} &= e^{-\Phi/3} m_{mn} m_{\alpha\beta} + 2 e^{2\Phi/3} m_{\alpha\gamma} m_{\beta\delta} m^{kp} \\ &\quad \times (m_{mn} m^{lq} b_{kl}{}^{\gamma} b_{pq}{}^{\delta} - 2 m^{kp} b_{mk}{}^{\gamma} b_{np}{}^{\delta}), \end{aligned} \quad (5.24)$$

etc., with $m_{mn} = (\nu_5)_m{}^a (\nu_5)_n{}^a$. From the inverse matrix $\mathcal{M}^{MN}$ we will in particular need the components

$$\mathcal{M}^{mn} = e^{4\Phi/3} m^{mn}. \quad (5.25)$$

With Eq. (5.5) we find for the covariant derivatives of the matrix parameters from Eq. (5.21)

$$\begin{aligned} \mathcal{D}_\mu \Phi &= D_\mu \phi + \frac{4}{5} \partial_k A_\mu{}^k, \\ \mathcal{D}_\mu m_{mn} &= D_\mu m_{mn} + \frac{2}{5} \partial_k A_\mu{}^k m_{mn}, \\ \mathcal{D}_\mu b_{mn}{}^{\alpha} &= D_\mu b_{mn}{}^{\alpha} - \epsilon^{\alpha\beta} \partial_{[m} A_{n]\beta\mu}, \\ \mathcal{D}_\mu c_{klmn} &= D_\mu c_{klmn} + 4\sqrt{2} \partial_{[k} A_{lmn]\mu} + 12 b_{[kl}{}^{\alpha} \partial_m A_{n]\alpha}, \end{aligned} \quad (5.26)$$

which will build the kinetic term of the Lagrangian.

As discussed above and similar to the analysis for the embedding of $D = 11$ supergravity, the precise map with type IIB supergravity requires some dualizations of the fields. To this end, we observe that in the Lagrangian the 2-form tensors $\tilde{B}_{\mu\nu}{}^{k\beta}$ appear only under a divergence, i.e. contracted with ∂_k, cf. Eq. (5.8), and with the algebraic field equations

$$\epsilon_{\alpha\beta} \partial_k \tilde{B}_{\mu\nu}{}^{k\beta} = (\mathcal{M}^{\alpha\beta})^{-1} \mathcal{M}^{\beta}{}_M \mathcal{F}^{\mu\nu M} - \frac{1}{6} \varepsilon^{\mu\nu\rho\sigma\tau} (\mathcal{M}^{\alpha\beta})^{-1} \tilde{\mathcal{H}}_{\rho\sigma\tau}{}^{\beta}. \quad (5.27)$$

By means of these equations, the fields $\tilde{B}_{\mu\nu}{}^{k\beta}$ can be eliminated from the Lagrangian. The gauge symmetry (5.15) shows that in the process, the vector fields $A_{\mu\alpha}$ also disappear. We infer from Eq. (5.27) that the kinetic term for the remaining vector fields changes into the form (4.96) with $\tilde{\mathcal{M}}_{MN}$ from Eq. (5.24). Moreover, the 2-forms $\tilde{B}_{\mu\nu}{}^{\alpha}$ are promoted into propagating fields with the kinetic term

$$-e^{-5/3\Phi} m_{\alpha\beta} \tilde{\mathcal{H}}_{\mu\nu\rho}{}^{\alpha} \tilde{\mathcal{H}}^{\mu\nu\rho\beta}, \quad (5.28)$$

and we note that the cross terms from Eq. (5.27) give rise to additional contributions to the topological term in Eq. (5.18).

Let us conclude by commenting on some of the properties of the resulting Lagrangian. At first sight, it may appear surprising that we can obtain a ten-dimensional Lagrangian describing the field equations of the full IIB supergravity, whereas it is known that the presence of a self-dual 4-form poses a severe obstruction to the construction of a Lorentz-covariant Lagrangian. It is the latter property which justifies the existence of our Lagrangian: what we have constructed is a ten-dimensional Lagrangian in which however only an $\mathrm{SO}(1,4) \times \mathrm{SO}(5)$ subgroup of the SO(1,9) Lorentz symmetry is realized. In this respect, its existence is no more surprising than the corresponding constructions of Refs. [86,87] in which Lorentz symmetry appears broken to SO(9) but is recovered on the level of the equations of motion. The self-dual 4-form is described by propagating degrees of freedom c_{klmn} and $A_{\mu kmn}$, yet the final Lagrangian also carries some of the dual degrees of freedom in the 2-forms $\tilde{B}_{\mu\nu mn}$. These do not appear with a kinetic term but couple by a topological term (5.18) such that their field equations precisely give rise to the first-order duality equations that relate their field strength to the field strength of the $A_{\mu kmn}$, thereby reproducing part of the ten-dimensional self-duality equations.

