Progress of Theoretical Physics, Vol. 61, No. 4, April 1979

Excitation-Modes and Their Interactions in a Many-Boson System. II

Shuichiro YAMASAKI,*) Takeji KEBUKAWA and Sigenobu SUNAKAWA

Department of Physics, College of General Education Osaka University, Osaka 560

(Received November 4, 1978)

By making use of the expressions for ρ_{-k} and $\boldsymbol{g}_k$ obtained in the preceding paper, the interaction Hamiltonian among the excitations described by the operators b_l and $b_l{}^*$ is derived up to the order of $N^{-3/2}$. It is shown that all interaction terms have convergent forms and they always give finite corrections to all physical quantities.

§ 1. Introduction

In the preceding paper[1] (which is hereafter referred to as A), the operators ρ_{-k} and $\boldsymbol{g}_k$ have been written in terms of the operators b_l and $b_l{}^*$ up to order of $N^{-3/2}$ by taking account of the A-1-type nature of the density-fluctuation operator and the momentum-density operator. The results are expressed in the following way. The density-fluctuation operator ρ_{-k} is written as

$$\frac{1}{\sqrt{N\lambda_k}}\rho_{-k}=b^*_{-k}+b_k+\frac{1}{\sqrt{N}}\sum_{p,q}\sigma_k{}^{11}(p;q)\,b_p{}^*b_{-q}+\frac{1}{N}\sum_{p,q,r}\sigma_k{}^{21}(p,q;r)\times(b_p{}^*b_q{}^*b_{-r}+b_r{}^*b_{-q}b_{-p})$$
$$+\left(\frac{1}{\sqrt{N}}\right)^3\{\sum_{p,q,r,s}\sigma_k{}^{22}(p,q;r,s)\,b_p{}^*b_q{}^*b_{-r}b_{-s}+\sum_{p,q}\Delta\sigma_k{}^{11}(p;q)\,b_p{}^*b_{-q}\}+\cdots, \qquad (1\cdot1)$$

where

$$\sigma_k{}^{11}(p,q)=\delta_{0,k+p+q}\sqrt{\lambda_k\lambda_p\lambda_q}\,,$$
$$\sigma_k{}^{21}(p,q;r)=\frac{1}{4}\delta_{0,k+p+q+r}\sqrt{\lambda_k\lambda_p\lambda_q\lambda_r}\,(\eta_{k+p}+\eta_{k+q}),$$
$$\sigma_k{}^{22}(p,q;r,s)=\frac{1}{8}\delta_{0,k+p+q+r+s}\sqrt{\lambda_k\lambda_p\lambda_q\lambda_r\lambda_s}\{x(p,q;r,s,k)+x(r,s;p,q,k)\} \qquad (1\cdot2)$$

and

*) College of Bio-Medical Technology, Osaka University, Osaka 560.

$$\varDelta\sigma_{\boldsymbol{k}}^{11}(\boldsymbol{p};\boldsymbol{q})=\frac{\Omega}{2}\delta_{0,\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}}\sqrt{\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}}\,y(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}).$$

The momentum-density operator $\boldsymbol{g}_{\boldsymbol{k}}$ is represented as

$$\begin{aligned}\frac{\boldsymbol{g}_{\boldsymbol{k}}}{\hbar\sqrt{N}}=&\frac{\boldsymbol{k}}{2\sqrt{\lambda_{\boldsymbol{k}}}}(b_{\boldsymbol{k}}-b^*_{-\boldsymbol{k}})+\frac{1}{\sqrt{N}}\sum_{\boldsymbol{p},\boldsymbol{q}}\{\boldsymbol{\gamma}_{\boldsymbol{k}}^{11}(\boldsymbol{p};\boldsymbol{q})b_{\boldsymbol{p}}^*b_{-\boldsymbol{q}}\\&+\boldsymbol{\gamma}_{\boldsymbol{k}}^{20}(\boldsymbol{p},\boldsymbol{q})(b_{\boldsymbol{p}}^*b_{\boldsymbol{q}}^*-b_{-\boldsymbol{q}}b_{-\boldsymbol{p}})\}\\&+\frac{1}{N}\sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}\boldsymbol{\gamma}_{\boldsymbol{k}}^{21}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{r})(b_{\boldsymbol{p}}^*b_{\boldsymbol{q}}^*b_{-\boldsymbol{r}}-b_{\boldsymbol{r}}^*b_{-\boldsymbol{q}}b_{-\boldsymbol{p}})\\&+\left(\frac{1}{\sqrt{N}}\right)^3\sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{s}}\{\boldsymbol{\gamma}_{\boldsymbol{k}}^{31}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r};\boldsymbol{s})(b_{\boldsymbol{p}}^*b_{\boldsymbol{q}}^*b_{\boldsymbol{r}}^*b_{-\boldsymbol{s}}-b_{\boldsymbol{s}}^*b_{-\boldsymbol{r}}b_{-\boldsymbol{q}}b_{-\boldsymbol{p}})\\&+\boldsymbol{\gamma}_{\boldsymbol{k}}^{22}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{r},\boldsymbol{s})b_{\boldsymbol{p}}^*b_{\boldsymbol{q}}^*b_{-\boldsymbol{r}}b_{-\boldsymbol{s}}\}\\&+\left(\frac{1}{\sqrt{N}}\right)^3\sum_{\boldsymbol{p},\boldsymbol{q}}\varDelta\boldsymbol{\gamma}_{\boldsymbol{k}}^{20}(\boldsymbol{p},\boldsymbol{q})(b_{\boldsymbol{p}}^*b_{\boldsymbol{q}}^*-b_{-\boldsymbol{q}}b_{-\boldsymbol{p}}),\end{aligned}\tag{1·3}$$

where

$$\boldsymbol{\gamma}_{\boldsymbol{k}}^{11}(\boldsymbol{p};\boldsymbol{q})=\frac{1}{2}\delta_{0,\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}}\sqrt{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}}\left(\frac{\boldsymbol{p}}{\lambda_{\boldsymbol{p}}}-\frac{\boldsymbol{q}}{\lambda_{\boldsymbol{q}}}\right),$$

$$\boldsymbol{\gamma}_{\boldsymbol{k}}^{20}(\boldsymbol{p},\boldsymbol{q})=-\frac{1}{4}\delta_{0,\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}}\sqrt{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}}\left(\frac{\boldsymbol{p}}{\lambda_{\boldsymbol{p}}}\eta_{\boldsymbol{p}}+\frac{\boldsymbol{q}}{\lambda_{\boldsymbol{q}}}\eta_{\boldsymbol{q}}\right),$$

