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EXCITATION OF NEUTRON STAR NORMAL MODES DURING BINARY INSPIRAL

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ABSTRACT

As a compact binary inspirals due to the emission of gravitational waves, its orbital period decreases continuously down to ~ 1 ms, its value at coalescence. During the last part of the inspiral, the two stars are close together, and their tidal interactions become strong. Neutron stars have many normal modes (core g-modes, crustal discontinuity modes, shear modes, etc.) whose periods lie in the range ($\sim$several ms) swept by the orbital period. Some of these modes are resonantly excited by the tidal force. The amount of energy a mode absorbs is proportional to the square of the overlap integral between its displacement field and the tidal force field. For all modes of interest, this overlap is poor, resulting in relatively weak excitation. For the best case, the absorbed energy is only a small fraction ($\sim 10^{-6}$) of the orbital energy, so the orbital phase shift is too weak to be detected by observations of the gravitational wave signal emitted by the inspiraling binary. However, with displacement amplitudes of excited quadrupole modes ranging up to 0.5% of the stellar radius, the possibility of a detectable electromagnetic signature cannot be dismissed. Both the periods of the modes and the energy they absorb depend quite strongly on the internal structure of the star. Their observation could shed light on the correct high-density equation of state.

Subject headings: binaries: close — gravitation — stars: neutron — stars: oscillations

1. INTRODUCTION

Inspiraling compact binaries (two neutron stars, two black holes, or one of each) are seen as "safe" sources of gravitational radiation (Schutz 1986; Narayan, Piran, & Shemi 1991; Phinney 1991) to be detected by proposed instruments such as LIGO (Abramovici et al. 1992) and VIRGO (Bradaschia et al. 1990), and as possible sources of gamma-ray bursts (see Narayan, Paczyński, & Piran 1992 and references therein). Because of low signal-to-noise ratios, the detection of gravitational waves from these events relies heavily on accurate prior knowledge of the class of waveforms in the frequency range (~ 50 Hz) to which the detectors are most sensitive (e.g., Thorne 1987; Cutler et al. 1993). Currently, only speculation connects coalescing binaries and gamma-ray bursts. Observational signatures that might distinguish between the "standard model" mentioned above and proposed alternatives (e.g., Blaes et al. 1989) should be of interest. We attempt to (partially) address both of these issues by studying the resonant tidal excitation of neutron star oscillation modes.

Recent theoretical investigations (McDermott et al. 1985; Finn 1987; Reisenegger & Goldreich 1992, hereafter RG) have uncovered a dense spectrum of neutron star normal modes, with many modes having periods in the range ($\sim$several ms) swept by the orbital motion of an inspiraling compact binary close to coalescence. Although no secure observations of such modes have been reported, their existence is on a fairly firm theoretical basis. This is apart from their precise forms and periods, which are in some cases quite sensitive to the state of matter at supernuclear densities, or to other details of neutron star structure.

We study the excitation of these modes in an inspiraling binary system. In § 2, we present a heuristic treatment of the problem, regarding the modes as harmonic oscillators subject to a force whose period decreases slowly (the tidal potential of the orbiting companion). In § 3, we apply the formalism of Press & Teukolsky (1977, hereafter PT), to relate the strength of the forcing to the overlap of the mode displacement field and the tidal force field, and argue that this overlap is small for modes whose periods are long in comparison to the dynamical time scale of the neutron star. Evaluations of the energy absorbed by the modes, and typical displacement amplitudes, are given in § 4. In § 5, we estimate the phase error made by calculating the orbital evolution neglecting resonant mode excitation. The significance of these results is discussed in the conclusions (§ 6).

