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Existence and classification of overtwisted contact structures in all dimensions

by

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To Misha Gromov with admiration.

1. Introduction

A *contact structure* on a $(2n+1)$-dimensional manifold M is a completely non-integrable hyperplane field $\xi\subset TM$. If we define ξ by a Pfaffian equation $\alpha=0$, where α is a 1-form, possibly with coefficients in a local system for a non-coorientable ξ, then the complete non-integrability is equivalent to $\alpha\wedge d\alpha^n$ being non-vanishing on M. An equivalent definition of the contact condition is that the complement of the 0-section of the total space of the conormal bundle $L_\xi\subset T^*M$ is a symplectic submanifold of T^*M with its canonical symplectic structure $d(p\,dq)$.

The corresponding formal homotopy counterpart of a contact structure is an *almost contact structure*, which is a hyperplane field $\xi\subset TM$ equipped with a conformal class of linear symplectic structures. Almost contact structures can be represented by a pair (α,ω), where α is a non-vanishing 1-form on M (again possibly with local coefficients in a non-trivial line bundle) and ω is a non-degenerate 2-form on the hyperplane field $\xi=\{\alpha=0\}$ (with coefficients in the same local system). In the coorientable case, i.e. when TM/ξ is trivialized by α, the existence of an almost contact structure is equivalent to

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the existence of a *stable almost complex structure* on M, i.e. a complex structure on the bundle $TM\oplus\varepsilon^1$, where ε^1 is the trivial line bundle over M.

The current paper is concerned with basic topological questions about contact structures: existence, extension, and homotopy. This problem has a long history. It was first explicitly formulated, probably, in S. S. Chern's paper [9]. In 1969 M. Gromov [28] proved a parametric h-principle for contact structures on an *open* manifold M: *any almost contact structure is homotopic to a genuine one*, *and two contact structures are homotopic if they are homotopic as almost contact structures*, see Theorem 7.1 below for a more precise formulation of Gromov's theorem.

For closed manifolds a lot of progress was achieved in the 3-dimensional case beginning from the work of J. Martinet [38] and R. Lutz [36] who solved the non-parametric existence problem for 3-manifolds. D. Bennequin [2] showed that the 1-parametric h-principle fails for contact structures on S^3 and Y. Eliashberg in [12] introduced a dichotomy of 3-dimensional contact manifolds into *tight* and *overtwisted* and established a parametric h-principle for overtwisted ones: *any almost contact homotopy class on a closed* 3*-manifold contains a unique*, *up to isotopy*, *overtwisted contact structure.* Tight contact structures were also classified on several classes of 3–manifolds, see e.g. [14], [24], [33], and [34]. V. Colin, E. Giroux, and K. Honda proved in [11] that any atoroidal contact 3-manifold admits at most finitely many non-isotopic tight contact structures.

Significant progress in the problem of construction of contact structures on closed manifolds was achieved in the 5-dimensional case beginning from the work of H. Geiges [19], [20] and H. Geiges and C. B. Thomas [22], [23], and followed by the work of R. Casals, D. M. Pancholi, and F. Presas [5] and J. Etnyre [18], where the existence of a contact structures in any homotopy class of almost contact structures was established. For manifolds of dimension greater than 5 the results are more scarce. The work [13] implied existence of contact structures on all closed $(2n+1)$-dimensional manifolds that bound almost complex manifolds with the homotopy type of $(n+1)$-dimensional cell complexes, provided $n\geqslant 2$. F. Bourgeois [3] proved that for any closed contact manifold M and any surface Σ with genus at least 1, the product $M\times\Sigma$ admits a contact structure, using work of E. Giroux [25]. This positively answered a long standing problem about existence of contact structures on tori of dimension $2n+1>5$ (a contact structure on T^5 was first constructed by R. Lutz in [37]).

Non-homotopic, but formally homotopic contact structures were constructed on higher-dimensional manifolds as well, see e.g. [46]. As far as we know, before the current paper there were no known general results concerning the extension of contact structures in dimension greater than 3.

Theorem 1.1. *Let M be a $(2n+1)$-manifold, $A\subset M$ be a closed set, and ξ be an almost contact structure on M. If ξ is genuine on $\mathcal{O}p\,A\subset M$ then ξ is homotopic relative to A to a genuine contact structure. In particular, any almost contact structure on a closed manifold is homotopic to a genuine contact structure.*

Here we are using Gromov's notation $\mathcal{O}p\,A$ for any unspecified open neighborhood of a closed subset $A\subset M$.

