CERN-TH.5256/88

Exceptional Groups in String Theory

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ABSTRACT

We review the occurrence of exceptional groups in string theory: their dual role as gauge symmetry and as a symmetry unifying space-time, superconformal ghost and internal degrees of freedom. In both cases the relation to the extended world-sheet supersymmetries is discussed in detail. This is used to construct the supermultiplet structure of the massless sectors of all supergravity theories possible in string theory in even space-time dimensions between four and ten.

CERN-TH.5256/88
December 1988

1. Introduction

Exceptional groups play a fundamental role in the construction of (closed, oriented) string theories. First, it was realized that the absence of space-time anomalies in 10-dimensional supersymmetric field theory[1] and the requirement of modular invariance of the heterotic string theory [2] leads to the gauge group $E_8 \times E_8$. Here the exceptional gauge group arises by the Frenkel-Kac construction [3] from a level one Kac-Moody algebra which can be constructed from 16 (left-moving, i.e. antiholomorphic) bosons moving on the root lattice of $E_8 \times E_8$. Space-time supersymmetry was guaranteed by the GSO projection[4] in the right-moving (holomorphic) fermionic string sector and seemed unrelated to the appearance of the exceptional group E_8.

However, the covariant lattice construction [5] of heterotic strings in less than ten dimensions [5-18] revealed an intimate relation between space-time supersymmetry and exceptional groups, in general not restricted to E_8. Now the weight lattices of the exceptional groups encode the space-time transformation properties of any d-dimensional string state after bosonizing the (right-moving) fermionic string. However, the connection between space-time supersymmetries and exceptional groups is not restricted to the covariant lattice approach, but is a general feature of any space-time supersymmetric string theory [19-23].

Using the map from the fermionic to the bosonic string (bosonic string map) [24-26] it is evident that the two ways exceptional groups appear in d-dimensional strings are in fact equivalent from the world-sheet point of view – in the bosonic as well as in the fermionic case there exists [27-31,20] an extended superconformal algebra [32]. Then it is only a matter of the interpretation we give to the exceptional group – gauge group or covariant lattice – to obtain either a purely bosonic, a heterotic or a type II string theory. The models with identical left- and right-moving exceptional algebras and thus with identical left- and right-moving n-extended world-sheet superconformal algebras are commonly denoted as $(n,n)^*$ models and correspond to Calabi-Yau ($n=2, d=4$), K_3 ($n=4, d=6$) or torus compactifications of the ten-dimensional string theory [33-35].

In section two of this paper we discuss the relation between the level one exceptional Kac-Moody algebras and the primary fields of the extended superconformal algebras, the action of the internal two-dimensional supersymmetry generators and the spectral flow. In section three we will present a complete discussion of all possible combinations of left- and

* Our notation is such that n denotes the number of world-sheet supersymmetries and N the number of gravitini.

right-moving exceptional groups for heterotic and type II string theories in even dimensions $4 \le d \le 10$. In this way we construct all possible supergravity theories which can emerge from string theories below ten dimensions. The (massless) supermultiplet structure of N-extended space-time supersymmetry follows immediately from the representations of the exceptional groups. Some (but not all) of these theories have already been constructed in previous papers [36-40,19,20]. However we believe that a systematic derivation from the exceptional groups is nevertheless illuminating and worth presenting.

2. World-sheet properties

The holomorphic world-sheet degrees of freedom of the fermionic string in d uncompactified space-time dimensions consist of d free bosons $X^\mu(z)$ and fermions $\psi^\mu(z)$ ($\mu = 1 \ldots d$) together with the conformal and superconformal ghosts $b(z)$, $c(z)$, $\beta(z)$ and $\gamma(z)$. Via bosonization one can consider instead of $\psi^\mu(z)$ $\frac{d}{2}$ bosonic fields $\phi_i(z)$ ($i = 1, \ldots, \frac{d}{2}$) and similarly instead of $\beta(z)$, $\gamma(z)$ a scalar field $\phi(z)$ [41]. These fields provide $\frac{3}{2}d - 15$ units to the central charge of the Virasoro algebra. Thus, the so far unspecified internal $n = 1$ superconformal field theory of the fermionic string must have $c = 15 - \frac{3}{2}d$ to cancel the conformal anomaly. The covariant vertex operators of the fermionic string can be written as (neglecting the bosonic coordinates $X^\mu(z)$ and oscillators)

$$\begin{aligned} V_{\vec{w}}(z) &= \exp\Big(i\vec{w} \cdot \vec{\phi}(z)\Big) V_{\text{int}}(z), \\ \vec{w} &= (\lambda_1, \ldots, \lambda_{\frac{d}{2}}; q), \;\; \vec{\phi} = (\phi_1, \ldots, \phi_{\frac{d}{2}}; -i\phi). \end{aligned} \tag{2.1}$$

$V_{\text{int}}(z)$ are operators of the internal $c = 15 - \frac{3}{2}d$ superconformal field theory. The conformal weight of $V_{\vec{w}}(z)$ is given by

$$h = \frac{1}{2}\vec{\lambda}^2 - \frac{1}{2}q^2 - q + h_{\text{int}} \tag{2.2}$$

where h_{int} is the conformal weight of $V_{\text{int}}(z)$. The $\vec{w}$'s are lattice vectors of the Lorentzian "covariant" lattice $D_{\frac{d}{2},1}$ with conjugacy classes 0, S, C, V (for a review with more details and references on covariant lattices see [42]). Decomposing $D_{\frac{d}{2},1}$ to $D_{\frac{d}{2}} \otimes D_1$ the factor $D_{\frac{d}{2}}$ corresponds to the level one Kac-Moody algebra $SO(d)_{\text{Lorentz}}$ with lattice vectors $\vec{\lambda} \in D_{\frac{d}{2}}$. $\vec{\lambda}$ describes the transformation properties under the space-time Lorentz group $SO(d)_{\text{Lorentz}}$ and $q \in D_1$ is the ghost charge. For space-time fermions $\vec{\lambda}$ is a weight vector of one of the two spinor conjugacy classes S, C of $D_{\frac{d}{2}}$ and q is half-integer. Space-time bosons have $\vec{\lambda} \in 0, V$ conjugacy classes of $D_{\frac{d}{2}}$ and q integer.

Physical light-cone states have fixed longitudinal component in $D_{\frac{d}{2}}$ and are conventionally described in the canonical ghost picture $q=-\frac{1}{2}$ for fermions and $q=-1$ for bosons. This amounts to decomposing $D_{\frac{d}{2},1}$ to $D_{\frac{d}{2}-1}\otimes D_{1,1}$. Then, physical light-cone states have fixed entries in $D_{1,1}$: $(0,-1)\in V$ of $D_{1,1}$ for space-time bosons and $(\frac{1}{2},-\frac{1}{2})\in C$ of $D_{1,1}$ for space-time bosons.

The picture changing operation [41] connects equivalent states in different ghost pictures. It is closely related to a two-dimensional supersymmetry transformation of vertex operators:

$$\begin{aligned} V_{q+1}(z) &= \oint \frac{dw}{2\pi i} P_F(z) V_q(w), \\ P_F(z) &= e^{\phi(z)} T_F(z). \end{aligned} \tag{2.3}$$

$T_F(z)$ is the supercurrent of the $n=1$ superconformal algebra. It splits into a space-time part $T_F^{\text{space-time}} = -\frac{1}{2}\partial X_\mu \psi^\mu$ and an internal supercurrent T_F^{int}. Since ψ_μ and e^ϕ correspond to vector weights of $D_{\frac{d}{2}}$ and D_1 respectively, the space-time picture changing operator is characterized by a root vector of $D_{\frac{d}{2},1}$: $\vec{w}_{PC}^{\text{space-time}} = (0,\ldots,\pm 1,0,\ldots;1)$. On the other hand, P_F^{int} corresponds to a vector weight $\vec{w}_{PC}^{\text{int}} = (0,\ldots;1)$ and its action on physical states therefore maps vectors of the 0 (S)-conjugacy class to vectors of the V (C)-conjugacy class and vice versa.

