Hiroshima Math. J.
44 (2014), 275–313

Existence and analyticity of solutions to the drift-diffusion equation with critical dissipation

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(Received November 29, 2013)
(Revised April 1, 2014)

Abstract. The initial value problem for the drift-diffusion equation arising from a model of semiconductor-devices is studied. The goal in this paper is to derive well-posedness and real analyticity of solutions of the initial value problem for the drift-diffusion equation with its dissipating term $\Lambda = (-\Delta)^{1/2}$. In the preceding works for some associated equations, the case corresponding to this is known as critical. In this case, the drift-diffusion equation with Λ is of elliptic type, so we may not apply the L^p-theory for parabolic partial differential equations used in the case that the dissipating term is $\Lambda^\theta = (-\Delta)^{\theta/2}$ with $1 < \theta \le 2$.

1. Introduction

We consider the following initial value problem for the drift-diffusion equation arising from a model of semiconductors:

$$\begin{cases} \partial_t u + \Lambda^\theta u - \nabla \cdot (u \nabla \psi) = 0, & t > 0,\ x \in \mathbf{R}^n, \\ -\Delta \psi = u, & t > 0,\ x \in \mathbf{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1}$$

where $n \ge 2$, $1 \le \theta \le 2$, $\partial_t = \partial/\partial t$, $\nabla = (\partial_1, \dots, \partial_n)$, $\partial_j = \partial/\partial x_j$ $(j = 1, \dots, n)$, $\Lambda^\theta \varphi = \mathscr{F}^{-1}[|\xi|^\theta \mathscr{F}[\varphi]]$, $\Delta = \sum_{j=1}^n \partial_j^2$, and $u_0 = u_0(x)$ is given real valued initial data. The unknown functions $u = u(t,x)$ and $\psi = \psi(t,x)$ stand for the density of electrons and the potential of electromagnetic-field in a semiconductor, respectively. When $\theta = 2$, the dissipative operator Λ^2 gives the positive Laplacian $-\Delta$. When $1 \le \theta < 2$, the fractional Laplacian Λ^θ involves the jumping-process in the stochastic-process and it gives the suitable dissipation to describe the dynamics of electrons in a semiconductor. This operator yields the anomalous diffusion in dissipative equations. For the basic properties of

2010 *Mathematics Subject Classification*. Primary 35A01, 35B65, 35J60, 76D03; Secondary 35L65, 35Q35.

Key words and phrases. drift-diffusion equation, anomalous diffusion, critical dissipation, analyticity.

the anomalous diffusion we refer to Brandolese and Karch [1], Karch [14], Metzler and Klafter [25], and references therein. In particular, when $\theta = 1$, the operator $\Lambda\varphi = \mathscr{F}^{-1}[|\xi|\mathscr{F}[\varphi]]$ is called the half-Laplacian and the first equation on (1) is of elliptic type.

The drift-diffusion equation was first considered as a Neumann problem on a bounded domain with $\theta = 2$ in Mock [27]. The first equation $\partial_t u - \Delta u - \nabla \cdot (u\nabla\psi) = 0$ is derived from the mass-conservation-law for electrons and the Poisson equation $-\Delta\psi = u$ provides the potential of an electromagnetic-field. For this Neumann problem, well-posedness and asymptotic-stability of time steady solutions were shown. For the initial value problem (1) with $1 < \theta \le 2$, well-posedness, global existence in time, decay and spatial analyticity of solutions were proved in Kawashima and Kobayashi [19], Kurokiba and Ogawa [21], Ogawa and the first author [29], Matsumoto and Tanaka [24] and the first author [35]. These facts were shown by employing the L^p-theory for equation of parabolic type. Unfortunately these arguments are difficult to extend into the case $\theta = 1$ since the dissipation balances the nonlinearity. Namely our equation is of elliptic type in the case $\theta = 1$ (see the remark after Theorem 1 in this section). Our goal is to derive well-posedness and analyticity of solutions of (1) when $\theta = 1$.

Before stating our results, we refer to some preceding works for involving equations. When $\theta = 2$, we see the Navier-Stokes equation describes the model of incompressible fluid flow and the Keller-Segel equation appearing in the model of chemotaxis (cf. Escudero [9], Giga, Miyakawa and Osada [11], Keller and Segal [20], and Nagai, Senba and Yoshida [28]). For those problems, well-posedness, global existence in time and decay of solutions are considered by many authors. Moreover spatial analyticity of solutions under several conditions was shown in Giga and Sawada [12], Kahane [13], Masuda [23], and Sawada [30]. When $1 \le \theta \le 2$, we refer to the following two-dimensional quasi-geostrophic equation with the fractional dissipation:

$$\begin{cases} \partial_t u + \Lambda^\theta u - \nabla^\perp\psi \cdot \nabla u = 0, & t > 0,\ x \in \mathbf{R}^2, \\ (-\Delta)^{1/2}\psi = u, & t > 0,\ x \in \mathbf{R}^2, \end{cases}$$

where $\nabla^\perp = (-\partial_2, \partial_1)$. We remark that, for the potential-term on this equation, the divergence-free condition $\nabla \cdot \nabla^\perp\psi = 0$ holds. The quasi-geostrophic equation is a model which corresponds to geophysical fluid dynamics. This is very much related to the three-dimensional incompressible Euler equation (see Chae, Constantin and Wu [4], and Constantin, Majda and Tabak [5]). For the initial value problem of this equation with $1 < \theta \le 2$, existence of smooth solutions was shown. On the preceding studies for the quasi-geostrophic equation, the case $\theta = 1$ is known as critical (cf. Caffarelli and

Vasseur [3], Constantin and Wu [7], Cordoba and Cordoba [8], Kiselev and Nazarov [17], Kiselev, Nazarov and Volberg [18] and references therein). Indeed the structure of the quasi-geostrophic equation with $\theta = 1$ is similar to that of the three-dimensional Navier-Stokes equation. For this fact we refer to Miura [26]. On the studies for the critical case by Constantin, Cordoba and Wu [6], smallness of the initial data in $L^\infty(\mathbf{R}^2)$ was assumed in order to derive existence of solutions. Moreover the structure of the potential-term, namely the divergence-free condition, was applied in those preceding works. Maekawa and Miura [22] also studied well-posedness of solutions to the fractional dissipative equation with a generalized potential-term which satisfies the divergence-free condition. Since our potential-term $\nabla\psi$ has not such a structure, the idea of the preceding studies might not work to our problem (1).

Before considering the critical case, we briefly review the results of the subcritical case for our problem. For the initial value problem (1) with $1 < \theta \le 2$, Ogawa and the first author [29] and Matsumoto and Tanaka [24] showed the following properties.

Proposition 1. *Let $n \ge 2$, $1 < \theta \le 2$, $n/\theta \le p < n$ and $u_0 \in L^p(\mathbf{R}^n)$. Then there exist a positive constant T and a unique solution u of* (1) *such that*

$$u \in C([0,T); L^p(\mathbf{R}^n)) \cap C((0,T); W^{\theta,p}(\mathbf{R}^n)) \cap C^1((0,T); L^p(\mathbf{R}^n)).$$

Moreover, if $u_0 \ge 0$ and $u_0 \in L^1(\mathbf{R}^n)$ are assumed, then $u(t,x) \ge 0$ and $\|u(t)\|_1 = \|u_0\|_1$ hold for any $t > 0$ and $x \in \mathbf{R}^n$. In addition, assume that $u_0 \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$, then the solution of (1) *exists globally in time and satisfies*

$$\|u(t)\|_{L^p(\mathbf{R}^n)} \le C(1+t)^{-(n/\theta)(1-1/p)}$$

for any $t > 0$ and $1 \le p \le \infty$.

Proposition 2. *Let $n \ge 2$, $1 < \theta \le 2$ and $u_0 \in L^{n/\theta}(\mathbf{R}^n)$. Assume that the solution u of* (1) *exists globally in time and satisfies*

$$\|u(t)\|_{L^p(\mathbf{R}^n)} \le C(1+t)^{-(n/\theta)(\theta/n-1/p)}$$

for any $t > 0$ and $n/\theta \le p \le \infty$. Then there exist positive constants K_1 and K_2 and $1/\theta < \delta < 1$ such that

$$\|\nabla^\beta u(t)\|_{L^p(\mathbf{R}^n)} \le K_1(K_2|\beta|)^{|\beta|-\delta} t^{-(n/\theta)(\theta/n-1/p)-|\beta|/\theta}$$

for any $n/\theta \le p \le \infty$, $\beta \in \mathbf{Z}^n_+ \backslash \{0\}$ and $t > 0$. Especially, the solution u is analytic in x.

Proposition 2 was proved in [35]. The above propositions are proved by employing the L^p-theory for equation of parabolic type. Unfortunately, in the case $\theta = 1$, we cannot extend those propositions since our equation is of elliptic type. Hereafter we consider this case. Namely we study

$$\partial_t u + \Lambda u - \nabla \cdot (u \nabla \psi) = 0.$$

In order to discuss our results, we introduce the following weighted-Sobolev spaces:

$$\begin{aligned} H_m^s(\mathbf{R}^n) &= \{\varphi \in \mathscr{D}'(\mathbf{R}^n) \mid \langle x \rangle^m J^s \varphi \in L^2(\mathbf{R}^n)\}, \\ \mathscr{H}_m^s(\mathbf{R}^n) &= \{\varphi \in \mathscr{D}'(\mathbf{R}^n) \mid |x|^m J^s \varphi \in L^2(\mathbf{R}^n)\}, \end{aligned} \tag{2}$$

where $J^s = (1 - \Delta)^{s/2}$ and $\langle x \rangle = \sqrt{1 + |x|^2}$. For simplicity, we represent $H_m(\mathbf{R}^n) = H_m^0(\mathbf{R}^n)$ and $\mathscr{H}_m(\mathbf{R}^n) = \mathscr{H}_m^0(\mathbf{R}^n)$. The inner-products on $H_m^s(\mathbf{R}^n)$ and $\mathscr{H}_m^s(\mathbf{R}^n)$ are given by

$$\begin{aligned} \langle f, g \rangle_{H_m^s(\mathbf{R}^n)} &= \langle \langle x \rangle^m J^s f, \langle x \rangle^m J^s g \rangle_{L^2(\mathbf{R}^n)}, \\ \langle f, g \rangle_{\mathscr{H}_m^s(\mathbf{R}^n)} &= \langle |x|^m J^s f, |x|^m J^s g \rangle_{L^2(\mathbf{R}^n)}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{L^2(\mathbf{R}^n)}$ is the standard inner-product on $L^2(\mathbf{R}^n)$. Then we give well-posedness of solutions of (1).

Theorem 1. *Let $n \geq 3$, $\theta = 1$, $s > \frac{n}{2} + 1$ and $u_0 \in H_2^s(\mathbf{R}^n)$. Then there exist a positive constant $T > 0$ and a unique solution u of* (1) *such that*

$$u \in L^\infty(0, T; H_2^s(\mathbf{R}^n)) \cap L^2(0, T; \dot{H}^{s+1/2}(\mathbf{R}^n)),$$

where $H_2^s(\mathbf{R}^n)$ is defined as (2).

In the assumption of this theorem the regularity $H^s(\mathbf{R}^n)$ with $s > n/2 + 1$ is not essential. Indeed it is possible that we solve (1) on some natural class in view of the invariant-scaling $u_\lambda(t, x) = \lambda u(\lambda t, \lambda x)$ $(\lambda > 0)$ but we will not develop this point here. Also we can prove the assertion of this theorem if the initial function is in $H_1^s(\mathbf{R}^n)$ instead of in $H_2^s(\mathbf{R}^n)$. However the proof would be longer in this case, because the estimate of the term $[\langle x \rangle J^s, \Lambda]$ is more complicated than that of $[|x|^2 J^s, \Lambda]$ in (20) for example. For simplicity, we assume that the initial function is in $H_2^s(\mathbf{R}^n)$ in this paper. A more complete claim will be proved in our forthcoming paper [32]. The proof of Theorem 1 is based on the energy method with weighted $L^2(\mathbf{R}^n)$-norm. When $1 < \theta \leq 2$, we have derived Proposition 1 by employing the energy method with usual $L^2(\mathbf{R}^n)$-norm and the Sobolev inequality (see [21, 29]). Unfortunately, when $\theta = 1$, we cannot apply the similar procedure as in the proof of Proposition 1 since the regularizing effect from Λ is too weak in this case. In Section 3, we prove Theorem 1 by employing a commutator estimate via Kato and Ponce [16] and Hardy's inequality.

Our second objective is established as follows.

Theorem 2. *Let $n \geq 3$, $\theta = 1$, $s > \frac{n}{2} + 1$ and $u_0 \in H_2^s(\mathbf{R}^n)$, where $H_2^s(\mathbf{R}^n)$ is defined as* (2). *Then the solution of* (1) *is real analytic with respect to both the space and the time variable on $(0, T) \times \mathbf{R}^n$, where T is the positive constant which is determined in Theorem* 1.

We do not use the decay property that $\langle x \rangle^2 u(t, x) \in L^2(\mathbf{R}^n)$ in the proof of Theorem 2. Our proof works when $u \in C^\infty((0, T); H^\infty(\mathbf{R}^n))$ is ensured. In [15], the second author showed analyticity of the solution to elliptic equations by using a cut-off function. In Section 4, we employ this idea in order to prove Theorem 2.

By using the similar arguments as in this paper, we can treat some fractional dissipative equations with a potential-term. For example we consider the following Keller-Segel equation of parabolic-elliptic type:

$$\begin{cases} \partial_t u + \Lambda^\theta u + \nabla \cdot (u \nabla \psi) = 0, & t > 0,\ x \in \mathbf{R}^n, \\ \psi - \Delta \psi = u, & t > 0,\ x \in \mathbf{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbf{R}^n. \end{cases} \tag{3}$$

Then we have well-posedness of solutions of (3) with $0 < \theta \leq 2$. In particular we obtain analyticity of the solution when $1 \leq \theta \leq 2$.

