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Exhausting domains of the symmetrized bidisc

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Abstract. We show that the symmetrized bidisc may be exhausted by strongly linearly convex domains. It shows in particular the existence of a strongly linearly convex domain that cannot be exhausted by domains biholomorphic to convex ones.

1. Introduction

In our paper we show that the symmetrized bidisc can be exhausted by strongly linearly convex domains. Additionally, the symmetrized bidisc cannot be exhausted by domains biholomorphic to convex ones. These two facts have many interesting consequences. In particular we get a solution to open problems and we get alternative proofs of known results for the symmetrized bidisc.

Recall that a domain $D\subset\mathbb{C}^n$ is $\mathbb{C}$-*convex* if for any complex line ℓ intersecting D the intersection $\ell\cap D$ is connected and simply connected. A bounded domain $D\subset\mathbb{C}^n$ with C^2-boundary is called *strongly linearly convex* (sometimes it is called *strongly* $\mathbb{C}$-*convex*) if the defining function r of D satisfies the inequality

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(z_0) X_j \overline{X}_k > \left| \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k}(z_0) X_j X_k \right| \tag{1}$$

for any boundary point z_0 and any non-zero vector X from the complex tangent space to ∂D at z_0.

Basic facts on $\mathbb{C}$-convex domains and strongly linearly convex ones that we use in the paper can be found in [2] and [6]. Let us recall only that strong linear convexity implies $\mathbb{C}$-convexity.

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For $\varepsilon\in[0,1)$ let us define

$$D_\varepsilon := \big\{(s,p)\in\mathbb{C}^2 : \sqrt{|s-\bar{s}p|^2+\varepsilon}+|p|^2<1\big\}. \tag{2}$$

Note that D_0 is the symmetrized bidisc $\mathbb{G}_2$ (see [1] for the above description of the symmetrized bidisc) and $D_\varepsilon\nearrow\mathbb{G}_2$ as $\varepsilon\to 0^+$. Moreover, $\overline{D}_\varepsilon\subset\mathbb{C}\times\mathbb{D}$, $\varepsilon\in(0,1)$.

Note that the mapping

$$\mathbb{C}\times\mathbb{D}\ni(s,p)\longmapsto(s-\bar{s}p,p)\in\mathbb{C}^2 \tag{3}$$

is an $\mathbb{R}$-diffeomorphism onto the image. It shows in particular that the set D_ε is $\mathbb{R}$-diffeomorphic to the convex domain $G_\varepsilon=\big\{(w,z)\in\mathbb{C}^2:\sqrt{|w|^2+\varepsilon}+|z|^2<1\big\}$. Moreover, it is elementary to see that the strongly convex domains G_ε, $\varepsilon\in(0,1)$, exhaust the (non-strongly) convex domain G_0.

We show that a similar result holds for the domains D_ε.

Theorem 1.1. *The domain D_ε is strongly linearly convex, $\varepsilon\in(0,1)$. Consequently, the symmetrized bidisc can be exhausted by an increasing sequence of strongly linearly convex domains.*

Combining Theorem 1.1 with the fact that the symmetrized bidisc cannot be exhausted by domains biholomorphic to convex ones (see [4]) we get the following corollary, which gives a negative answer to a long-standing open problem on the existence of a strongly linearly convex domain not biholomorphic to a convex domain. Note that examples of strongly linearly convex domains, which are not convex are well known (see [11] and also [2]).

Corollary 1.2. *The domains D_ε for $\varepsilon>0$ small enough are examples of strongly linearly convex domains that are not biholomorphic to convex ones* (*and even cannot be exhausted by such domains*).

Remark. Recall that the equality between the Lempert function and the Carathéodory distance (i.e. the Lempert theorem) holds for strongly linearly convex domains (see [9]). Therefore, Theorem 1.1 implies that the equality between the two functions on the symmetrized bidisc follows directly from the Lempert theorem. It gives an alternative proof of that fact (to that in [1] and [3]). Moreover, it also implies that the tetrablock (recall that the *tetrablock* is the image under the proper mapping $A\mapsto(a_{11},a_{22},\det A)$ of the Cartan domain of the first type $\mathcal{R}_I(2,2)$) is the only known non-trivial example of a domain (i.e. bounded and pseudoconvex) for which the fact that the Lempert theorem holds does not follow directly from the papers [8] and [9] (see [5]). It would be interesting to know whether the tetrablock can be exhausted by strongly linearly convex domains.

Remark. Recall that in two papers ([8] and [9]) Lempert showed the equality of the Lempert function and the Carathéodory distance for different classes of domains (convex ones and strongly linearly convex). It was however unclear whether domains from the second class were not (up to biholomorphisms) equivalent to domains from the first one. Theorem 1.1 shows that this is not the case.