The covariant Lorentzian lattice $D_{\frac{d}{2},1}$ completely describes the model independent part of the fermionic string which is connected to the space-time and superconformal ghost degrees of freedom. However, as explained in refs.[43,5] it is very useful to replace the Lorentzian lattice $D_{\frac{d}{2},1}$ by an Euclidian lattice $D_{\frac{d}{2}+3}$ keeping all conjugacy classes the same. This essentially amounts to replacing the fermionic string by a bosonic string (cf. below), while preserving modular invariance of the partition function at all loop orders [44]. So we are dealing now with a unitary, level one $SO(d+6)$ Kac-Moody algebra. Decomposing $D_{\frac{d}{2}+3}$ to the regular subalgebra $D_{\frac{d}{2}-1}\otimes D_4$, $D_{\frac{d}{2}-1}$ of course describes again the transverse Lorentz properties of any state and D_4 corresponds, instead of $D_{1,1}$, to the longitudinal and ghost degrees of freedom. Physical states in the canonical ghost picture are now conventionally characterized by fixed weight vectors $\vec{\lambda}=(0,0,0,-1)\in D_4$ for space-time bosons and $\vec{\lambda}=(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})\in D_4$ for space-time fermions. The conformal weight of an operator $\exp i\vec{w}\cdot\vec{X}$, $\vec{w}\in D_{\frac{d}{2}+3}$, is given by $\frac{1}{2}\vec{\lambda}^2$; this however coincides with the conformal weight of the covariant vertex operators eq.(2) only in the canonical ghost picture.

The space-time picture changing operator now corresponds to a root of $D_{\frac{d}{2}+3}$ and the internal picture changing operator to a vector weight of $D_{\frac{d}{2}+3}$. It is important to note that replacing $D_{\frac{d}{2},1}$ by $D_{\frac{d}{2}+3}$ only maps the four conjugacy classes onto each other but not individual lattice vectors. It means that the non-unitary ghost Hilbert space is not contained in the positive definite Hilbert space of the $SO(d+6)$ Kac-Moody algebra. Thus in order to calculate scattering amplitudes in the fermionic string one has to use the Lorentzian lattice $D_{\frac{d}{2},1}$.

The map from $D_{\frac{d}{2},1}$ to $D_{\frac{d}{2}+3}$ is already almost the bosonic string map described in [24-26]. The complete bosonic string map is realized by replacing the d dimensional fermionic string characterized by the Lorentzian lattice $D_{\frac{d}{2},1}$ by a "compactified" bosonic string characterized by the Euclidian lattice $D_{\frac{d}{2}+3} \otimes E_8$. Thus, the corresponding gauge symmetry of the bosonic string is $SO(d+6) \times E_8$. The additional level one E_8 Kac-Moody algebra is needed to cancel the conformal anomaly of the bosonic string and preserves modular invariance of the partition function. The internal sector of the theory with $c = 15 - \frac{3}{2}d$ is of course not touched by the bosonic string map. This implies that also the internal sector of the bosonic string theory obtained in this way possesses $n = 1$ superconformal invariance provided by the internal supercurrent. The corresponding bosonic "picture changing operator" has the form

$$P_F(z) \sim \psi^a(z) T_F^{int}(z) \tag{2.4}$$

where $\psi^a(z)$ is a vector of $SO(d+6)$. Now bosonic picture changing maps inequivalent states into each other, in contrast to the fermionic case where picture changing connects equivalent states in different ghost pictures. E.g. a state transforming as a vector of $SO(d+6)$ is transformed into a singlet of this group and spinors of $SO(d+6)$ change helicity.

So far we have only considered general features of a not necessarily space-time supersymmetric string theory. Now assume that there exist N holomorphic supersymmetry charges Q_α^A $(A = 1 \dots N)$ in the d-dimensional (d even) fermionic string theory. The supercharges are given by contour integrals of the holomorphic part of the N gravitini vertex operators at zero momentum:

$$Q^A_{\alpha(q)} = \oint \frac{dz}{2\pi i} V^A_{\alpha(q)}(z). \tag{2.5}$$

In the canonical ghost picture we have

$$V^A_{\alpha(-\frac{1}{2})}(z) = S_\alpha(z) e^{-\phi/2} \Sigma^A(z). \tag{2.6}$$

$S_\alpha(z)$ is a spin field of the Lorentz group $SO(d)$ characterized by the spinor weights of $D_{\frac{d}{2}}$:

$$\begin{aligned} S_\alpha(z) &= \exp\Big(i\vec{\lambda}_\alpha \cdot \vec{\phi}\Big)(z) \\ \vec{\lambda}_\alpha &= (\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}) \quad \text{even number of } " - ". \end{aligned} \tag{2.7}$$

Thus $V^A_{\alpha(-\frac{1}{2})}(z)$ is described by antispinor weights $\vec{w} \in C$ of $D_{\frac{d}{2}+3}$.

The fields $\Sigma^A(z)$ are the (not unique) Ramond groundstates of dimension $h_{\text{int}} = \frac{c}{24} = \frac{10-d}{16}$ of the internal superconformal field theory. The d-dimensional supersymmetry algebra translates immediately to the following operator product expansion between the Σ fields [29,30,20]:

$$\Sigma^A(z)\big(\Sigma^B(w)\big)^\dagger \sim (z-w)^{\frac{d-10}{8}}\delta^A_B + (z-w)^{\frac{d-2}{8}} J^A_B(w) + \ldots \tag{2.8}$$

The dimension one fields $J^A_B(z)$ are currents of an internal Kac-Moody algebra $\hat{g}$ of rank k ($k = 1, 2, 3$ for $N = 1, 2, 4$). Using the Frenkel-Kac construction, these currents can be explicitly expressed by k free internal bosons $H^{\text{int}}_i(z)$ ($i = 1\ldots k$). It follows that the internal vertex operators $V_{\text{int}}(z)$ always contain a factor which can be entirely written as exponentials of the free bosons $H^{\text{int}}_i(z)$ (neglecting derivatives of H^{int}_i):

$$V_{\text{int}}(z) = \exp\Big(i\vec{w}_{\text{int}} \cdot \vec{H}_{\text{int}}(z)\Big)\tilde{V}_{\text{int}}(z). \tag{2.9}$$

The vectors $\vec{w}_{\text{int}}$ are the weights of the internal algebra g spanning the weight lattice Γ_k of g. The $\tilde{V}_{\text{int}}(z)$ belong to the remaining conformal field theory with $\tilde{c} = 15 - \frac{3}{2}d - k$. Thus, a state of the fermionic string theory (cf. eq.(1)) is now characterized by a vector $\vec{W} = (\vec{w}, \vec{w}_{int})$ with $\vec{w} \in D_{\frac{d}{2}+3}$ and $\vec{w}_{\text{int}} \in \Gamma_k$.