Notation. In this paper, we use the following notation. For $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in \mathbf{R}^n$, we denote $x \cdot y = x_1 y_1 + \cdots + x_n y_n$, $|x| = \sqrt{x \cdot x}$ and $\langle x \rangle = \sqrt{1 + |x|^2}$. We define the Fourier transform and the Fourier inverse transform by $\mathscr{F}[\varphi](\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-ix\cdot\xi} \varphi(x) dx$, $\mathscr{F}^{-1}[\varphi](x) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \varphi(\xi) d\xi$, $\mathscr{F}_{t,x}[\varphi](\tau, \xi) = (2\pi)^{-(n+1)/2} \iint_{\mathbf{R}\times\mathbf{R}^n} e^{-it\tau - ix\cdot\xi} \varphi(t, x) dx dt$, $\mathscr{F}_{\tau,\xi}^{-1}[\varphi](t, x) = (2\pi)^{-(n+1)/2} \iint_{\mathbf{R}\times\mathbf{R}^n} e^{it\tau + ix\cdot\xi} \varphi(\tau, \xi) d\xi d\tau$, where $i = \sqrt{-1}$. For simplicity we denote $\hat{\varphi}(\xi) = \mathscr{F}[\varphi](\xi)$. The partial derivative operators are denoted by $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $(1 \leq j \leq n)$, $\nabla = (\partial_1, \dots, \partial_n)$, $\Delta = \partial_1^2 + \cdots + \partial_n^2$, $\Lambda^\theta \varphi = \mathscr{F}^{-1}[|\xi|^\theta \mathscr{F}[\varphi]]$ and $J^s \varphi = (1 - \Delta)^{s/2} \varphi = \mathscr{F}^{-1}[(1 + |\xi|^2)^{s/2} \mathscr{F}[\varphi]]$. The gamma function $\Gamma(p)$ for $p > 0$ is provided by $\Gamma(p) = \int_0^\infty e^{-t} t^{p-1}\, dt$. We write $[A, B] = AB - BA$ for operators A and B. For $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{Z}_+^n$, we use $\nabla^\alpha = \prod_{i=1}^n \partial_i^{\alpha_i}$ and $|\beta| = \sum_{i=1}^n \beta_i$, where $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$. We write $\alpha \leq \beta$ if $\alpha_j \leq \beta_j$ $(1 \leq j \leq n)$, and $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$. We denote $\binom{\beta}{\alpha} = \prod_{j=1}^n \frac{\beta_j!}{\alpha_j!(\beta_j - \alpha_j)!}$ for α and $\beta \in \mathbf{Z}_+^n$ such as $\alpha \leq \beta$. We denote L^p and H^s the Lebesgue spaces and the Sobolev spaces for $1 \leq p \leq \infty$ and $s \in \mathbf{R}$.

The norm of $L^p(\mathbf{R}^n)$ and $H^s(\mathbf{R}^n)$ are represented by $\|\cdot\|_{L^p(\mathbf{R}^n)}$ and $\|\cdot\|_{H^s(\mathbf{R}^n)}$. In particular the inner-products on $L^2(\mathbf{R}^n)$ and $H^s(\mathbf{R}^n)$ are denoted by $\langle f,g\rangle_{L^2(\mathbf{R}^n)} = \int_{\mathbf{R}^n} f(x)g(x)dx$ and $\langle f,g\rangle_{H^s(\mathbf{R}^n)} = \langle J^s f, J^s g\rangle_{L^2}$. The homogenous Sobolev spaces and their inner-products are represented by $\dot{H}^s(\mathbf{R}^n)$ and $\langle f,g\rangle_{\dot{H}^s(\mathbf{R}^n)} = \langle \Lambda^s f, \Lambda^s g\rangle_{L^2(\mathbf{R}^n)}$. We denote the set of all functions f on $\mathbf{R}^n$ whose Lipschitz-norm $\|f\|_{\mathrm{Lip}(\mathbf{R}^n)} = \|f\|_{L^\infty(\mathbf{R}^n)} + \|\nabla f\|_{L^\infty(\mathbf{R}^n)}$ is finite by $\mathrm{Lip}(\mathbf{R}^n)$. For $\mu \in \mathbf{R}$, the Gauss' symbol $[\mu]$ describes $[\mu] = \max\{m \in \mathbf{Z} \mid m \le \mu\}$. The dual space of a normed space X is denoted by X^*. We denote the coupling of $f \in X^*$ and $x \in X$ by $\langle f, x\rangle$. For normed spaces X and Y, we denote the set of linear bounded operators X to Y by $\mathscr{L}(X,Y)$. Especially we denote $\mathscr{L}(X) = \mathscr{L}(X,X)$ and $\|A\|_{\mathscr{L}(X)} = \sup_{x\neq 0}\|Ax\|_X/\|x\|_X$ for $A \in \mathscr{L}(X)$. Various constants are simply denoted by C.

2. Preliminaries

In this section, we prepare several lemmas to be used in the proof of our main conclusions.

LEMMA 1 (Kato-Ponce's inequality). *Let $s > 0$ and $f \in H^s(\mathbf{R}^n) \cap \mathrm{Lip}(\mathbf{R}^n)$ and $g \in H^{s-1}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$. Then the following inequality holds:*

$$\|[J^s, f]g\|_{L^2(\mathbf{R}^n)} \le C(\|f\|_{H^s(\mathbf{R}^n)}\|g\|_{L^\infty(\mathbf{R}^n)} + \|f\|_{\mathrm{Lip}(\mathbf{R}^n)}\|g\|_{H^{s-1}(\mathbf{R}^n)}),$$

where $J^s = (1-\Delta)^{s/2}$ and C is a positive constant which is independent of f and g.

PROOF. For the proof of this lemma, see [16]. □

LEMMA 2 (Hardy's inequality). *Let $0 \le s < n/2$. Then there exists a positive constant C such that the inequality*

$$\|\,|x|^{-s} f\|_{L^2(\mathbf{R}^n)} \le C\|\Lambda^s f\|_{L^2(\mathbf{R}^n)}$$

holds for any $f \in H^s(\mathbf{R}^n)$.

PROOF. The proof of this lemma is given in [33]. □

Hardy's inequality provides the following inequalities.

LEMMA 3. *Let $n \ge 3$. Then there exists a positive constant C such that the inequality*

$$\|\nabla(-\Delta)^{-1}\varphi\|_{L^2(\mathbf{R}^n)} \le C\|\varphi\|_{\mathscr{H}_1(\mathbf{R}^n)}$$

holds for any $\varphi \in \mathscr{H}_1(\mathbf{R}^n)$.

Proof. Using the Perseval identity, we see that

$$\|\nabla(-\Delta)^{-1}\varphi\|_{L^2(\mathbf{R}^n)} \le C\| \, |\xi|^{-1}\hat{\varphi}\|_{L^2(\mathbf{R}^n)}.$$

Applying Lemma 2, we obtain the desired inequality. □

Lemma 4. *Let $s > \frac{n}{2}+1$. Then there exists a positive constant C such that the inequality*

$$\|\nabla(-\Delta)^{-1}\varphi\|_{L^\infty(\mathbf{R}^n)} \le C(\|\varphi\|_{H^{s-1}(\mathbf{R}^n)} + \|\varphi\|_{\mathscr{H}_1(\mathbf{R}^n)})$$

holds for any $\varphi \in H_1^{s-1}(\mathbf{R}^n)$.

Proof. The Sobolev inequality yields

$$\|\nabla(-\Delta)^{-1}\varphi\|_{L^\infty(\mathbf{R}^n)} \le C\|\nabla(-\Delta)^{-1}\varphi\|_{H^s(\mathbf{R}^n)}.$$

Hence, employing the Perseval identity, we see that

$$\begin{aligned}\|\nabla(-\Delta)^{-1}\varphi\|_{H^s(\mathbf{R}^n)}^2 &\le C\int_{\mathbf{R}^n} \frac{(1+|\xi|^2)^s}{|\xi|^2}|\hat{\varphi}(\xi)|^2 d\xi \\ &= C\int_{|\xi|\le 1} \frac{(1+|\xi|^2)^s}{|\xi|^2}|\hat{\varphi}(\xi)|^2 d\xi \\ &\quad + C\int_{|\xi|>1} \frac{(1+|\xi|^2)^s}{|\xi|^2}|\hat{\varphi}(\xi)|^2 d\xi. \end{aligned} \tag{4}$$

Lemma 2 yields

$$\begin{aligned}\int_{|\xi|\le 1} \frac{(1+|\xi|^2)^s}{|\xi|^2}|\hat{\varphi}(\xi)|^2 d\xi &\le C\int_{\mathbf{R}^n} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|^2}\, d\xi \\ &\le C\|\varphi\|_{\mathscr{H}_1(\mathbf{R}^n)}^2.\end{aligned}$$

For the second term on the right hand side of (4), we have

$$\begin{aligned}\int_{|\xi|>1} \frac{(1+|\xi|^2)^s}{|\xi|^2}|\hat{\varphi}(\xi)|^2 d\xi &\le C\int_{\mathbf{R}^n} (1+|\xi|^2)^{s-1}|\hat{\varphi}(\xi)|^2 d\xi \\ &\le C\|\varphi\|_{H^{s-1}(\mathbf{R}^n)}^2.\end{aligned}$$

Thus we obtain that

$$\|\nabla(-\Delta)^{-1}\varphi\|_{H^s(\mathbf{R}^n)} \le C(\|\varphi\|_{H^{s-1}(\mathbf{R}^n)} + \|\varphi\|_{\mathscr{H}_1(\mathbf{R}^n)}). \tag{5}$$

Consequently we derive the desired inequality. □

The following lemma is well-known in functional analysis.

LEMMA 5 (Banach-Alaoglu's theorem). *Let X be a separable normed space, and let $\{f_n\}_{n\in\mathbf{N}} \subset X^*$ be a sequence which is norm bounded. Then there exists a subsequence of $\{f_n\}_{n\in\mathbf{N}}$ which converges weakly*. Namely there exists $f \in X^*$ such that*

$$\lim_{n\to\infty} \langle f_n, x\rangle = \langle f, x\rangle$$

holds for any $x \in X$.

PROOF. For the proof of Lemma 5, we refer [2]. □

On the Sobolev spaces, a product of functions is treated by the following inequality.

LEMMA 6. *Let $s > 0$. Then there exists a positive constant C such that the inequality*

$$\|uv\|_{H^s(\mathbf{R}^n)} \le C(\|u\|_{H^s(\mathbf{R}^n)}\|v\|_{L^\infty(\mathbf{R}^n)} + \|u\|_{L^\infty(\mathbf{R}^n)}\|v\|_{H^s(\mathbf{R}^n)})$$

is satisfied for any $u, v \in H^s(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$.

PROOF. The proof of this lemma is given in [34]. □

For a pseudo-differential operator, we obtain the following lemma.

LEMMA 7 (Calderón-Vaillancourt's theorem). *Let $p \in C^{2n+1}(\mathbf{R}^n \times \mathbf{R}^n)$ and the pseudo-differential operator $P(x, D_x)$ be defined by*

$$P(x, D_x)\varphi(x) = \int_{\mathbf{R}^n} e^{ix\cdot\xi} p(x, \xi)\hat{\varphi}(\xi)d\xi$$

for $\varphi \in L^2(\mathbf{R}^n)$. Assume that

$$\sum_{|\alpha+\beta|\le 2n+1} \|\nabla_x^\alpha \nabla_\xi^\beta p\|_{L^\infty(\mathbf{R}^n\times\mathbf{R}^n)} < \infty \tag{6}$$

is satisfied. Then

$$P(x, D_x) \in \mathscr{L}(L^2(\mathbf{R}^n))$$

holds.

PROOF. For the proof of Lemma 7, we refer to [10]. □

3. Proof of well-posedness of solutions

In order to prove Theorem 1, we represent our problem as

$$\begin{cases} \partial_t u + \Lambda u - \nabla(-\Delta)^{-1}u \cdot \nabla u + u^2 = 0, & t > 0,\ x \in \mathbf{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbf{R}^n, \end{cases}$$

where the operator $(-\Delta)^{-1}$ is defined by

$$(-\Delta)^{-1}\varphi(x) = \frac{\Gamma(n/2)}{2\pi^{n/2}(n-2)} \int_{\mathbf{R}^n} \frac{\varphi(y)}{|x-y|^{n-2}}\,dy.$$

For some $s > \frac{n}{2} + 1$ and $T > 0$, we introduce the following complete metric spaces X_T and Y_T:

$$X_T = L^\infty(0,T;H_2^s(\mathbf{R}^n)), \qquad \|u\|_{X_T} = \sup_{t\in(0,T)} \|\langle x\rangle^2 J^s u(t)\|_{L^2(\mathbf{R}^n)},$$

$$Y_T = L^\infty(0,T;H_2(\mathbf{R}^n)), \qquad \|u\|_{Y_T} = \sup_{t\in(0,T)} \|\langle x\rangle^2 u(t)\|_{L^2(\mathbf{R}^n)},$$

where $H_2^s(\mathbf{R}^n)$ and $H_2(\mathbf{R}^n)$ are defined as (2). For $M > 0$, we define

$$X_{T,M} = \{u \in X_T \mid \|u\|_{X_T} \le M\}, \tag{7}$$

in which we look for solutions.

Proposition 3. *Let the function spaces $X_{T,M}$ and Y_T be defined as above. Then $X_{T,M}$ is a closed subset in Y_T.*

Proof. Since Y_T is a Banach space, for any Cauchy sequence $\{u_m\}_{m\in\mathbf{N}}$ in Y_T with $\{u_m\}_{m\in\mathbf{N}} \subset X_{T,M}$, there exists $u \in Y_T$ satisfying $u_m \to u$ as $m \to \infty$ in Y_T. Namely

$$\lim_{m\to\infty} \langle x\rangle^2 u_m = \langle x\rangle^2 u \qquad \text{in } L^\infty(0,T;L^2(\mathbf{R}^n)).$$

We show $u \in X_{T,M}$. Since $u_m \in X_{T,M}$, we see that

$$\sup_{0<t<T} \|\langle x\rangle^2 J^s u_m(t)\|_{L^2(\mathbf{R}^n)} \le M.$$

In addition $L^\infty(0,T;L^2(\mathbf{R}^n))$ is the dual space of $L^1(0,T;L^2(\mathbf{R}^n))$. Hence Lemma 5 states that there exist a subsequence $\{u_{m_j}\}_{j\in\mathbf{N}} \subset \{u_m\}_{m\in\mathbf{N}}$ and $v \in L^\infty(0,T;L^2(\mathbf{R}^n))$ such that

$$\text{w}^*\text{-}\lim_{j\to\infty} \langle x\rangle^2 J^s u_{m_j} = v \qquad \text{in } L^\infty(0,T;L^2(\mathbf{R}^n)).$$

We define the function $\tilde{u}$ by $\tilde{u} = J^{-s}(\langle x\rangle^{-2}v)$. Then we see that

$$\tilde{u} \in L^\infty(0,T;H_2^s(\mathbf{R}^n))$$

and

$$\begin{aligned}\sup_{0<t<T} \|\langle x\rangle^2 J^s \tilde{u}(t)\|_{L^2(\mathbf{R}^n)} &= \sup_{0<t<T} \|v(t)\|_{L^2(\mathbf{R}^n)} \\ &\le \liminf_{j\to\infty} \sup_{0<t<T} \|\langle x\rangle^2 J^s u_{m_j}(t)\|_{L^2(\mathbf{R}^n)} \le M.\end{aligned}$$

Namely we have that $\tilde{u} \in X_{T,M}$. On the other hand, since the limit value of $\{u_{m_j}\}_{j\in\mathbf{N}}$ is unique, we obtain that $\tilde{u} = u$. Thus we complete the proof. □

For $v \in X_{T,M}$, we introduce the following linearized problem:

$$\begin{cases} \partial_t u + \Lambda u - \nabla(-\Delta)^{-1} v \cdot \nabla u + vu = 0, & 0 < t < T,\ x \in \mathbf{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbf{R}^n. \end{cases} \tag{8}$$

The solution of (8) can be constructed by using the following approximated problem of (8):

$$\begin{cases} \partial_t u_\varepsilon + \Lambda J_\varepsilon(D_x) u_\varepsilon - J_\varepsilon(D_x)\nabla(-\Delta)^{-1} v \cdot \nabla J_\varepsilon(D_x) u_\varepsilon + v u_\varepsilon = 0, \\ u_\varepsilon(0,x) = u_0(x), \end{cases} \tag{9}$$

where $J_\varepsilon(D_x)$ is a Friedrich's mollifier. (9) can be regarded as a linear ordinary differential equation for u_ε in Banach space $H^s(\mathbf{R}^n)$, which is solved for each ε. The same estimate as in the proof of Proposition 4 and the standard weak convergent argument show existence and uniqueness of solutions of (8) (see [34, Chapter 5] for more details). Then (8) has a unique solution

$$u \in L^\infty(0,T; H_2^s(\mathbf{R}^n)) \cap L^2(0,T; \dot{H}^{s+1/2}(\mathbf{R}^n))$$

associated with v. We define a map Φ by

$$\Phi[v] = u \tag{10}$$

for $v \in X_{T,M}$. Then the following proposition holds.

