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Excitations in a High Density Electron Gas.* I

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A new systematical method which enables us to describe an electron gas in terms of bosons is developed. This boson corresponds to an "exciton", *i.e.* a pair of an electron outside the Fermi sphere and a hole inside. The formalism is particularly suitable to the system at high density, as suggested by Sawada's discussion of the same system. As a straightforward application, the effect of electron exchange on the plasma frequency is calculated. The result coincides with that of a Hartree-Fock treatment.

§ 1. Introduction

In his reformulation of Sawada's theory[1] on the Coulomb interaction in a dense electron gas, Wentzel[2] used operators corresponding to processes where an electron with a momentum inside the Fermi sphere is excited to a momentum state outside, or *vice versa*. These operators were treated as boson creation and annihilation operators, which is justifiable at high density limit. Also a substitute Hamiltonian was adopted that gives the same commutators, or in other words, the same equation of motion of the aforesaid operators, as the basic relations of Sawada's argument. In view of Sawada's success in formulating the high-density problem in terms of this " boson " avoiding the perturbation expansion of Gell-Mann and Brueckner,[3] the boson (excited electron plus hole), which we shall loosely call an " exciton ", can be an effective concept, particularly at high density of electrons. It is the purpose of this paper to present a systematical formulation of the electron gas problem in terms of the excitons.

In order to simplify the presentation, we shall consider a system of N spinless fermions, which shall be called electrons in the following. Let us assume that they are confined in a box of volume Ω with periodic boundaries. The Hamiltonian of the system is such that the number of the fermions should be conserved.

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As a reference state we shall take the state in which the N electrons occupy the lowest N momentum states (orbitals), *i.e.* the states with momenta smaller than p_F are occupied and the other states are vacant. Let us denote this reference state by Φ_0.

Now any state of the system can be written in the form

$$\Pi a_P^* a_p \Phi_0, \tag{1}$$

where a and a^* are the fermion annihilation and creation operators and it is understood that p always stands for a momentum vector inside the Fermi sphere, while P always stands for one outside. In order to define these base states uniquely, we shall consider all the momentum vectors to be arranged in a certain order, and understand that the annihilation or creation operators in the expression (1) are arranged according to this prescribed order. Then the state above can be considered as a certain assembly of excitons, each exciton being a definite pair of a hole with momentum p and an electron with momentum P. This pairing is unique thanks to the prescription above. We are going to describe the fermion system as an ensemble of excitons of this type. Let us first note that excitons are of Bose type in a certain sense because

$$[a_P^* a_p,\ a_{P'}^* a_{p'}]=0. \tag{2}$$

§ 2. Transformation

Let us introduce here Bose operators C_p^P and $C^*{}_p^P$ which have two indices defined in the same region as above and obey the standard commutation relations:

$$\begin{aligned}&[C_p^P,\ C_{p'}^{*P'}]=\delta_{pp'}\delta_{PP'},\\ &[C_p^P,\ C_{p'}^{P'}]=[C_p^{*P},\ C_{p'}^{*P'}]=0.\end{aligned} \tag{3}$$

The vacuum state of the bosons will be denoted by Ψ_0.

Define the operator U in the product space of the fermions and the bosons by

$$U=A\cdot\exp\left(\sum\sum C_p^{*P} a_p^* a_P\right)\cdot\Gamma, \tag{4}$$

where A and Γ are the projection operators to Φ_0 and Ψ_0, respectively. Then one can easily show that

$$U\Pi a_P^* a_p \Phi_0 \Psi_0=\sum_{\mathrm{P}}(-)^{\mathrm{P}}\mathrm{P}\,\Pi C_p^{*P}\Psi_0\Phi_0, \tag{5}$$

where P is a permutation of the lower indices of C^*'s. This theorem shows that the operator U can effectuate a one-to-one correspondence between the fermion states and a certain sub-space of the boson states. Before accomplishing that, we have to introduce an ordering operator in accordance with the prescription stated in Section 1: Let O be an operator which operates on the indices of C or C^* and, by definition, its eigenvalue is one if the pairing of the indices P and p is in compliance with the prescription, and zero otherwise. Then the

product operator OU just effectuates the transformation of $\Pi a_P^* a_p \Phi_0$ to $\Pi C_p^{*P} \Psi_0$ with the same pairs of indices. It may be noted that the ordering operator defined above is a projection operator:

$$O^2 = O. \tag{6}$$

Inversely, the hermitian conjugate of U,

$$\widetilde{U} = \Gamma \cdot \exp\left(\sum\sum C_p^P a_P^* a_p\right) \cdot A, \tag{7}$$

transforms any boson state back to a corresponding fermion state,

$$\widetilde{U} \Pi C_p^{*P} \Psi_0 \Phi_0 = \Pi a_P^* a_p \Phi_0 \Psi_0. \tag{8}$$