The appearance of the additional free bosons H^{int}_i in the space-time supersymmetric fermionic string theories implies [19-21] that the $SO(d+6) \times g$ Kac-Moody algebra is extended to the level one Kac-Moody algebra of the exceptional groups E_6, E_7 or E_8:

$$E_i \supset SO(d+6) \times g \quad i = 6, 7, 8 \tag{2.10}$$

The reason for the extension of the $SO(d+6)$ Kac-Moody algebra is the existence of the gravitini vertex operators. Locality of the operator product expansion of any vertex operator with the gravitini vertex operators restricts the allowed combinations of $(\vec{w}, \vec{w}_{\text{int}})$ to the weight vectors of the exceptional algebras. Equivalently, the conjugacy classes of the lattice $D_{\frac{d}{2}+3}\otimes\Gamma_k$ are all those obtained by decomposing the weight lattice of the exceptional algebras to $D_{\frac{d}{2}+3}\times g$ where g is the subalgebra in E_i which commutes maximally with $D_{\frac{d}{2}+3}$. The supercharges (gravitini) correspond exactly to those root vectors of E_i which are needed for the extension of the $SO(d+6)\times g$ Kac-Moody algebra. Space-time supersymmetry transformations are performed adding these roots to a lattice vector within one specific conjugacy of $D_{\frac{d}{2}+3}\otimes\Gamma_k$. Then supersymmetric partners are obtained which correspond to vectors in different conjugacy classes of $D_{\frac{d}{2}+3}\otimes\Gamma_k$ which are in the decomposition of one E_i conjugacy class. Therefore the supermultiplet structure is encoded in the representations of the exceptional groups.

The supersymmetry algebra also leads to an extension of the internal superconformal algebra [29,30,20]. The Kac-Moody part of the extended superconformal algebra is generated by the currents J_A^B in eq.(8). According to eq.(9) the n supercurrents $T_{Fj}^{\rm int}$ $(j=1,\ldots,n)$ can be written as

$$T_{Fj}^{\rm int}(z)=\exp\Big(i\vec{t}_j\cdot\vec{H}_{\rm int}(z)\Big)\hat{T}_{Fj}^{\rm int}(z)\quad j=1,\ldots,n \tag{2.11}$$

Thus, also the n supercurrents correspond to specific weight vectors of the exceptional algebras. Therefore, reversing arguments, it is the appearance of the exceptional groups which implies the existence of the space-time supercharges and the extension of the world-sheet superconformal algebras. In the table we give an exhaustive list of the appearing exceptional groups E_i together with the number N of (holomorphic) supercharges and the internal n-extended superconformal algebras with their corresponding internal Kac-Moody algebras.

	E_6	E_7	E_8
$d=10$	–	–	$N=1$
$d=8$	–	–	$N=1$ $n=2$ $SO(2)$
$d=6$	–	$N=1$ $n=4$ $SU(2)$	$N=2$ $2\times(n=2)$ $SO(4)$
$d=4$	$N=1$ $n=2$ $U(1)$	$N=2$ $n=4\oplus n=2$ $SU(2)\times SO(2)$	$N=4$ $3\times(n=2)$ $SO(6)$

The bosonic string map takes above fermionic string theories to bosonic strings with gauge group $E_i\times E_8$ and the same extended superconformal symmetries as their fermionic counterparts. In symmetric (n, n) compactifications the holomorphic fermionic string and the anti-holomorphic bosonic string are related by the bosonic string map.

Let us now investigate in more detail the relation between the level one conformal families (=conjugacy classes) of E_6, E_7 and E_8 and the highest weight states of the internal extended superconformal algebras. We will restrict ourselves to the case of maximal space-time dimension with one (holomorphic) supercharge. The theories in dimensions $d_{max} - d'$ (d' even) have an additional $D_{\frac{d'}{2}}$ internal Kac-Moody algebra corresponding to d' internal world-sheet fermions. These fermions build, together with d' free bosons, d' free real two-dimensional superfields and thus d' copies of a $c = \frac{3}{2}$, $n = 2$ superconformal algebra. The internal momenta lie on the lattice $D_{\frac{d'}{2}}$ and lead to central charges in the space-time supersymmetry algebra [20].

First, E_8 in ten dimensions is rather trivial since there is no internal conformal field theory now. The E_8 root lattice decomposes into D_8 conjugacy classes as $0 \oplus S$. Using the physical state condition we recognize that the 0 conjugacy class of D_8 leads to odd rank tensors (containing a massless vector) of the transverse Lorentz group $SO(8)$ and the S conjugacy class to spinors of negative chirality (again with a massless groundstate).

Next consider E_7 in six dimensions. The E_7 weight lattice contains two conjugacy classes denoted by 0 with the singlet as lowest representation of E_7, and by 1 with the weights of the $\underline{56}$ as lowest dimensional representation. These two conjugacy classes decompose to $D_6 \otimes A_1$ as

$$\begin{aligned} 0 &= (0,0) \oplus (S,1) \\ 1 &= (V,1) \oplus (C,0) \end{aligned} \tag{2.12}$$

Here the two A_1 conjugacy classes 0 and 1 correspond to $SU(2)$ representations with integer and half-integer isospin I respectively. The space-time supercharge (gravitino vertex operator) is obtained from the $(S,1)$ conjugacy class of $D_6 \otimes A_1$; it transforms as a doublet of the internal symmetry group $SU(2)$:

$$V_\alpha^{\pm}(z) = S_\alpha e^{-\frac{\phi}{2}} \exp\Bigl(\pm i\frac{\sqrt{2}}{2} H_{\rm int}\Bigr)(z) \tag{2.13}$$

The gravitini are easily seen to satisfy a symplectic Majorana-Weyl condition, i.e. hermitian conjugation changes chirality and $SU(2)$ quantum number. This means that we have $N = 1$ supersymmetry in $d = 6$.

The internal sector of the theory is built by a $n = 4$, $c = 6$ superconformal field theory with internal level one A_1 Kac-Moody algebra. Thus the part of the theory which is described by the E_7 weight lattice has to be tensored together with a $\tilde{c} = 5$ conformal field theory. The NS groundstate of the internal $n = 4$ superconformal algebra has $h_{\text{int}} = 0$, $I = 0$. It is connected by the spectral flow of the superconformal algebra to the Ramond (R) groundstate with $h_{\text{int}} = 1/4$, $I = 1/2$. These two highest weight states build the massless universal sector of the theory. It corresponds to the 0 conjugacy class of E_7 in connection with the $\tilde{h} = 0$ groundstate of the $\tilde{c} = 5$ theory.

The massless matter fields are generated by the highest weight states $h_{\text{int}} = \frac{1}{2}$, $I = \frac{1}{2}$ in the NS sector and the field with $h_{int} = \frac{1}{4}$, $I = 0$ in the R sector of the theory. They are again connected by the spectral flow operator. These two highest weight states correspond to the $(V, 1)$ resp. $(C, 0)$ conjugacy class of $D_6 \otimes A_1$ orginating from the the 1 conjugacy class of E_7 tensored together with conformal fields of dimension $\tilde{h} = \frac{1}{4}$ of the $\tilde{c} = 5$ theory. In the canonical ghost picture the $(V, 1)$ conjugacy class leads to massless scalars transforming as a doublet of $SU(2)$, where the $(C, 0)$ conjugacy class leads to a $SU(2)$ singlet spinor.

On the other hand, when interpreting E_7 as gauge group the universal sector leads to massless states transforming as $\underline{133}$ of E_7 and the matter sector to states transforming as $\underline{56}$.

The internal supercurrent splits into two parts:

$$T_F^{\text{int}}(z) = T_F^1 \tilde{T}_F^1(z) + T_F^2 \tilde{T}_F^2(z) \tag{2.14}$$

T_F^i $(i = 1, 2)$ are a doublet under the internal $SU(2)$:

$$T_F^{1,2}(z) = \exp\Big(\pm \frac{i}{\sqrt{2}} H_{\text{int}}(z)\Big) \tag{2.15}$$

The $\tilde{T}_F^{1,2}$ are $\tilde{h} = \frac{5}{4}$ conformal fields of the $\tilde{c} = 5$ theory. Obviously, since the $SU(2)$ Kac-Moody currents $J^{\pm}(z)$ interpolate between T_F^1 and T_F^2, the field

$$\hat{T}_F^{int}(z) = T_F^1 \tilde{T}_F^2(z) + T_F^2 \tilde{T}_F^1(z) \tag{2.16}$$

must also be present. It can be equally well used as internal supercurrent instead of T_F^{int}. Their sum is, however, not a valid supercurrent of the string theory. On the other hand, the four components of T_F^{int} and $\hat{T}_F^{\text{int}}$ build the four supersymmetry generators of the $n = 4$ superconformal algebra.