PROPOSITION 4. *Let* $s > \frac{n}{2} + 1$ *and* $M = 4\|u_0\|_{H_2^s}$. *Let the function-class* $X_{T,M}$ *and the mapping* Φ *be defined by* (7) *and* (10). *Then the inequality*

$$\|\Phi[v]\|_{X_T} \le M \tag{11}$$

holds for all $v \in X_{T,M}$, *if* $T > 0$ *is sufficiently small. Moreover there exists a constant* $0 < L < 1$ *such that the inequality*

$$\|\Phi[v_1] - \Phi[v_2]\|_{Y_T} \le L\|v_1 - v_2\|_{Y_T} \tag{12}$$

is satisfied for all v_1 *and* $v_2 \in X_{T,M}$.

Proof. We show the inequality (11). Using (8), we see that

$$\langle \partial_t J^s u(t), J^s u(t)\rangle_{L^2(\mathbf{R}^n)} + \langle \Lambda J^s u(t), J^s u(t)\rangle_{L^2(\mathbf{R}^n)}$$
$$= \langle J^s(\nabla(-\Delta)^{-1} v \cdot \nabla u)(t), J^s u(t)\rangle_{L^2(\mathbf{R}^n)} - \langle J^s(vu)(t), J^s u(t)\rangle_{L^2(\mathbf{R}^n)}.$$

Hence we obtain that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H^s(\mathbf{R}^n)}^2 + \|\Lambda^{1/2}u(t)\|_{H^s(\mathbf{R}^n)}^2$$
$$= \langle \nabla(-\Delta)^{-1}v(t)\cdot\nabla u(t), u(t)\rangle_{H^s(\mathbf{R}^n)} - \langle v(t)u(t), u(t)\rangle_{H^s(\mathbf{R}^n)}, \tag{13}$$

since $\Lambda^{1/2}$ is self-adjoint on $L^2(\mathbf{R}^n)$. For the first term on the right hand side of this equality, we see that

$$\langle \nabla(-\Delta)^{-1}v(t)\cdot\nabla u(t), u(t)\rangle_{H^s(\mathbf{R}^n)} = \langle \nabla(-\Delta)^{-1}v(t)\cdot\nabla J^s u(t), J^s u(t)\rangle_{L^2(\mathbf{R}^n)}$$
$$+ \langle [J^s, \nabla(-\Delta)^{-1}v]\nabla u(t), J^s u(t)\rangle_{L^2(\mathbf{R}^n)}. \tag{14}$$

Using integration by parts for the first term on the right hand side of (14), we have that

$$\langle \nabla(-\Delta)^{-1}v(t)\cdot\nabla J^s u(t), J^s u(t)\rangle_{L^2(\mathbf{R}^n)} = \frac{1}{2}\int_{\mathbf{R}^n} \nabla((J^s u(t))^2)\cdot\nabla(-\Delta)^{-1}v(t)dx$$
$$= \frac{1}{2}\int_{\mathbf{R}^n} (J^s u(t))^2 v(t)dx.$$

Hence, applying the Hölder inequality and the Sobolev inequality, we obtain that

$$|\langle \nabla(-\Delta)^{-1}v(t)\cdot\nabla J^s u(t), J^s u(t)\rangle_{L^2(\mathbf{R}^n)}| \le C\|v(t)\|_{L^\infty(\mathbf{R}^n)}\|J^s u(t)\|_{L^2(\mathbf{R}^n)}^2$$
$$\le C\|v(t)\|_{H^s(\mathbf{R}^n)}\|u(t)\|_{H^s(\mathbf{R}^n)}^2.$$

Thus the condition of $v(t)$ concludes that

$$|\langle \nabla(-\Delta)^{-1}v(t)\cdot\nabla J^s u(t), J^s u(t)\rangle_{L^2(\mathbf{R}^n)}| \le CM\|u(t)\|_{H^s(\mathbf{R}^n)}^2.$$

Schwarz' inequality and Lemma 1 yield that the second term on the right hand side of (14) satisfies

$$|\langle [J^s, \nabla(-\Delta)^{-1}v]\nabla u(t), J^s u(t)\rangle_{L^2(\mathbf{R}^n)}|$$
$$\le C\|[J^s, \nabla(-\Delta)^{-1}v]\nabla u(t)\|_{L^2(\mathbf{R}^n)}\|J^s u(t)\|_{L^2(\mathbf{R}^n)}$$
$$\le C\|\nabla(-\Delta)^{-1}v(t)\|_{H^s(\mathbf{R}^n)}\|u(t)\|_{H^s(\mathbf{R}^n)}^2.$$

Thus, applying (5) to this inequality, we have that

$$|\langle [J^s, \nabla(-\Delta)^{-1}v]\nabla u(t), J^s u(t)\rangle_{L^2(\mathbf{R}^n)}| \le C\|v(t)\|_{H_2^s(\mathbf{R}^n)}\|u(t)\|_{H^s(\mathbf{R}^n)}^2$$
$$\le CM\|u(t)\|_{H^s(\mathbf{R}^n)}^2.$$

Consequently, employing those inequalities on (14), we conclude that the first term on the right hand side of (13) satisfies

$$|\langle \nabla(-\Delta)^{-1}v(t)\cdot\nabla u(t), u(t)\rangle_{H^s(\mathbf{R}^n)}| \le CM\|u(t)\|_{H^s(\mathbf{R}^n)}^2. \tag{15}$$

For the second term on the right hand side of (13), the Hölder inequality, Lemma 6 and the Sobolev inequality yield that

$$|\langle v(t)u(t), u(t)\rangle_{H^s(\mathbf{R}^n)}| \le C\|v(t)\|_{H^s(\mathbf{R}^n)}\|u(t)\|_{H^s(\mathbf{R}^n)}^2$$
$$\le CM\|u(t)\|_{H^s(\mathbf{R}^n)}^2. \tag{16}$$

Applying (15) and (16) into (13), we see that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H^s(\mathbf{R}^n)}^2 + \|\Lambda^{1/2}u(t)\|_{H^s(\mathbf{R}^n)}^2 \le CM\|u(t)\|_{H^s(\mathbf{R}^n)}^2.$$

Hence we obtain that the inequality

$$\|u(t)\|_{H^s(\mathbf{R}^n)}^2 + \int_0^t \|\Lambda^{1/2}u(\tau)\|_{H^s(\mathbf{R}^n)}^2 d\tau \le \|u_0\|_{H^s(\mathbf{R}^n)}^2 + CMT \sup_{0<\tau<T}\|u(\tau)\|_{H^s(\mathbf{R}^n)}^2$$

holds for any $t \in (0, T)$. Consequently, if $T > 0$ is sufficiently small, we conclude that

$$\sup_{0\le t\le T}\|u(t)\|_{H^s}^2 + \int_0^T \|\Lambda^{1/2}u(t)\|_{H^s}^2 dt \le M^2/4. \tag{17}$$

We estimate $\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}$ in order to conclude the inequality (11). Using (8), we obtain that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}^2 + \langle \Lambda u(t), u(t)\rangle_{\mathscr{H}_2^s(\mathbf{R}^n)}$$
$$= \langle \nabla(-\Delta)^{-1}v(t)\cdot\nabla u(t), u(t)\rangle_{\mathscr{H}_2^s(\mathbf{R}^n)} - \langle v(t)u(t), u(t)\rangle_{\mathscr{H}_2^s(\mathbf{R}^n)}. \tag{18}$$

The second term on the left hand side of this equality is split into

$$\langle \Lambda u(t), u(t)\rangle_{\mathscr{H}_2^s(\mathbf{R}^n)} = \langle |x|^2 J^s \Lambda u(t), |x|^2 J^s u(t)\rangle_{L^2(\mathbf{R}^n)}$$
$$= \langle \Lambda(|x|^2 J^s u)(t), |x|^2 J^s u(t)\rangle_{L^2(\mathbf{R}^n)}$$
$$+ \langle [|x|^2 J^s, \Lambda]u(t), |x|^2 J^s u(t)\rangle_{L^2(\mathbf{R}^n)}.$$

Since the operator $\Lambda^{1/2}$ is self-adjoint on $L^2(\mathbf{R}^n)$, the first term on the right hand side of this satisfies

$$\langle \Lambda(|x|^2 J^s u)(t), |x|^2 J^s u(t)\rangle_{L^2(\mathbf{R}^n)} = \|\Lambda^{1/2}(|x|^2 J^s u)(t)\|_{L^2(\mathbf{R}^n)}^2$$

and the equation (18) is represented as

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}^2 + \|\Lambda^{1/2}(|x|^2 J^s u)(t)\|_{L^2(\mathbf{R}^n)}^2 \\
&\quad = \langle \nabla(-\Delta)^{-1} v(t)\cdot \nabla u(t), u(t)\rangle_{\mathscr{H}_2^s(\mathbf{R}^n)} - \langle v(t)u(t), u(t)\rangle_{\mathscr{H}_2^s(\mathbf{R}^n)} \\
&\qquad - \langle [|x|^2 J^s, \Lambda]u(t), |x|^2 J^s u(t)\rangle_{L^2(\mathbf{R}^n)}. \qquad (19)
\end{aligned}$$

In the last term on the right hand side of (19), the commutator $[|x|^2 J^s, \Lambda]$ is represented as

$$\begin{aligned}
[|x|^2 J^s, \Lambda]u &= |x|^2 \mathscr{F}^{-1}[|\xi|(1+|\xi|^2)^{s/2}\mathscr{F}[u]] - \mathscr{F}^{-1}[|\xi|\mathscr{F}[|x|^2 J^s u]] \\
&= \mathscr{F}^{-1}[(-\Delta_\xi)(|\xi|(1+|\xi|^2)^{s/2}\mathscr{F}[u])] \\
&\quad - \mathscr{F}^{-1}[|\xi|(-\Delta_\xi)((1+|\xi|^2)^{s/2}\mathscr{F}[u])] \\
&= \mathscr{F}^{-1}[(-\Delta_\xi|\xi|)(1+|\xi|^2)^{s/2}\mathscr{F}[u]] \\
&\quad + 2\mathscr{F}^{-1}[(-\nabla_\xi|\xi|)\cdot\nabla_\xi((1+|\xi|^2)^{s/2}\mathscr{F}[u])]. \qquad (20)
\end{aligned}$$

The Plancherel identity and Lemma 2 yield that the first term on the right hand side of (20) satisfies

$$\begin{aligned}
&\|\mathscr{F}^{-1}[(-\Delta_\xi|\xi|)(1+|\xi|^2)^{s/2}\mathscr{F}[u]](t)\|_{L^2(\mathbf{R}^n)} \\
&\quad \le C\| \, |\xi|^{-1}(1+|\xi|^2)^{s/2}\mathscr{F}[u](t)\|_{L^2(\mathbf{R}^n)}.
\end{aligned}$$

Applying Lemma 2 and the Plancherel identity again, we obtain that

$$\begin{aligned}
&\|\mathscr{F}^{-1}[(-\Delta_\xi|\xi|)(1+|\xi|^2)^{s/2}\mathscr{F}[u]](t)\|_{L^2(\mathbf{R}^n)} \\
&\quad \le C(\|u(t)\|_{H^s(\mathbf{R}^n)} + \|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}). \qquad (21)
\end{aligned}$$

For the second term on the right hand side of (20), employing the Plancherel identity and the Hölder inequality, we have that

$$\begin{aligned}
&\|\mathscr{F}^{-1}[(-\nabla_\xi|\xi|)\cdot\nabla_\xi((1+|\xi|^2)^{s/2}\mathscr{F}[u])](t)\|_{L^2(\mathbf{R}^n)} \\
&\quad \le \|\nabla_\xi((1+|\xi|^2)^{s/2}\mathscr{F}[u])(t)\|_{L^2(\mathbf{R}^n)}.
\end{aligned}$$

Hence the Plancherel identity yields that

$$\begin{aligned}
&\|\mathscr{F}^{-1}[(-\nabla_\xi|\xi|)\cdot\nabla_\xi((1+|\xi|^2)^{s/2}\mathscr{F}[u])](t)\|_{L^2(\mathbf{R}^n)}\\
&\quad\le C\|u(t)\|_{\mathscr{H}_1^s(\mathbf{R}^n)}\\
&\quad\le C(\|u(t)\|_{H^s(\mathbf{R}^n)}+\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}). \qquad (22)
\end{aligned}$$

Applying the inequalities (21) and (22) into (20), we obtain that

$$\begin{aligned}
\|[|x|^2J^s,\Delta]u(t)\|_{L^2(\mathbf{R}^n)} &\le C(\|u(t)\|_{H_1(\mathbf{R}^n)}+\|u(t)\|_{H^s(\mathbf{R}^n)}+\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)})\\
&\le C(M+\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}).
\end{aligned}$$

Hence the last term on the right hand side of (19) satisfies that

$$\begin{aligned}
&|\langle[|x|^2J^s,\Delta]u(t),|x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}|\\
&\quad\le\|[|x|^2J^s,\Delta]u(t)\|_{L^2(\mathbf{R}^n)}\|\,|x|^2J^su(t)\|_{L^2(\mathbf{R}^n)}\\
&\quad\le C(M+\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)})\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}. \qquad (23)
\end{aligned}$$

We consider the first term on the right hand side of (19). The identity

$$\begin{aligned}
&|x|^2J^s(\nabla(-\Delta)^{-1}v\cdot\nabla u)\\
&\quad=[|x|^2,J^s](\nabla(-\Delta)^{-1}v\cdot\nabla u)+[J^s,\nabla(-\Delta)^{-1}v](|x|^2\nabla u)\\
&\qquad+\nabla(-\Delta)^{-1}v\cdot[J^s,|x|^2]\nabla u+|x|^2\nabla(-\Delta)^{-1}v\cdot J^s\nabla u
\end{aligned}$$

gives that

$$\begin{aligned}
&\langle\nabla(-\Delta)^{-1}v(t)\cdot\nabla u(t),u(t)\rangle_{\mathscr{H}_2^s(\mathbf{R}^n)}\\
&\quad=\langle[|x|^2,J^s](\nabla(-\Delta)^{-1}v\cdot\nabla u)(t),|x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}\\
&\qquad+\langle[J^s,\nabla(-\Delta)^{-1}v](|x|^2\nabla u)(t),|x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}\\
&\qquad+\langle\nabla(-\Delta)^{-1}v(t)\cdot[J^s,|x|^2]\nabla u(t),|x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}\\
&\qquad+\langle|x|^2\nabla(-\Delta)^{-1}v(t)\cdot J^s\nabla u(t),|x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}. \qquad (24)
\end{aligned}$$