VI. SUMMARY AND OUTLOOK

In this paper, we have presented the detailed construction of the $E_{6(6)}$ exceptional field theory recently announced in Ref. [33]. This theory is formally defined in $5+27$ dimensions, with 27 coordinates transforming in the fundamental representation of $E_{6(6)}$, subject to a covariant section constraint. This constraint, which implies that only a subset of the coordinates are physical, is the M-theory analogue of the strong constraint in double field theory, which in turn is a stronger version of the level-matching constraint in string theory. The constraint allows for different solutions, two of which we have discussed in detail. The first reduces the 27 coordinates to six, thereby breaking $E_{6(6)}$ to GL(6), leading to a $(5+6)$-dimensional formulation of the full (untruncated) 11-dimensional supergravity. The second solution of the constraint reduces the 27 coordinates to five, breaking $E_{6(6)}$ to $\mathrm{GL}(5)\times\mathrm{SL}(2)$, leading to a $(5+5)$-dimensional formulation of type IIB supergravity with manifest SL(2) S-duality. In this sense, the exceptional field theory (1.3) unifies M-theory and type IIB in that both are obtained on different "slices" of the generalized spacetime. This generalizes type II double field theory, in which type IIA and type IIB arise on different slices of the doubled spacetime [50,51]. As a byproduct, we have obtained an off-shell action for type IIB supergravity, at the cost of sacrificing ten-dimensional Lorentz invariance.

In this paper we have restricted ourselves to the purely bosonic theory, but we are confident that the extension to include fermions and the construction of a supersymmetric action is straightforward along the lines of the supersymmetric $D=5$ gauged supergravity [77]. The fermions will be $E_{6(6)}$ singlets transforming under the local generalized Lorentz group of the corresponding coset, i.e., in this case $\mathrm{H}=\mathrm{USp}(8)$, which will require a notion of generalized Lorentz connection. This should also clarify the relation of our construction to that of de Wit and Nicolai [5,6], who cast the 11-dimensional supersymmetry transformations into an H-covariant from. At first sight it may appear surprising that such a supersymmetric covariant construction is feasible at all. First we know that conventional supersymmetric theories are restricted to dimensions $D\leq 11$. Second, the resulting theory would encode both type IIA and type IIB, despite the crucial difference of their fermion chiralities. The first obstacle is circumvented by virtue of the section constraint, which implies that the additional coordinates are not physical in the same sense as the usual spacetime coordinates. In fact, in double field theory supersymmetric extensions are possible and beautifully simplify the usually rather involved $\mathcal{N}=1$ supergravities in $D=10$, with the supersymmetry transformations closing into the generalized diffeomorphisms [47]. The second obstacle is circumvented since the EFT formulation does not preserve the $D=10$ Lorentz invariance, so that the EFT fermions can consistently encode the fermions of type IIA and type IIB. This possibility is then no more surprising than the observation that both type IIA and type IIB give rise to the same supersymmetric theory in $D=5$ upon dimensional reduction.

A novel feature of the supersymmetric EFT is that usually it is supersymmetry which fixes the detailed form of even some of the purely bosonic couplings, most notably the presence and shape of the scalar potential. In contrast, in Eq. (1.3) all bosonic couplings are already uniquely determined by the bosonic gauge and duality symmetries. This points to a deep connection between the duality-covariant geometries of double and exceptional field theories on the one hand and supersymmetry on the other, as for instance illustrated by the striking economy of the supersymmetric double field theory. We leave a discussion of these matters and the detailed construction of supersymmetric EFT to a separate publication.