The internal picture changing operator is obtained by multiplying T_F^{int} with e^{ϕ} in the fermionic resp. with the $SO(12)$ vector $\psi^a(z)$ in the bosonic (gauge) case. Therefore, $P_F^{\rm int}$ corresponds to a weight vector in the $(V,1)$ conjugacy class of $D_6 \otimes A_1$ which belongs to the 1 conjugacy class of E_7. Thus, the picture changing operation transforms states in the 0 conjugacy class of E_7 into states of the 1 conjugacy class and vice versa. In the bosonic case this corresponds to inequivalent states. E.g. in the massless matter sector the $\underline{56}$ of E_7 is mapped by the picture changing operation to a E_7 singlet. Therefore any $\underline{56}$ is always accompanied by a $\underline{1}$; these two states form a two-dimensional supermultiplet (see also the discussion in the next paragraph in the context of E_6).

Finally, let us consider the exceptional algebra E_6 relevant in four-dimensional string theories. E_6 possesses three conjugacy classes denoted by 0, 1 and $\bar{1}$ with lowest representations $\underline{1}$, $\underline{27}$ resp. $\overline{\underline{27}}$ (note that the 1 and $\bar{1}$ conjugacy classes are the CPT conjugates of each other and therefore appear always simultaneously). These conjugacy classes decompose under $D_5 \otimes U(1)$ like:

$$\begin{aligned} 0 &= (0,0) \oplus (V, \sqrt{3}) \oplus (S, -\frac{\sqrt{3}}{2}) \oplus (C, \frac{\sqrt{3}}{2}) \\ 1 &= (0, \frac{2}{3}\sqrt{3}) \oplus (V, -\frac{\sqrt{3}}{3}) \oplus (S, \frac{\sqrt{3}}{6}) \oplus (C, -\frac{5}{6}\sqrt{3}) \\ \bar{1} &= (0, -\frac{2}{3}\sqrt{3}), \oplus (V, \frac{\sqrt{3}}{3}) \oplus (C, -\frac{\sqrt{3}}{6}) \oplus (S, \frac{5}{6}\sqrt{3}) \end{aligned} \tag{2.17}$$

In this notation the 12 $U(1)$ conjugacy classes q

$$q = -\sqrt{3}, -\frac{5}{6}\sqrt{3}, \ldots \frac{5}{6}\sqrt{3} \tag{2.18}$$

define the elements α_q of the one-dimensional $U(1)$ weight lattice by

$$\alpha_q = q + 2\sqrt{3}k \;\; (k \in Z) \tag{2.19}$$

The resulting level six $U(1)$ Kac-Moody algebra builds, together with two internal supercurrents and the energy momentum tensor, the internal $c = 9$, $n = 2$ superconformal algebra.

The NS vacuum in the universal sector has $h_{\rm int} = 0$, $Q = 0$ ($Q = \sqrt{3}\alpha_q$). The spectral flow (space-time supersymmetry) relates it to the R groundstate with $h_{\rm int} = 3/8$, $Q = \pm\frac{3}{2}$. In the fermionic description these two primary fields lead to a (holomorphic) vector resp. spinor the latter being the vertex operator of the four-dimensional supercharge. This massless universal sector of the theory is contained in the 0 conjugacy class of E_6 multiplied with the identity operator of the $\tilde{c} = 8$ conformal field theory. In addition there is a massive field in this sector corresponding to the $(0, \sqrt{3})$ conjugacy class of $D_5 \otimes U(1)$. It is obtained by acting twice with the spectral flow operator on the NS groundstate and corresponds to the vertex operator of the massive holomorphic 3-form field.

In the massless matter sector the NS primary fields have $h_{\rm int} = \frac{1}{2}$, $Q = \pm 1$ and are related by the spectral flow to the R fields with $h_{\rm int} = 3/8$, $Q = \pm\frac{1}{2}$. These fields appear in the 1 resp. $\bar{1}$ conjugacy class of E_6 and must be combined with conformal fields $\tilde{G}(z)$ with dimension $\tilde{h} = 1/3$.

The internal supercurrent of the string theory is the sum of the two supercharges $T_F^{\pm}$ of the internal $n = 2$ superconformal algebra:

$$T_F^{\pm}(z) = \exp\Big(\pm i\frac{\sqrt{3}}{3}H_{\rm int}\Big)\tilde{T}_F^{\pm}(z) \tag{2.20}$$

$\tilde{T}_F^{\pm}(z)$ are conformal field of the $\tilde{c} = 8$ conformal field theory with dimension $\tilde{h}_{int} = 4/3$. Comparison with eq.(18) clearly shows that the internal picture changing operator corresponds to weight vectors contained in the $(V, \pm\frac{\sqrt{3}}{3})$ conjugacy classes of $D_5 \otimes U(1)$ which appear in the decomposition of the 1, $\bar{1}$ conjugacy classes of E_6. Thus, just like in the E_7 case, states in the universal sector of the theory are mapped by the action of the internal supercurrent onto weight vectors of the 1, $\bar{1}$ conjugacy classes of E_6 now being multiplied with the $\tilde{h} = 4/3$ fields $\tilde{T}_F^{\pm}$. On the other hand, the matter sector finds its image under two-dimensional supersymmetry transformations in the 0 conjugacy class multiplied by fields of conformal dimension one in the $\tilde{c} = 8$ part of the theory. In the case where E_6 takes the role of a gauge symmetry, the world-sheet symmetry therefore implies that any massless state transforming under the $\underline{27}$ of E_6 is accompanied by a singlet under this group which builds the highest component of the common two-dimensional supermultiplet [45,46]. However, the reverse argument is not true. Under special circumstances there exist more gauge singlets than matter fields in the $\underline{27}$ of E_6. To understand this feature consider the picture changing operation acting on h massless (holomorphic) space-time scalars in the fermionic interpretation of E_6. Their vertex operators in the -1 ghost picture have the following form

$$V^a_{(-1)}(z) = e^{-\phi}\exp\Big(i\frac{\sqrt{3}}{3}H_{\rm int}\Big)\tilde{G}^a_{(-1)}(z) \quad a = 1, \dots, h \tag{2.21}$$

Acting with $P_F^-(z) \sim e^{\phi}\exp\Big(-i\frac{\sqrt{3}}{3}H_{\rm int}\Big)\tilde{T}_F^-(z)$ on $V^a_{(-1)}(w)$ leads to the scalar vertex operators in the 0 ghost picture:

$$V^a_{(0)}(z) \sim \tilde{G}^a_{(0)}(z) \tag{2.22}$$

$\tilde{G}^a_{(0)}(z)$ is a field of dimension one obtained from the operator product between $\tilde{T}^-_F(z)$ and $\tilde{G}^a_{(-1)}(w)$ (BRST invariance of $V^a_{(-1)}$ forbids a simple pole in its OPE with P^+_F):

$$\tilde{T}^-_F(z)\tilde{G}^a_{(-1)}(w) \sim \frac{1}{(z-w)^{2/3}}\tilde{G}^a_{(0)}(w) + \dots \tag{2.23}$$

In fact, $\exp\left(i\frac{\sqrt{3}}{3}H\right)\tilde{G}^a_{(-1)}(z)$ and $\tilde{G}^a_{(0)}(z)$ build together a two-dimensional superfield of the internal superconformal algebra. The BRST invariant operators $\tilde{G}^a_{(0)}(z)$ are the truly marginal fields of the internal (right-moving) superconformal field theory and correspond in a left-right symmetric model to the moduli [45,46] of the underlying Calabi-Yau manifold. It is however important to realize that for specific values of the moduli, the multicritical points of the superconformal field theory, there exist extra marginal operators. These massless scalars which are also singlets under E_6 are however not the highest components of two-dimensional superfields. Vacuum expectation values of these fields break the two-dimensional supersymmetry [47]. Therefore they do not correspond to moduli of the underlying Calabi-Yau manifold.

