

Research Note

Excitation of the Normal Modes of the Earth by Earthquake Sources

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(Received 1970 July 28)

Summary

Some century-old results, due to Rayleigh and Routh, are used to derive a very compact and simple representation for the transient response of the Earth to earthquakes. In particular, it is shown how the residual static displacement field is naturally represented in terms of the normal mode eigenfunctions.

If one can calculate the excitation of the normal modes of the Earth due to a particular earthquake source, one can use such calculations in an attempt to infer the earthquake mechanism and total moment. Some general results in normal mode theory, due to Rayleigh and Routh about a century ago, make the excitation calculations remarkably simple. First we treat a system of N particles and then pass to the continuum limit.

Consider a conservative system of N particles in small oscillation about a state of stable equilibrium. Let the α -th particle, $\alpha = 1, \dots, N$ have mass m_α , displacement \mathbf{u}_α , and let a force \mathbf{f}_α be applied to it. Then, if $\mathbf{V}_{\alpha\beta}$ is the symmetric, positive definite potential energy matrix, the linearized equation for the conservation of linear momentum is

$$m_\alpha \frac{d^2 \mathbf{u}_\alpha}{dt^2} + \sum_{\beta=1}^N \mathbf{V}_{\alpha\beta} \cdot \mathbf{u}_\beta = \mathbf{f}_\alpha. \quad (1)$$

The Laplace transform of (1) is

$$m_\alpha p^2 \bar{\mathbf{u}}_\alpha + \sum_{\beta} \mathbf{V}_{\alpha\beta} \cdot \bar{\mathbf{u}}_\beta = \bar{\mathbf{f}}_\alpha \quad (2)$$

where

$$\bar{\mathbf{u}}_\alpha(p) = \int_0^\infty \mathbf{u}_\alpha(t) \exp(-pt) dt; \quad \text{Re } p > 0 \quad (3)$$

and

$$\mathbf{u}_\alpha(0) = \frac{d}{dt} \mathbf{u}_\alpha(0) = 0.$$

We recall that there are $3N$ eigenfrequencies ω_n ; $n = 1, \dots, 3N$, and $3N$ eigenvectors, or normal modes, $\mathbf{s}_{\alpha,n} \exp(i\omega_n t)$ for the dynamical system.

The normal modes can be orthonormalized (* denotes complex conjugate)

$$\sum_{\alpha} m_\alpha \mathbf{s}_{\alpha,n}^* \cdot \mathbf{s}_{\alpha,l} = \delta_{n,l} \quad (4)$$

and, since there are $3N$ degrees of freedom, they form a complete basis for the system. Consequently, $\bar{\mathbf{u}}_\alpha$ has the representation

$$\bar{\mathbf{u}}_\alpha = \sum_n a_n \mathbf{s}_{\alpha, n} \tag{5}$$

and the use of (4) gives

$$a_n = \sum_\alpha m_\alpha \mathbf{s}_{\alpha, n}^* \cdot \bar{\mathbf{u}}_\alpha. \tag{6}$$

If we take the scalar product of (2) with $\mathbf{s}_{\alpha, n}^*$ and sum over α we get

$$p^2 a_n + \sum_{\alpha\beta} \mathbf{s}_{\alpha, n}^* \cdot \mathbf{V}_{\alpha\beta} \cdot \left(\sum_l a_l \mathbf{s}_{\beta l} \right) = \sum_\alpha \mathbf{s}_{\alpha, n} \cdot \bar{\mathbf{f}}_\alpha. \tag{7}$$

We can simplify (7) by using a form of Rayleigh's principle. The normal mode $\mathbf{s}_{\alpha, n}$ is a solution to

$$-\omega_n^2 m_\alpha \mathbf{s}_{\alpha, n} + \sum_\beta \mathbf{V}_{\alpha\beta} \cdot \mathbf{s}_{\beta, n} = 0. \tag{8}$$

Taking the scalar product of (8) with $\mathbf{s}_{\alpha, l}$, summing over α , and using (4) gives

$$\sum_{\alpha\beta} \mathbf{s}_{\alpha, l}^* \cdot \mathbf{V}_{\alpha\beta} \cdot \mathbf{s}_{\beta, n} = \omega_n^2 \bar{\delta}_{nl}. \tag{9}$$