By the similar calculation as in (20), the commutator on the first term on the right hand side of this equality is represented by

$$\begin{aligned}
[|x|^2,J^s](\nabla(-\Delta)^{-1}v\cdot\nabla u) &= \mathscr{F}^{-1}[(-\Delta_\xi(1+|\xi|^2)^{s/2})\mathscr{F}[\nabla(-\Delta)^{-1}v\cdot\nabla u]]\\
&\quad-2\mathscr{F}^{-1}[(\nabla_\xi(1+|\xi|^2)^{s/2})\cdot\nabla_\xi\mathscr{F}[\nabla(-\Delta)^{-1}v\cdot\nabla u]].
\end{aligned}$$

Since $\nabla_\xi(1+|\xi|^2)^{s/2} = s\xi(1+|\xi|^2)^{s/2-1}$ and $\Delta_\xi(1+|\xi|^2)^{s/2} = sn(1+|\xi|^2)^{s/2-1} + s(s-2)|\xi|^2(1+|\xi|^2)^{s/2-2}$, we have that

$$\|[|x|^2, J^s](\nabla(-\Delta)^{-1}v\cdot\nabla u)(t)\|_{L^2(\mathbf{R}^n)} \le C\|J^{s-2}(\nabla(-\Delta)^{-1}v\cdot\nabla u)(t)\|_{L^2(\mathbf{R}^n)} + C\|J^{s-1}(x\nabla(-\Delta)^{-1}v\cdot\nabla u)(t)\|_{L^2(\mathbf{R}^n)}.$$

Thus Lemma 6, the Sobolev inequality and (5) give

$$\begin{aligned}\|[|x|^2, J^s](\nabla(-\Delta)^{-1}v\cdot\nabla u)(t)\|_{L^2(\mathbf{R}^n)} &\le C(\|v(t)\|_{H_2^s(\mathbf{R}^n)} + \|u(t)\|_{H_1^s(\mathbf{R}^n)})\\ &\le C(M + \|u(t)\|_{H_1^s(\mathbf{R}^n)})\end{aligned}$$

and we see that the first term on the right hand side of (24) satisfies

$$\begin{aligned}&|\langle[|x|^2, J^s](\nabla(-\Delta)^{-1}v\cdot\nabla u)(t), |x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}|\\ &\quad\le C(M + \|u(t)\|_{H_1^s(\mathbf{R}^n)})\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}.\end{aligned}\tag{25}$$

The Hölder inequality and Lemma 1 give that the second term on the right hand side of (24) satisfies

$$\begin{aligned}&|\langle[J^s, \nabla(-\Delta)^{-1}v](|x|^2\nabla u)(t), |x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}|\\ &\quad\le \|[J^s, \nabla(-\Delta)^{-1}v](|x|^2\nabla u)(t)\|_{L^2(\mathbf{R}^n)}\|\,|x|^2J^su(t)\|_{L^2(\mathbf{R}^n)}\\ &\quad\le C\|\nabla(-\Delta)^{-1}v(t)\|_{H^s(\mathbf{R}^n)}\|\,|x|^2\nabla u(t)\|_{H^{s-1}(\mathbf{R}^n)}\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}.\end{aligned}$$

Thus the inequality (5) and the condition of $v(t)$ provide that

$$\begin{aligned}&|\langle[J^s, \nabla(-\Delta)^{-1}v](|x|^2\nabla u)(t), |x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}|\\ &\quad\le C\|v(t)\|_{H_2^s(\mathbf{R}^n)}(\|u(t)\|_{H^{s-2}(\mathbf{R}^n)} + \|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)})\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}\\ &\quad\le CM(M + \|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)})\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}.\end{aligned}\tag{26}$$

The commutator on the third term on the right hand side of (24) is represented as

$$[J^s, |x|^2]\nabla u = -\mathscr{F}^{-1}[(-\Delta_\xi(1+|\xi|^2)^{s/2})\mathscr{F}[\nabla u]] + 2\mathscr{F}^{-1}[(\nabla_\xi(1+|\xi|^2)^{s/2})\cdot\nabla_\xi\mathscr{F}[\nabla u]].$$

Hence we see that

$$\begin{aligned}\|[J^s, |x|^2]\nabla u(t)\|_{L^2(\mathbf{R}^n)} &\le C\|J^{s-2}\nabla u(t)\|_{L^2(\mathbf{R}^n)} + C\|J^{s-1}(x\nabla u)(t)\|_{L^2(\mathbf{R}^n)}\\ &\le C\|u(t)\|_{H_1^{s-1}(\mathbf{R}^n)}.\end{aligned}$$

A combination of this and Lemma 4 gives that the third term on the right hand side of (24) satisfies

$$\begin{aligned}
&|\langle \nabla(-\Delta)^{-1}v(t)\cdot[J^s,|x|^2]\nabla u(t),|x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}|\\
&\quad\le \|\nabla(-\Delta)^{-1}v(t)\|_{L^\infty(\mathbf{R}^n)}\|[J^s,|x|^2]\nabla u(t)\|_{L^2(\mathbf{R}^n)}\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}\\
&\quad\le C\|v(t)\|_{H_2^s(\mathbf{R}^n)}\|u(t)\|_{H^{s-1}(\mathbf{R}^n)}\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}.
\end{aligned}$$

Thus, using (17) and the condition of $v(t)$, we conclude that

$$|\langle \nabla(-\Delta)^{-1}v(t)\cdot[J^s,|x|^2]\nabla u(t),|x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}|\le CM^2\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}. \tag{27}$$

Applying integration by parts, we have that the last term on the right hand side of (24) is split into

$$\begin{aligned}
&\langle |x|^2\nabla(-\Delta)^{-1}v(t)\cdot J^s\nabla u(t),|x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}\\
&\quad=-2\langle(\nabla|x|^2)\cdot\nabla(-\Delta)^{-1}v(t)J^su(t),|x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}\\
&\qquad+\langle |x|^2v(t)J^su(t),|x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}\\
&\qquad-\langle |x|^2\nabla(-\Delta)^{-1}v(t)J^su(t),|x|^2J^s\nabla u(t)\rangle_{L^2(\mathbf{R}^n)}.
\end{aligned}\tag{28}$$

We transport the last term on the right hand side of (28) into the left hand side of (28), then we obtain

$$\begin{aligned}
&\langle |x|^2\nabla(-\Delta)^{-1}v(t)\cdot J^s\nabla u(t),|x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}\\
&\quad=-\langle(\nabla|x|^2)\cdot\nabla(-\Delta)^{-1}v(t)J^su(t),|x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}\\
&\qquad+\frac{1}{2}\langle |x|^2v(t)J^su(t),|x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}.
\end{aligned}$$

Hence the Hölder inequality and Lemma 4 provide that

$$\begin{aligned}
&|\langle |x|^2\nabla(-\Delta)^{-1}v(t)\cdot J^s\nabla u(t),|x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}|\\
&\quad\le C(\|\nabla(-\Delta)^{-1}v(t)\|_{L^\infty(\mathbf{R}^n)}\|u(t)\|_{\mathscr{H}_1^s(\mathbf{R}^n)}\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}\\
&\qquad+\|v(t)\|_{H^{s-1}(\mathbf{R}^n)}\|u(t)\|^2_{\mathscr{H}_2^s(\mathbf{R}^n)})\\
&\quad\le C\|v(t)\|_{H_2^s(\mathbf{R}^n)}\|u(t)\|^2_{\mathscr{H}_2^s(\mathbf{R}^n)}.
\end{aligned}$$

Thus the condition of $v(t)$ gives that

$$|\langle |x|^2\nabla(-\Delta)^{-1}v(t)\cdot J^s\nabla u(t),|x|^2J^su(t)\rangle_{L^2(\mathbf{R}^n)}|\le CM\|u(t)\|^2_{\mathscr{H}_2^s(\mathbf{R}^n)}. \tag{29}$$

Applying (25)–(27) and (29) into (24), we see that the first term on the right hand side of (19) satisfies

$$
\begin{aligned}
&|\langle \nabla(-\Delta)^{-1} v(t) \cdot \nabla u(t), u(t) \rangle_{\mathscr{H}_2^s(\mathbf{R}^n)}| \\
&\qquad \le C(1+M)(M + \|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)})\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}. \qquad (30)
\end{aligned}
$$

Schwarz' inequality gives that the second term on the right hand side of (19) satisfies

$$
|\langle v(t)u(t), u(t) \rangle_{\mathscr{H}_2^s(\mathbf{R}^n)}| \le \|v(t)u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}.
$$

Since the equality

$$
|x|^2 J^s(vu) = \mathscr{F}^{-1}[(-\Delta_\xi)((1+|\xi|^2)^{s/2}\mathscr{F}[vu])]
$$

is satisfied, we see by Lemma 6 and the Sobolev inequality that

$$
\begin{aligned}
\|v(t)u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)} &\le C(\|v(t)u(t)\|_{H^{s-2}(\mathbf{R}^n)} + \|(\nabla_\xi(1+|\xi|^2)^{s/2}) \cdot \mathscr{F}[xvu](t)\|_{L^2(\mathbf{R}^n)} \\
&\qquad + \| \, |x|^2 v(t)u(t)\|_{H^s(\mathbf{R}^n)}) \\
&\le CM(\|u(t)\|_{H^s(\mathbf{R}^n)} + \|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}).
\end{aligned}
$$

Thus we obtain that

$$
|\langle v(t)u(t), u(t) \rangle_{\mathscr{H}_2^s(\mathbf{R}^n)}| \le CM(M + \|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)})\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}. \qquad (31)
$$

Applying the inequalities (23), (30) and (31) into (19), we obtain that

$$
\begin{aligned}
&\frac{d}{dt}\|u(t)\|^2_{\mathscr{H}_2^s(\mathbf{R}^n)} + \|\Lambda^{1/2}(|x|^2 J^s u)(t)\|^2_{L^2(\mathbf{R}^n)} \\
&\qquad \le C(1+M)(M + \|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)})\|u(t)\|_{\mathscr{H}_2^s(\mathbf{R}^n)}. \qquad (32)
\end{aligned}
$$

Using Gronwall's inequality, we have that

$$
\sup_{0<t<T} \|u(t)\|^2_{\mathscr{H}_2^s(\mathbf{R}^n)} \le e^{C(1+M)T}\|u_0\|^2_{\mathscr{H}_2^s(\mathbf{R}^n)} + C(1+M)M^2(e^{C(1+M)T} - 1).
$$

Hence we derive that

$$
\sup_{0<t<T} \|u(t)\|^2_{\mathscr{H}_2^s(\mathbf{R}^n)} \le M^2/4
$$

holds if $T > 0$ is sufficiently small. Summing up this inequality and (17), we obtain the desired inequality (11).

Next we show the inequality (12). For v_1 and $v_2 \in X_{T,M}$, we denote $u_j = \Phi[v_j]$ $(j = 1, 2)$, $\tilde{v} = v_1 - v_2$ and $\tilde{u} = u_1 - u_2$. Then we obtain

$$\begin{cases} \partial_t \tilde{u} + \Lambda \tilde{u} = \nabla(-\Delta)^{-1} v_1 \cdot \nabla \tilde{u} - v_1 \tilde{u} \\ \qquad\qquad + \nabla(-\Delta)^{-1} \tilde{v} \cdot \nabla u_2 - \tilde{v} u_2, \quad t > 0,\ x \in \mathbf{R}^n, \\ \tilde{u}(0, x) = 0, \qquad\qquad\qquad\qquad\qquad x \in \mathbf{R}^n. \end{cases} \tag{33}$$

We multiply the first equation on this by $\tilde{u}$ and obtain

$$\begin{aligned} &\frac{d}{dt} \|\tilde{u}(t)\|_{L^2(\mathbf{R}^n)}^2 + \|\Lambda^{1/2} \tilde{u}(t)\|_{L^2(\mathbf{R}^n)}^2 \\ &\quad = \frac{1}{2} \int_{\mathbf{R}^n} \tilde{u}(t)^2 v_1(t) dx - \langle v_1(t) \tilde{u}(t), \tilde{u}(t) \rangle_{L^2(\mathbf{R}^n)} \\ &\qquad + \langle \nabla(-\Delta)^{-1} \tilde{v}(t) \cdot \nabla u_2(t), \tilde{u}(t) \rangle_{L^2(\mathbf{R}^n)} - \langle \tilde{v}(t) u_2(t), \tilde{u}(t) \rangle_{L^2(\mathbf{R}^n)}. \end{aligned} \tag{34}$$

The first, the second and the last terms on the right hand side of this equality are estimated in the similar way as in the derivation of (17). Indeed the inequalities

$$\int_{\mathbf{R}^n} \tilde{u}(t)^2 v_1(t) dx \le CM \|\tilde{u}(t)\|_{L^2(\mathbf{R}^n)}^2$$

and

$$\begin{aligned} &|\langle v_1(t) \tilde{u}(t), \tilde{u}(t) \rangle_{L^2(\mathbf{R}^n)}| + |\langle \tilde{v}(t) u_2(t), \tilde{u}(t) \rangle_{L^2(\mathbf{R}^n)}| \\ &\quad \le CM \|\tilde{u}(t)\|_{L^2(\mathbf{R}^n)} \|\tilde{v}(t)\|_{L^2(\mathbf{R}^n)} \end{aligned}$$

hold. Since $s > n/2 + 1$, the Hölder inequality and Lemma 4 imply that the third term on the right hand side of (34) satisfies

$$\begin{aligned} &|\langle \nabla(-\Delta)^{-1} \tilde{v}(t) \cdot \nabla u_2(t), \tilde{u}(t) \rangle_{L^2(\mathbf{R}^n)}| \\ &\quad \le \|\nabla(-\Delta)^{-1} \tilde{v}(t)\|_{L^2(\mathbf{R}^n)} \|\nabla u_2(t)\|_{L^\infty(\mathbf{R}^n)} \|\tilde{u}(t)\|_{L^2(\mathbf{R}^n)} \\ &\quad \le C \|\tilde{v}(t)\|_{\mathscr{H}_1(\mathbf{R}^n)} \|u_2(t)\|_{H^s(\mathbf{R}^n)} \|\tilde{u}(t)\|_{L^2(\mathbf{R}^n)} \\ &\quad \le CM \|\tilde{v}(t)\|_{\mathscr{H}_1(\mathbf{R}^n)} \|\tilde{u}(t)\|_{L^2(\mathbf{R}^n)}. \end{aligned}$$

Hence we obtain that

$$\begin{aligned} &\frac{d}{dt} \|\tilde{u}(t)\|_{L^2(\mathbf{R}^n)}^2 + \|\Lambda^{1/2} \tilde{u}(t)\|_{L^2(\mathbf{R}^n)}^2 \\ &\quad \le C(\|\tilde{u}(t)\|_{L^2(\mathbf{R}^n)} + \|\tilde{v}(t)\|_{H_1(\mathbf{R}^n)}) \|\tilde{u}(t)\|_{L^2(\mathbf{R}^n)}. \end{aligned} \tag{35}$$