Let us illustrate the appearance of additional marginal operators by the untwisted sector of the well-known Z-orbifold [7]. $V^a_{(-1)}(z)$ can be written as

$$V^a_{(-1)}(z) = e^{-\phi}\psi_a(z) \tag{2.24}$$

where the $\psi_a(z)$ are the three internal complex fermions which can be represented in a $SU(3) \times U(1)$ basis as

$$\psi_a(z) = \exp\left(i\frac{\sqrt{3}}{3}H_{\rm int}\right)\exp\left(i\vec{w}_a\vec{H}_{\rm int}\right)(z) \tag{2.25}$$

$\vec{w}_a$ $(a = 1,2,3)$ are the weight vectors of the fundamental representation of $SU(3)$ normalized as $\vec{w}_a^2 = 2/3$. The internal supercurrent has the form

$$T^\pm_F(z) \sim \sum_{a=1}^{3}\psi^{\pm a}\partial X_{\mp a}(z) \tag{2.26}$$

The $X_{\mp a}$ are just the three complex coordinates of the underlying six-torus. Then the vertex operators in the 0 ghost picture are simply

$$\tilde{G}^{a}_{(0)}(z) \sim \partial X_{-a}(z) \tag{2.27}$$

As discussed in [48,49], for special values of the background parameters (metric and anti-symmetric tensor fields) the lattice of the compactified momenta contains root vectors $\vec{\alpha}_a^b$ satisfying $\vec{\alpha}^2 = 2$. Then the orbifold coordinates $\partial X_{\pm a}$ can be rebosonized in terms of six free bosons $Y_i(z)$ $(i = 1, \ldots, 6)$ like:

$$\partial X_a(z) \sim \sum_b C_a^b \exp i\vec{\alpha}_a^b \cdot \vec{Y}(z) \tag{2.28}$$

Now, any dimension one operator $\exp i\vec{\alpha}_a^b \vec{Y}(z)$ is itself not the highest component of a two-dimensional superfield and therefore does not correspond to a BRST invariant operator in the fermionic string. However, in the bosonic description these operators correspond to physical states and lead to massless gauge singlet scalar fields in the heterotic Z-orbifold. In addition to these massless scalars there exist also massless gauge bosons (in addition to the E_6 gauge bosons) corresponding to the new Cartan subalgebra generators $\partial Y_i(z)$. For arbitrary background fields, the additional gauge bosons as well as scalars become massive and realize together the stringy Higgs effect.

3. Space-time properties

In this chapter we will list the various possible supergravity theories arising as low energy field theories in heterotic and type II (A and B) string theories. We will be discussing model independent features only; building concrete string models requires the specification of the internal conformal field theory which of course has to pass additional consistency requirements such as modular invariance. We restrict ourselves to looking at the massless spectrum [50,51] for which only the singlet, adjoint and fundamental representations of the exceptional groups are relevant. As discussed in the previous chapter we need the regular embeddings of $SO(6+d)$ in E_{6+i}; the following possibilities arise: (i) $d = 4$: $E_6 \subset SO(10) \times U(1)$, $E_7 \subset SO(10) \times SU(2) \times SO(2)$ and $E_8 \subset SO(10) \times SO(6)$; (ii) $d = 6$: $E_7 \subset SO(12) \times SU(2)$ and $E_8 \subset SO(12) \times SO(4)$; (iii) $d = 8$: $E_8 \subset SO(14) \times SO(2)$. For $d = 10$ we get of course the ordinary heterotic string with $N = 1$ supersymmetry or the type II (A and B) superstring with $N = 2$ supersymmetry. Below we give the relevant decompositions of the fundamental and adjoint representations of E_{6+i} after the truncation to the light-cone degrees of freedom $SO(d+6) \times g \to SO(d-2) \times g$ where $SO(d-2)$ is the little group for massless states in d dimensions.

$$
\begin{aligned}
\underline{248}_{E_8} &\to \begin{cases} [(\underline{6},0)+(\underline{1},\pm 1)+(\bar{\underline{4}},+\frac{1}{2})+(\underline{4},-\frac{1}{2})]_{SO(6)\times SO(2)} \\ [(\underline{4},\underline{1})+(\underline{1},\underline{4})+(\bar{\underline{2}},\underline{2})+(\underline{2},\bar{\underline{2}})]_{SO(4)\times SO(4)} \\ [(\pm 1,\underline{1})+(0,\underline{6})+(-\frac{1}{2},\underline{4})+(+\frac{1}{2},\bar{\underline{4}})]_{SO(2)\times SO(6)} \end{cases} \\
\underline{133}_{E_7} &\to \begin{cases} [(\underline{4},\underline{1})+(\underline{2},\underline{2})]_{SO(4)\times SU(2)} \\ [(\pm 1,\underline{1},0)+(0,\underline{1},\pm 1)+(-\frac{1}{2},\underline{2},-\frac{1}{2})+(+\frac{1}{2},\underline{2},+\frac{1}{2})]_{SO(2)\times SU(2)\times SO(2)} \end{cases} \\
\underline{56}_{E_7} &\to \begin{cases} [(\underline{1},\underline{2})+(\bar{\underline{2}},\underline{1})]_{SO(4)\times SU(2)} \\ [(0,\underline{2},0)+(-\frac{1}{2},\underline{1},+\frac{1}{2})+(+\frac{1}{2},\underline{1},-\frac{1}{2})]_{SO(2)\times SU(2)\times SO(2)} \end{cases} \\
\underline{78}_{E_6} &\to [(\pm 1,0)+(-\frac{1}{2},-\frac{\sqrt{3}}{2})+(+\frac{1}{2},\frac{\sqrt{3}}{2})]_{SO(2)\times U(1)} \\
\underline{27}_{E_6} &\to [(0,\frac{1}{\sqrt{3}})+(-\frac{1}{2},\frac{\sqrt{3}}{6})]_{SO(2)\times U(1)}
\end{aligned}
\tag{3.1}
$$

For case $d=4$ we have given the helicity of the physical states.

The appearance of the exceptional groups not only guarantees a space-time supersymmetric spectrum; it also gives information about the supersymmetry multiplets into which the states fall. This information is however not complete, as some of the multiplets are reducible representations of supersymmetry. However, from a string point of view these reducible multiplets are more natural. The appearance of some of the multiplets is universal. In the heterotic theories they arise from the combination $(\bar{\partial}X^\mu)_L \times (\text{adjoint of } E_{6+i})_R$ whereas in type II theories they are contained in $(\text{adjoint of } E_{6+i})_L \times (\text{adjoint of } E_{6+j})_R$ both multiplied with the identity operator of the internal conformal field theory. For heterotic strings only the right-moving (holomorphic) realization of the exceptional groups bears any significance for the occurence of space-time supersymmetry. Also, e.g. the number of matter multiplets in $N=1$ theories is model dependent. Let us now treat the various cases in turn. In what follows, all fields will be real unless stated otherwise.