In §3 we will define the notion of an *overtwisted* contact structure for any odd-dimensional manifold. Deferring the definition until §3.2, we will say here that a contact manifold (M^{2n+1},ξ) is called *overtwisted* if it admits a contact embedding of a piecewise smooth $2n$-disc $D_{\rm ot}$ with a certain model germ $\zeta_{\rm ot}$ of a contact structure. In the 3-dimensional case this notion is equivalent to the standard notion introduced in [12]. See §10 for further discussion of the overtwisting property.

Given a $(2n+1)$-dimensional manifold M, let A be a closed subset such that $M\setminus A$ is connected, and let ξ_0 be an almost contact structure M that is a genuine contact structure on $\mathcal{O}p\,A$. Define $\mathfrak{Cont}_{\rm ot}(M;A,\xi_0)$ to be the space of contact structures on M that are overtwisted on $M\setminus A$ and coincide with ξ_0 on $\mathcal{O}p\,A$. The notation $\mathfrak{cont}(M;A,\xi_0)$ stands for the space of *almost* contact structures that agree with ξ_0 on $\mathcal{O}p\,A$. Let

$$j\colon \mathfrak{Cont}_{\rm ot}(M;A,\xi_0)\longrightarrow \mathfrak{cont}(M;A,\xi_0)$$

be the inclusion map. For an embedding $\phi\colon D_{\rm ot}\to M\setminus A$, let $\mathfrak{Cont}_{\rm ot}(M;A,\xi_0,\phi)$ and $\mathfrak{cont}_{\rm ot}(M;A,\xi_0,\phi)$ be the subspaces of $\mathfrak{Cont}_{\rm ot}(M;A,\xi_0)$ and $\mathfrak{cont}_{\rm ot}(M;A,\xi_0)$ of contact and almost contact structures for which $\phi\colon(D_{\rm ot},\zeta_{\rm ot})\to(M,\xi)$ is a contact embedding.

Theorem 1.2. *The inclusion map induces an isomorphism*

$$j_*\colon \pi_0(\mathfrak{Cont}_{\rm ot}(M;A,\xi_0))\longrightarrow \pi_0(\mathfrak{cont}(M;A,\xi_0)),$$

and moreover the map

$$j\colon \mathfrak{Cont}_{\rm ot}(M;A,\xi_0,\phi)\longrightarrow \mathfrak{cont}_{\rm ot}(M;A,\xi_0,\phi)$$

is a (weak) homotopy equivalence.

As an immediate corollary, we have the following result.

Corollary 1.3. *On any closed manifold M, any almost contact structure is homotopic to an overtwisted contact structure which is unique up to isotopy.*

We also have the following corollary (see §3.6 for the proof) concerning isocontact embeddings into an overtwisted contact manifold.

Corollary 1.4. *Let* (M^{2n+1},ξ) *be a connected overtwisted contact manifold and* (N^{2n+1},ζ) *be an open contact manifold of the same dimension. Let* $f\colon N\to M$ *be a smooth embedding covered by a contact bundle homomorphism* $\Phi\colon TN\to TM$, *that is* $\Phi(\zeta_x)=\xi|_{f(x)}$ *and* Φ *preserves the conformal symplectic structures on* ζ *and* ξ. *If* df *and* Φ *are homotopic as injective bundle homomorphisms* $TN\to TM$, *then* f *is isotopic to a contact embedding* $\tilde{f}\colon(N,\zeta)\to(M,\xi)$. *In particular, an open ball with any contact structure embeds into any overtwisted contact manifold of the same dimension.*

We note that there were many proposals for defining the overtwisting phenomenon in dimension greater than 3. Inspired by an obstruction to symplectic fillability of a contact manifold described in Gromov's seminal paper [30], K. Niederkrüger introduced in [40] a notion of a *plastikstufe*, see §10 for the definition and further discussion of this notion and its relation to the overtwisting. A technique for construction of closed contact manifolds with plastikstufes was developed in the papers [44], [42], and [43].

We claim that our notion of overtwisting is stronger than any other possible notions, in the sense that any exotic phenomenon, e.g. a plastikstufe can be found in any overtwisted contact manifold. Indeed, suppose we are given some exotic model (A,ζ), which is an open contact manifold, and assume it formally embeds into an equidimensional (M,ξ_{ot}), then by Corollary 1.4 we know that (A,ζ) admits a genuine contact embedding into (M,ξ_{ot}). See §10 for a more detailed discusson about embeddings of plastikstufes.