Before concluding the estimate for $\|\tilde{u}(t)\|_{L^2(\mathbf{R}^n)}$, we estimate $\|\tilde{u}(t)\|_{\mathscr{H}_2(\mathbf{R}^n)} = \| |x|^2\tilde{u}(t)\|_{L^2(\mathbf{R}^n)}$. We multiply the first equation on (33) by $|x|^2\tilde{u}|x|^2$ and obtain

$$\begin{aligned}
&\frac{d}{dt}\|\tilde{u}(t)\|^2_{\mathscr{H}_2(\mathbf{R}^n)} + \langle \Lambda\tilde{u}(t), \tilde{u}(t)\rangle_{\mathscr{H}_2(\mathbf{R}^n)} \\
&\quad = \langle \nabla(-\Delta)^{-1}v_1(t)\cdot\nabla\tilde{u}(t), \tilde{u}(t)\rangle_{\mathscr{H}_2(\mathbf{R}^n)} - \langle v_1(t)\tilde{u}(t), \tilde{u}(t)\rangle_{\mathscr{H}_2(\mathbf{R}^n)} \\
&\qquad + \langle \nabla(-\Delta)^{-1}\tilde{v}(t)\cdot\nabla u_2(t), \tilde{u}(t)\rangle_{\mathscr{H}_2(\mathbf{R}^n)} - \langle \tilde{v}(t)u_2(t), \tilde{u}(t)\rangle_{\mathscr{H}_2(\mathbf{R}^n)}. \qquad (36)
\end{aligned}$$

The first, the second and the last terms on the right hand side and the second term on the left hand side of this equality are treated in the same way as in the derivation of (32). Indeed the first and the second term on the right hand side of (36) can be estimated as

$$|\langle \nabla(-\Delta)^{-1}v_1(t)\cdot\nabla\tilde{u}(t), \tilde{u}(t)\rangle_{\mathscr{H}_2(\mathbf{R}^n)}| \le CM\|\tilde{u}(t)\|^2_{\mathscr{H}_2(\mathbf{R}^n)} \qquad (37)$$

and

$$\begin{aligned}
&|\langle v_1(t)\tilde{u}(t), \tilde{u}(t)\rangle_{\mathscr{H}_2(\mathbf{R}^n)}| + |\langle \tilde{v}(t)u_2(t), \tilde{u}(t)\rangle_{\mathscr{H}_2(\mathbf{R}^n)}| \\
&\quad \le CM\|\tilde{u}(t)\|_{\mathscr{H}_2(\mathbf{R}^n)}(\|\tilde{u}(t)\|_{\mathscr{H}_2(\mathbf{R}^n)} + \|\tilde{v}(t)\|_{\mathscr{H}_2(\mathbf{R}^n)}). \qquad (38)
\end{aligned}$$

The second term on the left hand side of (36) satisfies

$$\langle \Lambda\tilde{u}(t), \tilde{u}(t)\rangle_{\mathscr{H}_2(\mathbf{R}^n)} = \|\Lambda^{1/2}(|x|^2\tilde{u}(t))\|^2_{L^2(\mathbf{R}^n)} + \langle [|x|^2, \Lambda]\tilde{u}(t), |x|^2\tilde{u}(t)\rangle_{L^2(\mathbf{R}^n)} \qquad (39)$$

and

$$|\langle [|x|^2, \Lambda]\tilde{u}(t), |x|^2\tilde{u}(t)\rangle_{L^2(\mathbf{R}^n)}| \le C\|\tilde{u}(t)\|^2_{H_2(\mathbf{R}^n)}. \qquad (40)$$

For the third term on the right hand side of (36), using the Hölder inequality, we see that

$$\begin{aligned}
&|\langle \nabla(-\Delta)^{-1}\tilde{v}(t)\cdot\nabla u_2(t), \tilde{u}(t)\rangle_{\mathscr{H}_2(\mathbf{R}^n)}| \\
&\quad = |\langle |x|^2\nabla(-\Delta)^{-1}\tilde{v}(t)\cdot\nabla u_2(t), |x|^2\tilde{u}(t)\rangle_{L^2(\mathbf{R}^n)}| \\
&\quad \le \|\nabla(-\Delta)^{-1}\tilde{v}(t)\|_{L^2(\mathbf{R}^n)}\| |x|^2\nabla u_2(t)\|_{L^\infty(\mathbf{R}^n)}\| |x|^2\tilde{u}(t)\|_{L^2(\mathbf{R}^n)}.
\end{aligned}$$

Employing Lemma 3 and the Sobolev inequality, we obtain that

$$\begin{aligned}
&|\langle \nabla(-\Delta)^{-1}\tilde{v}(t)\cdot\nabla u_2(t), \tilde{u}(t)\rangle_{\mathscr{H}_2(\mathbf{R}^n)}| \\
&\quad \le C\|\tilde{v}(t)\|_{\mathscr{H}_1(\mathbf{R}^n)}\| |x|^2\nabla u_2(t)\|_{H^{s-1}(\mathbf{R}^n)}\|\tilde{u}(t)\|_{\mathscr{H}_2(\mathbf{R}^n)} \\
&\quad \le CM\|\tilde{v}(t)\|_{\mathscr{H}_1(\mathbf{R}^n)}\|\tilde{u}(t)\|_{\mathscr{H}_2(\mathbf{R}^n)}. \qquad (41)
\end{aligned}$$

Hence, applying (37)–(41) into (36), we have that

$$\frac{d}{dt}\|u(t)\|_{\mathscr{H}_2(\mathbf{R}^n)}^2 + \|\Lambda^{1/2}(|x|^2u)(t)\|_{L^2(\mathbf{R}^n)}^2$$
$$\leq C(\|u(t)\|_{H_2(\mathbf{R}^n)} + \|v(t)\|_{H_2(\mathbf{R}^n)})\|u(t)\|_{H_2(\mathbf{R}^n)}.$$

By summing up this and (35), we obtain that

$$\frac{d}{dt}\|\tilde{u}(t)\|_{H_2(\mathbf{R}^n)}^2 \leq C(\|\tilde{u}(t)\|_{H_2(\mathbf{R}^n)} + \|\tilde{v}(t)\|_{H_2(\mathbf{R}^n)})\|\tilde{u}(t)\|_{H_2(\mathbf{R}^n)}.$$

Thus Gronwall's inequality provides that

$$\|\tilde{u}\|_{Y_T}^2 \leq C(e^{CT} - 1)\|\tilde{v}\|_{Y_T}^2.$$

Consequently we conclude (12) if $T > 0$ is sufficiently small. □

We remark that the norm $\|u(t)\|_{H_2^s}^2$ may not be differentiated with the time variable. The proof of Proposition 4 is justified by employing a mollifier. In the following, we give our proofs with formal calculus, but we can justify the proofs by applying the argument with mollifier.

Proposition 4 provides the proof of Theorem 1.

PROOF OF THEOREM 1. Since $X_{T,M}$ is a closed subset on Y_T, Proposition 4 and Banach's contraction mapping theorem imply that there exists $u \in X_{T,M}$ such that $\Phi[u] = u$ if we take $T > 0$ as sufficiently small. Especially this u is a unique solution of (1) with $\theta = 1$. Employing (17), we see that

$$\int_0^T \|\Lambda^{1/2}u(t)\|_{H^s(\mathbf{R}^n)}^2 dt < \infty.$$

In particular we have $u \in L^2(0, T; \dot{H}^{s+1/2}(\mathbf{R}^n))$. Thus we complete the proof. □

4. Proof of analyticity of the solution

Ellipticity of the equation (1) with respect to (t, x) gives the following proposition which is proved in the same way as in Proposition 7.1.B in Taylor's textbook [34].

PROPOSITION 5. *Let $n \geq 3$, $s > \frac{n}{2} + 1$ and $u_0 \in H_2^s(\mathbf{R}^n)$. Then the solution of* (1) *satisfies*

$$u \in C^\infty((0, T); H^\infty(\mathbf{R}^n)),$$

where T is the positive constant which appears in Theorem 1.

Proof. We prove this proposition by applying the formal argument, which can be justified by employing a mollifier. We put

$$M = \sup_{0<t<T} \|u(t)\|_{H_2^{s_0}(\mathbf{R}^n)}$$

for some $s_0 \in \left(1 + \frac{n}{2}, s\right)$. Since $u \in L^\infty(0, T; L^2(\mathbf{R}^n)) \cap L^2(0, T; \dot{H}^{s+1/2}(\mathbf{R}^n))$, we see that $u(t) \in H^{s+1/2}(\mathbf{R}^n)$ for almost all $t \in (0, T)$. We show that $u(t) \in H^{s+1/2}(\mathbf{R}^n)$ for any $t \in (0, T)$ and

$$u \in L^2(0, T; H^{s+1}(\mathbf{R}^n)) \cap L^\infty(0, T; H^{s+1/2}(\mathbf{R}^n)).$$

In the similar way as in the proof of Theorem 1, we have that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{H^{s+1/2}(\mathbf{R}^n)} + \|\Lambda^{1/2}u(t)\|^2_{H^{s+1/2}(\mathbf{R}^n)} \\
&\quad = \langle \nabla(-\Delta)^{-1}u(t) \cdot \nabla u(t), u(t)\rangle_{H^{s+1/2}(\mathbf{R}^n)} - \langle u(t)^2, u(t)\rangle_{H^{s+1/2}(\mathbf{R}^n)}. \qquad (42)
\end{aligned}$$

The first term on the right hand side of this equality is split into

$$\begin{aligned}
&\langle \nabla(-\Delta)^{-1}u(t) \cdot \nabla u(t), u(t)\rangle_{H^{s+1/2}(\mathbf{R}^n)} \\
&\quad = \langle \nabla(-\Delta)^{-1}u(t) \cdot \nabla J^{s+1/2}u(t), J^{s+1/2}u(t)\rangle_{L^2(\mathbf{R}^n)} \\
&\qquad + \langle [J^{s+1/2}, \nabla(-\Delta)^{-1}u] \cdot \nabla u(t), J^{s+1/2}u(t)\rangle_{L^2(\mathbf{R}^n)}. \qquad (43)
\end{aligned}$$

Using integration by parts, we have that

$$\begin{aligned}
&\langle \nabla(-\Delta)^{-1}u(t) \cdot \nabla J^{s+1/2}u(t), J^{s+1/2}u(t)\rangle_{L^2(\mathbf{R}^n)} \\
&\quad = \frac{1}{2}\langle u(t)J^{s+1/2}u(t), J^{s+1/2}u(t)\rangle_{L^2(\mathbf{R}^n)}.
\end{aligned}$$

Hence the Hölder inequality and the Sobolev inequality give that the first term on the right hand side of (43) satisfies

$$\begin{aligned}
\langle \nabla(-\Delta)^{-1}u(t) \cdot \nabla J^{s+1/2}u(t), J^{s+1/2}u(t)\rangle_{L^2(\mathbf{R}^n)} &\le C\|u(t)\|_{L^\infty(\mathbf{R}^n)}\|J^{s+1/2}u(t)\|^2_{L^2(\mathbf{R}^n)} \\
&\le C\|u(t)\|_{H^{s_0}(\mathbf{R}^n)}\|u(t)\|^2_{H^{s+1/2}(\mathbf{R}^n)}.
\end{aligned}$$

Applying the Schwarz inequality and Lemma 1 into the second term on the right hand side of (43), we have that

$$\begin{aligned}
&\langle [J^{s+1/2}, \nabla(-\Delta)^{-1}u] \cdot \nabla u(t), J^{s+1/2}u(t)\rangle_{L^2(\mathbf{R}^n)} \\
&\quad \le C(\|\nabla(-\Delta)^{-1}u(t)\|_{\mathrm{Lip}(\mathbf{R}^n)}\|\nabla u(t)\|_{H^{s-1/2}(\mathbf{R}^n)} \\
&\qquad + \|\nabla(-\Delta)^{-1}u(t)\|_{H^{s+1/2}(\mathbf{R}^n)}\|\nabla u(t)\|_{L^\infty(\mathbf{R}^n)})\|u(t)\|_{H^{s+1/2}(\mathbf{R}^n)}.
\end{aligned}$$

Thus Lemma 4, the inequality (5) and the Sobolev inequality provide that

$$\langle [J^{s+1/2}, \nabla(-\Delta)^{-1}u] \cdot \nabla u(t), J^{s+1/2}u(t)\rangle_{L^2(\mathbf{R}^n)} \le CM(1 + \|u(t)\|^2_{H^{s+1/2}(\mathbf{R}^n)}).$$

Consequently, by those inequalities on (43), the following estimate holds for the first term on the right hand side of (42):

$$\langle \nabla(-\Delta)^{-1}u(t) \cdot \nabla u(t), u(t)\rangle_{H^{s+1/2}(\mathbf{R}^n)} \le CM(1 + \|u(t)\|^2_{H^{s+1/2}(\mathbf{R}^n)}). \tag{44}$$

Employing Lemma 6, we see that the second term on the right hand side of (42) satisfies

$$\langle u(t)^2, u(t)\rangle_{H^{s+1/2}(\mathbf{R}^n)} \le CM^2\|u(t)\|_{H^{s+1/2}(\mathbf{R}^n)}. \tag{45}$$

Applying (44) and (45) into (42), we obtain that

$$\frac{d}{dt}\|u(t)\|^2_{H^{s+1/2}(\mathbf{R}^n)} + \|\Lambda^{1/2}u(t)\|^2_{H^{s+1/2}(\mathbf{R}^n)} \le CM(1 + \|u(t)\|^2_{H^{s+1/2}(\mathbf{R}^n)}). \tag{46}$$

Here we choose $t_0 \in (0, T)$ such that $u(t_0) \in H^{s+1/2}(\mathbf{R}^n)$. Then Gronwall's inequality gives that

$$\|u(t)\|^2_{H^{s+1/2}(\mathbf{R}^n)} + \int_{t_0}^{t} e^{CM(t-s)}\|\Lambda^{1/2}u(s)\|^2_{H^{s+1/2}(\mathbf{R}^n)}ds \le (1 + \|u(t_0)\|^2_{H^{s+1/2}(\mathbf{R}^n)})e^{CMt}.$$

Especially we have that $u(t) \in H^{s+1/2}(\mathbf{R}^n)$ for any $t \in (t_0, T)$ and

$$u \in L^2(t_0, T; H^{s+1}(\mathbf{R}^n)).$$

Since we can choose $t_0 > 0$ sufficiently small, we conclude that $u(t) \in H^{s+1/2}(\mathbf{R}^n)$ for any $t \in (0, T)$ and $u \in L^2(0, T; H^{s+1}(\mathbf{R}^n))$. From (46), we have that for any $t, \tau \in (0, T)$,

$$|\,\|u(t)\|^2_{H^{s+1/2}(\mathbf{R}^n)} - \|u(\tau)\|^2_{H^{s+1/2}(\mathbf{R}^n)}| + \int_{\tau}^{t} \|\Lambda^{1/2}u(s)\|^2_{H^{s+1/2}(\mathbf{R}^n)}ds \le C|t - \tau|,$$

where the positive constant C depends on M and T, which implies that

$$u \in C((0, T); H^{s+1/2}(\mathbf{R}^n)) \cap C^1((0, T); H^{s-1/2}(\mathbf{R}^n)),$$

by the equations in (1) and the fact that $u \in L^2(0, T; H^{s+1}(\mathbf{R}^n))$. By repeating this procedure, we obtain the desired conclusion. □

In order to prove Theorem 2, we introduce a cut-off function as follows. We choose $t_0 \in (0, T)$ arbitrarily. For this t_0, we introduce a real-valued function $r_1 \in C_0^\infty(0, T)$ such that $0 \le r_1(t) \le 1$ and $r_1 = 1$ on a neighborhood of t_0. We prove the following proposition.