2. HARMONIC OSCILLATOR MODEL

We consider two stars in a nearly circular orbit, one of mass M, regarded as a point mass for simplicity, and another, a neutron star of mass M_* and radius R_* whose normal modes are being tidally excited. We assume that the neutron star does not rotate with respect to an inertial reference frame, and therefore counterrotates at the orbital frequency in a (noninertial) frame tied to the orbit. The validity of our results requires only that the rotation frequency be much smaller than the orbital frequency. Kochanek (1992) and Bildsten & Cutler (1992) show that unreasonably large viscosities are needed to bring a neutron star's spin into corotation with its orbital motion during inspiral. We assume further that the unperturbed neutron star is spherically symmetric. As shown below, the tidal bulge is small at the time when resonances are encountered, even smaller than the amplitudes of some of the excited modes. Moreover, intrinsic asymmetries of the neutron star, e.g., those produced by strong magnetic fields or elastic stresses, are expected to be even smaller.

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We give a heuristic discussion of this problem, into which we insert exact results based on the formalism of PT, who study essentially the same physical problem (orbital excitation of stellar oscillation modes) in a different astronomical context (tidal capture), and make the same assumptions as described above.

A normal mode with "quantum numbers" n, l, and m can be thought of as a harmonic oscillator with mass M_* and natural frequency ω_{nl}.[3] If such an oscillator is resonantly driven by a sinusoidal external force of amplitude F during a time interval $\delta t \gg 2\pi/\omega_{nl}$, it ends up oscillating with an amplitude[4]

$$\xi = \frac{F\,\delta t}{2M_*\omega_{nl}}\,, \tag{1}$$

and storing a total energy

$$\Delta E = \frac{1}{2}\,M_*\,\xi^2\omega_{nl}^2 = \frac{(F\,\delta t)^2}{8M_*}\,. \tag{2}$$

The effective resonance time interval for an inspiraling binary system is given by the Fourier integral (see, e.g., PT)

$$\delta t \approx \left|\int_{-\infty}^{+\infty} dt e^{i[\phi(t)-\omega_{nl}t]}\right|\,, \tag{3}$$

where $\phi(t)$ is the phase of the external force. Choosing the origin of time at resonance, $\dot\phi(t=0)=\omega_{nl}$, and taking account of the slow inspiral, i.e.,

$$\Gamma \equiv \frac{\dot\phi(0)^2}{2\pi\ddot\phi(0)} \gg 1\,, \tag{4}$$

one can expand $\phi(t) = \phi(0) + \omega_{nl}t + \frac{1}{2}\ddot\phi(0)t^2 + O(t^3)$, simplifying the integral to

$$\delta t \approx \left|\int_{-\infty}^{+\infty} dt e^{i\ddot\phi(0)t^2/2}\right| = \left[\frac{2\pi}{\ddot\phi(0)}\right]^{1/2} = \Gamma^{1/2}\,\frac{2\pi}{\omega_{nl}}\,. \tag{5}$$

Terms neglected in this approximation are smaller than the leading term by a factor $\sim\Gamma^{-1/2} \ll 1$.

For an inspiraling binary system, the mode frequency, ω_{nl}, is related to the orbital frequency at resonance, $\Omega(t=0)=\Omega_{nlm}$, by $\omega_{nl} = m\Omega_{nlm}$. Thus, the orbital phase may be identified as $\Phi(t) = \phi(t)/m$, and

$$\frac{\Gamma}{m} = \frac{\Omega^2}{2\pi\dot\Omega} = \frac{5}{192\pi}\,\frac{[1+(M/M_*)]^{1/3}}{M/M_*}\left(\frac{\Omega_g}{\Omega}\right)^{5/3}\,, \tag{6}$$

where $\dot\Omega$ expresses the orbital decay due to the emission of gravitational waves to lowest post-Newtonian order, and $\Omega_g \equiv c^3/(GM_*)$ (Peters & Mathews 1963).

[3] l and m are the usual spherical harmonic indices, and n is some arbitrary label for the radial dependence of the mode, usually given as a letter (f, p, g, etc.) indicating a general class to which the mode belongs (considering its main restoring mechanism), together with a number indicating the number of radial nodes it possesses.