$$\begin{aligned}\boldsymbol{\gamma}_{\boldsymbol{k}}^{21}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{r})=&\frac{1}{4}\delta_{0,\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}}\sqrt{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{r}}}\left\{\frac{\boldsymbol{k}}{2}(\eta_{\boldsymbol{k}+\boldsymbol{p}}+\eta_{\boldsymbol{k}+\boldsymbol{q}})+\frac{\boldsymbol{p}}{\lambda_{\boldsymbol{p}}}(\eta_{\boldsymbol{k}+\boldsymbol{p}}-\eta_{\boldsymbol{p}})\right.\\&\left.+\frac{\boldsymbol{q}}{\lambda_{\boldsymbol{q}}}(\eta_{\boldsymbol{k}+\boldsymbol{q}}-\eta_{\boldsymbol{q}})\right\},\end{aligned}$$

$$\begin{aligned}\boldsymbol{\gamma}_{\boldsymbol{k}}^{31}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r};\boldsymbol{s})=&\frac{1}{24}\delta_{0,\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}+\boldsymbol{s}}\sqrt{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{r}}\lambda_{\boldsymbol{s}}}\left\{\boldsymbol{k}x(\boldsymbol{k},\boldsymbol{s};\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\right.\\&+\frac{\boldsymbol{p}}{\lambda_{\boldsymbol{p}}}(\eta_{\boldsymbol{k}+\boldsymbol{p}}-\eta_{\boldsymbol{p}})(\eta_{\boldsymbol{q}+\boldsymbol{s}}+\eta_{\boldsymbol{r}+\boldsymbol{s}})+\frac{\boldsymbol{q}}{\lambda_{\boldsymbol{q}}}(\eta_{\boldsymbol{k}+\boldsymbol{q}}-\eta_{\boldsymbol{q}})(\eta_{\boldsymbol{p}+\boldsymbol{s}}+\eta_{\boldsymbol{r}+\boldsymbol{s}})\\&\left.+\frac{\boldsymbol{r}}{\lambda_{\boldsymbol{r}}}(\eta_{\boldsymbol{k}+\boldsymbol{r}}-\eta_{\boldsymbol{r}})(\eta_{\boldsymbol{p}+\boldsymbol{s}}+\eta_{\boldsymbol{q}+\boldsymbol{s}})\right\},\end{aligned}$$

$$\begin{aligned}\boldsymbol{\gamma}_{\boldsymbol{k}}^{22}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{r},\boldsymbol{s})=&\frac{1}{16}\delta_{0,\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}+\boldsymbol{s}}\sqrt{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{r}}\lambda_{\boldsymbol{s}}}\left\{\frac{\boldsymbol{p}}{\lambda_{\boldsymbol{p}}}(\eta_{\boldsymbol{k}+\boldsymbol{p}}-\eta_{\boldsymbol{p}})(\eta_{\boldsymbol{q}+\boldsymbol{r}}+\eta_{\boldsymbol{q}+\boldsymbol{s}})\right.\\&+\frac{\boldsymbol{q}}{\lambda_{\boldsymbol{q}}}(\eta_{\boldsymbol{k}+\boldsymbol{q}}-\eta_{\boldsymbol{q}})(\eta_{\boldsymbol{p}+\boldsymbol{r}}+\eta_{\boldsymbol{p}+\boldsymbol{s}})-\frac{\boldsymbol{r}}{\lambda_{\boldsymbol{r}}}(\eta_{\boldsymbol{k}+\boldsymbol{r}}-\eta_{\boldsymbol{r}})(\eta_{\boldsymbol{s}+\boldsymbol{p}}+\eta_{\boldsymbol{s}+\boldsymbol{q}})\\&\left.-\frac{\boldsymbol{s}}{\lambda_{\boldsymbol{s}}}(\eta_{\boldsymbol{k}+\boldsymbol{s}}-\eta_{\boldsymbol{s}})(\eta_{\boldsymbol{r}+\boldsymbol{p}}+\eta_{\boldsymbol{r}+\boldsymbol{q}})\right\}\end{aligned}$$

and

$$\Delta\gamma_k^{20}(p, q) = \frac{\Omega}{16}\delta_{0,k+p+q}\sqrt{\lambda_p\lambda_q}(py(p)+qy(q)). \tag{1·4}$$

If one makes use of (2·13) in A and the expressions (2·16), (2·22) and (2·23) in A, one can write the velocity operator v_k in terms of the operators b_l and b_l^*. The result is given by

$$v_k = \frac{\hbar k}{2m}\frac{1}{\sqrt{N\lambda_k}}\phi_k \tag{1·5}$$

and

$$\begin{aligned}\phi_k = b_k - b^*_{-k} &+ \frac{1}{\sqrt{N}}\sum_{p,q}\psi_k^{20}(p, q)(b_p^*b_q^* - b_{-q}b_{-p})\\ &+\frac{1}{N}\sum_{p,q,r}\{\psi_k^{30}(p, q, r)(b_p^*b_q^*b_r^* - b_{-r}b_{-q}b_{-p})\\ &\qquad + \psi_k^{21}(p, q; r)(b_p^*b_q^*b_{-r} - b_r^*b_{-q}b_{-p})\}\\ &+\left(\frac{1}{\sqrt{N}}\right)^3\sum_{p,q,r,s}\{\psi_k^{40}(p, q, r, s)(b_p^*b_q^*b_r^*b_s^* - b_{-s}b_{-r}b_{-q}b_{-p})\\ &\qquad + \psi_k^{31}(p, q, r; s)(b_p^*b_q^*b_r^*b_{-s} - b_s^*b_{-r}b_{-q}b_{-p})\}\\ &+\left(\frac{1}{\sqrt{N}}\right)^3\sum_{p,q}\Delta\psi_k^{20}(p, q)(b_p^*b_q^* - b_{-q}b_{-p}),\end{aligned} \tag{1·6}$$

where

$$\psi_k^{20}(p, q) = \frac{1}{2}\delta_{0,k+p+q}\sqrt{\lambda_k\lambda_p\lambda_q},$$

$$\psi_k^{30}(p, q, r) = -\frac{1}{3}\delta_{0,k+p+q+r}\sqrt{\lambda_p\lambda_q\lambda_r\lambda_k},$$

$$\psi_k^{21}(p, q; r) = \frac{1}{4}\delta_{0,k+p+q+r}\sqrt{\lambda_k\lambda_p\lambda_q\lambda_r}(\eta_{k+p}+\eta_{k+q}),$$

$$\psi_k^{40}(p, q, r, s) = \frac{1}{4}\delta_{0,k+p+q+r+s}\sqrt{\lambda_k\lambda_p\lambda_q\lambda_r\lambda_s},$$

$$\psi_k^{31}(p, q, r; s) = \frac{1}{12}\delta_{0,k+p+q+r+s}\sqrt{\lambda_k\lambda_p\lambda_q\lambda_r\lambda_s} \times\{x(k, s; p, q, r) - 2x(s; p, q, r)\}$$

and

$$\Delta\psi_k^{20}(p, q) = -\frac{\Omega}{4}\delta_{0,k+p+q}\sqrt{\lambda_k\lambda_p\lambda_q}\{y(p)+y(q)+y\}. \tag{1·7}$$