Hence, the transformations OU and $\widetilde{U}O$ complete the one-to-one correspondence of the exciton states in the fermion representation and in the boson representation. It is evident that $\widetilde{U}OU$ is unity in the fermion state space.

§ 3. The Hamiltonian in the boson representation

Let the Hamiltonian of the fermion system be $\mathcal{H}$. Then the equation of motion for any fermion state Φ is

$$i\frac{\partial}{\partial t}\Phi = \mathcal{H}\Phi. \tag{9}$$

Now according to the last statement in Section 2, this equation may be written in the form:

$$i\frac{\partial}{\partial t}\Phi\Psi_0 = \mathcal{H}\widetilde{U}OU\Phi\Psi_0, \tag{10}$$

where we assumed that the energy of the state Ψ_0 is zero. Hereby we see that we can investigate the dynamics of the original fermion system in the boson representation if we take $OU\mathcal{H}\widetilde{U}O$ as its Hamiltonian. In other words, we can transcribe the dynamics of the fermion system into that of the boson system, where, however, we have to limit ourselves to the subspace $O\Psi$ and hence the Hamiltonian $U\mathcal{H}\widetilde{U}$ should also be cut in such a way that it connects only the states in this subspace.

Now let us proceed to work out the corresponding boson Hamiltonian assuming that the Hamiltonian of the fermion system is given by

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 \tag{11}$$

where

$$\mathcal{H}_1 = \sum \epsilon_\lambda a_\lambda^* a_\lambda$$

and

$$\mathcal{H}_2 = \frac{1}{2}\sum\sum\sum V_\lambda a_{\kappa+\lambda}^* a_\mu^* a_{\mu+\lambda} a_\kappa.$$

The general principle of the calculation which we make in the following may be

stated as follows. Assume first that the resulting Hamiltonian is expanded in powers of C and C^* operators. Then we can find its coefficients by evaluating the expectation values of the multiple commutators of $U\mathcal{H}\tilde{U}$ with C and C^*, in the product state $\Phi_0\Psi_0$.

The first term, which is a constant, is just the expectation value of $U\mathcal{H}\tilde{U}$ itself. We shall denote the expectation value of any operator Q in the state $\Phi_0\Psi_0$ by $\langle Q\rangle$. The result is

$$\langle U\mathcal{H}\tilde{U}\rangle=\langle\mathcal{H}\rangle=\sum^{F}\epsilon_p+\tfrac{1}{2}\sum_{\neq}^{F}\sum(V_0-V_{p-p'}). \tag{12}$$

The coefficient of C_p^P is given by $\langle[U\mathcal{H}\tilde{U},\ C_p^{*P}]\rangle$, which is equal to

$$\langle[U\mathcal{H}\tilde{U},\ C_p^{*P}]\rangle=\langle[\mathcal{H},\ a_P{}^*a_p]\rangle=0, \tag{13}$$

because of the translation invariance of the Hamiltonian. The hermitian property of $U\mathcal{H}\tilde{U}$ ensures that the coefficient of C_p^{*P} also vanishes. Following a similar procedure, we can get the coefficients of the next higher terms. We shall not write down the details of the calculation, which is straightforward, but just list the results:

$$\begin{aligned}\langle[C_{p_1}^{P_1},\ [U\mathcal{H}\tilde{U},\ C_p^{*P}]]\rangle&=\langle[a_{p_1}^*a_{P_1},\ [\mathcal{H},\ a_P{}^*a_p]]\rangle\\&=\{(\epsilon_P-\epsilon_p)+\sum_{p'}(V_{p-p'}-V_{P-p'})\}\delta_{p_1p}\delta_{P_1P}+(V_{P-p}-V_{P_1-P})\delta_{p+P_1,p_1+P},\end{aligned} \tag{14}$$

and

$$\langle[[U\mathcal{H}\tilde{U},\ C_p^{*P}],\ C_{p'}^{*P'}]\rangle=\langle[[\mathcal{H},\ a_P{}^*a_p],\ a_{P'}^*a_{p'}]\rangle=(V_{P-p}-V_{P'-p})\delta_{p+p',P+P'}. \tag{15}$$