[4] For a normal mode of a neutron star, this is the density-weighted rms amplitude. For low l, low n, f-modes, p-modes, and core g-modes, where much of the star vibrates with a similar amplitude, this is a good measure of the typical oscillation amplitude. For modes concentrated in or near the crust, such as the discontinuity modes described by Finn (1987; see also RG), or the shear and interface modes and thermal g-modes discussed by McDermott et al. (1985) and McDermott, Van Horn, & Hansen (1988), a small amount of mass oscillates with a much larger amplitude than the bulk of the star.

The amplitude of the forcing can be written in terms of the orbital separation at resonance, a_{nlm}, as

$$F = \frac{GMM_*}{R_*^2}\left(\frac{R_*}{a_{nlm}}\right)^{l+1} S_{nlm}\,, \tag{7}$$

i.e., as the 2^l-pole moment of the tidal force exerted by the binary companion on the neutron star, multipled by the dimensionless factor, $S_{nlm} \lesssim 1$, whose properties are described in the following section.

3. THE OVERLAP INTEGRAL

We define a scalar product for the displacement fields of two modes $\boldsymbol{\xi}_{nlm}(x) = \xi_{nl}^r(r)Y_{lm}(\theta,\phi)\hat{r} + \xi_{nl}^{\perp}(r)r\nabla Y_{lm}(\theta,\phi)$, by (e.g., PT)

$$\begin{aligned}\langle nlm|n'l'm'\rangle &\equiv \int_{M_*} d^3x\,\rho\boldsymbol{\xi}_{nlm}^*\cdot\boldsymbol{\xi}_{n'l'm'}\\ &= \delta_{ll'}\,\delta_{mm'}\int_0^1 r^2\,dr\,\rho[\xi_{nl}^{r*}\xi_{n'l'}^r + l(l+1)\xi_{nl}^{\perp*}\xi_{n'l'}^{\perp}]\\ &= \delta_{nn'}\,\delta_{ll'}\,\delta_{mm'}\,,\end{aligned} \tag{8}$$

where r is the radial coordinate in units of the stellar radius, R_*, and $\rho(r)$ is the stellar density in units of M_*/R_*^3. The first equality defines the scalar product, the second arises from the orthogonality of spherical harmonics, and the third follows from the orthonormality of the modes. That stellar, normal modes are eigenstates of a linear, Hermitian operator is shown for a purely fluid star, e.g., by Unno et al. (1989), and is easily generalized to a solid star.

Substituting equation (7) into equation (2), and comparing the result to equations (18), (37), (38), and (39) of PT, one can read off

$$S_{nlm} = 2W_{lm}\langle P_{lm}|nlm\rangle\,, \tag{9}$$

where the vector $|P_{lm}\rangle = \nabla[r^l Y_{lm}(\theta,\phi)]$ contains the spatial dependence of the 2^l-pole component of the tidal field. The numerical factor

$$W_{lm} = (-)^{(l+m)/2}\,\frac{[4\pi(l-m)!\,(l+m)!/(2l+1)]^{1/2}}{2^l[(l-m)/2]!\,[(l+m)/2]!}\,, \tag{10}$$

where the symbol $(-)^k$ is to be interpreted as zero if k is not an integer. We note that W_{lm} is of order unity for modes with small l and even $l+m$, and also that $|W_{lm}| \le [4\pi/(2l+1)]^{1/2}$ for all l and m.

The overlap integral, $\langle P_{lm}|nlm\rangle$, is the only term in these calculations that cannot be evaluated analytically. It appears in the expression for the mode damping time due to emission of gravitational waves,

$$\tau_{nl} = \frac{(l-1)l}{(l+1)(l+2)}\,\frac{[(2l+1)!!]^2}{2\pi|\langle P_{lm}|nlm\rangle|^2}\,\frac{c^2R_*}{GM_*}\left(\frac{c}{R_*\omega_{nl}}\right)^{2l-1}\omega_{nl}^{-1} \tag{11}$$

(Balbinski & Schutz 1982; see also RG). The overlap integral may be estimated from damping times which have been tabulated for several types of neutron star oscillation modes (RG; Finn 1987; McDermott et al. 1988).[5]

[5] However, the damping times of some modes in the last of these references have been severely underestimated (Reisenegger 1994).