The functions y and x in the above expressions are defined by

$$y=\frac{1}{\Omega}\sum_{\boldsymbol{k}}\eta_{\boldsymbol{k}}\,,$$

$$y(\boldsymbol{p})=\frac{1}{\Omega}\sum_{\boldsymbol{k}}\eta_{\boldsymbol{k}}\eta_{\boldsymbol{k}+\boldsymbol{p}}\,,$$

$$y(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})=\frac{1}{3\Omega}\sum_{\boldsymbol{k}}\{\eta_{\boldsymbol{p}+\boldsymbol{k}}\eta_{\boldsymbol{k}}\eta_{\boldsymbol{q}-\boldsymbol{k}}+\eta_{\boldsymbol{q}+\boldsymbol{k}}\eta_{\boldsymbol{k}}\eta_{\boldsymbol{r}-\boldsymbol{k}}+\eta_{\boldsymbol{r}+\boldsymbol{k}}\eta_{\boldsymbol{k}}\eta_{\boldsymbol{p}-\boldsymbol{k}}\}$$

$$\text{for}\quad \boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=0\,,$$

$$x(\boldsymbol{p},\boldsymbol{q};\boldsymbol{r},\boldsymbol{s},\boldsymbol{l})=\eta_{\boldsymbol{p}+\boldsymbol{r}}\eta_{\boldsymbol{q}+\boldsymbol{s}}+\eta_{\boldsymbol{p}+\boldsymbol{r}}\eta_{\boldsymbol{q}+\boldsymbol{l}}+\eta_{\boldsymbol{p}+\boldsymbol{s}}\eta_{\boldsymbol{q}+\boldsymbol{r}}+\eta_{\boldsymbol{p}+\boldsymbol{s}}\eta_{\boldsymbol{q}+\boldsymbol{l}}$$
$$+\eta_{\boldsymbol{p}+\boldsymbol{l}}\eta_{\boldsymbol{q}+\boldsymbol{r}}+\eta_{\boldsymbol{p}+\boldsymbol{l}}\eta_{\boldsymbol{q}+\boldsymbol{s}}\,,$$

$$x(\boldsymbol{s};\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})=\eta_{\boldsymbol{s}+\boldsymbol{p}}+\eta_{\boldsymbol{s}+\boldsymbol{q}}+\eta_{\boldsymbol{s}+\boldsymbol{r}}\quad\text{and}\quad \eta_{\boldsymbol{k}}=\lambda_{\boldsymbol{k}}-1\,. \tag{1·8}$$

One should note here that the velocity operator in (1·6) does not have the A-1-type structure in contrast to the operators $\rho_{-\boldsymbol{k}}$ and $\boldsymbol{g}_{\boldsymbol{k}}$ in (1·1) and (1·3).

The main purpose of the present paper is to derive the interaction Hamiltonian among the excitations described by the mode-operators $b_{\boldsymbol{l}}$ and $b_{\boldsymbol{l}}^*$ up to order of $N^{-3/2}$, and to show that the interaction terms have convergent forms and always give finite corrections to all physical quantities in the perturbation theoretical calculations. Before the derivation of the interaction Hamiltonian, we discuss in § 2 the relation between the old methods in the previous papers[2)] (referred to as I, II and III, respectively) to derive the transformations (1·1) and (1·3) up to order of N^{-1} and clarify the reason why the interaction terms have convergent forms.

§ 2. Density correlations

The working hypothesis in the previous paper II to determine the coefficients in the expansion of $\rho_{-\boldsymbol{k}}$ and $\boldsymbol{g}_{\boldsymbol{k}}$ in terms of the operators $b_{\boldsymbol{l}}$ and $b_{\boldsymbol{l}}^*$ was to take notice of the asymptotic conditions like (2·19) in A. In this section, we show that the asymptotic conditions are also satisfied by the expression determined as (1·1) and (1·3), and that the procedure to determine the unknown coefficients in the preceding paper is a sort of extension of the previous working hypothesis.

When we take the expectation values of the operators $\rho_{\boldsymbol{k}}\rho_{\boldsymbol{p}}\rho_{\boldsymbol{q}}$ and $\rho_{\boldsymbol{q}}\rho_{-\boldsymbol{q}}$, one has (2·20) and (2·21) of A in the crudest approximation. If one takes account of the new higher-order corrections of $\rho_{\boldsymbol{l}}$ in the expectation values, one has a new contribution $(\Omega/2)\delta_{0,\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}}\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}y(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$ to (2·20) in A and no higher-order correction for (2·21) in A. Since the function $y(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$ given in (1·8) vanishes whenever any one of the arguments goes to infinity, we see that the asymptotic condition below (2·21) in A still remains valid.

As will be seen from the condition (2·19) in A written in the first quantized form, however, the conditions should be satisfied irrespective of the states by which expectation values are taken. This means that the asymptotic conditions should hold as operator relations in the sense of weak convergence. Therefore, our next task is to demonstrate that the operator ρ_k expressed in terms of b_l and b_l^* satisfies the asymptotic conditions.

This problem can be analysed more extensively by investigating the operator $U_{k,l}$ in (2·9) of A. We consider first the operator $U_{k,l}=\rho_k\rho_l-\rho_{k+l}$. When one writes $\rho_k\rho_l$ and $U_{k,l}$ in terms of the operators a_k and a_k^*, one has

$$\rho_k\rho_l=\sum_{p,q} a^*_{p+k}a_p a^*_{q+l}a_q \tag{2·1}$$

and

$$U_{k,l}=\sum_{p,q} a^*_{p+k}a^*_{q+l}a_q a_p\,, \tag{2·2}$$

respectively. Both operators (2·1) and (2·2) forbid that the transferred momentum $\boldsymbol{k}$ (or $\boldsymbol{l}$) is divided into large pieces and is imparted to more than two atoms. In the case of (2·2), however, the momenta $\boldsymbol{k}$ and $\boldsymbol{l}$ are always transferred independently to two atoms. On the other hand, in (2·1), the momenta are not necessarily shared with two atoms, but the sum $\boldsymbol{k}+\boldsymbol{l}$ may be transferred to a single atom, since the terms like $a^*_{q+l+k}a_{q+l}a^*_{q+l}a_l$ are involved in the sum over $\boldsymbol{p}$ and they transfer $\boldsymbol{k}+\boldsymbol{l}$ to a single atom. This is an essential difference between two operators in (2·1) and (2·2). This fact indicates that the operator $U_{k,l}$ expressed in terms of b_l and b_l^* also has property similar to (2·2), and that such an operator can be called an asymptotic two momentum-transfer operator (hereafter, referred to as an A-2-type operator). The A-2-type operator is defined more precisely in the following way. Consider an operator like