In particular, the known results about contact manifolds with a plastikstufe apply to overtwisted manifolds as well:

- Overtwisted contact manifolds are not (semi-positively) symplectically fillable [40].
- The Weinstein conjecture holds for any contact form defining an overtwisted contact structure on a closed manifold [1].
- Any Legendrian submanifold whose complement is overtwisted is loose [39]. Conversely, any loose Legendrian in an overtwisted ambient manifold has an overtwisted complement.

As customary in the h-principle type framework, a parametric h-principle yields results about leafwise structures on foliations, see e.g. [28]. In particular, in [6] the parametric h-principle [12] for overtwisted contact structures on a 3-manifold was used for the construction of leafwise contact structures on codimension-1 foliations on 4-manifolds.

Let $\mathcal{F}$ be a smooth $(2n+1)$-dimensional foliation on a manifold V of dimension $m=2n+1+q$.

Theorem 1.5. *Any leafwise almost contact structure on* $\mathcal{F}$ *is homotopic to a genuine leafwise contact structure.*

A leafwise contact structure ξ on a codimension-q foliation $\mathcal{F}$ on a manifold V of dimension $2n+1+q$ is called *overtwisted* if there exist disjoint embeddings

$$h_i\colon T_i\times B\longrightarrow V,\quad i=1,...,N,$$

where (B,ζ) is a $(2n+1)$-dimensional overtwisted contact ball and each T_i is a compact q-dimensional manifold with boundary, such that

- each leaf of $\mathcal{F}$ is intersected by one of these embeddings;
- for each $i=1,...,N$ and $\tau\in T_i$ the restriction $h_i|_{\tau\times B}$ is a contact embedding of (B,ζ) into some leaf of $\mathcal{F}$ with its contact structure.

The set of embeddings $h_1,...,h_N$ is called an *overtwisted basis* of the overtwisted leafwise contact structure ξ on $\mathcal{F}$.

For a closed subset $A\subset V$, let ξ_0 be a leafwise contact structure on $\mathcal{F}|_{\mathcal{O}p\,A}$, and let $h_i\colon T_i\times B\to V\setminus A$, $i=1,...,N$, be a collection of disjoint embeddings. Define

$$\mathfrak{Cont}_{\mathrm{ot}}(\mathcal{F};A,\xi_0,h_1,...,h_N)$$

to be the space of leafwise contact structures $\mathcal{F}$ that coincide with ξ_0 over $\mathcal{O}p\,A$ and such that $\{h_i\}_{i=1}^N$ is an overtwisted basis for $\mathcal{F}_{V\setminus A}$. Define

$$\mathfrak{cont}_{\mathrm{ot}}(\mathcal{F};A,\xi_0,h_1,...,h_N)$$

to be the analogous space of leafwise almost contact structures on $\mathcal{F}$.

THEOREM 1.6. *The inclusion map*

$$\mathfrak{Cont}_{\mathrm{ot}}(\mathcal{F};A,\xi_0,h_1,...,h_N)\longrightarrow\mathfrak{cont}_{\mathrm{ot}}(\mathcal{F};A,\xi_0,h_1,...,h_N)$$

is a (*weak*) *homotopy equivalence.*

Remark 1.7. If V is closed then an analog of Gray–Moser's theorem still holds even though the leaves could be non-compact. Indeed, the leafwise vector field produced by Moser's argument is integrable because V is compact, and hence it generates the flow realizing the prescribed deformation of the leafwise contact structure. Therefore, a homotopical classification of leafwise contact structures coincides with their isotopical classification.

Plan of the paper. Because of Gromov's h-principle for contact structures on open manifolds, the entire problem can be reduced to a local extension problem of when a germ of a contact structure on the $2n$-sphere ∂B^{2n+1} can be extended to a contact structure on B^{2n+1}. Our proof is based on the two main results: Proposition 3.1, which reduces the extension problem to a unique model in every dimension, and Proposition 3.10, which provides an extension of the connected sum of this universal model with a neighborhood of an overtwisted $2n$-disc D_{ot} defined in §3.2. We formulate Propositions 3.1 and 3.10 in §3, and then deduce Theorem 1.1 from them. We then continue §3 with Propositions 3.11 and 3.12, which are parametric analogs of the preceding propositions, and then prove Theorem 1.2 and Corollary 1.4. The proofs of Theorems 1.5 and 1.6, concerning leafwise contact structures on a foliation, are postponed till §9.