Since f-modes have significant Eulerian density perturbations and displacement eigenfunctions that are similar to the tidal force field, their overlap integrals are rather large ($S_{nlm} \sim 1$). However, their frequencies are high, so f-modes are not resonantly forced until, at best, very close to coalescence. From the similarity of the f-modes to the tidal force field, together with the orthogonality of the modes, it follows that the overlap integrals for all other modes must be small, $S_{nlm} \ll 1$. This means, in other words, that the crests and troughs in the eigenfunctions corresponding to the other modes cancel each other quite precisely in the overlap integral (RG, § 3.2). That the overlap integral for g-modes has to be small can also be argued by observing (RG) that the fractional Eulerian density variation in a g-mode is smaller than the fractional radial displacement of matter by a factor $\sim(N/\Omega_d)^2$, where N is the Brunt-Väisälä frequency, and

$$\Omega_d \equiv (GM_*/R_*^3)^{1/2} \tag{12}$$

is approximately the maximum rotation rate or inverse dynamical time of the star.[6] The effective force that excites the modes is reduced by (at least) the same factor. A numerical evaluation shows, however, that S_{nlm} is larger for core g-modes than for other modes (crustal discontinuity modes, shear modes) in the period range of interest. For this reason, we focus mainly on the former. In a future paper (Reisenegger 1994), an attempt will be made to justify more rigorously some of the properties of stellar oscillation modes that are invoked here.

4. AMPLITUDE AND ENERGY OF THE EXCITED MODES

Combining the results derived so far, and using Kepler's law,

$$\left(\frac{a}{R_*}\right)^3 = \left(1 + \frac{M}{M_*}\right)\left(\frac{\Omega_d}{\Omega}\right)^2 , \tag{13}$$

we obtain the fractional displacement[7]

$$\frac{\xi}{R_*} \approx \left(\frac{5\pi}{192m^3}\right)^{1/2} \frac{(M/M_*)^{1/2}}{[1 + (M/M_*)]^{(2l+1)/6}} \left(\frac{c^2R_*}{GM_*}\right)^{5/4} \times \left(\frac{\Omega_{nlm}}{\Omega_d}\right)^{(4l-13)/6} S$$

$$\approx 5.3 \times 10^{-3}\left(\frac{P_{\rm orb}}{10 \text{ ms}}\right)^{5/6}\left(\frac{S}{10^{-3}}\right) , \tag{14}$$

and the absorbed energy

$$\Delta E = \frac{5\pi}{384m} \frac{M/M_*}{[1 + (M/M_*)]^{(2l+1)/3}} \left(\frac{c^2R_*}{GM_*}\right)^{5/2} \times \left(\frac{\Omega_{nlm}}{\Omega_d}\right)^{(4l-7)/3} \frac{GM_*^2}{R_*} S^2$$

$$\approx 6.2 \times 10^{46}\left(\frac{P_{\rm orb}}{10 \text{ ms}}\right)^{-1/3}\left(\frac{S}{10^{-3}}\right)^2 \text{ ergs} . \tag{15}$$

[6] For core g-modes, $(N/\Omega_d)^2$ is of the order of the fraction of particles other than neutrons contained in the stellar core.

[7] For g-modes, the horizontal surface displacement is of similar magnitude to the typical interior displacement, but the surface boundary condition of vanishing Lagrangian pressure perturbation forces the radial surface displacement to be a factor $(\omega/\Omega_d)^2 \sim 10^{-2}$ smaller.