Heterotic string: In addition to the universal sector we also have the gauge sector which arises from $(\text{gauge-current})_L \times$ $(\text{adjoint of } E_{6+i})_R$ and a matter sector $(\Phi)_L \times (\text{fundamental of } E_{6+i})_R$. Here Φ describes the transformation properties of the states under the left-moving internal symmetries and the fundamental representation of E_{6+i} has to be multiplied by a field $\tilde{G}(z)$ in the internal coformal field theory. Note that the combination $(\bar{\partial}X^\mu)_L \times (\text{fundamental of } E_{6+i})$ leads to extra gravitini and hence to a larger exceptional group. $(\Phi)_L \times (\text{adjoint of } E_{6+i})$ gives extra gauge bosons. Since we assume that the universal and gauge sectors are complete we do not have to consider these cases. The vectors in the universal sector are all associated with central charges of the supersymmetry algebra. The holomorphic part of their vertex operators coincides with the central charges whose $q = -1$ and $q = 0$ versions are the components of a real free $(0, \frac{1}{2})$ superfield, each of which generates a $SO(2)$ super-Kac-Moody algebra. The presence of d' central charges in the d dimensional theory indicates that it is is a holomorphic d'-torus compactification of a $d + d' = d_{max}$ dimensional theory and the central charges are nothing but the internal momenta.* They are gauged by the vectors in the universal sector. Obviously, only fields with internal torus excitations are charged under these $SO(2)$ symmetries; in particular, massless fields are neutral.

Of particular interest are the "left-right symmetric" heterotic string theories with (n, n) world-sheet supersymmetry. They have (like the type II theories) the structure $(E_{6+i})_L \times (E_{6+i})_R$ multiplied with an internal conformal field theory with $\tilde{c}_L = \tilde{c}_R = 15 - \frac{3}{2} - k$. These theories correspond to compactifications on Ricci flat manifolds with $SU(n)$-holonomy [33,35]. Obviously, the gauge sector now follows from (adjoint of $E_{6+i})_L \times$(adjoint of $E_{6+i})_R$, the left-moving gauge group always contains $E_{6+i} \times E_8$. The matter sector is obtained by the product of the fundamental representations of $(E_{6+i})_L$ and $(E_{6+i})_R$ multiplied with internal conformal fields $\tilde{G}(z, \bar{z})$. Thus the number of matter fields in the fundamental representation of E_{6+i} is determined by the number of $\tilde{G}(z, \bar{z})'$s and reflects the topological properties of the underlying manifolds. These fields will be always accompanied by at least the same number of singlets due to the left- moving world-sheet supersymmetry. Therefore this part of the matter fields which corresponds to the moduli of the underlying manifold is contained in the sector (singlet of $E_{i+6})_L \times$ (fundamental of $E_{6+i})_R$. It follows that the partition function of these types of left-right symmetric heterotic string theories is not diagonal in the product of the left and right characters [52,53,21] of the exceptional groups. (The same argument holds of course for the corresponding type II theories.)

$d = 8$: The only possibility leading to space-time supersymmetry is the occurrence of E_8 in the left-moving sector. We get the following $N = 1$ multiplets:

universal sector : gravity multiplet: $(g_{\mu\nu}, A_{\mu\nu}, \phi, 2A_\mu, \psi_\mu, \lambda)$

---

* In fact, the massless multiplets of all theories in $d < d_{max}$ based on a particular exceptional group are simply dimensional reductions of the theory in d_{max}. This also applies to type II theories and to theories in odd dimensions.

gauge sector : vector multiplet: $(A_\mu^a, \lambda^a, 2\phi^a)$

All fermions are Majorana. The fields in the gauge sector transform in the adjoint of the gauge group. There is no matter sector. Here and in the following we use a notation where $g_{\mu\nu}$ denotes a graviton, $A_{\mu_1\ldots\mu_n}$ a rank n antisymmetric tensor, ϕ a scalar, ψ_μ a gravitino (spin 3/2) and λ a spin 1/2 field. From our general discussion it follows from the presence of two vectors, the graviphotons, in the universal sector that $N = 1$, $d = 8$ theories are holomorphic torus compactifications of $N = 1$, $d = 10$ theories. Note that there is no increase in the number of gravitini as they are Majorana-Weyl in $d = 10$ and Majorana in $d = 8$.

$d = 6$: There are two possibilities:
(i) $E_7 \to N = 1$ supersymmetry:

universal sector : gravity multiplet: $(g_{\mu\nu}, \psi_{\mu Li}, A_{\mu\nu}^{(-)})$

antisymmetric tensor multiplet: $(A_{\mu\nu}^{(+)}, \lambda_{Ri}, \phi)$

gauge sector : vector multiplet: $(A_\mu^a, \lambda_{Li}^a)$

matter sector : hypermultiplet: (λ_R, ϕ_i)

i is an $SU(2)$ index. The spinors in the universal and the gauge sectors are $SU(2)$ Majorana-Weyl. The fields in the matter sector are complex, i.e. the hypermultiplet consists of four real scalars and two Majorana spinors. In supersymmetric field theory this is required by CPT invariance; in string theory it follows from the fact that the vertex operators for the massless hypermultiplet contain a complex field of conformal dimension $\frac{1}{4}$ which lives in the internal right-moving $\hat{c} = 5$ conformal field theory. We want to point out that the connection with the representations of E_7 leads to the correct chirality assignments for the various spinors as required by supersymmetry; i.e. not only the multiplet structure but also some necessary properties of their interactions can be obtained. Anomalies (gravitational, gauge and mixed) can be cancelled by the Green-Schwarz mechanism if one for instance considers compactifications of the ten-dimensional theory on K_3. In this case we have $(E_7)_L \times (E_7)_R$ and there must be 20 matter hypermultiplets which transform as $\underline{56}$ of E_7 [39]. They are accompanied by 20 hypermultiplets which are singlet under E_7 containing the 80 moduli of the K_3 manifold with torsion.
(ii) $E_8 \to N = 2$ supersymmetry:

universal sector : gravity multiplet: $(g_{\mu\nu}, \psi^i_{\mu L}, \psi_{\mu Ri}, A^\pm_{\mu\nu}, A_\mu{}^i{}_j, \lambda_{Li}, \lambda^i_R, \phi)$

gauge sector : vector multiplet: $(A_\mu^a, \lambda^a_{Ri}, \lambda^{ai}_L, \phi^{ai}{}_j)$;

There is no matter sector. This is the multiplet structure of the non-chiral type IIA theory in six dimensions (cf. below). Upper and lower indices i, j refer to the two internal $SU(2) \times SU(2) \sim SO(4)$ and the subscripts L, R denote left- and right-handed spinors respectively. The labels for the antisymmetric tensors indicate that their field strengths are selfdual (+) or anti-selfdual (−). All spinors are $SU(2)$ Majorana-Weyl. The presence of the four vectors in the universal sector stems from the fact that the theories are holomorphic torus compactifications of the $N = 1$, $d = 10$ theory. Furthermore, if one considers a true (left plus right) torus compactification [8,55] of the ten-dimensional string theory the rank of the gauge group must be 20 so that there are 20 vector multiplets representing the Cartan subalgebra. They contain the 80 moduli which descibe the torus compactification. In fact, the moduli space of the torus compactification coincides with the one of the K_3 manifold [39].

$d = 4$: there exist three possibilities for arriving at supersymmetric theories:

(i) $E_8 \to N = 4$ supersymmetry:

$$\begin{aligned} \text{universal sector}: &\ \text{gravity multiplet: } (g_{\mu\nu}, A_{\mu\nu}, \psi^i_\mu, A^{[ij]}_\mu, \lambda_i, \phi) \\ \text{gauge sector}: &\ \text{vector multiplet: } (A^a_\mu, \lambda^{ai}, \phi^{a[ij]}) \end{aligned}$$

There are no matter fields. Upper and lower indices refer to the two fundamental representations of $SU(4)$ respectively. These theories are holomorphic torus compactifications of $N = 1$, $d = 10$ theories and for the true torus compactifications the rank of the gauge group is 22.