In §4 we study the notion of domination of contact shells and prove Proposition 4.8 and its corollary Proposition 4.9, which can be thought of as certain disorderability results for the group of contactomorphisms of a contact ball. These results are used in an essential way in the proofs of Propositions 3.1 and 3.11 in §8. We prove the main extension results, Propositions 3.10 and 3.12, in §5.

Propositions 3.1 and 3.11 are proved in §8. This is done by gradually standardizing the extension problem in §6 and §7. First, in §6 we reduce it to extension of germs of contact structures induced by a certain family of immersions of S^{2n} into the standard contact $\mathbb{R}^{2n+1}$. This part is fairly standard, and the proof uses the traditional h-principle type techniques going back to Gromov's papers [28], [29] and Eliashberg–Mishachev's paper [16]. In §6 we show how the extension problem of §6 can be reduced to the extension of some special models determined by contact Hamiltonians. Finally, to complete the proofs of Propositions 3.1 and 3.11 we introduce in §8 equivariant coverings and use them to further reduce the problem to just one universal extension model in any given dimension.

The final §10 is devoted to further comments regarding the overtwisting property. We also provide an explicit classification of overtwisted contact structures on spheres.

The diagram in Figure 1 outlines the logical dependency of the major propositions in the paper. Notice that the left three columns together give the proof of Theorem 1.1, whereas the right three columns together prove Theorem 1.2. The double arrow between Propositions 6.12 and 3.1 indicates that Proposition 6.12 is used in the proof of Proposition 3.1 twice in an essential way. The diagram is symmetrical about the central column, in the sense that any two propositions which are opposite of each other are parametric/non-parametric versions of the same result.

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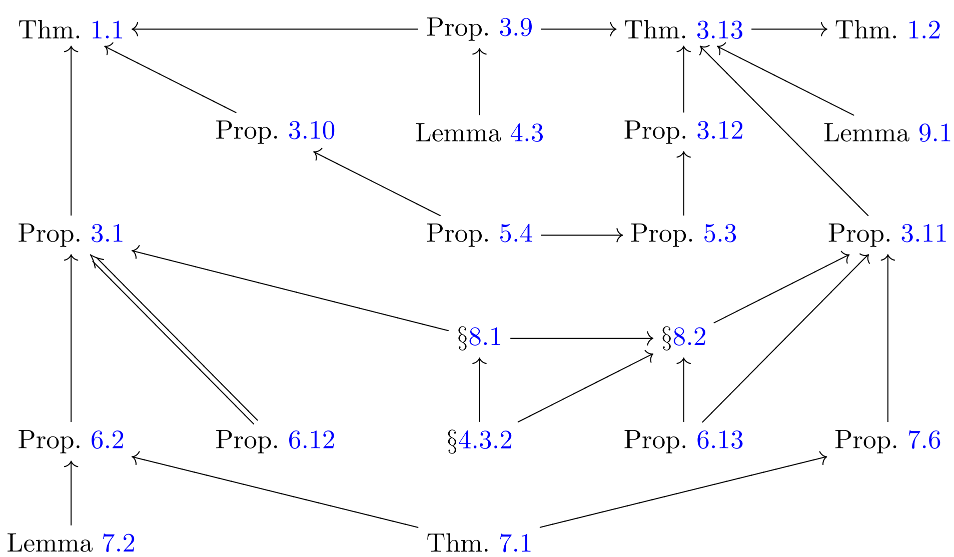

Figure 1.1. Results of the paper, with logical dependencies.

2. Basic notions

2.1. Notation and conventions

Throughout the paper, we will often refer to discs of dimension $2n-1$, $2n$, and $2n+1$. For the sake of clarity, we will always use the convention $\dim B=2n+1$, $\dim D=2n$, and $\dim \Delta=2n-1$. When we occasionally refer to discs of other dimensions we will explicitly write their dimension as a superscript, e.g. D^m. All discs will be assumed diffeomorphic to closed balls, *with possibly piecewise smooth boundary*.

Functions, contact structures, etc., on a subset A of a manifold M will always be assumed given on a neighborhood $\mathcal{O}p\, A\subset M$. Throughout the paper, the notation I stands for the interval $I=[0,1]$ and S^1 for the circle $S^1=\mathbb{R}/\mathbb{Z}$. The notation $A\Subset B$ stands for compact inclusion, meaning that $\bar{A}\subset \operatorname{Int} B$.