Hereafter explicit evaluations are made for $l = m = 2$, with "typical" parameters $M = M_* = 1.4\ M_\odot$ and $R_* = 10$ km. $P_{\rm orb}$ is the orbital period at resonance (i.e., m times the mode's period). The reference numbers $P_{\rm orb} = 10$ ms and $S = 10^{-3}$ are chosen to agree with the lowest order quadrupole g-mode, as calculated for Model 2 ($M_* = 1.405\ M_\odot$) in RG. For the first quadrupole discontinuity mode calculated by Finn (1987), $S \sim 6 \times 10^{-5}$ (from the numbers given by Finn, and confirmed to order of magnitude by RG), so that $\Delta E \sim 6 \times 10^{43}$ ergs, much smaller than for core modes. However, $\xi/R_* \sim 7 \times 10^{-2}$ in the thin layer where the energy of the discontinuity mode is concentrated. We emphasize that numerical results involve relatively uncertain parameters such as the neutron star radius, the mode period, and especially the overlap parameter S, all of which depend upon the state of matter above nuclear density.

The total orbital energy and its fractional change due to mode excitation are

$$E = -\frac{GMM_*}{2a} = -\frac{M/M_*}{2[1 + (M/M_*)]^{1/3}}\left(\frac{\Omega}{\Omega_d}\right)^{2/3}\frac{GM_*^2}{R_*}$$

$$\approx -2.7 \times 10^{52}\left(\frac{P_{\rm orb}}{10 \text{ ms}}\right)^{-2/3} \text{ ergs} , \tag{16}$$

and

$$-\frac{\Delta E}{E} = \frac{5\pi}{192m} S^2\left(1 + \frac{M}{M_*}\right)^{-2l/3}\left(\frac{c^2R_*}{GM_*}\right)^{5/2}\left(\frac{\Omega_{nlm}}{\Omega_d}\right)^{(4l-9)/3}$$

$$\approx 2.3 \times 10^{-6}\left(\frac{S}{10^{-3}}\right)^2\left(\frac{P_{\rm orb}}{10 \text{ ms}}\right)^{1/3} . \tag{17}$$

By comparison, the fraction of the orbital energy carried away by gravitational waves during the effective resonance time interval δt is

$$-\frac{\delta E}{E} = \frac{2}{3}\frac{\dot{\Omega}}{\Omega}\delta t = \frac{2}{3}\Gamma^{-1/2}$$

$$= 16\left(\frac{\pi}{15m}\right)^{1/2}\frac{(M/M_*)^{1/2}}{[1 + (M/M_*)]^{1/6}}\left(\frac{\Omega}{\Omega_g}\right)^{5/6}$$

$$\approx 5.0 \times 10^{-2}\left(\frac{P_{\rm orb}}{10 \text{ ms}}\right)^{-5/6} . \tag{18}$$

That $\delta E \gg \Delta E$ justifies our neglect of the mode's effect on $\dot{\Omega}$ or Γ.

However, the typical displacement associated with an excited mode (eq. [14]) is comparable to the quasi-static tidal deformation

$$\frac{\xi_{\rm static}}{R_*} = \frac{3}{4}\frac{M}{M_*}\left(\frac{R_*}{a}\right)^3 = \frac{3}{4}\frac{M/M_*}{1 + (M/M_*)}\left(\frac{\Omega}{\Omega_d}\right)^2$$

$$\approx 7.9 \times 10^{-4}\left(\frac{P_{\rm orb}}{10 \text{ ms}}\right)^{-2} . \tag{19}$$

The static tidal deformation differs from the distortions due to g-modes in that it involves radial and horizontal surface displacements of similar magnitude.

As is apparent from equations (14), (15), and (17), quadrupole ($l = 2$) modes are the most strongly excited. For example, the small ratio $\Omega_{nlm}/\Omega_d \lesssim 1/20$ appears raised to the $(4l - 7)/3$ power in the relation giving ΔE.