(ii) $E_7 \to N = 2$ supersymmetry:

$$\begin{aligned} \text{universal sector}: &\ \text{gravity multiplet: } (g_{\mu\nu}, \psi^i_\mu, A_\mu) \\ &\ \text{vector multiplet: } (A_\mu, \lambda^i, 2\phi) \\ \text{gauge sector}: &\ \text{vector multiplet: } (A^a_\mu, \lambda^{ia}, \phi^a) \\ \text{matter sector}: &\ \text{hypermultiplet: } (\lambda, \phi^i) \end{aligned}$$

i is again an $SU(2)$ index. The pseudoscalar component of the complex scalar in the universal sector is the duality transformed antisymmetric tensor, which always accompanies, together with the dilaton, the graviton. The splitting of the universal sector into a gravity and a vector multiplet is very unnatural from the string point of view. In fact, the vectors in these multiplets are linear combinations of the two vectors which are present because the $N = 2$ heterotic theories in $d = 4$ are holomorphic torus compactifications of the $N = 1$ theories in $d = 6$ discussed above. This also entails that the fields in the hypermultiplet are again complex.

(iii) $E_6 \to N = 1$ supersymmetry:

$$\begin{aligned}
\text{universal sector}: &\ \text{gravity multiplet: } (g_{\mu\nu}, \psi_\mu) \\
&\ \text{linear multiplet: } (A_{\mu\nu}, \lambda, \phi) \\
\text{gauge sector}: &\ \text{vector multiplet: } (A^a_\mu, \lambda^a) \\
\text{matter sector}: &\ \text{chiral multiplets: } (\lambda, 2\phi)
\end{aligned}$$

The gauge groups and the number of matter multiplets (if present) are model dependent. In chiral theories there are constraints from anomaly freedom. For the case of symmetric (2, 2) Calabi-Yau compactification the matter sector contains $h_{1,1}$ chiral multiplets in $\underline{27}$ of E_6 and $h_{2,1}$ chiral multiplets in the $\overline{\underline{27}}$ representation ($h_{1,1}$ ($h_{2,1}$) are the number of harmonic (1, 1) ((2, 1))-forms on the Calabi-Yau manifold). These fields originate from the products $\underline{27}_L \times \overline{\underline{27}}_R + c.c$ resp. $\underline{27}_L \times \underline{27}_R + c.c$ multiplied by $h_{1,1}$ resp. $h_{2,1}$ different fields $\tilde{G}(z, \bar{z})$. In addition there are at least $h_{1,1} + h_{2,1}$ E_6 singlet chiral multiplets containing the moduli of the Calabi-Yau manifold.

This exhausts all the possible multiplets appearing in the massless spectrum of heterotic string theories.

*Type II string:** There are more possibilities than in the heterotic case as we can now get gravitini from both the holomorphic and the antiholomorphic sectors. That also means that we have to consider the combinations $(E_{6+i})_R \times (E_{6+j})_L$ where $i, j = 0, 1, 2$. The universal sector now arises from (adjoint of $E_{6+i})_R \times$ (adjoint of $E_{6+j})_L$. Especially, it is an immediate consequence of the connection with the representations of the exceptional groups that theories with extended left- and right-moving world-sheet symmetries which lead to extended space-time symmetries, have an abelian gauge group $U(1)^n$. There are two types of graviphotons, those from the (R,R) sector and those from the (NS,NS) sector. The latter are always associated with central charges of the supersymmetry algebra each of which gives rise to a $(\frac{1}{2}, 0)$ (or $(0, \frac{1}{2})$) real free superfield, analogous to the heterotic case. All massless particles are neutral with respect to $U(1)$'s generated by them (cf. above) as well as with respect to the $U(1)$'s generated by the (R,R) vectors. Here is an important difference between heterotic and type II theories. Whereas in the former theories all vectors of the universal sector are associated with central charges of the supersymmetry algebra, this is only true for some of them, namely the (NS,NS) vectors of the latter theories. Since the multiplet structure for the massless fields is the same in both cases (given the same N, d and chirality assignment) we conclude that the number of central charges must be

* Type II strings in dimensions lower than ten were discussed in [17,16,14,54].

different. This is in fact trivial to see from the conformal field theory point of view. In the heterotic case all supercharges are holomorphic whereas in the type II theories there are both holomorphic and anti-holomorphic supercharges. Clearly these two sets anticommute with each other and central charges can only arise within each set, thus reducing their total number as compared to the heterotic case with the same N. This in particular means that the supermultiplet structure of the massive states will be different for heterotic and type II theories [56].

The matter sector arises from $(\text{fundamental of} E_{6+i})_L \times (\text{fundamental of} E_{6+j})_R$ multiplied by $\tilde{G}(z,\bar{z})$. The scalars can, like the vectors, arise in the (NS,NS) and (R,R) sectors. Since in the bosonic theory the moduli are associated with singlets of E_{6+i} it follows that in type II theories they correspond to (NS,NS) scalars. Combinations of the form $(\text{fundamental of} E_{6+i})_L \times (\text{adjoint of} E_{6+j})_R$ multiplied with $\tilde{G}(\bar{z})$ in the left part and by the identity operator in the right part of the internal conformal field theory always lead to extra gravitini and the multiplets combine to fall into representations of a larger exceptional group. In addition to the theories mentioned so far there is also the possibility of having unextended superconformal symmetry in one of the sectors. We then get the combination $(D_{\frac{d}{2}+3})_R \times (E_{6+i})_L$. Only these theories can lead to chiral fermions in $d=4$ (for $i=0$) and non-abelian gauge groups with gauge bosons from the (NS,NS) sector. Again, (R,R) gauge bosons only generate abelian gauge groups. The multiplet structure of these theories is very similar to the corresponding heterotic theories. The universal sector is contained in $(\text{adjoint of} D_{\frac{d}{2}+3})_L \times (\text{adjoint of} E_{6+i})_R$ whereas $(\text{vector of} D_{\frac{d}{2}+3})_L \times (\text{adjoint of} E_{6+i})_R$ generates the gauge sector which lead to the same multiplets as the heterotic theories. Only the matter sector differs. It consists of two parts: $(\text{vector of} D_{\frac{d}{2}+3})_L \times (\text{fundamental of} E_{6+i})_R$ leads to the fundamental supersymmetry multiplets and coincides with the heterotic matter sector. In addition we have the combination $(\text{spinor of} D_{\frac{d}{2}+3})_L \times (\text{fundamental of} E_{6+i})_R$, which leads to extra (R,R) gauge bosons. All other combinations are excluded as they give additional gravitini. Let us now go through the different cases.

$d=8$: there are only two possibilities:
(i) $D_7 \times E_8 \rightarrow N=1$ supersymmetry with the same multiplets as the $d=8$ heterotic theory.

(ii) $E_8 \times E_8 \rightarrow N=2$ supersymmetry:

$\text{universal sector}:$ gravity multiplet: $(g_{\mu\nu}, 3A_{\mu\nu}, 6A_\mu, 7\phi, 2\psi_\mu, 6\lambda)$

All fields are real, i.e. fermions are Majorana.

$d = 6$: there are four possibilities:

(i) $D_6 \times E_8 \rightarrow N = 2$ supersymmetry: cf. $N = 2$ heterotic.

(ii) $D_6 \times E_7 \rightarrow N = 1$ supersymmetry: the universal and gauge sectors coincide with the $N = 1$ heterotic theory; the matter sector contains hypermultiplets and additional antisymmetric tensor and $U(1)$ gauge multiplets with (R,R) vectors.

(iii) $E_7 \times E_7 \rightarrow N = 2$ supersymmetry: these theories have $(4,4)$ world-sheet supersymmetry, so we assume that they describe K_3 compactifications of the ten-dimensional supersymmetric type II theories (compare with ref.[39]). We have to distinguish between two types, namely type IIB and IIA, depending on whether the representations of the transverse Lorentz group in the left- and right-moving sectors are chosen to be the same or reversed.

$$\begin{aligned} \mathrm{IIB}:\ \text{universal sector}: &\ \text{chiral gravity multiplet: } (g_{\mu\nu}, 5A^{(+)}_{\mu\nu}, 4\psi_{\mu R}) \\ &\ \text{antisymmetric tensor multiplet: } (A^{(-)}_{\mu\nu}, 5\phi, 2\lambda_L). \\ \text{matter sector}: &\ 20 \text{ antisymmetric tensor multiplets: } (A^{(-)}_{\mu\nu}, 5\phi, 2\lambda_L). \end{aligned}$$

Again, as in the heterotic case, the fermions are symplectic Majorana-Weyl.