Remark. Theorem 1.1 also implies that $\mathbb{G}_2$ is a $\mathbb{C}$-convex domain—it gives an alternative proof to that in [10].

2. Proofs

Proof of Theorem 1.1. Let us fix $\varepsilon\in(0,1)$ and choose one of the possible (global) defining C^∞ functions for the domain D_ε,

$$r_\varepsilon(s,p):=r(s,p):=|s-\bar{s}p|^2+\varepsilon-(1-|p|^2)^2,\quad (s,p)\in\mathbb{C}\times\mathbb{D}. \tag{4}$$

Note that the defining function is even real-analytic.

First we note that the gradient of r does not vanish on ∂D_ε (we shall calculate the complex tangent below).

Now for a point $(s_0,p_0)\in\partial D_\varepsilon$ and (s,p) being a non-zero tangent vector to ∂D_ε (in the complex sense), we shall show that $\rho_{\lambda\bar\lambda}(0)>|\rho_{\lambda\lambda}(0)|$, where $\rho(\lambda):=r(s_0+\lambda s,p_0+\lambda p)$, $\lambda\in\mathbb{C}$. Note that for $\rho(s_0,p_0)=0$ and arbitrary (s,p) we have by Taylor expansion

$$\begin{aligned}\rho(\lambda) &= 2\operatorname{Re}(((\bar{s}_0-s_0\bar{p}_0)(s-\bar{s}_0p)-(s_0-\bar{s}_0p_0)s\bar{p}_0+2\bar{p}_0p-2|p_0|^2\bar{p}_0p)\lambda)\\ &\quad +|\lambda|^2(|s-\bar{s}_0p|^2+|s|^2|p_0|^2-2\operatorname{Re}((\bar{s}_0-s_0\bar{p}_0)\bar{s}p)+2|p|^2-2|p_0|^2|p|^2)\\ &\quad -\operatorname{Re}(2(s-\bar{s}_0p)s\bar{p}_0\lambda^2)-(\operatorname{Re}(2\bar{p}_0p\lambda))^2+o(\lambda^2).\end{aligned} \tag{5}$$

The above formula shows in particular that tangent vectors (s,p) to ∂D_ε are given by the formula

$$s(\bar{s}_0-s_0\bar{p}_0-\bar{p}_0(s_0-\bar{s}_0p_0))=p(\bar{s}_0(\bar{s}_0-s_0\bar{p}_0)-2\bar{p}_0+2|p_0|^2\bar{p}_0). \tag{6}$$

It is also elementary to see that for a C^2-function

$$v(\lambda)=\operatorname{Re}(A\lambda)+a|\lambda|^2+\operatorname{Re}(b\lambda^2)-(\operatorname{Re}(c\lambda))^2+o(\lambda^2),$$

where $a\in\mathbb{R}$, $A,b,c\in\mathbb{C}$, the condition for $v_{\lambda\bar\lambda}(0)>|v_{\lambda\lambda}(0)|$ is

$$a-\frac{|c|^2}{2}>\left|b-\frac{c^2}{2}\right|. \tag{7}$$

Applying this information to the function ρ we get the following inequality

$$|s-\bar{s}_0p|^2+|s|^2|p_0|^2-2\operatorname{Re}((\bar{s}_0-s_0\bar{p}_0)\bar{s}p)+2|p|^2-2|p_0|^2|p|^2-\frac{|2\bar{p}_0p|^2}{2}$$
$$(8)\qquad >\left|2(s-\bar{s}_0p)s\bar{p}_0+\frac{(2\bar{p}_0p)^2}{2}\right|$$

that when proven for boundary points (s_0,p_0) and non-zero tangents (s,p) will finish the proof of the theorem.

Substitute the condition on the tangency of the vector (s,p). Since the inequality is trivial when $s_0=0$ we may neglect this case. Then we divide both sides by $|p|^2$ and after reductions we get the inequality

$$\begin{aligned}&\big|2|p_0|^2\bar{p}_0-2\bar{p}_0+\bar{s}_0\bar{p}_0(s_0-\bar{s}_0p_0)\big|^2\\&\quad+|p_0|^2\big|\bar{s}_0(\bar{s}_0-s_0\bar{p}_0)-2\bar{p}_0+2|p_0|^2\bar{p}_0\big|^2\\&\quad-2\operatorname{Re}((\bar{s}_0-s_0\bar{p}_0)(s_0(s_0-\bar{s}_0p_0)-2p_0+2|p_0|^2p_0)(\bar{s}_0-s_0\bar{p}_0-\bar{p}_0(s_0-\bar{s}_0p_0)))\\&\quad+2|\bar{s}_0-s_0\bar{p}_0-\bar{p}_0(s_0-\bar{s}_0p_0)|^2-4|p_0|^2|\bar{s}_0-s_0\bar{p}_0-\bar{p}_0(s_0-\bar{s}_0p_0)|^2\\ (9)\qquad&\quad>\big|2(2|p_0|^2\bar{p}_0-2\bar{p}_0+\bar{s}_0\bar{p}_0(s_0-\bar{s}_0p_0))(\bar{s}_0(\bar{s}_0-s_0\bar{p}_0)-2\bar{p}_0+2|p_0|^2\bar{p}_0)\bar{p}_0\\&\quad+2\bar{p}_0^2(\bar{s}_0-s_0\bar{p}_0-\bar{p}_0(s_0-\bar{s}_0p_0))^2\big|.\end{aligned}$$