There are many open questions and possible generalizations. An obvious question is about the physical significance of the 27 coordinates. Beyond the six coordinates of $D=11$ supergravity, are they a purely formal device, or do they have a deeper role to play? A comparison with string theory is illuminating. Here the doubled coordinates, at least on toroidal backgrounds, are undoubtedly physical and real, as made explicit by closed string field theory, subject only to the weaker level-matching constraint that allows for solutions depending locally both on coordinates and their duals [38]. Thus, although the currently understood double field theory is subject to the strong constraint, the latter constraint is well motivated from string theory, implementing the level-matching constraint in stronger form. The section constraint of exceptional field theory has been postulated by analogy to the strong constraint, but since there is no analogue to string field theory in M-theory, it cannot be "derived" in a similar fashion. However, we may consider a partial solution of the $E_{6(6)}$-covariant section constraint that breaks the symmetry to the T-duality group of string theory. Specifically, we can embed the SO (5,5) T-duality group that is appropriate for a $(5+5)$-dimensional decomposition of type II string theory into $E_{6(6)}$. The fundamental representation then decomposes as

$$\mathrm{SO}(5,5)\subset E_{6(6)}\colon\ \mathbf{27}\to 10\oplus 16\oplus 1, \qquad (6.1)$$

where 10 and 16 are the vector and spinor representation of SO(5,5), respectively. Thus, under this decomposition we obtain the Neveu-Schwarz-Neveu-Schwarz (NS-NS) fields transforming as a vector (or rather, in the generalized metric formulation, as a 2-tensor) but also the Ramond-Ramond fields transforming as a spinor. The resulting theory will be a Kaluza-Klein-type decomposition of the original type II double field theory of Refs. [50,51], in the sense of Ref. [73]. The decomposition of the d symbol is then such that the section constraint implies the independence of all fields on the $1 + 16$ variables, and further restricts the field dependence on the remaining ten variables in the fundamental vector representation of SO(5,5) as

$$d^{MNK}\partial_M\partial_N = 0 \Rightarrow \eta^{\check{M}\check{N}}\partial_{\check{M}}\partial_{\check{N}} = 0, \qquad (6.2)$$

with the SO(5,5) vector indices denoted by $\check{M}, \check{N}$; see e.g. Eqs. (3.27) and (3.29) in Ref. [90]. Thus, the section constraint reduces precisely to the strong constraint in double field theory. Since in string theory the strong constraint is relaxed so that the doubled coordinates are physical and real, U-duality covariance strongly suggests the same for the 27 coordinates of the $E_{6(6)}$ EFT, and similarly for the extended coordinates of the higher EFTs with respect to $E_{7(7)}$ and $E_{8(8)}$.

A related question is about the most general solutions of the section constraint (1.1), in particular whether there are solutions beyond the known $D = 10$ and $D = 11$ supergravity. While we do not have a proof that there are no solutions with $D > 11$, this appears unlikely. However, it is certainly important to classify all solutions, in particular in order to see whether or not there may be any "nongeometric" solutions, for any $D > 5$. For instance, one may imagine that the gauged diffeomorphisms (3.27) and the generalized internal diffeomorphisms do not organize into conventional diffeomorphisms of a D-dimensional theory, thereby escaping the conventional classifications. We leave this for future work. Even if such more general solutions of the section constraint may be excluded, it is still likely that there are nongeometric solutions of the EFT field equations that locally depend on the subset of coordinates corresponding to one solution of the constraint, but that patch together inequivalent solutions in a globally nontrivial manner, as happens in double field theory [69]. Perhaps the most intriguing, but also most involved question is about a genuine relaxation of the section constraint, which would truly transcend the framework of supergravity.

Another fascinating prospect is to generalize the presently known EFT to include higher-derivative M-theory corrections along the lines of the recent results on double field theory [68]. This would entail a deformation of the $E_{n(n)}$ generalized Lie derivatives and other structures. If possible, this would give a scheme to compute the α' corrections of type II string theories and the higher-derivative M-theory corrections in a unified manner.