As the standard model contact structure on $\mathbb{R}^{2n-1}=\mathbb{R}\times(\mathbb{R}^2)^{n-1}$, we choose

$$\xi_{\mathrm{st}}:=\left\{\lambda_{\mathrm{st}}^{2n-1}:=dz+\sum_{i=1}^{n-1}u_i\, d\varphi_i=0\right\},$$

where (r_i,φ_i) are polar coordinates in $n-1$ copies of $\mathbb{R}^2$ with $\varphi_i\in S^1$ and $u_i:=r_i^2$ for $i=1,...,n-1$. We always use the contact form $\lambda_{\mathrm{st}}^{2n-1}$ throughout the paper. On $\mathbb{R}^{2n+1}$

we will use two equivalent contact structures, both defined by

$$\xi_{\rm st} := \{\lambda_{\rm st}^{2n-1} + v\, dt = 0\},$$

where the coordinates (v,t) have two possible meanings. For $\mathbb{R}^{2n-1}\times\mathbb{R}^2$ we will take $v:=r^2$ and $t\in S^1$, where (r,t) are polar coordinates on $\mathbb{R}^2$, while for $\mathbb{R}^{2n-1}\times T^*\mathbb{R}$ we will take $v:=-y_n$ and $t:=x_n$. In each case it will be explicitly clarified which model contact structure is considered.

A compact domain in $(\mathbb{R}^{2n-1},\xi_{\rm st})$ will be called *star-shaped* if its boundary is transverse to the contact vector field

$$Z = z\frac{\partial}{\partial z} + \sum_{i=1}^{n-1} u_i \frac{\partial}{\partial u_i}.$$

An abstract contact $(2n-1)$-dimensional closed ball will be called *star-shaped* if it is contactomorphic to a star-shaped domain in $(\mathbb{R}^{2n-1},\xi_{\rm st})$.

A hypersurface $\Sigma\subset(M,\xi=\ker\lambda)$ in a contact manifold has a singular 1-dimensional *characteristic distribution* $\ell\subset T\Sigma\cap\xi$, defined to be the kernel of the 2-form $d\lambda|_{T\Sigma\cap\xi}$, with singularities where $\xi=T\Sigma$. The distribution ℓ integrates to a singular characteristic foliation $\mathcal{F}$ with a transverse contact structure, that is a contact structure on a hypersurface $Y\subset\Sigma$ transverse to $\mathcal{F}$, which is invariant with respect to monodromy along the leaves of $\mathcal{F}$. The characteristic foliation $\mathcal{F}$ and its transverse contact structure determines the germ of ξ along Σ up to a diffeomorphism fixed on Σ.

2.2. Shells

Below we will need some specific models for germs of contact structures along the boundary sphere of a $(2n+1)$-dimensional ball B with *piecewise smooth* (i.e. stratified by smooth submanifolds) boundary, extended to B as *almost contact* structures.[1]

A *contact shell* will be an almost contact structure ξ on a ball B such that ξ is genuine near ∂B. A contact shell (B,ξ) is called *solid* if ξ is a genuine contact structure. An *equivalence* between two contact shells (B,ξ) and (B',ξ') is a diffeomorphism $g: B\to B'$ such that $g_*\xi$ coincides with ξ' on $\mathcal{O}p\,\partial B'$ and $g_*\xi$ is homotopic to ξ' through almost contact structures fixed on $\mathcal{O}p\,\partial B'$.

Given two shells $\zeta_+=(B_+,\xi_+)$ and $\zeta_-=(B_-,\xi_-)$, we say that ζ_+ *dominates* ζ_- if there exist both

- a shell $\tilde\zeta=(B,\xi)$ with an equivalence $g: (B,\xi)\to(B_+,\xi_+)$ of contact shells;

([1]) We always view these balls as domains in a larger manifold, so the germs of contact structures along ∂B are assumed to be slightly extended outside of B.

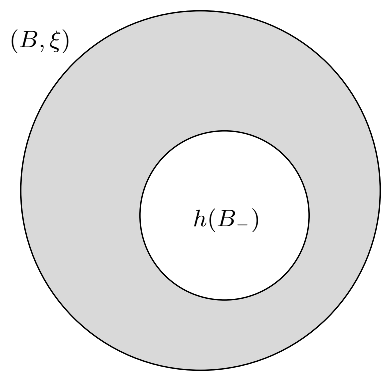

Figure 2.1. Domination of contact shells, where ξ is genuine in a neighborhood of the gray region and $\xi|_{h(B_-)}\cong\xi_-$ as almost contact structures.

• an embedding $h: B_-\to B$ such that $h^*\xi=\xi_-$ and ξ is a genuine contact structure on $B\setminus\operatorname{Int} h(B_-)$.