Proposition 6. *Let* $n \geq 3$, $s > \frac{n}{2} + 1$, $u_0 \in H_2^s(\mathbf{R}^n)$ *and* u *be the solution of* (1). *Let* $r_1 = r_1(t)$ *be defined as above. Let* σ *be an even integer which satisfies* $\sigma \geq \left[\frac{n+1}{2}\right] + 1$. *Then there exist positive constants* C^* *and* K *such that*

$$\|r_1(t)^{m+|\alpha|+1} \partial_t^m \nabla^\alpha u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \leq K^{m+|\alpha|}(m+|\alpha|)! \tag{47}$$

and

$$\|r_1(t)^{m+|\alpha|} \partial_t^m \nabla^\alpha \nabla\psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \leq C^* K^{m+|\alpha|-1}(m+|\alpha|-1)! \tag{48}$$

hold for any $(m,\alpha) \in \mathbf{Z}_+ \times \mathbf{Z}_+^n \backslash \{(0,0)\}$.

Proof. By the definition of r_1, there exists a small constant $\delta > 0$ such that $\operatorname{supp} r_1 \subset (t_0 - \delta, t_0 + \delta)$. We introduce a cut-off function $\zeta_0 \in C_0^\infty(\mathbf{R})$ such that $0 \leq \zeta_0(t) \leq 1$ and

$$\begin{cases} \zeta_0(t) = 1, & t \in (t_0 - \delta, t_0 + \delta), \\ \zeta_0(t) = 0, & t \in (-\infty, t_0 - 2\delta) \cup (t_0 + 2\delta, \infty). \end{cases}$$

Since we estimate $u(t,x)$ only on $\operatorname{supp} r_1$, we can identify $u(t,x)$ with $\zeta_0(t)u(t,x)$. By Proposition 5, we see that (47) with $(m,\alpha) = (0,0)$ holds. We prove (47) and (48) by induction in three steps. For $N \in \mathbf{Z}_+$, we assume that (47) and (48) hold for any $(m,\alpha) \in \mathbf{Z}_+ \times \mathbf{Z}_+^n$ with $m + |\alpha| \leq N$. Under this assumption, we show that (47) for any $(m,\alpha) \in \mathbf{Z}_+ \times \mathbf{Z}_+^n$ with $|\alpha| \neq 0$ for $m + |\alpha| = N + 1$ in the first step, (47) with $|\alpha| = 0$ for $m + |\alpha| = N + 1$ in the second step and (48) for $m + |\alpha| = N + 1$ in the third step. Those are done if we take C^* and K sufficiently large, which are determined later.

First Step. We take and fix $\alpha, \beta \in \mathbf{Z}_+^n$ and $m \in \mathbf{Z}_+$ with $m + |\alpha| = N$ and $|\beta| = 1$. We estimate

$$r_1(t)^{m+|\alpha|+2} \nabla^\beta \partial_t^m \nabla^\alpha u = \nabla^\beta (r_1(t)^{m+|\alpha|+2} \partial_t^m \nabla^\alpha u).$$

The crucial point of the first step is to replace ∇^β by $\partial_t + \Lambda - \nabla\psi \cdot \nabla$, which is carried out in (60). Putting

$$U(t,x) = r_1(t)^{m+|\alpha|+2} \partial_t^m \nabla^\alpha u(t,x), \tag{49}$$

we prove that

$$\|\nabla^\beta U\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \leq K^{N+1}(N+1)!. \tag{50}$$

We introduce a positive function $\zeta_1 \in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$ such that $0 \leq \zeta_1 \leq 1$ and

$$\zeta_1(\tau,\xi) = \begin{cases} 1 & \text{for } |(\tau,\xi)| \leq 1, \\ 0 & \text{for } |(\tau,\xi)| \geq 2, \end{cases}$$

where $|(\tau,\xi)| = \sqrt{\tau^2 + |\xi|^2}$. We put

$$\zeta_2(\tau,\xi) = 1 - \zeta_1(\tau,\xi). \tag{51}$$

Then, employing the Plancherel identity, we obtain that

$$\begin{aligned}\|\nabla^\beta U\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} &= \|\mathscr{F}_{\tau,\xi}^{-1}[\xi^\beta \mathscr{F}_{t,x}[U]]\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \\ &\le \|\mathscr{F}_{\tau,\xi}^{-1}[|(\tau,\xi)|\zeta_1(\tau,\xi)\mathscr{F}_{t,x}[U]]\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \\ &\quad + \left\|\mathscr{F}_{\tau,\xi}^{-1}\left[\frac{|(\tau,\xi)|\xi^\beta\zeta_2(\tau,\xi)}{\sqrt{1+|(\tau,\xi)|^2}}\mathscr{F}_{t,x}[U]\right]\right\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}. \end{aligned} \tag{52}$$

Since $|(\tau,\xi)|\zeta_1 \le 2$ holds, the assumption of induction shows

$$\|\mathscr{F}_{\tau,\xi}^{-1}[|(\tau,\xi)|\zeta_1\mathscr{F}_{t,x}[U]]\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le 2K^N N! \le \frac{1}{2}K^{N+1}(N+1)!, \tag{53}$$

if K satisfies $K \ge 4$.

Putting $L = \partial_t + \Lambda - \nabla\psi\cdot\nabla$, integration by parts yields that

$$\begin{aligned}\mathscr{F}_{t,x}[LU](\tau,\xi) &= (i\tau+|\xi|)\mathscr{F}_{t,x}[U](\tau,\xi) - \mathscr{F}_{t,x}[uU](\tau,\xi) \\ &\quad - (2\pi)^{-(n+1)/2} i\xi\cdot\iint_{\mathbf{R}\times\mathbf{R}^n} e^{-is\tau - iy\cdot\xi}(U\nabla\psi)(s,y)\,dyds. \end{aligned} \tag{54}$$

We divide the first term of the right hand side of the above into

$$\begin{aligned}&(i\tau+|\xi|)\mathscr{F}_{t,x}[U](\tau,\xi) \\ &\quad = (i\tau+|\xi|\zeta_4(\xi))\mathscr{F}_{t,x}[U](\tau,\xi) + |\xi|\zeta_3(\xi)\mathscr{F}_{t,x}[U](\tau,\xi),\end{aligned} \tag{55}$$

where $\zeta_3 \in C_0^\infty(\mathbf{R}^n)$ is a positive function such that $0 \le \zeta_3 \le 1$ and

$$\begin{aligned}&\zeta_3(\xi) = \begin{cases} 1 & \text{for } |\xi| \le \frac{1}{8}\min\{1, (\sup_{[0,T]}\|\nabla\psi\|_{L^\infty})^{-1}\}, \\ 0 & \text{for } |\xi| \ge \frac{1}{4}\min\{1, (\sup_{[0,T]}\|\nabla\psi\|_{L^\infty})^{-1}\}, \end{cases} \\ &\zeta_4(\xi) = 1 - \zeta_3(\xi).\end{aligned} \tag{56}$$

Here we assume that $\nabla\psi \not\equiv 0$ without of loss of generality. Indeed, when $\nabla\psi \equiv 0$, our problem is trivial. Since the Taylor expansion of ψ is represented by

$$\begin{aligned}\nabla\psi(s,y) &= \sum_{k+|\gamma|\le l}\frac{\partial_t^k\nabla^\gamma\nabla\psi(t,x)}{k!\gamma!}(s-t)^k(y-x)^\gamma \\ &\quad + \sum_{k+|\gamma|=l+1}\int_0^1 \frac{\partial_t^k\nabla^\gamma\nabla\psi(t+\lambda(s-t), x+\lambda(y-x))}{k!\gamma!} \\ &\quad \times (1-\lambda)^l d\lambda (s-t)^k(y-x)^\gamma,\end{aligned} \tag{57}$$

the equalities (54) and (55) provide that

$$\begin{aligned}
&\mathscr{F}_{t,x}[LU](\tau,\xi)\\
&\quad = (i\tau + |\xi|\zeta_4(\xi) - i\xi\cdot\nabla\psi(t,x))\mathscr{F}_{t,x}[U](\tau,\xi) - \mathscr{F}_{t,x}[uU](\tau,\xi)\\
&\qquad + |\xi|\zeta_3(\xi)\mathscr{F}_{t,x}[U](\tau,\xi) + \sum_{1\le k+|\gamma|\le l}\tilde{R}_{k,\gamma}(t,\tau,x,\xi) + \tilde{R}_l(t,\tau,x,\xi), \qquad (58)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{R}_{k,\gamma}(t,\tau,x,\xi) &= -(2\pi)^{-(n+1)/2}\frac{i\xi\cdot\partial_t^k\nabla_x^\gamma\nabla_x\psi(t,x)}{k!\gamma!}\\
&\quad\times\iint_{\mathbf{R}\times\mathbf{R}^n} e^{-is\tau-iy\cdot\xi}U(s,y)(s-t)^k(y-x)^\gamma dyds,\\
\tilde{R}_l(t,\tau,x,\xi) &= -(2\pi)^{-(n+1)/2}\iint_{\mathbf{R}\times\mathbf{R}^n} e^{-is\tau-iy\cdot\xi}U(s,y)i\xi\\
&\quad\cdot\sum_{k+|\gamma|=l+1}\int_0^1\frac{\partial_t^k\nabla_x^\gamma\nabla_x\psi(t+\lambda(s-t),x+\lambda(y-x))}{k!\gamma!}(1-\lambda)^l d\lambda\\
&\quad\times(s-t)^k(y-x)^\gamma dyds. \qquad (59)
\end{aligned}$$

We determine $l\in\mathbf{Z}_+$ on (57) later. Dividing the both sides of (58) by $(i\tau+|\xi|\zeta_4(\xi)-i\xi\cdot\nabla\psi(t,x))$, we obtain that

$$\begin{aligned}
\mathscr{F}_{t,x}[U](\tau,\xi) &= (i\tau+|\xi|\zeta_4(\xi)-i\xi\cdot\nabla\psi(t,x))^{-1}\Bigg(\mathscr{F}_{t,x}[LU](\tau,\xi)+\mathscr{F}_{t,x}[uU](\tau,\xi)\\
&\quad - |\xi|\zeta_3(\xi)\mathscr{F}_{t,x}[U](\tau,\xi) - \sum_{1\le k+|\gamma|\le l}\tilde{R}_{k,\gamma}(t,\tau,x,\xi) - \tilde{R}_l(t,\tau,x,\xi)\Bigg).
\end{aligned}$$

The second term on the right hand side of (52) is represented by

$$\begin{aligned}
&(1+|(\tau,\xi)|^2)^{\sigma/2}\rho(\tau,\xi)\mathscr{F}_{t,x}[U](\tau,\xi)\\
&\quad = (1+|(\tau,\xi)|^2)^{\sigma/2}S(t,\tau,x,\xi)\Bigg(\mathscr{F}_{t,x}[LU](\tau,\xi)+\mathscr{F}_{t,x}[uU](\tau,\xi)\\
&\qquad - |\xi|\zeta_3(\xi)\mathscr{F}_{t,x}[U](\tau,\xi) - \sum_{1\le k+|\gamma|\le l}\tilde{R}_{k,\gamma}(t,\tau,x,\xi) - \tilde{R}_l(t,\tau,x,\xi)\Bigg), \qquad (60)
\end{aligned}$$

where

$$\rho(\tau,\xi) = \frac{|(\tau,\xi)|\xi^{\beta}\zeta_2(\tau,\xi)}{\sqrt{1+|(\tau,\xi)|^2}}$$

and

$$S(t,\tau,x,\xi) = \rho(\tau,\xi)(i\tau + |\xi|\zeta_4(\xi) - i\xi\cdot\nabla\psi(t,x))^{-1}.$$

The symbol $S(t,\tau,x,\xi)$ is smooth since this contains the cut-off functions $\zeta_2(\tau,\xi)$ and $\zeta_4(\xi)$ which are defined by (51) and (56). The symbol $S(t,\tau,x,\xi)$ satisfies the assumption on Lemma 7 with $p = S$. Namely the inequality

$$\sum_{m_1+m_2+|\alpha+\beta|\leq 2n+3} \|\partial_t^{m_1}\partial_\tau^{m_2}\nabla_x^{\alpha}\nabla_\xi^{\beta} S\|_{L^\infty(\mathbf{R}\times\mathbf{R}\times\mathbf{R}^n\times\mathbf{R}^n)} < \infty$$

holds. The symbol satisfies that

$$|\partial_\tau^k \nabla_\xi^\gamma S(t,\tau,x,\xi)| \leq C(1+\tau^2)^{-1}(1+|\xi|^2)^{-(\sigma+n+1)/2} \tag{61}$$

for any $k\in\mathbf{Z}_+$ and $\gamma\in\mathbf{Z}_+^n$ with $k+|\gamma|\geq l$ if we choose $l\in\mathbf{Z}_+$ as sufficiently large. We remark that l depends only on $S(t,\tau,x,\xi)$. Thus l is independent of N. The inequality (61) can be checked easily by using the inequality

$$|(\tau,\xi)| \leq C|i\tau + |\xi|\zeta_4(\xi) - i\xi\cdot\nabla\psi(t,x)|,$$

where $\zeta_4(\xi)$ is defined by (56). We remark that the symbol $S(t,\tau,x,\xi)$ does not belong to the standard class $S^0_{1,0}$, where the definition of $S^0_{1,0}$ is given in [10]. Indeed the coefficient $(i\tau + |\xi|\zeta_4(\xi) - i\xi\cdot\nabla\psi(t,x))^{-1}$ on $S(t,\tau,x,\xi)$ is not included in $S^0_{1,0}$. Putting $\tilde{S}(t,\tau,x,\xi) = (1+|(\tau,\xi)|^2)^{\sigma/2}S(t,\tau,x,\xi)$, we have that the equality (60) derives

$$\begin{aligned}&\mathscr{F}^{-1}_{\tau,\xi}[(1+|(\tau,\xi)|^2)^{\sigma/2}\rho\mathscr{F}_{t,x}[U]](t)\\ &\quad = \tilde{S}(t,D_t,x,D_x)(LU + uU - \Lambda_x\zeta_3(D_x)U)(t) + R(U)(t),\end{aligned} \tag{62}$$

where $\tilde{S}(t,D_t,x,D_x)$ is the pseudo-differential operator provided by

$$\tilde{S}(t,D_t,x,D_x)v(t,x) = (2\pi)^{-(n+1)/2}\iint_{\mathbf{R}\times\mathbf{R}^n} e^{it\tau+ix\cdot\xi}\tilde{S}(t,\tau,x,\xi)\mathscr{F}_{t,x}[v](\tau,\xi)d\xi d\tau,$$

$$R(U) = R_1(U) + R_2(U) \tag{63}$$

and

$$R_1(U)(t,x) = -(2\pi)^{-(n+1)/2} \sum_{1 \le k+|\gamma| \le l} \iint_{\mathbf{R}\times\mathbf{R}^n} e^{it\tau+ix\cdot\xi}(1+|(\tau,\xi)|^2)^{\sigma/2}$$
$$\times \tilde{S}(t,\tau,x,\xi)\tilde{R}_{k,\gamma}(t,\tau,x,\xi)d\xi d\tau,$$
$$R_2(U)(t,x) = -(2\pi)^{-(n+1)/2} \iint_{\mathbf{R}\times\mathbf{R}^n} e^{it\tau+ix\cdot\xi}(1+|(\tau,\xi)|^2)^{\sigma/2}$$
$$\times \tilde{S}(t,\tau,x,\xi)\tilde{R}_l(t,\tau,x,\xi)d\xi d\tau.$$