IIA : here the universal sector is identical to the one of the E_8 heterotic theory and the matter sector contains 20 vector multiplets originating from 20 fields $\tilde{G}(z, \bar{z})$ of conformal dimension $(\frac{1}{4}, \frac{1}{4})$.

Note that all vectors of the type IIa and IIB theories are (R,R).

(iv) $E_7 \times E_8 \rightarrow N = 3$ supersymmetry:

$$\text{universal sector}: \text{ gravity multiplet: } (g_{\mu\nu}, 5A^{(+)}_{\mu\nu}, A^{(-)}_{\mu\nu}, 8A_\mu, 5\phi, 4\psi_{\mu L}, 2\psi_{\mu R}, 4\lambda_R, 10\lambda_L).$$

This theory is clearly chiral and the anomalies must be cancelled. Of the eight vectors, four are of the (R,R) and four of the (NS,NS) type. The latter are the components of four real right-moving free superfields whose presence is equivalent to the presence of four central charges in the holomorphic sector of the operator product algebra of the supercharges. If we add the matter sector we automatically enlarge the supersymmetry to $N = 4$. That means that the internal left-moving conformal field theory must not lead to massless fields in the matter sector in order to have $N = 3$ supersymmetry.

(v) $E_8 \times E_8 \rightarrow N = 4$ supersymmetry:

$$\text{universal sector}: \text{ gravity multiplet: } (g_{\mu\nu}, 5A^{(+)}_{\mu\nu}, 5A^{(-)}_{\mu\nu}, 16A_\mu, 25\phi, 4(\psi_{\mu R}, \psi_{\mu L}), 20(\lambda_R, \lambda_L)).$$

This theory is non-chiral. Eight of the graviphotons are (NS,NS) and are again components of free superfields, four holomorphic and four antiholomorphic ones, whose components correpond to the central charges in the $q=-1$ and $q=0$ ghost picture.

$d=4$: there are nine possibilities altogether:

(i) $D_5 \times E_6 \rightarrow N=1$ supersymmetry: cf. $N=1$ heterotic theory, with additional $U(1)$ gauge multiplets in the matter sector.

(ii) $D_5 \times E_7 \rightarrow N=2$ supersymmetry: here the same modifications with respect to the corresponding heterotic theory as above apply.

(iii) $D_5 \times E_8 \rightarrow N=4$ supersymmetry: this leads to the same multiplet structure as the $N=4$ heterotic theory.

(iv) $E_6 \times E_6 \rightarrow N=2$ supersymmetry:
These theories have $(2,2)$ world-sheet supersymmetry and correspond to Calabi-Yau compactifications of the ten-dimensional theory (compare also with ref.[40]). The universal sector consists of a gravity and a hypermultiplet. For the type IIA theories, the matter sector gives vector multiplets from $\underline{27}_L \times \overline{\underline{27}}_R + c.c.$ multiplied by fields $\tilde{G}(z,\bar{z})$ of conformal dimension $(\frac{1}{3},\frac{1}{3})$, of which there are $h_{1,1}$. All vectors are (R,R). The $h_{1,1}$ complex scalars of these vectormultiplets correspond to the moduli of the Calabi-Yau manifold. On the other hand, the product $\underline{27}_L \times \underline{27}_R + c.c$ leads to $h_{2,1}$ hypermultiplets containing an equal number of moduli of the Calabi-Yau manifold. For the type IIB theories, the vector and hypermultiplets in the matter sector are simply exchanged.

(v) $E_6 \times E_7 \rightarrow N=3$ supersymmetry:*

$$\begin{aligned}
\text{universal sector}: \ & \text{gravity multiplet: } (g_{\mu\nu}, 3A_\mu, 3\psi_\mu, 3\lambda) \\
& \text{vector multiplet: } (A_\mu, 6\phi, 4\lambda) \\
\text{matter sector}: \ & \text{vector multiplets}
\end{aligned}$$

Two of the graviphotons are (NS,NS). The other graviphotons as well as the vectors in the vector multiplets are (R,R). The number of vector multiplets in the matter sector is model dependent.

(vi) $E_6 \times E_8 \rightarrow N=5$ supersymmetry:

$$\text{universal sector}: \ \text{gravity multiplet: } (g_{\mu\nu}, 10A_\mu, 10\phi, 5\psi_\mu, 11\lambda)$$

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* Within the fermionic construction, $d=4$ type II superstring theories with $N=3,5$ and 6 space-time supersymmetry have been considered by S. Ferrara and C. Kounnas (private communication).

Six of the graviphotons are (NS,NS), again reflecting the presence of central charges. The matter sector must again not lead to massless states since this would generate extra gravitini which would result in the $N = 8$ theory.

(*vii*) $E_7 \times E_7 \rightarrow N = 4$ supersymmetry:

$$\begin{aligned} \text{universal sector:}\ & \text{gravity multiplet: } (g_{\mu\nu}, 6A_\mu, 2\phi, 4\psi_\mu, 4\lambda) \\ & \text{2 vector multiplets: } 2(A_\mu, 6\phi, 4\lambda) \\ \text{matter sector:}\ & \text{vector multiplets.} \end{aligned}$$

Four of the graviphotons are (NS,NS), all other vectors are (R,R).

(*viii*) $E_7 \times E_8 \rightarrow N = 6$ supersymmetry:

$$\text{universal sector: gravity multiplet: } (g_{\mu\nu}, 16A_\mu, 30\phi, 6\psi_\mu, 26\lambda).$$

Eight of the vectors are (NS,NS). Again, massless fields from the $(\underline{56}_{E_7})_L \times (\underline{248}_{E_8})_R$ would lead to extra gravitini.

(*ix*) $E_8 \times E_8 \rightarrow N = 8$ supersymmetry:

$$\text{universal sector: gravity multiplet: } (g_{\mu\nu}, 28A_\mu, 70\phi, 8\psi_\mu, 56\lambda)$$

with 12 (NS,NS) graviphotons.

4. Summary

We have reviewed the connection between space-time and world-sheet symmetries in heterotic and type II string theories with special emphasis on the role of the exceptional groups as the unifying symmetries. Their appearance in both the bosonic as well as the fermionic sectors has been discussed. We have seen how the presence of extended world-sheet symmetries ($n \geq 1\, SCA$) implies the existence of the picture-changing operator in the fermionic case, which interpolates between equivalent states. In the bosonic case the equivalent operator maps the massless fields transforming under the fundamental representation of the exceptional gauge group, whose existence is guaranteed by the extended world-sheet symmetry, to gauge singlets, which, in the case of bosons, correspond to the moduli of the compactification space. Finally, we have given the supermultiplet structure of the massless states of all possible heterotic and type II string theories. Here the exceptional groups contain all the information about the number of supersymmetries and the corresponding supermultiplet structure. Due to the different number of central charges in the type II and heterotic theories their multiplet structure will differ once we consider massive multiplets. This is already suggested by the different way in which the internal symmetry which labels supersymmetry multiplets is realized in the two cases [56]. In four dimensions, $N = 1, 2, 3, 4, 5, 6, 8$ supersymmetries are possible, however $N = 7$ is excluded in analogy with supersymmetric field theory. We like to emphasize that our considerations are totally model independent, i.e. independent of any particular way to construct string theories in arbitrary dimensions.

ACKNOWLEDGEMENTS

We wish to thank C. Kounnas and B. Schellekens for helpful comments and especially S. Ferrara for valuable discussions.

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