We will refer to the composition $g\circ h: (B_-,\xi_-)\to(B_+,\xi_+)$ as a *subordination* map. Notice that, if (B_+,ξ_+) dominates (B_-,ξ_-) and (B_-,ξ_-) is solid, then (B_+,ξ_+) is equivalent to a solid shell. If both shells (B_-,ξ_-) and (B_+,ξ_+) are solid, then the subordination map is called *solid* if it is a contact embedding.

A *gluing place* on a contact shell (B,ξ) is a smooth point $p\in\partial B$ where $T_p\partial B=\xi|_p$. Given two gluing places $p_i\in(B_i,\xi_i)$ on contact shells, the standard topological boundary connected sum construction can be performed in a straightforward way at the points p_i to produce a contact shell $(B_0\#B_1,\xi_0\#\xi_1)$, which we will call the *boundary connected sum* of the shells (B_i,ξ_i) at the boundary points p_i. We refer the reader to §5.1 for precise definitions, and only say here that we can make the shells (B_i,ξ_i) isomorphic near p_i via an orientation-reversing diffeomorphism by a C^1-perturbation of the shells that fixes the contact planes $\xi_i|_{p_i}$.

2.3. Circular model shells

Here we will describe a contact shell model associated with contact Hamiltonians, which will play a key role in this paper for it is these models that we will use to define overtwisted discs.

Let $\Delta\subset\mathbb{R}^{2n-1}$ be a compact star-shaped domain and consider a smooth function

$$K: \Delta\times S^1\longrightarrow\mathbb{R},\quad \text{with } K|_{\partial\Delta\times S^1}>0. \tag{1}$$

Throughout the paper we will use the notation (K,Δ) to refer to such a contact Hamiltonian on a star-shaped domain.

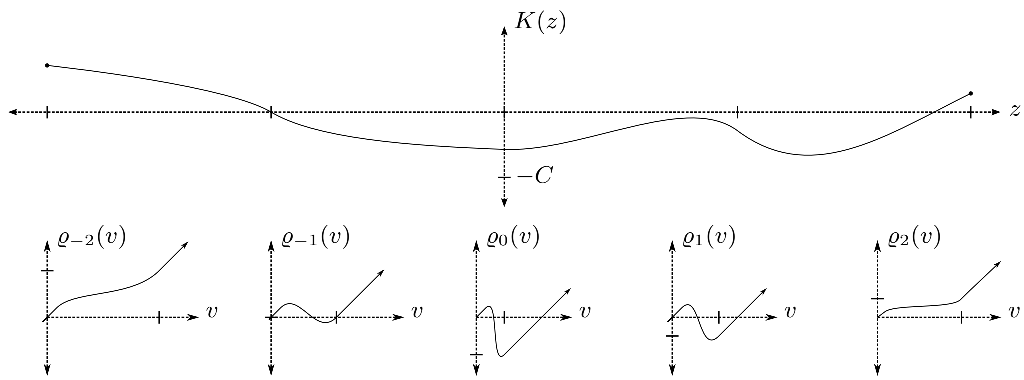

Figure 2.2. A family of functions ϱ_z for the Hamiltonian $K:[-2,2]\to\mathbb{R}$. The hash mark on the vertical axis is at $\varrho_z=K(z)$ and the hash mark on the horizontal axis is at $v=K(z)+C$.

For a constant $C\in\mathbb{R}$, we can define a piecewise smooth $(2n+1)$-dimensional ball associated with (K,Δ) by

$$B_{K,C} := \{(x,v,t)\in\Delta\times\mathbb{R}^2 : v\leqslant K(x,t)+C\}\subset\mathbb{R}^{2n-1}\times\mathbb{R}^2, \tag{2}$$

provided $C+\min_{\Delta\times S^1} K>0$. Pick a smooth family of functions

$$\varrho_{(x,t)}:\mathbb{R}_{\geqslant 0}\longrightarrow\mathbb{R},\quad (x,t)\in\Delta\times S^1, \tag{3}$$

such that

(i) $\varrho_{(x,t)}(v)=v$ when $(x,v,t)\in\mathcal{O}p\{v=0\}$;

(ii) $\varrho_{(x,t)}(v)=v-C$ when $(x,v,t)\in\mathcal{O}p\{v=K(x,t)+C\}$;

(iii) $\partial_v\varrho_{(x,t)}(v)>0$ when $(x,v,t)\in\mathcal{O}p\{v\leqslant K(x,t)+C$ and $x\in\partial\Delta\}$.