Since S_{nlm} decreases rapidly with increasing n, g-modes of given l and m are progressively less excited as n increases. A rough fit gives $S \propto \omega^{2.75}$ for the first five quadrupole g-modes of Model 2 in RG. Taking this result at face value, one obtains (for $l = 2$) $\xi \propto \omega^{23/12}$, $\Delta E \propto \omega^{35/6}$, and $\Delta E/E \propto \omega^{31/6}$.

Applying the WKB dispersion relation for g-modes (e.g., RG), one can show that the number of modes per unit frequency (for a given l) varies as ω^{-2} for $\omega \ll N$. Thus the total energy transferred to modes whose angular frequencies fall below ω scales proportionally to $\omega^{29/6}$. From this we see that the energy absorbed by the lowest order quadrupole mode (numerical estimate in eq. [15]) is a large fraction of the total energy absorbed by all modes.

5. PHASE ERROR

Let $\Omega_e(t)$ describe the evolution of the orbital angular velocity as it would unfold if affected exclusively by the emission of gravitational waves. Suppose that the resonant tidal excitation of a normal mode (or something else) produces a sudden perturbation of the angular velocity by $\Delta\Omega_{\rm int} \ll \Omega$ at time $t_{\rm int}$. Since $\dot{\Omega}_e$ is a function of Ω only (i.e. it does not depend explictly on time), after this interaction $\Omega_a(t) = \Omega_e(t + \Delta t)$, where $\Delta t \approx \Delta\Omega_{\rm int}/\dot{\Omega}_e(t_{\rm int})$. Thus at time $t > t_{\rm int}$, the phase shift due to the interaction amounts to

$$\Delta\Phi(t) = \int_{t_{\rm int}}^{t} \Omega_a(t')dt' - \int_{t_{\rm int}}^{t} \Omega_e(t')dt' \approx [\Omega_e(t) - \Omega_e(t_{\rm int})] \frac{\Delta\Omega_{\rm int}}{\dot{\Omega}_e(t_{\rm int})} . \quad (20)$$

(This formula, with reversed sign, holds for the phase error at a time $t < t_{\rm int}$ if the actual wave form is matched to the expected wave form at times $> t_{\rm int}$.)

Since $t_{\rm int}$ is the time at which resonance occurs,

$$\Delta\Phi(t) \approx \frac{3\pi\Gamma}{m}\left[\frac{\Omega_e(t)}{\Omega_{nlm}} - 1\right]\left(-\frac{\Delta E}{E}\right)$$
$$\approx 2.0 \times 10^{-3}\left[\frac{\Omega_e(t)}{\Omega_{nlm}} - 1\right]\left(\frac{P_{\rm orb}}{10\ {\rm ms}}\right)^2\left(\frac{S}{10^{-3}}\right)^2 . \quad (21)$$

The small size of $\Delta\Phi(t)$ implies that it will be below the threshold for detection by LIGO and VIRGO.

Arguments analogous to those at the end of the previous section imply that the effect of modes with $l > 2$ is negligible compared to that from modes $l = 2$, and that the cumulative phase shift due to all $l = 2$ g-modes having frequencies $< \omega$ is (approximately) $\propto \omega^{2.5}$. Thus, a large fraction of the total phase shift comes from the lowest order, quadrupole g-mode.

6. CONCLUSIONS

Neutron star oscillation modes are only weakly excited by resonant tidal forces in an inspiraling binary because the overlap between their displacement fields and the tidal force field is small. In particular, they absorb a negligible fraction of the total orbital energy, and their effect on the orbit is too weak to be detected by instruments such as LIGO and VIRGO. Nevertheless, the absorbed energy is quite high compared to the energies involved in phenomena observed electromagnetically, such as radio pulses and X-ray bursts. Moreover, the oscillation amplitudes of excited modes can be as large as $\sim 0.5\%$ of the radius for core g-modes and several percent for crustal g-modes. Thus, the possibility that tidally excited oscillation modes might reveal themselves through electromagnetic signals cannot be easily dismissed.

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