Let us finally note that the details for the remaining finite-dimensional groups $E_{7(7)}$ and $E_{8(8)}$ will be presented in a separate publication. The general construction proceeds along the same lines as the one presented here, with a $4 + 56$- and $3 + 248$-dimensional formulation, respectively. One novel feature of these cases is that additional field components need to be introduced which, from an 11-dimensional perspective, play the role of the dual graviton, a field for which a local field theory formulation is usually considered impossible on the grounds of the no-go theorems in Refs. [91,92]. We have shown in Ref. [72] how to handle this problem in the covariant approach via introducing constrained compensator fields, extending the approach of Ref. [93]. In three dimensions, the components of the higher-dimensional dual graviton figure among the coordinates of the scalar target space. The Lagrangian of Ref. [72] carries these fields in a duality-covariant way and yields the first-order duality equations which relate them to the corresponding components of the higher-dimensional metric, all while retaining full higher-dimensional coordinate dependence. The construction hinges on the introduction of the covariantly constrained compensator fields, which can be viewed as extra gauge potentials, however, satisfying the analogue of the section constraint, but for the field components, so that effectively only a subset of fields survives, cf. Eq. (2.34) of Ref. [72]. In fact, these additional gauge fields appear among the $(D-2)$-forms in the covariant formulation in all dimensions and neatly fit in the structure of the tensor hierarchy. For instance, although such fields are not visible in the $E_{6(6)}$ action (1.3) presented in this paper, they would show up when extending the tensor hierarchy on-shell to the full set of 2-forms $B_{\mu\nu M}$ in the form of compensating gauge fields $C_{\mu\nu\rho M}$ among the 3-forms. For our action, they are irrelevant thanks to the extra gauge redundancy corresponding to $\mathcal{O}_{\mu\nu M}$ [see Eq. (2.34)], whose 3-form gauge potential does not enter the action. For the $D = 4$ decomposition, however, the compensating gauge field is a 2-form and thus enters explicitly the gauge-covariant field strength of the gauge vectors $A_\mu{}^M$. Finally, in the $D = 3$ decomposition the compensating gauge fields are among the vectors entering the covariant derivatives, as discussed for the Ehlers $\mathrm{SL}(2,\mathbb{R})$ subgroup in Ref. [72]. This mechanism also circumvents the seeming problem of nonclosure of the $E_{8(8)}$ generalized Lie derivatives [29]. Summarizing, we have arrived at a satisfying homogeneous picture of the exceptional field theory formulations for $E_{n(n)}$, $n = 6, 7, 8$. It is a fascinating question whether and if so how these constructions can be extended to even larger groups, possibly starting with the infinite-dimensional $E_{9(9)}$ and lifting the action functional of Ref. [94], but here we can only speculate.

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APPENDIX: TRUNCATIONS OF EXCEPTIONAL FIELD THEORY

In this appendix we discuss possible truncations of the EFT action (1.3) in order to relate it to results in the literature on duality-covariant formulations of subsectors of 11-dimensional supergravity [20,23–26,30]. In particular, in these formulations all off-diagonal field components and the external components of the 3-form are set to zero, and it is assumed that all fields depend only on internal coordinates. In terms of the fields and coordinates of the $E_{6(6)}$ EFT presented here this truncation therefore assumes

$$A_\mu{}^M = 0, \qquad B_{\mu\nu M} = 0, \qquad \partial_\mu = 0. \tag{A1}$$

For the action (1.3) this truncation implies

$$\hat{R} \to 0, \qquad g^{\mu\nu}\mathcal{D}_\mu \mathcal{M}^{MN}\mathcal{D}_\nu \mathcal{M}_{MN} \to 0,$$
$$\mathcal{M}_{MN}\mathcal{F}^{\mu\nu M}\mathcal{F}_{\mu\nu}{}^N \to 0, \qquad \mathcal{L}_{\text{top}} \to 0, \tag{A2}$$

such that the only surviving term is a truncation of the potential term $V(\mathcal{M}_{MN}, g_{\mu\nu})$. The available formulations in the literature differ in the treatment of the remaining fields, i.e., the external metric $g_{\mu\nu}$ and the generalized metric $\mathcal{M}_{MN}$ encoding the internal field components. The original work by Hillmann on $E_{7(7)}$ covariance [20] sets the external metric to the flat Minkowski metric,

$$g_{\mu\nu} = \eta_{\mu\nu} \Rightarrow \sqrt{-g} = e = 1, \tag{A3}$$

so that the volume factor becomes unity. In the analogous truncation of the $E_{6(6)}$ EFT, the action (1.3) reduces to the "potential term" only,