See Figure 2.2 for a schematic picture of such a family of functions. Given ϱ, pick a 1-form β on $B_{K,C}$ such that

$$\beta\left(\frac{\partial}{\partial v}\right)>0 \text{ on } B_{K,C} \quad\text{and}\quad \beta=d\varrho \text{ on } \mathcal{O}p\,\partial B_{K,C},$$

which is possible as $d\varrho(\partial/\partial v)>0$ on $\mathcal{O}p\,\partial B_{K,C}$. For example $\beta_g=(1-g)\,dv+g\,d\varrho$, where $g: B_{K,C}\to[0,1]$ is a bump function such that $\partial_v\varrho>0$ on its support and $g\equiv 1$ on $\mathcal{O}p\,\partial B_{K,C}$.

Define the contact shell structure $\eta_{K,\varrho}:=(\alpha_\varrho,\omega_\beta)$ on $B_{K,C}$ by

$$\alpha_\varrho := \lambda_{\mathrm{st}}+\varrho\,dt \quad\text{and}\quad \omega_\beta := d\lambda_{\mathrm{st}}+\beta\wedge dt, \tag{4}$$

which is indeed an almost contact structure since

$$\alpha_\varrho\wedge\omega_\beta^n=(n-1)\,\beta\left(\frac{\partial}{\partial v}\right)\lambda_{\mathrm{st}}\wedge(d\lambda_{\mathrm{st}})^{n-1}\wedge dv\wedge dt>0. \tag{5}$$

As the conditions on β are convex, up to homotopy relative to $\partial B_{K,C}$, the symplectic structure ω_β on $\ker\alpha_\varrho$ is independent of the choice of β, which is why we suppressed it from the notation $\eta_{K,\varrho}$. More generally we have the following result.

LEMMA 2.1. *Up to equivalence, the contact shell* $(B_{K,C}, \eta_{K,\varrho})$ *is independent of the choices of* β, ϱ, *and* C.

Proof. Consider the special case of two choices (ϱ_0, β_0) and (ϱ_1, β_1) for the same C. We can pick a family of diffeomorphisms $\phi_{(x,t)}: \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that

$$\phi_{(x,t)}(v) = (\varrho_1^{-1} \circ \varrho_0)_{(x,t)}(v) \quad \text{on } \mathcal{O}p\{(x,v,t) : v = K(x,t) + C\} \cup \mathcal{O}p\, \partial\Delta,$$

and this family induces a diffeomorphism $\Phi: B_{K,C} \to B_{K,C}$ such that

$$\Phi^*\alpha_{\varrho_1} = \alpha_{\varrho_1 \circ \phi} \quad \text{and} \quad \Phi^*\omega_{\beta_1} = \omega_{\beta_1 \circ \phi}.$$

Since on $\mathcal{O}p\, \partial B_{K,C}$ we have both $\alpha_{\varrho_1 \circ \phi} = \alpha_{\varrho_0}$ and $\omega_{\beta_1 \circ \phi} = \omega_{\beta_0}$, we can connect $\Phi^*\eta_{K,\varrho_1}$ and η_{K,ϱ_0} via a straight line homotopy that is fixed on the boundary.

Given two choices $(C_0, \varrho_0, \beta_0)$ and $(C_1, \varrho_1, \beta_1)$, we can pick a family of diffeomorphisms $\psi_{(x,t)}: \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that

$$\psi_{(x,t)}(v) = v + (C_1 - C_0) \quad \text{on } \mathcal{O}p\{(x,v,t) : v = C_0 + K(x,t)\}$$

and consider the induced diffeomorphism $\Psi: B_{K,C_0} \to B_{K,C_1}$. Pulling back $(\alpha_{\varrho_1}, \omega_{\beta_1})$ by Ψ reduces the problem to the special case. □

We will use the notation $(B_{K,C}, \eta_{K,\varrho})$ throughout the paper for this specific construction, though we will usually drop C and ϱ from the notation and write (B_K, η_K) when the particular choice will be irrelevant. We will refer to this contact shell as the *circle model associated with* (K, Δ).

Remark 2.2. It follows from (5) and conditions (i) and (ii) on ϱ that α_ϱ never can be a contact form if $K \leqslant 0$ somewhere. Conversely if $K > 0$ everywhere, then picking $\varrho(v) = v$ makes α_ϱ a contact form on $B_{K,0}$.