Concerning these two formulas, there are some points to be noted. First there appears no term with V_0. For example, the sum $\sum_{p'}V_{p-p'}$ should be understood to exclude the term with $p=p'$. The reason is the operator $\sum\sum a_\kappa{}^*a_\mu{}^*a_\mu a_\kappa$, which is associated with V_0 in the expression for $\mathcal{H}_2$, always commutes with $a_L^*a_p$ Secondly, the expression of the type (15) always vanishes when $P'=P$ and/or $p'=p$. This is just a consequence of the Pauli principle.

Leaving the calculation of higher terms to a later section, let us write down the exciton Hamiltonian to the second order. To this we shall introduce an abbreviation for the exchange energy, defined by

$$A(\lambda)=\sum^{F}V_{\lambda-p'}, \tag{16}$$

where we understand that if λ lies inside the Fermi sphere we have to omit the term with $p'=\lambda$. We shall also use the following notation:

$$\omega_p{}^P=\epsilon_P-\epsilon_p. \tag{17}$$

Then it is easy to see the relevant Hamiltonian is given by

$$U\mathcal{H}\tilde{U}=\sum\epsilon_p+\tfrac{1}{2}V_0N(N-1)-\tfrac{1}{2}\sum A(p)+\sum\sum\{\omega_p{}^P+A(p)-A(P)\}C_p^{*P}C_p^P$$
$$+\sum\sum\sum\{V_{P-p}-V_{p_1-p}\}C_{p_1}^{*p_1+P-p}C_p^P+\tfrac{1}{2}\sum\sum\sum\{V_{P-p}-V_{P-p'}\}$$
$$\times(C_{p'}^{p'+p-P}C_p^P+\text{complex conj.}). \tag{18}$$

To obtain the Hamiltonian for the excitons we remember that we still have to take the ordering operator into account. This may be done conveniently in each actual application we meet. We shall discuss an example in the next section. Here, however, we shall give some remarks concerning new features appearing in the Hamiltonian above.

The expression in the first line of Eq. (18) is just the expectation value of the Hamiltonian for the reference state Φ_0. The first sum is the Fermi energy, the next the average "Coulomb" interaction and the last sum is the exchange energy. Therefore, by definition, the remainder represents the so-called correlation energy in a certain approximation. The present correlation Hamiltonian, as a matter of course, has the same structure in the main as that of Wentzel.[2] However, in the present Hamiltonian there appear several new terms which were neglected by Sawada and Wentzel because of their less importance at high density. Namely, $\{A(p)-A(P)\}$'s in the second line, $-V_{p_1-p}$'s in the third and $-V_{P-p'}$'s in the last line. These are all originated from the exchange effect. For example, the term $-V_{p_1-p}C_{p_1}^{*p_1+P-p}C_p^P$ corresponds to the following process. One electron with momentum P makes a transition to the state p_1+P-p, while another electron makes a transition from p_1 to p. The appropriate matrix element is V_{p_1-p}. This type of transition was consistently neglected in Sawada's and Wentzel's calculations. However, in the present formulation we do not discard any process in the mid-way, sticking consistently to the exciton picture. Hence, we describe the process above in such a way, as the creation and annihilation operators show, that one electron is excited from p_1 to p_1+P-p and another is deexcited from P to p, the final state being physically the same as above. The phase, however, is different from the former by π, because the electrons in the states p and p_1+P-p have been exchanged. This is the reason for the appearance of the minus sign.

In short and formally stated, the present Hamiltonian has a structure consistent with the Fermi statistics of the original particles. Besides, it is obtained straightforwardly by a mathematical apparatus. (In fact, one can supplement Wentzel's Hamiltonian in accordance with the Fermi statistics and get to the result (18). This procedure, however, is apparently limited in its applicability.)