Let us get rid of subscripts. After elementary calculations we get the inequality

$$\begin{aligned}&|p|^2\big|2|p|^2-2+\bar{s}(s-\bar{s}p)\big|^2+|p|^2\big|\bar{s}(s-s\bar{p})+2|p|^2\bar{p}-2\bar{p}\big|^2\\&\quad-2\operatorname{Re}((\bar{s}-s\bar{p})(s(s-\bar{s}p)-2p+2|p|^2p)(\bar{s}-s\bar{p}-\bar{p}(s-\bar{s}p)))\\&\quad+2|\bar{s}-s\bar{p}-\bar{p}(s-\bar{s}p)|^2-4|p|^2|\bar{s}-s\bar{p}-\bar{p}(s-\bar{s}p)|^2\\ (10)\qquad&\quad>2|p|^2\big|(2|p|^2-2+\bar{s}(s-\bar{s}p))(\bar{s}(\bar{s}-s\bar{p})-2\bar{p}+2|p|^2\bar{p})\\&\quad+(\bar{s}-s\bar{p}-\bar{p}(s-\bar{s}p))^2\big|.\end{aligned}$$

Note that the above function is invariant with respect to the mapping $(s,p)\mapsto(e^{it}s,e^{i2t}p)$, which means that we may assume that $s\geq0$. Since $\rho(s,p)=0$ we get that

$$s^2=\frac{(1-|p|^2)^2-\varepsilon}{|1-p|^2}$$

(and p may be an arbitrary complex number satisfying the inequality $\varepsilon\leq(1-|p|^2)^2$). Therefore, we get that

$$\begin{aligned}
&|p|^2\big|2(|p|^2-1)(1-\bar p)+(1-|p|^2)^2-\varepsilon\big|^2\\
&\quad+|p|^2\big|(1-|p|^2)^2-\varepsilon-2\bar p(1-|p|^2)(1-p)\big|^2\\
&\quad-2((1-|p|^2)^2-\varepsilon)\,\mathrm{Re}\biggl((1-\bar p)\biggl(\frac{(1-|p|^2)^2-\varepsilon}{1-\bar p}-2p(1-|p|^2)\biggr)(1-2\bar p+|p|^2)\biggr)\\
&\quad+2((1-|p|^2)^2-\varepsilon)\big|1-2\bar p+|p|^2\big|^2-4|p|^2((1-|p|^2)^2-\varepsilon)\big|1-2\bar p+|p|^2\big|^2\\
&\quad>2|p|^2\big|(2(|p|^2-1)(1-\bar p)+(1-|p|^2)^2-\varepsilon)((1-|p|^2)^2-\varepsilon-2\bar p(1-|p|^2)(1-p))\\
&\quad+((1-|p|^2)^2-\varepsilon)(1-2\bar p+|p|^2)^2\big|,
\end{aligned}\tag{11}$$

which is equivalent to the inequality

$$\begin{aligned}
&\big|1-2p+|p|^2\big|^2 2|p|^2\varepsilon+2|p|^2\varepsilon^2+2\varepsilon((1-|p|^2)^2-\varepsilon)\,\mathrm{Re}(1-2p+|p|^2)\\
&\qquad\qquad>2|p|^2\big|\varepsilon^2-\varepsilon(1-2p+|p|^2)^2\big|.
\end{aligned}\tag{12}$$

Note that $\mathrm{Re}(1-2p+|p|^2)=|1-p|^2>0$, which easily implies that the above inequality holds for all possible p (i.e. satisfying the inequality $(1-|p|^2)^2\geq\varepsilon$). $\square$

Remark. Let us recall some of the open questions concerning the strongly linearly convex and $\mathbb{C}$-convex domains that still remain open and that can be found in [2] and [12]:

(a) Does the Lempert theorem hold for any bounded $\mathbb{C}$-convex domain?

(b) Can any bounded $\mathbb{C}$-convex domain be exhausted by strongly linearly convex ones? The answer is positive under an additional assumption of smoothness of D, see [7].

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