Since $\mathbf{V}_{\alpha\beta}$ is symmetric we can interchange α and β , as well as n and l , in (9). Thus, using (9) in (7) gives

$$(p^2 + \omega_n^2) a_n = \sum_\alpha \mathbf{s}_{\alpha, n} \cdot \bar{\mathbf{f}}_\alpha \tag{10}$$

and we can write (5) as

$$\bar{\mathbf{u}}_\alpha = \sum_n s_{\alpha, n} \left[\frac{\sum_\beta \mathbf{s}_{\beta, n}^* \cdot \bar{\mathbf{f}}_\beta}{(p^2 + \omega_n^2)} \right]. \tag{11}$$

Most earthquake sources are modelled as step functions so that $\bar{\mathbf{f}} = p^{-1} \mathbf{f}$, then the Laplace inversion of (11), for $t > 0$, is

$$\mathbf{u}_\alpha(t) = \sum_n \mathbf{s}_{\alpha, n} \left(\sum_\beta \mathbf{s}_{\beta, n}^* \cdot \mathbf{f}_\beta \right) \frac{1 - \cos \omega_n t}{\omega_n^2}. \tag{12}$$

When there is a small amount of dissipation ($Q \gg 1$) we modify (12)

$$\mathbf{u}_\alpha(t) = \sum_n \mathbf{s}_{\alpha, n} \left(\sum_\beta \mathbf{s}_{\beta, n}^* \cdot \mathbf{f}_\beta \right) \frac{1 - \cos \omega_n t \exp(-\omega_n t/2Q_n)}{\omega_n^2}. \tag{13}$$

All that remains after a long time is the static displacement

$$\lim_{t \rightarrow \infty} \mathbf{u}_\alpha(t) = \sum_n \mathbf{s}_{\alpha, n} \left(\sum_\beta \mathbf{s}_{\beta, n}^* \cdot \mathbf{f}_\beta \right) \frac{1}{\omega_n^2}. \tag{14}$$

It appears to be unappreciated by many geophysicists that the static response of a mechanical system can be expressed in terms of the normal modes of that system. Thus, the static or dynamical theory of bodily tides, all Love number calculations, the excitation of the chandlerian nutation are all expressible in terms of the Earth's normal mode eigenfunctions, as is the static displacement field caused by an earthquake.

For the Earth, which we approximate as a classical continuum, a particle sum, such as

$$\sum_{\beta=1}^N \mathbf{s}_{\beta,n}^* \cdot \mathbf{f}_{\beta},$$

becomes a volume integral, such as $\int dV \mathbf{s}_n^*(\mathbf{r}) \cdot \mathbf{f}(\mathbf{r})$. The body force \mathbf{f}_{β} is replaced by the body force per unit volume $\mathbf{f}(\mathbf{r})$. For (13) we have

$$u(r, t) = \sum_n \mathbf{s}_n(\mathbf{r}) \left(\int dV_0 \mathbf{s}_n(\mathbf{r}_0) \cdot \mathbf{f}(\mathbf{r}_0) \right) \times \frac{1 - \cos \omega_n t \exp(-\omega_n t/2Q_n)}{\omega_n^2}. \quad (15)$$

The sum in (15) is now an infinite sum but, as shown by Rayleigh, it converges as ω_n^{-2} . The normal modes $\mathbf{s}_n(\mathbf{r})$ in (15) are normalized (4),

$$\int dV \rho(\mathbf{r}) \mathbf{s}_n^*(\mathbf{r}) \cdot \mathbf{s}_n(\mathbf{r}) = 1 \quad (16)$$

and ρ is the density. In many applications the body force $\mathbf{f}(\mathbf{r})$ is represented by δ -functions so that the evaluation of the volume integral in (15) is particularly simple.

For earthquake sources, a commonly used concept is that of a stress release mechanism. Let the difference between the initial static stress, before the earthquake, and the final static stress, after the earthquake, be \mathbf{T} so that the static stress field, \mathcal{T} , has the representation

$$\mathcal{T} = \mathcal{T}_0 - \mathbf{T}\mathbf{H}(t) \quad (17)$$

and let the source region be confined to a volume V_S . Then the action tensor, \mathbf{M} , for the source is

$$\mathbf{M} = \int dV_S \mathbf{T}. \quad (18)$$

When the source dimensions are small compared to wavelengths of interest

$$\mathbf{T} = \mathbf{M}\delta(\mathbf{r} - \mathbf{r}_S) \quad (19)$$

where \mathbf{r}_S is in the source volume V_S .