§ 4. The plasma oscillation

In this section we shall investigate the effect of the exchange terms on the plasma frequency. Sawada has already argued that the effect is small. However, as a specific example of application, we shall here derive a detailed expression for the plasma frequency of electrons in a neutralizing positive charge sea, including

the aforesaid exchange effect. Since the present author has not been able to find out any method to diagonalize the Hamiltonian (18), we are going to restrict ourselves to the discussion of the plasma oscillation and resort to the perturbation method.

The equation of motion for the annihilation operator of an exciton with momentum transfer q is, according to Eq. (18),

$$i\dot{C}_{p,q}=\tilde{\omega}_{p,q}C_{p,q}+\sum_{p'}(V_q-V_{p-p'})C_{p',q}+\sum_{p_1}^{0}(V_q-V_{p_1+p+q})C^*_{-p_1,-q}, \tag{19}$$

where we have used the slightly altered notations defined as follows:

$$C_{p,q}\equiv C_p^{p+q}$$

and

$$\tilde{\omega}_{p,q}\equiv\omega_p^{p+q}+A(p)-A(p+q); \tag{20}$$

and the summation denoted by $\sum$ with a superscript " 0 " means that one has to sum only those terms with the indices that are compatible with the prescription stated in Section 1. Hence in the last sum there appear only such p_1's that the combination (p, q) and $(-p_1, -q)$ is in compliance with the ordering prescription. This may be conveniently done in the following way: Given a certain q, one may prescribe the order of the momenta according to the magnitude of $p\cdot q$ or $P\cdot q$. Correspondingly we shall divide the momentum space relevant to Eq. (19) into the five regions indicated in Fig. 1. Then in the second line of the equation, p belongs to the region r_0+r, $p+q$ to R while $-p_1$ to r_1+r_0 and $-p_1-q$ to R_1. Hence in the summation we have only to be careful when both p and $-p_1$ run in the region r_0. Here, and everywhere in the following, the summation sign $\sum^0$ of Eq. (19) will just mean that one should sum the terms in compliance with the condition $p\cdot q>-p_1\cdot q$ or $(p+p_1)\cdot q>0$.

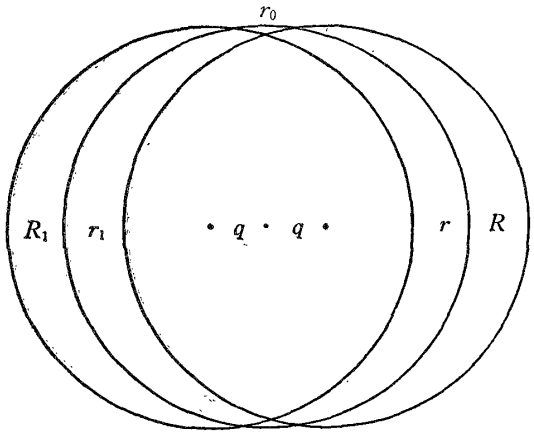

Fig. 1.

Then the equation above can be rewritten in the following way:

$$\begin{aligned}i\dot{C}_{p,q}=\tilde{\omega}_{p,q}C_{p,q}+V_q\sum(C_{p',q}+C^*_{-p',-q})-\sum V_{p-p'}C_{p',q}\\ -V_q\sum_{(p+p')\cdot q<0}C^*_{-p',-q}-\sum_{(p+p')\cdot q>0}V_{p+p'+q}C^*_{-p',-q}.\end{aligned} \tag{21}$$

The complex conjugate equation is

$$\begin{aligned}i\dot{C}^*_{-p,-q}=-\tilde{\omega}_{p,q}C^*_{-p,-q}-V_q\sum(C_{p',q}+C^*_{-p',-q})+\sum V_{p-p'}C^*_{-p',-q}\\ +V_q\sum C_{p',q}+\sum V_{p+p'+q}C_{p',q}.\end{aligned} \tag{21'}$$

To get the eigen-frequency, let us regard C or C^* as non-quantized amplitudes. Since we are treating progressive waves, we must set these expressions equal to $\nu C_{p,q}$ or $\nu C^*_{-p,-q}$, respectively, where ν is the eigen-frequency of the wave. The first lines in the equations are just those terms which were discussed in Sawada's or Wentzel's papers. As mentioned before, we shall treat the remainder as perturbations.