Lemma 7 implies that

$$S(t,D_t,x,D_x)(1+|(D_t,D_x)|^2)^{\sigma/2} \in \mathscr{L}(H^\sigma(\mathbf{R}\times\mathbf{R}^n), L^2(\mathbf{R}\times\mathbf{R}^n)),$$

from which, the second term of the right hand side of (52) can be estimated as

$$\left\| \mathscr{F}^{-1}_{\tau,\xi}\left[\frac{|(\tau,\xi)|\xi^\beta\zeta_2}{\sqrt{1+|(\tau,\xi)|^2}}\mathscr{F}_{t,x}[U]\right]\right\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}$$
$$= \|\mathscr{F}^{-1}_{\tau,\xi}[(1+|(\tau,\xi)|^2)^{\sigma/2}\rho\mathscr{F}_{t,x}[U]]\|_{L^2(\mathbf{R}\times\mathbf{R}^n)}$$
$$\le \|\tilde{S}(t,D_t,x,D_x)LU\|_{L^2(\mathbf{R}\times\mathbf{R}^n)} + \|\tilde{S}(t,D_t,x,D_x)(uU)\|_{L^2(\mathbf{R}\times\mathbf{R}^n)}$$
$$+ \|\tilde{S}(t,D_t,x,D_x)\Lambda_x\zeta_3(D_x)U\|_{L^2(\mathbf{R}\times\mathbf{R}^n)} + \|R(U)\|_{L^2(\mathbf{R}\times\mathbf{R}^n)}$$
$$\le C_1(\|LU\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} + \|uU\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} + \|\Lambda_x\zeta_3(D_x)U\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)})$$
$$+ \|R(U)\|_{L^2}, \tag{64}$$

where

$$C_1 = \|\tilde{S}(t,D_t,x,D_x)\|_{\mathscr{L}(H^\sigma(\mathbf{R}\times\mathbf{R}^n), L^2(\mathbf{R}\times\mathbf{R}^n))}. \tag{65}$$

The first term on the right hand side of (64) is rewritten by (49) and the equation $Lu = u^2$ as

$$LU = [L, r_1(t)^{m+|\alpha|+2}\partial_t^m\nabla_x^\alpha]u + r_1(t)^{m+|\alpha|+2}\partial_t^m\nabla_x^\alpha(u^2)$$
$$= [L, r_1(t)^{m+|\alpha|+2}]\partial_t^m\nabla_x^\alpha u + r_1(t)^{m+|\alpha|+2}[L,\partial_t^m]\nabla_x^\alpha u$$
$$+ r_1(t)^{m+|\alpha|+2}\partial_t^m[L,\nabla_x^\alpha]u + r_1(t)^{m+|\alpha|+2}\partial_t^m\nabla_x^\alpha(u^2). \tag{66}$$

Since

$$[L, r_1(t)^{m+|\alpha|+2}]\partial_t^m\nabla_x^\alpha u = [\partial_t, r_1(t)^{m+|\alpha|+2}]\partial_t^m\nabla_x^\alpha u$$
$$= \partial_t(r_1(t)^{m+|\alpha|+2})\partial_t^m\nabla_x^\alpha u,$$

by the assumption of induction, the first term on the right hand side of (66) satisfies

$$\begin{aligned}
&\|[L, r_1(t)^{m+|\alpha|+2}]\partial_t^m \nabla_x^\alpha u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \\
&\quad = (m+|\alpha|+2)\|r_1(t)^{m+|\alpha|+1} r_1'(t)\partial_t^m \nabla_x^\alpha u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \\
&\quad \le C_2(m+|\alpha|+2)\|r_1(t)^{m+|\alpha|+1}\partial_t^m \nabla_x^\alpha u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \\
&\quad \le C_2(m+|\alpha|+2)K^{m+|\alpha|}(m+|\alpha|)!,
\end{aligned}$$

where C_2 is a positive constant such that

$$r_1'(t) \le C_2 \tag{67}$$

holds. If K satisfies $K \ge 64C_1C_2$, then we have

$$C_1\|[L, r_1(t)^{m+|\alpha|+2}]\partial_t^m \nabla_x^\alpha u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le \frac{1}{32}K^{m+|\alpha|+1}(m+|\alpha|+1)!, \tag{68}$$

where the constant C_1 is appeared in (64). Since $[L, \partial_t^m] = [\nabla_x\psi\cdot\nabla_x, \partial_t^m]$, the second term on the right hand side of (66) is represented by

$$\begin{aligned}
r_1(t)^{m+|\alpha|+2}[L, \partial_t^m]\nabla_x^\alpha u &= r_1(t)^{m+|\alpha|+2}[\nabla_x\psi\cdot\nabla_x, \partial_t^m]\nabla_x^\alpha u \\
&= r_1(t)^{m+|\alpha|+2}\sum_{k=0}^{m-1}\binom{m}{k}\partial_t^{m-k}\nabla_x\psi\cdot\partial_t^k\nabla_x\nabla_x^\alpha u.
\end{aligned}$$

Hence we have by Lemma 6, the Sobolev inequality and the assumption of induction that

$$\begin{aligned}
&\|r_1(t)^{m+|\alpha|+2}[L, \partial_t^m]\nabla_x^\alpha u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \\
&\quad \le \sum_{k=0}^{m-1}\binom{m}{k}\|r_1(t)^{m+|\alpha|+2}\partial_t^{m-k}\nabla_x\psi\cdot\partial_t^k\nabla_x\nabla_x^\alpha u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \\
&\quad \le C_3\sum_{k=0}^{m-1}\binom{m}{k}\|r_1(t)^{m-k}\partial_t^{m-k}\nabla_x\psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}\|r_1(t)^{k+|\alpha|+2}\partial_t^k\nabla_x\nabla_x^\alpha u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \\
&\quad \le C_3C^*K^{m+|\alpha|}\sum_{k=0}^{m-1}\binom{m}{k}(m-k-1)!(k+|\alpha|+1)!,
\end{aligned}$$

where C_3 is a positive constant such that the following product estimate

$$\|fg\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le C_3\|f\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}\|g\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \tag{69}$$

holds. Thus, if K satisfies $K \ge 64C_1C_3C^*$, then we derive that

$$C_1\|r_1(t)^{m+|\alpha|+2}[L,\partial_t^m]\nabla_x^\alpha u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le \frac{1}{32}K^{m+|\alpha|+1}(m+|\alpha|+1)!. \tag{70}$$

Similarly the third term on the right hand side of (66) is rewritten by

$$\begin{aligned}
&r_1(t)^{m+|\alpha|+2}\partial_t^m[L,\nabla_x^\alpha]u \\
&\quad = r_1(t)^{m+|\alpha|+2}\partial_t^m[\nabla_x\psi\cdot\nabla_x,\nabla_x^\alpha]u \\
&\quad = -r_1(t)^{m+|\alpha|+2}\sum_{k=0}^{m}\sum_{\gamma<\alpha}\binom{m}{k}\binom{\alpha}{\gamma}\partial_t^{m-k}\nabla_x^{\alpha-\gamma}\nabla_x\psi\cdot\partial_t^k\nabla_x^\gamma\nabla_x u.
\end{aligned}$$

This gives that

$$\begin{aligned}
&\|r_1(t)^{m+|\alpha|+2}\partial_t^m[L,\nabla_x^\alpha]u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \\
&\quad \le C_3\sum_{k=0}^{m}\sum_{\gamma<\alpha}\binom{m}{k}\binom{\alpha}{\gamma}\|r_1(t)^{m-k+|\alpha-\gamma|}\partial_t^{m-k}\nabla_x^{\alpha-\gamma}\nabla_x\psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \\
&\qquad \times\|r_1(t)^{k+|\gamma|+2}\partial_t^k\nabla_x^\gamma\nabla_x u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \\
&\quad \le C_3C^*K^{m+|\alpha|}\sum_{k=0}^{m}\sum_{\gamma<\alpha}\binom{m}{k}\binom{\alpha}{\gamma}(m-k+|\alpha-\gamma|-1)!(k+|\gamma|+1)!.
\end{aligned}$$

Hence the last term on the right hand side of (66) satisfies

$$C_1\|r_1(t)^{m+|\alpha|+2}\partial_t^m[L,\nabla_x^\alpha]u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le \frac{1}{32}K^{m+|\alpha|+1}(m+|\alpha|+1)!, \tag{71}$$

if K satisfies $K \ge 64C_1C_3C^*$. Applying (68), (70) and (71) into (66), we conclude that the first term on the left hand side of (66) satisfies

$$C_1\|[L,r_1(t)^{m+|\alpha|+2}\partial_t^m\nabla_x^\alpha]u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le \frac{3}{32}K^{m+|\alpha|+1}(m+|\alpha|+1)!. \tag{72}$$

The last term on the right hand side of (66) can be estimated as

$$\begin{aligned}
&\|r_1(t)^{m+|\alpha|+2}\partial_t^m\nabla_x^\alpha(u^2)\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}\\
&\quad\le C_3\sum_{k=0}^m\sum_{\gamma\le\alpha}\binom{m}{k}\binom{\alpha}{\gamma}\|\partial_t^k\nabla_x^\gamma u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}\|\partial_t^{m-k}\nabla_x^{\alpha-\gamma}u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}\\
&\quad\le C_3K^{m+|\alpha|}\sum_{k=0}^m\sum_{\gamma\le\alpha}\binom{m}{k}\binom{\alpha}{\gamma}(|\alpha-\gamma|+m-k)!(|\gamma|+k)!\\
&\quad= C_3K^{m+|\alpha|}(m+|\alpha|)!\sum_{k=0}^m\sum_{\gamma\le\alpha}\binom{m}{k}\binom{\alpha}{\gamma}\binom{|\alpha|+m}{|\gamma|+k}^{-1}\\
&\quad= C_3K^{m+|\alpha|}(m+|\alpha|)!\sum_{j=0}^{m+|\alpha|}\sum_{\substack{\gamma\le\alpha,k\le m\\|\gamma|+k=j}}\binom{m}{k}\binom{\alpha}{\gamma}\binom{|\alpha|+m}{|\gamma|+k}^{-1}.
\end{aligned}$$

Here we used Lemma 6 and the assumption of induction. From the fact that

$$\sum_{\substack{\gamma\le\alpha,k\le m\\|\gamma|+k=j}}\binom{m}{k}\binom{\alpha}{\gamma}=\binom{|\alpha|+m}{j},$$

we have that

$$C_1\|r_1(t)^{m+|\alpha|+2}\partial_t^m\nabla_x^\alpha(u^2)\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}\le\frac{3}{32}K^{m+|\alpha|+1}(m+|\alpha|+1)!,\tag{73}$$

if K satisfies $K\ge 32C_1C_3$.

This estimate is given by employing Lemma 6 and the assumption of induction. Using (72) and (73) on (66), we estimate the first term on the right hand side of (64) as

$$C_1\|LU\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}\le\frac{1}{8}K^{N+1}(N+1)!,\tag{74}$$

where $N=m+|\alpha|$. Lemma 6 and Sobolev's inequality provide that the inequality

$$C_1\|uU\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}\le\frac{1}{8}K^{N+1}(N+1)!.\tag{75}$$

Employing the Plancherel identity, the definition (56) and the assumption of induction, the third term on the right hand side of (64) satisfies

$$C_1\|\Lambda_x\zeta_3(D_x)U\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}\le C_1C_4\|U\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}\le\frac{1}{8}K^{N+1}(N+1)!,\tag{76}$$

where

$$C_4 = \sup_{\mathbf{R}^n} |\xi|\zeta_3(\xi) \tag{77}$$

and we take $K \geq 8C_1C_4$.

Applying integration by parts, we have that

$$\begin{aligned}
R_1(U) &= (2\pi)^{-(n+1)} \sum_{1\leq k+|\gamma|\leq l} \frac{i\partial_t^k \nabla^\gamma \nabla\psi(t,x)}{k!\gamma!} \cdot \iint_{\mathbf{R}\times\mathbf{R}^n} \iint_{\mathbf{R}\times\mathbf{R}^n} e^{i(t-s)\tau + i(x-y)\cdot\xi} \\
&\quad \times \xi(1+|(\tau,\xi)|^2)^{\sigma/2} S(t,\tau,x,\xi) U(s,y)(s-t)^k (y-x)^\gamma dyds d\xi d\tau \\
&= (2\pi)^{-(n+1)} \sum_{1\leq k+|\gamma|\leq l} \frac{(-i)^{k+|\gamma|+1}\partial_t^k \nabla^\gamma \nabla\psi(t,x)}{k!\gamma!} \\
&\quad \cdot \iint_{\mathbf{R}\times\mathbf{R}^n} \iint_{\mathbf{R}\times\mathbf{R}^n} e^{i(t-s)\tau + i(x-y)\cdot\xi} \partial_\tau^k \nabla_\xi^\gamma (\xi(1+|(\tau,\xi)|^2)^{\sigma/2} S(t,\tau,x,\xi)) \\
&\quad \times U(s,y) dyds d\xi d\tau.
\end{aligned}$$

If we put

$$P_{k,\gamma}(t,\tau,x,\xi) = \partial_\tau^k \nabla_\xi^\gamma (\xi(1+|(\tau,\xi)|^2)^{\sigma/2} S(t,\tau,x,\xi)),$$

then we obtain that

$$|R_1(U)| = \left| \sum_{1\leq k+|\gamma|\leq l} \frac{1}{k!\gamma!} P_{k,\gamma}(t,D_t,x,D_x) U(t,x) \cdot \partial_t^k \nabla_x^\gamma \nabla\psi(t,x) \right|.$$

By the similar argument as in the derivation of (64), we obtain that

$$P_{k,\gamma}(t,D_t,x,D_x) \in \mathscr{L}(H^\sigma(\mathbf{R}\times\mathbf{R}^n), L^2(\mathbf{R}\times\mathbf{R}^n)^n).$$

We put

$$C_5 = \max_{1\leq k+|\gamma|\leq l} \frac{1}{k!\gamma!} \|P_{k,\gamma}(t,D_t,x,D_x)\|_{\mathscr{L}(H^\sigma(\mathbf{R}\times\mathbf{R}^n), L^2(\mathbf{R}\times\mathbf{R}^n)^n)}. \tag{78}$$