$$\sum_{\substack{p,q,\cdots\\ u,w,\cdots}} f_{k,l}(\boldsymbol{p},\boldsymbol{q},\cdots;\boldsymbol{u},\boldsymbol{w},\cdots)\,b_p{}^*b_q{}^*\cdots b_{-w}b_{-u}\cdots \tag{2·3}$$

with some coefficient $f_{k,l}$. When we group the momenta appearing in (2·3) so as to contain at most one momentum from each of three sets $\{\boldsymbol{k},\boldsymbol{l}\}$, $\{\boldsymbol{p},\boldsymbol{q},\cdots\}$ and $\{\boldsymbol{u},\boldsymbol{w},\cdots\}$, one has a partition like $[(\boldsymbol{k},\boldsymbol{p}),(\boldsymbol{l},\boldsymbol{w}),(\boldsymbol{q},\boldsymbol{u}),\cdots]$. When the coefficient $f_{k,l}$ vanishes whenever the sum of the momenta belonging to any group for the fixed partition tends to infinity, the operator (2·3) is called an A-2-type operator. The sum of A-2-type operators for various partitions is also an A-2-type operator $O_{k,l}$. Since the momenta $\boldsymbol{k}$ and $\boldsymbol{l}$ are not involved simultaneously in one group in any partition, the A-2-type operator inhibits a coalescence of large momentum-transfer as $(\boldsymbol{k}+\boldsymbol{l})$ to a single excitation and the "transferred" large momenta are always carried by two excitations. This structure of the A-2-type operator requires the following asymptotic conditions. When one takes an arbitrary matrix-element of the operator, the momenta $(\boldsymbol{p},\boldsymbol{q},\cdots;\boldsymbol{u},\boldsymbol{w},\cdots)$ attached to every operator in (2·3) are fixed by those in the state-vector. In this situation, the A-2-type operator $O_{k,l}$ satisfies the condition that all coefficients $f_{k,l}$ involved in the operator vanish for large $\boldsymbol{k}$ (or $\boldsymbol{l}$); i.e.,

$$\lim_{k\to\infty} O_{k,l}=0 \quad \text{or} \quad \lim_{l\to\infty} O_{k,l}=0 \tag{2·4}$$

in the sense of weak convergence, by which we mean that every matrix-element vanishes in these limits.

We now investigate the operator $U_{k,l}$ expressed in terms of b_p and b_q^*. Introducing (1·1) and (1·2) into $U_{k,l}=\rho_k\rho_l-\rho_{k+l}$, one can write a complicated expression as

$$\begin{aligned}
U_{-k,-l}=&N\delta_{0,k+l}\eta_k+\frac{N}{2}\sum_{q,r}\sqrt{\lambda_k\lambda_l}(\delta_{0,k+q}\delta_{0,l+r}+\delta_{0,k+r}\delta_{0,l+q})[b_q^*b_r^*+b_{-r}b_{-q}\\
&\qquad+2b_q^*b_{-r}]\\
&+\sqrt{N}\sum_{q}\delta_{0,k+l+q}(\lambda_k\lambda_l-1)\sqrt{\lambda_q}[b_q^*+b_{-q}]\\
&+\sqrt{N}\sum_{q,r,s}\delta_{0,k+l+q+r+s}\sqrt{\lambda_q\lambda_r\lambda_s}(\lambda_k\delta_{0,l+q}+\lambda_l\delta_{0,k+q})[b_q^*b_r^*b_{-s}+b_s^*b_{-r}b_{-q}]\\
&+\sum_{q,r}\delta_{0,k+l+q+r}\frac{1}{2}\lambda_k\lambda_l\sqrt{\lambda_q\lambda_r}\eta_{l+q}[b_q^*b_r^*+b_{-r}b_{-q}]\\
&+\sum_{q,r}\delta_{0,k+l+q+r}\{\lambda_k\lambda_l(\eta_{l+q}+\eta_{k+q})+\lambda_{k+l}(\lambda_k\lambda_l-1)\}\sqrt{\lambda_q\lambda_r}b_q^*b_{-r}\\
&+\sum_{q,r,s,t}\delta_{0,k+l+q+r+s+t}\frac{1}{2}\sqrt{\lambda_q\lambda_r\lambda_s\lambda_t}(\lambda_k\delta_{0,l+q}+\lambda_l\delta_{0,k+q})\eta_{s+t}[b_q^*b_r^*b_s^*b_{-t}\\
&\qquad+b_t^*b_{-s}b_{-r}b_{-q}]\\
&+\sum_{q,r,s,t}\delta_{0,k+l+q+r+s+t}\frac{1}{2}\sqrt{\lambda_q\lambda_r\lambda_s\lambda_t}[\{\lambda_k(\delta_{0,l+t}+\delta_{0,l+q})+\lambda_l(\delta_{0,k+t}+\delta_{0,k+q})\}\eta_{s+r}\\
&\qquad+2\delta_{0,k+q+s}\lambda_k\lambda_l]b_q^*b_r^*b_{-s}b_{-t}\\
&+\frac{1}{\sqrt{N}}\sum_{p,q,r,s,t}\delta_{0,k+l+p+q+r+s+t}\frac{1}{8}\sqrt{\lambda_p\lambda_q\lambda_r\lambda_s\lambda_t}\{\lambda_k\delta_{0,p+l}(x(\boldsymbol{q},\boldsymbol{r};\boldsymbol{k},\boldsymbol{s},\boldsymbol{t})\\
&\qquad+x(\boldsymbol{s},\boldsymbol{t};\boldsymbol{k},\boldsymbol{q},\boldsymbol{r}))+\lambda_l\delta_{0,p+k}(x(\boldsymbol{q},\boldsymbol{r};\boldsymbol{l},\boldsymbol{s},\boldsymbol{t})+x(\boldsymbol{s},\boldsymbol{t};\boldsymbol{l},\boldsymbol{q},\boldsymbol{r}))\\
&\qquad+2\lambda_k\lambda_l(\delta_{0,l+p+t}+\delta_{0,k+p+t})\}[b_p^*b_q^*b_r^*b_{-s}b_{-t}+b_t^*b_s^*b_{-r}b_{-q}b_{-p}]\\
&+\frac{1}{\sqrt{N}}\sum_{q,r,s}\delta_{0,k+l+q+r+s}\frac{1}{4}\sqrt{\lambda_q\lambda_r\lambda_s}\{2\lambda_k\lambda_l(x(\boldsymbol{k},\boldsymbol{l};\boldsymbol{q},\boldsymbol{r})+x(\boldsymbol{k},\boldsymbol{l};\boldsymbol{r},\boldsymbol{s})\\
&\qquad+(2\lambda_{r+s}-1)x(\boldsymbol{k},\boldsymbol{l};\boldsymbol{q}))+\lambda_{k+l}(\lambda_k\lambda_l-1)x(\boldsymbol{s};\boldsymbol{q},\boldsymbol{r})\\
&\qquad+2\delta_{0,l+q}\Omega\lambda_k y(\boldsymbol{k},\boldsymbol{r},\boldsymbol{s})+2\delta_{0,k+q}\Omega\lambda_l y(\boldsymbol{l},\boldsymbol{r},\boldsymbol{s})\}\\
&\qquad\times[b_q^*b_r^*b_{-s}+b_s^*b_{-r}b_{-q}]\\
&+\frac{1}{\sqrt{N}}\sum_{q}\delta_{0,k+l+q}\frac{\Omega}{2}\sqrt{\lambda_q}\lambda_k\lambda_l y(\boldsymbol{k},\boldsymbol{l},\boldsymbol{q})[b_q^*+b_{-q}],
\end{aligned} \tag{2·5}$$