First the unperturbed amplitudes are immediately given by these equations as follows:

$$C_{p,q}=\frac{V_q}{\nu-\tilde{\omega}_{p,q}}C_\nu \tag{22}$$

and

$$C^*_{-p,-q}=\frac{-V_q}{\nu+\tilde{\omega}_{p,q}}C_\nu, \tag{22'}$$

where

$$C_\nu=\sum_p{}'(C_{p,q}+C^*_{-p,-q}). \tag{23}$$

In these expressions C_ν is actually a non-perturbed amplitude. However, as long as the first perturbational approximation is intended, it makes no difference whether we take it to be nonperturbed or perturbed. Hence we shall regard it as the perturbed amplitude. Inserting these expressions into the perturbing terms in Eq. (21) and Eq. (21'), *i.e.* into the second lines, and solving them for $C_{p,q}$ and $C^*_{-p,-q}$, and then inserting these results back into the defining equation for C_ν, (23), we get

$$\frac{1}{2V_q}=\sum_p{}'\frac{\tilde{\omega}_{p,q}}{\nu^2-\tilde{\omega}^2_{p,q}}-\sum_p{}'\sum_{p'}{}'\frac{V_{p-p'}(\nu^2+\tilde{\omega}\tilde{\omega}')}{(\nu^2-\tilde{\omega}^2)(\nu^2-\tilde{\omega}'^2)}+\sum_{(p+p')\cdot q<0}{}'\sum{}'\frac{V_q(\nu^2-\tilde{\omega}\tilde{\omega}')}{(\nu^2-\tilde{\omega}^2)(\nu^2-\tilde{\omega}'^2)}$$
$$+\sum_{(p+p')\cdot q>0}{}'\sum{}'\frac{V_{p+p'+q}(\nu^2-\tilde{\omega}\tilde{\omega}')}{(\nu^2-\tilde{\omega}^2)(\nu^2-\tilde{\omega}'^2)}. \tag{24}$$

Since we are treating the plasma oscillation, we can assume that

$$\nu\gg\tilde{\omega}$$

and simplify the secular equation for ν by expansion. We are going to get the dispersion formula up to the second order in q.

The first terms in the expansion give

$$\nu^2=2V_q\{\sum^q\omega_{p,q}+\sum_{(p+p')\cdot q<0}^{q}\sum^{q}(V_q-V_{p+p'+q})\}. \tag{25}$$

Here the summation sign $\sum^q$ denotes that one should sum over those p's inside a beret-shaped region which is bounded by the surfaces of the Fermi sphere and of the sphere obtained by displacing it by the amount $-q$. We have simplified the expression by using the equality:

$$\sum_p^q\sum_{p'}^q V_{p-p'}-\sum_p^q\sum_{p'}^q V_{p-p'}=\sum_p^q\sum_{p'}^q V_{p+q-p'}-\sum_p^q\sum_{p'}^q V_{p+q+p'}. \tag{26}$$

By using a similar equality, one can easily see that the next terms in the expansion contribute

$$\Delta\nu^2=\frac{2V_q}{\nu^2}\left\{\sum^{q}\omega_{p,q}^3+\frac{1}{2}\sum_{p}^{q}\sum_{p'}^{q}[(\omega-\omega')^2V_{p-p'}-(\omega+\omega')^2V_{p+p'+q}]\right.$$
$$\left.+\sum_{(p+p')\cdot q<0}^{q}\sum^{q}(V_q-V_{p+p'+q})(\omega^2+\omega'^2-\omega\omega')\right\}. \tag{27}$$

It is readily seen that the last sum in Eq. (27) is higher by q^2 than the last one in Eq. (25).