Employing the assumption of induction and Sobolev's inequality, we have that

$$\begin{aligned}
\|R_1(U)\|_{L^2(\mathbf{R}\times\mathbf{R}^n)} &\leq C_5 \sum_{1\leq k+|\gamma|\leq l} \|U\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \|\partial_t^k \nabla_x^\gamma \nabla\psi\|_{L^\infty(\mathbf{R}\times\mathbf{R}^n)} \\
&\leq C_5C_6 \sum_{1\leq k+|\gamma|\leq l} \|\partial_t^k \nabla_x^\gamma \nabla\psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} K^N N!, \qquad (79)
\end{aligned}$$

where C_6 is a constant such that the following Sobolev's inequality

$$\|f\|_{L^\infty(\mathbf{R}\times\mathbf{R}^n)} \le C_6\|f\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \tag{80}$$

holds. We consider the term $R_2(t)$ on (63). Since

$$e^{i(t-s)\tau+i(x-y)\cdot\xi}(1+|t-s|^2)(1+|x-y|^2)^n(s-t)^k(y-x)^\gamma$$
$$= (-i)^{k+|\gamma|}(1-\partial_\tau^2)(1-\Delta_\xi)^n\partial_\tau^k\nabla_\xi^\gamma e^{i(t-s)\tau+i(x-y)\cdot\xi},$$

using integration by parts, we have that

$$|R_2(U)| \le (2\pi)^{-(n+1)}\sum_{k+|\gamma|=l+1}\bigg|\iint_{\mathbf{R}\times\mathbf{R}^n}\iint_{\mathbf{R}\times\mathbf{R}^n} e^{i(t-s)\tau+i(x-y)\cdot\xi}$$
$$\times(1+|t-s|^2)^{-1}(1+|x-y|^2)^{-n}$$
$$\times(1-\partial_\tau^2)(1-\Delta_\xi)^n\partial_\tau^k\nabla_\xi^\gamma(\xi(1+|(\tau,\xi)|^2)^{\sigma/2}S(t,\tau,x,\xi))$$
$$\cdot\,\partial_s^k\nabla_y^\gamma\nabla\tilde{\psi}(t,s,x,y)U(s,y)dydsd\xi d\tau\bigg|,$$

where

$$\nabla\tilde{\psi}(t,s,x,y) = \int_0^1 \nabla_x\psi(t+\lambda(s-t),x+\lambda(y-x))d\lambda.$$

Similarly we obtain that

$$|R_2(U)| \le (2\pi)^{-(n+1)}\sum_{k+|\gamma|=l+1}\bigg|\iint_{\mathbf{R}\times\mathbf{R}^n}\iint_{\mathbf{R}\times\mathbf{R}^n} e^{i(t-s)\tau+i(x-y)\cdot\xi}$$
$$\times(1+|x-y|^2)^{-n}(1+\tau^2)^{-\sigma/2}$$
$$\times(1-\partial_\tau^2)(1-\Delta_\xi)^n\partial_\tau^k\nabla_\xi^\gamma(\xi(1+|(\tau,\xi)|^2)^{\sigma/2}S(t,\tau,x,\xi))$$
$$\cdot\,(1-\partial_s^2)^{\sigma/2}((1+|t-s|^2)^{-1}\partial_s^k\nabla_y^\gamma\nabla\tilde{\psi}(t,s,x,y)U(s,y))dydsd\xi d\tau\bigg|.$$

The inequality (61) provides that

$$\max_{k+|\gamma|=l+1}|(1-\partial_\tau^2)(1-\Delta_\xi)^n\partial_\tau^k\nabla_\xi^\gamma(\xi(1+|(\tau,\xi)|^2)^{\sigma/2}S(t,\tau,x,\xi))|$$
$$\le C_7(1+\tau^2)^{\sigma/2-1}(1+|\xi|^2)^{-(n+1)/2}, \tag{81}$$

since l is sufficiently large. Hence, since σ is an even integer, we see that $|R_2(U)|$

$$\leq (2\pi)^{-(n+1)} C_7 C_8 \sum_{k+|\gamma|=l+1} \sum_{m_1+m_2 \leq \sigma} \iint_{\mathbf{R}\times\mathbf{R}^n} \iint_{\mathbf{R}\times\mathbf{R}^n} (1+\tau^2)^{-1}(1+|\xi|^2)^{-(n+1)/2}$$

$$\times (1+|t-s|^2)^{-1}(1+|x-y|^2)^{-n} |\partial_s^{k+m_1} \nabla_y^\gamma \nabla \tilde{\psi}(t,s,x,y)|$$

$$\times |\partial_s^{m_2} U(s,y)| dy ds d\xi d\tau$$

$$= (2\pi)^{-(n+1)} C_7 C_8 C_9 \sum_{k+|\gamma|=l+1} \sum_{m_1+m_2 \leq \sigma} \iint_{\mathbf{R}\times\mathbf{R}^n} (1+|t-s|^2)^{-1}(1+|x-y|^2)^{-n}$$

$$\times |\partial_s^{k+m_1} \nabla_y^\gamma \nabla \tilde{\psi}(t,s,x,y)| \, |\partial_s^{m_2} U(s,y)| dy ds,$$

where C_8 is a positive constant such that

$$|(1-\partial_s^2)^{\sigma/2} f(s) g(s)| \leq C_8 \sum_{m_1+m_2 \leq \sigma} |\partial_s^{m_1} f(s)| \, |\partial_s^{m_2} g(s)| \tag{82}$$

holds and

$$C_9 = \iint_{\mathbf{R}\times\mathbf{R}^n} (1+\tau^2)^{-1}(1+|\xi|^2)^{-(n+1)/2} d\xi d\tau. \tag{83}$$

Thus, applying the Hausdorff-Young inequality and the Hölder inequality, we obtain that

$$\|R_2(U)\|_{L^2(\mathbf{R}\times\mathbf{R}^n)} \leq (2\pi)^{-(n+1)} C_7 C_8 C_9 \sum_{k+|\gamma|=l+1} \sum_{m_1+m_2 \leq \sigma}$$

$$\|(1+t^2)^{-1}(1+|x|^2)^{-n} \partial_t^{k+m_1} \nabla_x^\gamma \nabla_x \psi\|_{L^1(\mathbf{R}\times\mathbf{R}^n)} \|\partial_t^{m_2} U\|_{L^2(\mathbf{R}\times\mathbf{R}^n)}$$

$$\leq (2\pi)^{-(n+1)} C_7 C_8 C_9 C_{10} \|U\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}$$

$$\times \sum_{k+|\gamma|=l+1} \sum_{m_1+m_2 \leq \sigma} \|\partial_t^{k+m_1} \nabla_x^\gamma \nabla_x \psi\|_{L^2(\mathbf{R}\times\mathbf{R}^n)},$$

where

$$C_{10} = \left(\iint_{\mathbf{R}\times\mathbf{R}^n} (1+t^2)^{-2}(1+|x|^2)^{-2n} dx dt \right)^{1/2}. \tag{84}$$

Consequently the assumption of induction gives that

$$\|R_2(U)\|_{L^2(\mathbf{R}\times\mathbf{R}^n)} \leq C_7 C_8 C_9 C_{10} C_{11} \sum_{1 \leq k+|\gamma| \leq l+1} \|\partial_t^k \nabla_x^\gamma \nabla \psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} K^N N!, \tag{85}$$

where

$$C_{11} = (2\pi)^{-(n+1)} \sum_{m_1+m_2 \le \sigma} = (2\pi)^{-(n+1)} \frac{\sigma(\sigma+1)}{2}. \tag{86}$$

In (79) and (85), if K satisfies that

$$K \ge \max\{C_5 C_6, C_7 C_8 C_9 C_{10} C_{11}\} \sum_{1 \le k+|\gamma| \le l+1} \|\partial_t^k \nabla_x^\gamma \nabla \psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)},$$

then we have that

$$\|R(U)\|_{L^2(\mathbf{R}\times\mathbf{R}^n)} \le \frac{1}{8} K^{N+1}(N+1)!. \tag{87}$$

Applying (74), (75), (76) and (87) into (64), we derive that

$$\left\| \mathscr{F}_{\tau,\xi}^{-1}\left[\frac{|(\tau,\xi)|\xi^\beta \zeta_2}{\sqrt{1+|(\tau,\xi)|^2}} \mathscr{F}_{t,x}[U] \right] \right\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le \frac{1}{2} K^{N+1}(N+1)!. \tag{88}$$

Using (53) and (88) on (52), consequently we conclude that the desired estimate (50) holds for any $m \in \mathbf{Z}_+$ and $\alpha, \beta \in \mathbf{Z}_+^n$ with $m + |\alpha| = N$ and $|\beta| = 1$.

Second Step. We show (47) with $|\alpha| = 0$ for $m + |\alpha| = N+1$. Since u is the solution to $\partial_t u + \Lambda u - \nabla \cdot (u \nabla \psi) = 0$, we see that

$$\begin{aligned} &\|r_1(t)^{N+2} \partial_t^{N+1} u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \\ &\quad \le \|r_1(t)^{N+2} \partial_t^N \Lambda u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} + \|r_1(t)^{N+2} \partial_t^N (u^2)\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \\ &\qquad + \|r_1(t)^{N+2} \partial_t^N (\nabla u \cdot \nabla \psi)\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}. \end{aligned} \tag{89}$$

In the same way as in the proof of (73), the assumption of induction implies that the second term on the right hand side of this inequality is estimated as

$$\|r_1(t)^{N+2} \partial_t^N (u^2)\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le \frac{1}{3} K^{N+1}(N+1)!. \tag{90}$$

The Plancherel identity gives that the first-term on the right hand side of (89) satisfies

$$\|r_1(t)^{N+2} \partial_t^N \Lambda u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le \sum_{|\beta|=1} \|r_1(t)^{N+2} \partial_t^N \nabla^\beta u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}.$$

In the similar way as in first step we conclude that

$$\|r_1(t)^{N+2}\partial_t^N \Lambda u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le \frac{1}{3}K^{N+1}(N+1)!. \tag{91}$$

The Leibnitz rule on the third term on the right hand side of (89) yields that

$$\|r_1(t)^{N+2}\partial_t^N(\nabla u\cdot\nabla\psi)\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le \sum_{j=0}^{N-1}\binom{N}{j}\|r_1(t)^{N+2}\partial_t^j\nabla u\cdot\partial_t^{N-j}\nabla\psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}$$

$$+\|r_1(t)^{N+2}\partial_t^N\nabla u\cdot\nabla\psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}.$$

In the same way as in the proof of (73), the first term on the right hand side of this inequality is estimated as

$$\sum_{j=0}^{N-1}\binom{N}{j}\|r_1(t)^{N+2}\partial_t^j\nabla u\cdot\partial_t^{N-j}\nabla\psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le C_3C^*K^N N!$$

$$\le \frac{1}{6}K^{N+1}(N+1)!,$$

if K satisfies that $K \ge 6C_3C^*$. By the similar argument as in the first step, the second term is estimated as

$$\|r_1(t)^{N+2}\partial_t^N\nabla u\cdot\nabla\psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le \frac{1}{6}K^{N+1}(N+1)!.$$

Hence we obtain that

$$\|r_1(t)^{N+2}\partial_t^N(\nabla u\cdot\nabla\psi)\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le \frac{1}{3}K^N N!. \tag{92}$$

By substituting (90), (91) and (92) into (89), we derive that (50) holds for $|\alpha| = 0$ and $m = N+1$.

Third Step. Using (47), we can show (48) with $m + |\alpha| = N+1$. Indeed, when $\alpha \ne 0$, we have that

$$\|r_1(t)^{m+|\alpha|}\partial_t^m\nabla^\alpha\nabla\psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le C\|r_1(t)^{m+|\alpha|}\partial_t^m\nabla^\beta u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}$$

for some $\beta \in \mathbf{Z}_+^n$ with $|\beta| = |\alpha| - 1$. Here we used the Plancherel identity. Thus the assumption (47) concludes (48) with $m + |\alpha| = N+1$ and $\alpha \ne 0$. We consider the case $m = N+1$ and $\alpha = 0$. Since

$$\partial_t\nabla\psi = \nabla(-\Delta)^{-1}(\nabla\cdot(u\nabla\psi) - \Lambda u),$$

using the Plancherel identity, we have that

$$\|r_1(t)^{N+1}\partial_t^{N+1}\nabla\psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le \|r_1(t)^{N+1}\partial_t^N(u\nabla\psi)\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} + \|r_1(t)^{N+1}\partial_t^N u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}. \tag{93}$$

The second term on the right hand side of this inequality is treated by the assumption of induction. Namely

$$\|r_1(t)^{N+1}\partial_t^N u\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le K^N N! \tag{94}$$

is satisfied. The first term on the right hand side of (93) can be estimated as

$$\|r_1(t)^{N+1}\partial_t^N(u\nabla\psi)\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le \|r_1(t)^{N+1}\partial_t^N u\nabla\psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} + \sum_{l=1}^{N}\binom{N}{l}\|r_1(t)^{N+1}\partial_t^{N-l}u\partial_t^l\nabla\psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}.$$

Hence, by the similar argument as in (92), we derive that

$$\|r_1(t)^{N+1}\partial_t^N(u\nabla\psi)\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le C_3(C^*+K)K^{N-1}N!. \tag{95}$$

Thus, by substituting (94) and (95) into (93), we obtain that

$$\|r_1(t)^{N+1}\partial_t^{N+1}\nabla\psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le (1+C_3)(1+C^*+K)K^{N-1}N!.$$

Consequently, if $K \ge \max\{1, C^*\}$ and $C^* \ge 3(C_3+1)$, then we have that

$$\|r_1(t)^{N+1}\partial_t^{N+1}\nabla\psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)} \le C^*K^N N!.$$

Thus we conclude that the inequality (48) holds for any $m \in \mathbf{Z}_+$ and $\alpha \in \mathbf{Z}_+^n$ with $m+|\alpha| = N+1$.

If we take $C^* \ge 3(C_3+1)$ and

$$K \ge \max\{4, 64C_1C_2, 64C_1C_3C^*, 8C_1C_4, C_{12}, C_{13}\},$$

from the above three steps, (47) and (48) for $m+|\alpha| \le N$ yield that (47) and (48) for $m+|\alpha| = N+1$, where

$$C_{12} = C_5C_6\sum_{1\le k+|\gamma|\le l+1}\|\partial_t^k\nabla_x^\gamma\nabla\psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}$$

and

$$C_{13} = C_7C_8C_9C_{10}C_{11}\sum_{1\le k+|\gamma|\le l+1}\|\partial_t^k\nabla_x^\gamma\nabla\psi\|_{H^\sigma(\mathbf{R}\times\mathbf{R}^n)}.$$

From (65), (67), (69), (77), (78), (80), (81), (82), (83), (84) and (86), K and C^* can be taken to be independent of N. Consequently, employing the induction with N, we complete the proof. □

Proof of Theorem 2. Proposition 6 states that the solution $u(t,x)$ of (1) is analytic on $(0,T) \times \mathbf{R}^n$ (see [15]). Hence we complete the proof. □

Acknowledgement

The authors would like to express their thanks to Professor Yoshiyuki Kagei, Professor Takayuki Kobayashi and Professor Hideyuki Miura for their valuable suggestions. The authors are also grateful to the referee and the editorial committee of the Hiroshima Mathematical Journal for their polite readings and appropriate comments for our manuscript. The first author is grateful to Professor Takayoshi Ogawa for his encouragement.

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