where

$$x(\boldsymbol{k},\boldsymbol{l};\boldsymbol{q},\boldsymbol{r})\equiv\eta_{k+q}\eta_{l+r}+\eta_{k+r}\eta_{l+q} \quad \text{and} \quad x(\boldsymbol{s};\boldsymbol{q},\boldsymbol{r})\equiv\eta_{s+q}+\eta_{s+r}\,.$$

By a detailed inspection of the rather complicated expression (2·5), one can conclude that the operator $U_{\boldsymbol{k},\boldsymbol{l}}$ in (2·5) has the structure as an A-2-type operator, that is, the large momenta $\boldsymbol{k}$ and $\boldsymbol{l}$ are not divided into more than two excitations and the sum $\boldsymbol{k}+\boldsymbol{l}$ is not carried by a single excitation. This latter property comes from the subtraction of $\rho_{-\boldsymbol{k}-\boldsymbol{l}}$ in $U_{-\boldsymbol{k}-\boldsymbol{l}}$. From (2·5), one can readily see that

$$\lim_{\boldsymbol{k}\ (\text{or}\ \boldsymbol{l})\to\infty} U_{-\boldsymbol{k}-\boldsymbol{l}} = \lim_{\boldsymbol{k}\ (\text{or}\ \boldsymbol{l})\to\infty} [\rho_{-\boldsymbol{k}}\rho_{-\boldsymbol{l}} - \rho_{-\boldsymbol{k}-\boldsymbol{l}}] = 0 \tag{2·6}$$

in the sense of the limit in (2·4). From (2·6), we see that the asymptotic condition (2·19) in A is satisfied as an operator relation, since $\lim_{\boldsymbol{p}\to\infty}\rho_{-\boldsymbol{p}-\boldsymbol{q}}\rho_{\boldsymbol{p}}=\rho_{-\boldsymbol{q}}$. The generalized asymptotic condition like

$$\lim_{\boldsymbol{p}\to\infty}\rho_{\boldsymbol{q}}\rho_{\boldsymbol{r}}\cdots\rho_{\boldsymbol{p}}\rho_{-\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{r}\cdots} = \rho_{\boldsymbol{q}}\rho_{\boldsymbol{r}}\cdots\rho_{-\boldsymbol{q}-\boldsymbol{r}\cdots} \tag{2·7}$$

is also satisfied in the sense of weak convergence.

Before closing this section, let us discuss the convergence properties of $W(\rho)$ in (2·9) of A. Since all terms in $W(\rho)$ are expressed in terms of the operators U multiplied by the factor $\boldsymbol{p}^2$ and have sums over the relevant momenta, some amount of worry is caused about the convergence of the sums. We may argue, however, that the sums can give convergent results on account of the condition (2·6). The first term on the right-hand side of (2·8) in A will converge thanks to the condition (2·6). The operator $U_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}$ in the second term can be written as

$$\begin{aligned} U_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} &= \rho_{\boldsymbol{p}}U_{\boldsymbol{q},\boldsymbol{r}} - U_{\boldsymbol{p}+\boldsymbol{q},\boldsymbol{r}} - U_{\boldsymbol{q},\boldsymbol{r}+\boldsymbol{p}} = \rho_{\boldsymbol{q}}U_{\boldsymbol{r},\boldsymbol{p}} - U_{\boldsymbol{q}+\boldsymbol{r},\boldsymbol{p}} - U_{\boldsymbol{r},\boldsymbol{p}+\boldsymbol{q}} \\ &= \rho_{\boldsymbol{r}}U_{\boldsymbol{p},\boldsymbol{q}} - U_{\boldsymbol{r}+\boldsymbol{p},\boldsymbol{q}} - U_{\boldsymbol{p},\boldsymbol{q}+\boldsymbol{r}}\,. \end{aligned} \tag{2·8}$$

If one takes the limit $\boldsymbol{r}\to\infty$ (or $\boldsymbol{p}\to\infty$, $\boldsymbol{q}\to\infty$) in the first expression (or in the second and third expressions), we see that

$$\lim_{\boldsymbol{p}\to\infty} U_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} = \lim_{\boldsymbol{q}\to\infty} U_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} = \lim_{\boldsymbol{r}\to\infty} U_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} = 0\,. \tag{2·9}$$

In virtue of these conditions, the second term of $W(\rho)$ gives convergent result. We note here that the operator $U_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}$ is the A-3-type operator, by which we mean that the division of momenta $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$ as well as the coalescence of these momenta is prohibited. This is seen from (2·8). Namely, although the operator $U_{\boldsymbol{q},\boldsymbol{r}}$ in (2·8) is certainly the A-2-type operator, the operator $\rho_{\boldsymbol{p}}U_{\boldsymbol{q},\boldsymbol{r}}$ may produce coalescence of momenta $\boldsymbol{p}$ and $\boldsymbol{q}$ (or $\boldsymbol{r}$). But these terms are subtracted by the second and third terms in (2·8), and the operator $U_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}$ does not produce coalescence of the momenta.

The same line of arguments makes one conclude that the U-operator with M indices $\boldsymbol{p}, \boldsymbol{q}, \cdots$ and $\boldsymbol{w}$ is an A-M type operator, since this operator can be rewritten as

$$U_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\cdots,\boldsymbol{w}} = \rho_{\boldsymbol{p}}U_{\boldsymbol{q},\boldsymbol{r},\cdots,\boldsymbol{w}} - U_{\boldsymbol{p}+\boldsymbol{q},\boldsymbol{r},\cdots,\boldsymbol{w}} - U_{\boldsymbol{q},\boldsymbol{p}+\boldsymbol{r},\cdots,\boldsymbol{w}} - \cdots - U_{\boldsymbol{q},\boldsymbol{r},\cdots,\boldsymbol{p}+\boldsymbol{w}}\,. \tag{2·10}$$

The A-M type structure of this operator is responsible for the convergence of higher order contributions to $W(\rho)$.