Let us evaluate the expressions for ν^2 and $\Delta\nu^2$ to the order q^2 in the case of spinless electrons in a positive charge sea. For this case

$$V_\lambda=\frac{4\pi e^2}{\lambda^2}\Omega^{-1} \tag{28}$$

and

$$\omega_{p,q}=\frac{p\cdot q}{m}+\frac{q^2}{2m}. \tag{29}$$

We shall give the values of the separate sums in Appendix 1. The final result is

$$\nu^2=\frac{2e^2p_F^3}{3\pi m}+\frac{e^4}{16\pi^2}q^2, \tag{30}$$

and with this value of ν^2 inserted,

$$\Delta\nu^2=\frac{3}{5}\frac{p_F^2}{m^2}q^2-\frac{1}{5\pi}\frac{e^2p_F}{m}q^2. \tag{31}$$

Transcribing these results in terms of the usual notations:

$$p_F^3=6\pi^2n,$$
$$\frac{1}{n}=\frac{4\pi}{3}r_s^3a_B^3,$$

and

$$a_B=(e^2m)^{-1}, \tag{32}$$

where n is the number density of the electrons and a_B is the Bohr radius, we obtain as the required dispersion formula for the plasma oscillation

$$\nu^2=\frac{4\pi e^2n}{m}+\frac{3}{5}\frac{p_F^2}{m^2}q^2\left\{1-\frac{1}{3\pi}\left(\frac{2}{9\pi}\right)^{1/3}r_s+\frac{5}{48\pi^2}\left(\frac{2}{9\pi}\right)^{2/3}r_s^2\right\}. \tag{33}$$

It is to be noted here that a correction to the non-dispersive term, which would otherwise contain a term proportional down to r_s, has been cancelled out by virtue of the relation (26). It may also be noted that the last term in the dispersion coefficient, which is proportional to e^4 or the correction factor proportional to r_s^2, has arisen from our taking account of the Pauli principle in the case

of doubly annihilation or creation processes in the region r_0.

The first and the second terms in the dispersion coefficient, (31), may also be written in the form:

$$q^2\left\{\left\langle\left(\frac{p}{m}\right)^2\right\rangle+\frac{4}{15}\frac{\epsilon_X}{m}\right\}, \tag{34}*$$

where the average is to be taken over the Fermi sphere and ϵ_X is the exchange energy per electron in the Fermi ground state, *i.e.*

$$\epsilon_X=-\frac{3}{4\pi}e^2 p_F. \tag{35}$$

This, including the result about the non-dispersive term, is just what one obtains on the basis of Hartree-Fock treatment of the plasma oscillation. We shall give the analysis in the Appendix 2.

§ 5. Interaction between the excitons and concluding remarks

One can extend the calculation to obtain the interaction Hamiltonian in a similar way. The three exciton interaction can be obtained by evaluating the expression

$$\langle[C_{p_1}^{P_1},\ [[U\mathcal{H}\tilde{U},\ C_p^{*P}],\ C_{p'}^{*P'}]]\rangle,$$

which, transformed into the fermion representation, is equal to

$$\langle[a_{p_1}^* a_{P_1},\ [[\mathcal{H},\ a_P{}^* a_p],\ a_{P'}^* a_{p'}]]\rangle.$$

The evaluation of this latter expression is straightforward, and one thus obtains

$$\begin{aligned}&\langle[C_{p_1}^{P_1},\ [[U\mathcal{H}\tilde{U},\ C_p^{*P}],\ C_{p'}^{*P'}]]\rangle\\&\quad=(V_{P'-p'}-V_{P-p'})\,\delta_{p_1,p}\delta_{p'+P_1,P+P'}+(V_{P-p}-V_{P'-p})\,\delta_{p_1,p'}\delta_{p+P_1,P+P'}\\&\quad+(V_{P-p'}-V_{P-p})\,\delta_{P_1,P'}\delta_{p_1+P,p+p'}+(V_{P'-p}-V_{P'-p'})\,\delta_{P_1,P}\delta_{p_1+P',p+p'}.\end{aligned} \tag{36}$$

Here one may note that

$$[[\mathcal{H}_1,\ a_P{}^* a_p],\ a_{P'}^* a_{p'}]=0, \tag{37}$$

so that $\mathcal{H}_1$ contributes only to the two-exciton Hamiltonian. Hence, the three-exciton interaction is given by

$$\begin{aligned}&\sum\sum(V_{P-p}-V_{P'-p})\{C_{p'}^{*P'+P-p}C_{p'}^{P'}C_p^P+C_p^{*P}C_{p'}^{*P'}C_{p'}^{P'+P-p}\}\\&\quad-\sum\sum(V_{P-p}-V_{P-p'})\{C_{p'+p-P}^{*P'}C_{p'}^{P'}C_p^P+C_p^{*P}C_{p'}^{*P'}C_{p'+p-P}^{P'}\}.\end{aligned} \tag{38}$$