$$S_{\text{EFT}} \to -\int d^{27}Y V(\mathcal{M}), \tag{A4}$$

with $V(\mathcal{M})$ obtained from Eq. (1.4) by setting $g_{\mu\nu} = \eta_{\mu\nu}$. It is useful to investigate what are the residual gauge symmetries after this truncation. Of course, the $(4+1)$-dimensional diffeomorphisms are broken, but also the "internal" generalized diffeomorphisms are not completely preserved, for the presence of g-dependent terms in the potential was crucial for gauge invariance, as discussed in Sec. III B. In particular, the volume factor e with the appropriate weight is needed. Requiring that the condition $e = 1$ be preserved under gauge transformations we obtain

$$\delta_\Lambda e = \Lambda^N \partial_N e + \frac{5}{3} e \partial_N \Lambda^N \stackrel{!}{=} 0 \Rightarrow \partial_N \Lambda^N = 0. \tag{A5}$$

In fact, Hillmann found that his formulation matches the considered truncation of $D = 11$ supergravity only in the "unimodular gauge" of the internal metric [19], for which the residual gauge transformations are indeed compatible with Eq. (A5).

For a proper duality-covariant truncation of Eq. (1.3), the volume factor of the internal metric has to be kept as a separate degree of freedom, as already noted in Ref. [19]. Specifically, Eq. (A3) is relaxed to

$$g_{\mu\nu} = e^{2\Delta}\eta_{\mu\nu}, \tag{A6}$$

with a warp factor that in accordance with Eq. (A1) is a function of Y only and transforms as a scalar density of weight $\lambda = \frac{2}{3}$ under the Λ gauge transformations (2.4). For this truncation, the EFT action (1.3) again reduces to its potential term, now with extra contributions in Δ,

$$\begin{aligned} S_{\text{EFT}} \to \int d^{27}Y e^{5\Delta}\bigg(&\frac{1}{24}\mathcal{M}^{MN}\partial_M \mathcal{M}^{KL}\partial_N \mathcal{M}_{KL} \\ &-\frac{1}{2}\mathcal{M}^{MN}\partial_M \mathcal{M}^{KL}\partial_L \mathcal{M}_{NK} \\ &-5\partial_M \Delta \partial_N \mathcal{M}^{MN} - 20\mathcal{M}^{MN}\partial_M \Delta \partial_N \Delta\bigg). \end{aligned} \tag{A7}$$

This truncated action is duality and Λ^M gauge invariant. Note that Δ is a separate degree of freedom that transforms independently of the 42 scalars parametrizing the $E_{6(6)}$ matrix $\mathcal{M}_{MN}$. It may be convenient to combine $\mathcal{M}_{MN}$ and Δ into a single object

$$\hat{\mathcal{M}}_{MN} = e^{\gamma\Delta}\mathcal{M}_{MN}, \tag{A8}$$

with some factor γ, and rewrite the potential in terms of $\hat{\mathcal{M}}$ only. This rescaled matrix is no longer an element of the duality group $E_{n(n)}$, but can rather be thought of as taking values in $E_{n(n)} \times \mathbb{R}^+$, which is the starting point in the approach of Ref. [26]. The formulations of Refs. [23–25] employ the object (A8) (with different choices for γ), but identify Δ with one of the internal components of $\mathcal{M}_{MN}$, which breaks the $E_{n(n)}$ covariance of Eq. (A7) down to the subgroup commuting with that parameter, as pointed out in Refs. [23,30]. The resulting truncation for the $E_{6(6)}$ case [25] coincides with Eqs. (A7) and (A8) (choosing $\gamma = -5$).

We close by pointing out that, in principle, one may also separate the $\mathbb{R}^+$ factor in the full, untruncated EFT in Eq. (1.3), by redefining $g_{\mu\nu} = e^{2\Delta}\hat{g}_{\mu\nu}$, with unimodular metric $\hat{g}$, and then rescaling the generalized metric $\mathcal{M}_{MN}$ as in Eq. (A8). This has various technical disadvantages, however, as for instance the Einstein-Hilbert and scalar-kinetic terms start mixing in an intricate fashion, thereby obscuring the manifest $E_{6(6)}$ covariance of the current formulation.

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