The closed expression of $U_{p,q,\cdots,w}$ in terms of the ρ's is given by

$$U_{p,q,\cdots,w}=(-1)^M\sum_{[m_n]}(\prod_n\{-(n-1)!\}^{m_n})\sum_{\text{perm}}\rho_{p'+q'+\cdots}\rho_{t'+\cdots}\cdots\rho_{\cdots+w'}\,, \tag{2·11}$$

where $(\boldsymbol{p}',\boldsymbol{q}',\cdots,\boldsymbol{w}')$ represents a permutation of $(\boldsymbol{p},\boldsymbol{q},\cdots,\boldsymbol{w})$ and m_n is the number of factors ρ whose indices are written as sums of n letters out of $\boldsymbol{p},\boldsymbol{q},\cdots$ and $\boldsymbol{w}$. Moreover, $\sum_{\text{perm}}$ implies the sum of different products of ρ that results from possible permutations $(\boldsymbol{p}',\boldsymbol{q}',\cdots,\boldsymbol{w}')$ of $(\boldsymbol{p},\boldsymbol{q},\cdots,\boldsymbol{w})$, and $\sum_{[m_n]}$ indicates summation over all sets $[m_n]$ of non-negative integers m_n satisfying $\sum_{n=1}^{\infty}nm_n=M$.

§ 3. Hamiltonian for elementary excitations

The elementary excitations in a Bose system can be described in an appropriate way by the operators $b_{\boldsymbol{k}}$ and $b_{\boldsymbol{k}}^*$. Introducing (1·1), (1·3) and (1·5) into (2·6) and (2·10) in A, one obtains after a straightforward and lengthy calculation the following expression for the Hamiltonian up to order of $N^{-3/2}$:

$$H=H_0+H_{\text{I}}+H_{\text{II}}+H_{\text{III}}+\cdots\,, \tag{3·1}$$

where the first three terms were already given in II; i.e.,

$$H_0=\frac{N^2}{2\Omega}\nu(0)-\sum_{\boldsymbol{k}}\frac{\hbar^2\boldsymbol{k}^2}{8m\lambda_{\boldsymbol{k}}^2}\eta_{\boldsymbol{k}}^2+\sum_{\boldsymbol{p}}E_{\boldsymbol{p}}^Bb_{\boldsymbol{p}}^*b_{\boldsymbol{p}}\,,$$

$$H_{\text{I}}=\frac{1}{2!\sqrt{N}}\sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}\delta_{0,\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}}F_{\text{I}}^{(2,1)}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{r})[b_{\boldsymbol{p}}^*b_{\boldsymbol{q}}^*b_{-\boldsymbol{r}}+b_{\boldsymbol{r}}^*b_{-\boldsymbol{q}}b_{-\boldsymbol{p}}]$$
$$+\frac{1}{3!\sqrt{N}}\sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}\delta_{0,\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}}F_{\text{I}}^{(3,0)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})[b_{\boldsymbol{p}}^*b_{\boldsymbol{q}}^*b_{\boldsymbol{r}}^*+b_{-\boldsymbol{r}}b_{-\boldsymbol{q}}b_{-\boldsymbol{p}}]$$

and

$$H_{\text{II}}=\frac{1}{N}F_{\text{II}}^{(0,0)}+\frac{1}{N}\sum_{\boldsymbol{p}}F_{\text{II}}^{(1,1)}(\boldsymbol{p})b_{\boldsymbol{p}}^*b_{\boldsymbol{p}}+\frac{1}{2N}\sum_{\boldsymbol{p}}F_{\text{II}}^{(2,0)}(\boldsymbol{p})[b_{\boldsymbol{p}}^*b_{-\boldsymbol{p}}^*+b_{-\boldsymbol{p}}b_{\boldsymbol{p}}]$$
$$+\frac{1}{(2!)^2N}\sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{s}}\delta_{0,\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}+\boldsymbol{s}}F_{\text{II}}^{(2,2)}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{r},\boldsymbol{s})b_{\boldsymbol{p}}^*b_{\boldsymbol{q}}^*b_{-\boldsymbol{r}}b_{-\boldsymbol{s}}$$
$$+\frac{1}{3!N}\sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{s}}\delta_{0,\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}+\boldsymbol{s}}F_{\text{II}}^{(3,1)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r};\boldsymbol{s})[b_{\boldsymbol{p}}^*b_{\boldsymbol{q}}^*b_{\boldsymbol{r}}^*b_{-\boldsymbol{s}}+b_{\boldsymbol{s}}^*b_{-\boldsymbol{r}}b_{-\boldsymbol{q}}b_{-\boldsymbol{p}}], \tag{3·2}$$

where the coefficients F in (3·2) are given by (see (3·25) in II)

$$F_{\text{I}}^{(2,1)}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{r})=\frac{\hbar^2}{4m}\frac{1}{\sqrt{\lambda_{\boldsymbol{p}}\lambda_{\boldsymbol{q}}\lambda_{\boldsymbol{r}}}}[(\boldsymbol{p}\cdot\boldsymbol{q})\lambda_{\boldsymbol{r}}\eta_{\boldsymbol{p}}\eta_{\boldsymbol{q}}+(\boldsymbol{q}\cdot\boldsymbol{r})\lambda_{\boldsymbol{p}}\eta_{\boldsymbol{q}}(\lambda_{\boldsymbol{r}}+1)$$
$$+(\boldsymbol{p}\cdot\boldsymbol{r})\lambda_{\boldsymbol{q}}\eta_{\boldsymbol{p}}(\lambda_{\boldsymbol{r}}+1)],$$

$$F_{\mathrm{I}}^{(3,0)}(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) = \frac{\hbar^2}{4m} \frac{1}{\sqrt{\lambda_p \lambda_q \lambda_r}} [(\boldsymbol{p}\cdot\boldsymbol{q})\lambda_r \eta_p \eta_q + (\boldsymbol{q}\cdot\boldsymbol{r})\lambda_p \eta_q \eta_r + (\boldsymbol{p}\cdot\boldsymbol{r})\lambda_q \eta_p \eta_r],$$

$$F_{\mathrm{II}}^{(0,0)} = -\frac{\hbar^2}{16m} \sum_{k,l,m} \delta_{0,k+l+m} k^2 \eta_k \eta_l \eta_m ,$$

$$F_{\mathrm{II}}^{(2,0)}(\boldsymbol{p}) = F_{\mathrm{II}}^{(1,1)}(\boldsymbol{p}) = -\frac{\hbar^2}{8m} \sum_{l,m} \delta_{0,p+l+m} (p^2 + l^2 + m^2) \eta_l \eta_m , \tag{3·3}$$

$$F_{\mathrm{II}}^{(2,2)}(\boldsymbol{p}, \boldsymbol{q}; \boldsymbol{r}, \boldsymbol{s}) = \frac{\hbar^2}{2m} \{(\boldsymbol{p}\cdot\boldsymbol{q}) + (\boldsymbol{r}\cdot\boldsymbol{s})\} \sqrt{\lambda_p \lambda_q \lambda_r \lambda_s} (\eta_{p+s} + \eta_{p+r})$$

$$+ \frac{\hbar^2}{4m} \Big[(\boldsymbol{p}\cdot\boldsymbol{s}) \sqrt{\frac{\lambda_r \lambda_q}{\lambda_p \lambda_s}} \{(\lambda_p \lambda_s - 1)(\lambda_{p+s} - 1) + \lambda_p \lambda_s (\eta_{p+r} + \eta_{q+s}) - \eta_p \eta_s\}$$