The four-exciton interaction is rather heavy to work out, but the calculation goes in quite a similar straightforward way. However, we shall refrain from

* The author learned later that DuBois obtained the same answer through a different analysis. D. F. DuBois, thesis, California Institute of Technology, 1958.

writing it down here. Instead we shall only note that in the above expansion of the fermion Hamiltonian on the basis of exciton picture the interaction Hamiltonian will involve evaluation of a finite number of coefficients, since the multiple commutator of $\mathcal{H}_2$ with various $a_p^* a_p$'s vanishes beyond the fourth.

We shall defer the investigation of the interactions to a later paper.

We showed once[4] that any system of interacting Bose particles can be completely described in terms of Bogoliubov excitations.[5] There appeared interactions involving up to six excitations. The formulation was analogous to the spin wave theory of Dyson.[6] The same procedure was followed for the fermion case first, obtaining essentially the same Hamiltonian. It is, however, not hermitian and, besides, it is not straightforward to take full account of the Pauli principle. The utilization of the transformation (4) is certainly superior in this respect. The present method may have a more versatile applicability.

We presented here only one simple application and laid a stress on the formal aspect of the method. However, the investigation of the diamagnetism and the Froehlich interaction of metals on the basis of this formalism is interesting and, in fact, the work on these lines is under way in our laboratory.

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Appendix 1

We shall list here the various sums involved in Eqs. (25) and (27) :

$$\sum^{q} \omega_{p,q} = \frac{\Omega}{(2\pi)^3} 2\pi \frac{1}{m} \frac{1}{3} p_F^3 q^2 \tag{A·1}$$

$$\sum^{q} \omega_{p,q}^3 = \frac{\Omega}{(2\pi)^3} 2\pi \frac{1}{m^3} \frac{1}{5} p_F^5 q^4 + \cdots \tag{A·2}$$

$$\sum_{(p+p';\,q<0)}^{q} \sum^{q} (V_q - V_{p+p'+q}) \approx \sum \sum V_q = \frac{4\pi e^2}{\Omega} \left[\frac{\Omega}{(2\pi)^3} \right]^2 \frac{\pi^2}{8} q^4 + \cdots \tag{A·3}$$

$$\frac{1}{2} \sum^{q} \sum^{q} [(\omega_{p,q} - \omega_{p',q})^2 V_{p-p'} - (\omega_{p,q} + \omega_{p',q})^2 V_{p+p'+q}]$$

$$= -\frac{4}{15} \frac{4\pi e^2}{\Omega} \frac{q^2}{m^2} \left[\frac{\Omega}{(2\pi)^3} 2\pi \frac{1}{2} p_F^2 q \right]^2 + \cdots. \tag{A·4}$$

Appendix 2

Here we shall give a brief discussion based on Hartree-Fock's self-consistent field.

First transform the density matrix defined by the orthogonal oribitals equal in number to the number density of the electrons, to the "Wigner representation":

$$\rho(q, p)=\int\left(q+\frac{1}{2}x|\rho|q-\frac{1}{2}x\right)e^{-ixp/\hbar}dx, \tag{A·5}$$

with

$$(q'|\rho|q'')=\sum_{r}^{n}(q'|r)\,(r|q''). \tag{A·5′}$$

This obeys the equation of motion of the type,

$$\frac{\partial}{\partial t}\rho(q, p)=\frac{2}{\hbar}\sin\frac{\hbar}{2}\left[\frac{\partial}{\partial p_\rho}\frac{\partial}{\partial q_{\mathcal{H}}}-\frac{\partial}{\partial p_{\mathcal{H}}}\frac{\partial}{\partial q_\rho}\right]\mathcal{H}(q, p)\rho(q, p)$$

$$\approx\frac{\partial\rho}{\partial p}\frac{\partial\mathcal{H}}{\partial q}-\frac{\partial\mathcal{H}}{\partial p}\frac{\partial\rho}{\partial q}, \tag{A·6}$$

with the Hamiltonian

$$\mathcal{H}(q, p)=\frac{p^2}{2m}+e^2\int\frac{dq'}{|q-q'|}\int\rho(q', p')\frac{dp'}{(2\pi\hbar)^3}-e^2\int\frac{4\pi\hbar^2}{|p-p'|^2}\rho(q, p')\frac{dp'}{(2\pi\hbar)^3}. \tag{A·6′}$$ [7]