The body force caused by the stress drop, \mathbf{T} , is $\mathbf{f} = -\nabla \cdot \mathbf{T}$ and the volume integral in (15) becomes

$$\int dV_0 \mathbf{s}_n^*(\mathbf{r}_0) \cdot \mathbf{f}(\mathbf{r}_0) = - \int dV_0 \mathbf{s}_n^*(\mathbf{r}_0) \cdot (\nabla_0 \cdot \mathbf{T}) \quad (20)$$

integrating (20) by parts, using Gauss' divergence theorem plus the symmetry of \mathbf{T} gives

$$\int dV_0 \mathbf{s}_n^*(\mathbf{r}_0) \cdot (\nabla_0 \cdot \mathbf{T}) = \int dA_0 \hat{\mathbf{n}} \cdot (\mathbf{T} \cdot \mathbf{s}_n^*) - \int dV_0 \mathbf{T} : \boldsymbol{\varepsilon}_n^* \quad (21)$$

where $\boldsymbol{\varepsilon}$ is the strain tensor for the displacement vector \mathbf{s}_n , A_0 is the surface of V_0 , and $\hat{\mathbf{n}}$ is the outward unit normal to V_0 . For sources within V_0 , the surface integral in (21) vanishes. Substituting (19) into the remaining volume integral in (21) gives

$$\int dV_0 \mathbf{s}_n^*(\mathbf{r}_0) \cdot (\nabla_0 \cdot \mathbf{T}) = -\mathbf{M} : \boldsymbol{\varepsilon}_n^*(\mathbf{r}_S) \quad (22)$$

where

$$\mathbf{M} : \boldsymbol{\varepsilon} = \sum_{i=1}^3 \sum_{j=1}^3 M_{ij} \varepsilon_{ij}.$$

With the aid of (22) we can write (15) as

$$\mathbf{u}(\mathbf{r}, t) = \sum_n \mathbf{s}_n(\mathbf{r}) (\mathbf{M} : \boldsymbol{\varepsilon}_n^*(\mathbf{r}_S)) \frac{1 - \cos \omega_n t \exp(-\omega_n t/2Q_n)}{\omega_n^2}. \quad (23)$$

A special kind of action tensor is the one that represents a shear dislocation, or fault plane model. Let the unit normal to the fault plane be $\hat{\mathbf{p}}$ and let the slip direction be $\hat{\mathbf{d}}$. Then $\hat{\mathbf{p}} \cdot \hat{\mathbf{d}} = 0$ and

$$\mathbf{M} = M(\hat{\mathbf{p}}\hat{\mathbf{d}} + \hat{\mathbf{d}}\hat{\mathbf{p}}) \tag{24}$$

for a shear dislocation. The symmetry of the action tensor is a consequence of the conservation of angular momentum.

The normal modes of the Earth are of two kinds: spheroidal, $\mathbf{s} = {}_n\sigma_l^m$

$${}_n\sigma_l^m = \hat{\mathbf{r}} {}_nU_l(r) Y_l^m(\theta, \phi) + {}_nV_l(r) \nabla_1 Y_l^m(\theta, \phi) \tag{25}$$

and toroidal, $\mathbf{s} = {}_n\tau_l^m$

$${}_n\tau_l^m = -{}_nW_l(r) \hat{\mathbf{r}} \times \nabla_1 Y_l^m(\theta, \phi). \tag{26}$$

In (25) and (26)

$$\nabla_1 = \hat{\boldsymbol{\theta}} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}} \csc \theta \frac{\partial}{\partial \phi}$$

and $Y_l^m(\theta, \phi)$ is a normalized surface harmonic

$$Y_l^m(\theta, \phi) = (-1)^m \left(\frac{2l+1}{4\pi} \right)^{\frac{1}{2}} \left(\frac{(l-m)!}{(l+m)!} \right)^{\frac{1}{2}} P_l^m(\cos \theta) \exp(im\phi). \tag{27}$$

The scalars U, V, W are solutions to the ordinary differential equations that represent the stress-strain relations and the conservation of linear momentum. Once the scalars U, V, W , have been obtained, and this is, by far, the most time consuming part of the numerical calculations, they can be stored for repeated retrieval, so that the evaluation of (23) for a particular \mathbf{M}, \mathbf{r}_S , and \mathbf{r} can be extremely rapid and inexpensive.

Several applications of the techniques presented here will be published elsewhere.

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Reference

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