+ (three terms obtained by exchanging variables as $(\boldsymbol{p}\leftrightarrow\boldsymbol{q})$, $(\boldsymbol{r}\leftrightarrow\boldsymbol{s})$ and $\begin{pmatrix}\boldsymbol{p}\leftrightarrow\boldsymbol{q}\\ \boldsymbol{r}\leftrightarrow\boldsymbol{s}\end{pmatrix}$)$\Big]$

and

$$F_{\mathrm{II}}^{(3,1)}(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}; \boldsymbol{s}) = \frac{\hbar^2}{4m} \Big[\Big\{ (\boldsymbol{p}\cdot\boldsymbol{q}) \sqrt{\frac{\lambda_r \lambda_s}{\lambda_p \lambda_q}} (\lambda_{p+q} \eta_p \eta_q + \lambda_q \eta_p \eta_{p+r} + \lambda_p \eta_q \eta_{q+r})$$

$$+ (\boldsymbol{p}\cdot\boldsymbol{s}) \sqrt{\frac{\lambda_q \lambda_r \lambda_s}{\lambda_p}} \eta_p (\eta_{p+q} + \eta_{p+r}) \Big\}$$

+ {two terms obtained by cyclic exchange of $\boldsymbol{p}, \boldsymbol{q}$ and $\boldsymbol{r}$}$\Big]$.

The interaction Hamiltonian H_{III} in the new higher-order term obtained in the present paper is given by

$$H_{\mathrm{III}} = \frac{1}{4!\,N^{3/2}} \sum_{p,q,r,s,t} \delta_{0,p+q+r+s+t} F_{\mathrm{III}}^{(4,1)}(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}; \boldsymbol{t}) [b_p{}^* b_q{}^* b_r{}^* b_s{}^* b_{-t} + b_t{}^* b_{-s} b_{-r} b_{-q} b_{-p}]$$

$$+ \frac{1}{3!\,2!\,N^{3/2}} \sum_{p,q,r,s,t} \delta_{0,p+q+r+s+t} F_{\mathrm{III}}^{(3,2)}(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}; \boldsymbol{s}, \boldsymbol{t}) \times [b_p{}^* b_q{}^* b_r{}^* b_{-s} b_{-t} + b_t{}^* b_s{}^* b_{-r} b_{-q} b_{-p}]$$

$$+ \frac{1}{3!\,N^{3/2}} \sum_{p,q,r} \delta_{0,p+q+r} F_{\mathrm{III}}^{(3,0)}(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) [b_p{}^* b_q{}^* b_r{}^* + b_{-r} b_{-q} b_{-p}]$$

$$+ \frac{1}{2!\,N^{3/2}} \sum_{p,q,r} \delta_{0,p+q+r} F_{\mathrm{III}}^{(2,1)}(\boldsymbol{p}, \boldsymbol{q}; \boldsymbol{r}) [b_p{}^* b_q{}^* b_{-r} + b_r{}^* b_{-q} b_{-p}]. \tag{3·4}$$

The coefficients in (5·3) are expressed as

$$F_{\mathrm{III}}^{(4,1)}(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}; \boldsymbol{t}) = \frac{1}{4!} [\mathscr{F}_{\mathrm{III}}^{(4,1)}(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}; \boldsymbol{t}) + (\text{symmetrized terms concerning } (\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}))],$$

$$F_{\text{III}}^{(3,2)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r};\boldsymbol{s},\boldsymbol{t})=\frac{1}{3!\,2!}[\mathcal{F}_{\text{III}}^{(3,2)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r};\boldsymbol{s},\boldsymbol{t})+(\text{symmetrized terms concerning } (\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) \text{ and } (\boldsymbol{s},\boldsymbol{t})],$$

$$F_{\text{III}}^{(3,0)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})=\frac{1}{3!}[\mathcal{F}_{\text{III}}^{(3,0)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})+(\text{symmetrized terms concerning } (\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})]$$

and

$$F_{\text{III}}^{(2,1)}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{r})=\frac{1}{2!}[\mathcal{F}_{\text{III}}^{(2,1)}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{r})+\mathcal{F}_{\text{III}}^{(2,1)}(\boldsymbol{q},\boldsymbol{p};\boldsymbol{r})], \tag{3·5}$$

where

$$\mathcal{F}_{\text{III}}^{(4,1)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{s};\boldsymbol{t})=-\frac{3\hbar^2}{2m}\sqrt{\lambda_p\lambda_q\lambda_r\lambda_s\lambda_t}\,\eta_{s+t}\Big[\eta_{p+q}\Big\{\frac{2\boldsymbol{p}^2}{\lambda_p}\eta_p+\frac{(\boldsymbol{p}\cdot\boldsymbol{q})}{\lambda_p\lambda_q}(\lambda_p\lambda_q-1)\Big\} -\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{\lambda_p\lambda_q}\eta_p\eta_q\Big], \tag{3·6a}$$

$$\mathcal{F}_{\text{III}}^{(3,2)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r};\boldsymbol{s},\boldsymbol{t})=\frac{3\hbar^2}{4m}\sqrt{\lambda_p\lambda_q\lambda_r\lambda_s\lambda_t}\,\eta_{r+s}\Big[\eta_{q+t}\Big\{2\Big(\boldsymbol{q}-\frac{\boldsymbol{t}}{\lambda_t}\Big)\cdot\Big(\boldsymbol{t}+\frac{\boldsymbol{q}}{\lambda_q}\Big)+\frac{2\boldsymbol{p}^2}{\lambda_p} -\boldsymbol{p}^2-4(\boldsymbol{q}+\boldsymbol{t})^2\Big\}-2\frac{\boldsymbol{p}}{\lambda_p}\eta_p\Big\{(\boldsymbol{p}+\boldsymbol{q})\eta_{p+q}+\frac{\boldsymbol{t}}{\lambda_t}\eta_t\Big\}+\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{\lambda_p\lambda_q}\lambda_{p+q}\eta_p\eta_q\Big], \tag{3·6b}$$