Let us assume that

$$\rho(\boldsymbol{q}, \boldsymbol{p}, t)=\rho_0(p)+\rho_1(\boldsymbol{q}, \boldsymbol{p}, t) \tag{A·7}$$

and linearize the equation. Here the assumed positive charge background cancels the Coulomb field due to the homogeneous distribution of the electrons $\rho_0(p)$.

$$\frac{\partial}{\partial t}\rho_1(q, p, t)=e^2\left(\int\frac{\partial}{\partial q}\frac{dq'}{|q-q'|}\int\rho_1(q', p', t)\frac{dp'}{(2\pi)^3}\right.$$

$$\left.-\int\frac{4\pi}{|p-p'|^2}\frac{\partial}{\partial q}\rho_1(q, p', t)\frac{dp'}{(2\pi)^3}\right)\frac{\partial\rho_0}{\partial p}$$

$$-\left(\frac{p}{m}-e^2\int\frac{4\pi}{|p-p'|^2}\frac{\partial}{\partial p'}\rho_0(p')\frac{dp'}{(2\pi)^3}\right)\frac{\partial\rho_1}{\partial q}. \tag{A·8}$$

Putting the fluctuating part of the Wigner distribution function in the form:

$$\rho_1=g(p)\,e^{i(kq-\nu t)}, \tag{A·9}$$

we obtain the eigen-value equation. The same perturbational method as in the text gives as the dispersion formula

$$1=-\frac{4\pi e^2}{mk^2}\int\frac{(k\cdot p)^2\frac{1}{p}\frac{d\rho_0}{dp}}{\nu^2-(k\cdot p/m)^2}\frac{d^3p}{(2\pi)^3}$$
$$+\frac{4\pi e^2}{k^2}\frac{2\pi e^2}{m^2}\iint\frac{k\cdot p\frac{1}{p}\frac{d\rho_0}{dp}}{(\nu-k\cdot p/m)^2}\frac{k\cdot p'\frac{1}{p'}\frac{d\rho_0}{dp'}}{(\nu-k\cdot p'/m)^2}\frac{(k\cdot(p-p'))^2}{|p-p'|^2}\frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}. \tag{A·10}$$

Assuming that $\nu^2\gg(k\cdot p/m)^2$ for the plasma oscillation and expanding in power of k^2, we can reduce Eq. (A. 10) to

$$\nu^2=\frac{4\pi e^2 n}{m}+k^2\left\langle\left(\frac{p}{m}\right)^2\right\rangle$$
$$+k^2\cdot\frac{2\pi e^2}{nmk^4}\iint\frac{(k\cdot p)(k\cdot p')(k\cdot(p-p'))^2}{|p-p'|^2}\frac{1}{p}\frac{d\rho_0}{dp}\frac{1}{p'}\frac{d\rho_0}{dp'}\frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}. \tag{A·11}$$

The last integral, divided by the factor k^2, can be shown to be equal to

$$\frac{4\pi e^2}{nm}\frac{1}{15}\iint\frac{1}{pp'}\log\left|\frac{p-p'}{p+p'}\right|\rho_0\rho_0'\frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}. \tag{A·12}$$

Hence Eq. (A. 11) can be written in the form

$$\nu^2=\frac{4\pi e^2 n}{m}+k^2\left\{\left\langle\left(\frac{p}{m}\right)^2\right\rangle+\frac{4}{15}\frac{\epsilon_x}{m}\right\}, \tag{A·13}$$

where ϵ_x is the exchange energy per electron in the state Ψ_0. Ferrell[8] gave a result of the same form except for a difference of the factor 1 instead of 4/15. The same result as Eq. (A. 13) was given by Silin.[9]

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