$$\mathcal{F}_{\text{III}}^{(3,0)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})=-\frac{3\hbar^2}{16m}\Omega\sqrt{\lambda_p\lambda_q\lambda_r}\Big[2z(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})+\frac{5}{3}\boldsymbol{p}^2y(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})+2\eta_r\frac{(\boldsymbol{r}\cdot\boldsymbol{p})}{\lambda_r}y(\boldsymbol{p})\Big], \tag{3·6c}$$

$$\mathcal{F}_{\text{III}}^{(2,1)}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{r})=\frac{\hbar^2}{4m}\Omega\sqrt{\lambda_p\lambda_q\lambda_r}\Big[-2\lambda_p z(\boldsymbol{p})-\frac{1}{2}\boldsymbol{p}\cdot\Big\{3\lambda_p\boldsymbol{p}+\Big(1+\frac{1}{\lambda_r}\Big)\boldsymbol{r}\Big\}y(\boldsymbol{p}) -\frac{5}{2}z(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})-\frac{1}{6}(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{z}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) +\Big\{\frac{\boldsymbol{p}^2}{\lambda_p}-\frac{1}{36}(29\boldsymbol{p}^2+29\boldsymbol{q}^2+17\boldsymbol{r}^2)\Big\}y(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\Big]. \tag{3·6d}$$

The functions y are given in (1·8) and the new functions z are defined by

$$z(\boldsymbol{p})=\Omega^{-1}\sum_{\boldsymbol{l}}\Big(\boldsymbol{l}+\frac{\boldsymbol{p}}{2}\Big)^2\eta_l\eta_{l+p},$$

$$z(\boldsymbol{p},\boldsymbol{q},-\boldsymbol{p}-\boldsymbol{q})=\Omega^{-1}\sum_{\boldsymbol{l}}\Big(\boldsymbol{l}-\frac{\boldsymbol{p}-\boldsymbol{q}}{3}\Big)^2\eta_{l-p}\eta_l\eta_{l+q} \tag{3·7}$$

and

$$\boldsymbol{z}(\boldsymbol{p},\boldsymbol{q},-\boldsymbol{p}-\boldsymbol{q})=\Omega^{-1}\sum_{\boldsymbol{l}}\Big(\boldsymbol{l}-\frac{\boldsymbol{p}-\boldsymbol{q}}{3}\Big)\eta_{l-p}\eta_l\eta_{l+q},$$

where $z(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})$ and $\boldsymbol{z}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})$ are defined only for the case $\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=0$ on account of the existence of the delta-function in (3·4).

We now inspect more closely the properties of the interaction terms in (3·4) by taking account of the rapid convergence of the factor $\eta_{\boldsymbol{k}}$ for large momentum $\boldsymbol{k}$. Firstly the coefficient $\mathscr{F}_{\rm III}^{(4,1)}$ in (3·6a) does vanish whenever any one of the four momenta $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ and $\boldsymbol{s}+\boldsymbol{t}$ becomes large, where one should take notice of the delta-function in (3·4). Thus, the first term in the interaction Hamiltonian $H_{\rm III}$ can be regarded as an asymptotic zero-momentum transfer operator (referred to as an A-0-type operator) concerning the partition $[(\boldsymbol{p}), (\boldsymbol{q}), (\boldsymbol{r}), (\boldsymbol{s}, \boldsymbol{t})]$. By the A-0-type operator, we mean here of course that any momentum-"transfer" produced by the operators b and b^* is effectively within the critical momentum k_c. Other terms in the first term of (3·4) which come from the symmetrized terms in (3·5) are also A-0-type operators concerning their corresponding partitions. In a similar way, the coefficient $\mathscr{F}_{\rm III}^{(3,2)}$ in (3·6·b) vanishes whenever any one of $\boldsymbol{p}, \boldsymbol{p}+\boldsymbol{q}$ and $\boldsymbol{r}+\boldsymbol{s}$ becomes large. This indicates that the second term of the interaction (3·4) with the coefficient $\mathscr{F}_{\rm III}^{(3,2)}$ is also an A-0-type operator corresponding to the partition $[(\boldsymbol{p}), (\boldsymbol{q}, \boldsymbol{t}), (\boldsymbol{r}, \boldsymbol{s})]$. Other symmetrized terms are also A-0-type operators for their corresponding partitions. The interaction terms which have the coefficients $\mathscr{F}_{\rm III}^{(3,0)}$ and $\mathscr{F}_{\rm III}^{(2,1)}$ are A-0-type ones concerning the partitions $[(\boldsymbol{p}), (\boldsymbol{q}), (\boldsymbol{r})]$ and $[(\boldsymbol{p}), (\boldsymbol{q}, \boldsymbol{r})]$, respectively. Therefore, we can conclude that the interaction Hamiltonian $H_{\rm III}$ is an A-0-type operator.

If one inspects the interaction Hamiltonians $H_{\rm I}$ and $H_{\rm II}$ in (3·2) and (3·3) from the present point of view, one can readily see that these interactions are also A-0-type operators. In virtue of the fact that the momentum-"transfer" in any A-0-type operator is restricted to smaller value than k_c, the A-0-type structure of the interactions described by the operators $b_{\boldsymbol{k}}$ and $b_{\boldsymbol{k}}{}^*$ assures to give convergent results for the perturbation-theoretical treatment of the interactions, since the momenta in intermediate states are limited so as to give finite results. This structure of the interactions among excitation modes indicates that the excitation modes described by the operators $b_{\boldsymbol{k}}$ and $b_{\boldsymbol{k}}{}^*$ are adequate to the collective description of an interacting boson system. The number of the convergence factor $\eta_{\boldsymbol{k}}$ to be attached to an A-0-type operator must be at least $n-1$, where n stands for the larger of the number of creation operators and that of annihilation operators. From this fact we see that the Hamiltonian (3·1) is not only a simple expansion in powers of $N^{-1/2}$, but also an expansion concerning the number of the convergence factor $\eta_{\boldsymbol{k}}$. Finally, we give a notice to the last two terms in (3·4). The coefficients in these interaction terms involve one more factor $\eta_{\boldsymbol{k}}/N$ than the corresponding terms in $H_{\rm I}$, and thus these terms are regarded as corrections for $H_{\rm I}$.

References

1) S. Yamasaki, T. Kebukawa and S. Sunakawa, Prog. Theor. Phys. **61** (1979), 996. (This paper is referred to as A).

2) S. Yamasaki, T. Kebukawa and S. Sunakawa, Prog. Theor. Phys. **53** (1975), 1243; **54** (1975), 348; **56** (1976), 415. (These papers are referred to as I, II and III, respectively.)