The contact germ $(\partial B_K, \eta_K)$ without its almost contact extension can be described more directly in the following way. Consider the contact germs on the hypersurfaces

$$\widetilde{\Sigma}_{1,K} = \{(x,v,t) : v = K(x,t)\} \subset (\Delta \times T^*S^1, \ker(\lambda_{\mathrm{st}} + v\, dt)),$$
$$\widetilde{\Sigma}_{2,K} = \{(x,v,t) : 0 \leqslant v \leqslant K(x,t) \text{ and } x \in \partial\Delta\} \subset (\Delta \times \mathbb{R}^2, \ker(\lambda_{\mathrm{st}} + v\, dt)).$$

These germs can be glued together via the natural identification between neighborhoods of their boundaries, to form a contact germ $\tilde{\eta}_K$ on $\widetilde{\Sigma}_K := \widetilde{\Sigma}_{1,K} \cup \widetilde{\Sigma}_{2,K}$.

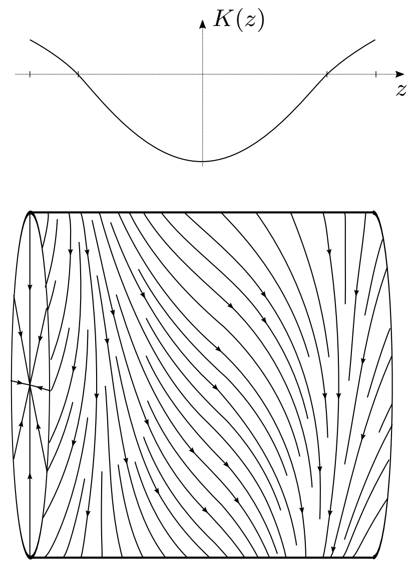

Figure 2.3. The characteristic foliation on the piecewise smooth sphere ∂B_K.

LEMMA 2.3. *The contact germs* $(\partial B_K, \eta_K)$ *and* $(\widetilde{\Sigma}_K, \tilde{\eta}_K)$ *are contactomorphic.*

Proof. We have that the boundary $\partial B_{K,C} = \Sigma_{1,K,C} \cup \Sigma_{2,K,C}$, where

$$\Sigma_{1,K,C} := \{(x,v,t) \in \Delta \times \mathbb{R}^2 : v = K(x,t) + C\},$$
$$\Sigma_{2,K,C} := \{(x,v,t) \in \Delta \times \mathbb{R}^2 : 0 \leqslant v \leqslant K(x,t) + C \text{ and } x \in \partial\Delta\}.$$

Recalling that the 1-form $\alpha_\varrho = \lambda_{\mathrm{st}} + \varrho\, dt$ is a contact form near $\partial B_{K,C} \subset \Delta \times \mathbb{R}^2$, just note that ϱ induces contactomorphisms of neighborhoods

$$(\mathcal{O}p\, \Sigma_{j,K,C}, \ker \alpha_\varrho) \longrightarrow (\mathcal{O}p\, \widetilde{\Sigma}_{j,K}, \ker(\lambda_{\mathrm{st}} + v\, dt))$$

for $j = 0, 1$, by construction. □

2.4. The cylindrical domain

Throughout the paper, we will often use the following star-shaped cylindrical domain:

$$\Delta_{\mathrm{cyl}} := D^{2n-2} \times [-1,1] = \{(q,z) : u \leqslant 1 \text{ and } |z| \leqslant 1\} \subset (\mathbb{R}^{2n-1}, \xi_{\mathrm{st}}),$$

where

$$D^{2n-2} := \left\{ q : u = \sum_{i=1}^{n-1} u_i \leqslant 1 \right\} \subset \mathbb{R}^{2n-2}$$

is the unit ball and $q = (u_1, \phi_1, \dots, u_{n-1}, \phi_{n-1}) \in \mathbb{R}^{2n-2}$.

Also observe for any contact Hamiltonian $(K, \Delta_{\mathrm{cyl}})$ the *north* and *south poles*

$$P_{\pm 1} := \{(u, z, v) = (0, \pm 1, 0)\} \in (\partial B_K, \eta_K)$$

are gluing places in the sense of §2.2. When performing a boundary connected sum of such models $(B_K \# B_{K'}, \eta_K \# \eta_{K'})$ we will always use the north pole of B_K and the south pole of $B_{K'}$. See §5.1 for more details on the gluing construction.

3. Proof of Theorems 1.1 and 1.2

3.1. Construction of contact structures with universal holes

Proposition 3.1, which we prove in §8.1, and which represents one half of the proof of Theorem 1.1, constructs from an almost contact structure a contact structure in the complement of a finite number of disjoint $(2n+1)$-balls, where the germ of the contact structure on the boundaries of the balls has a *unique universal* form.

PROPOSITION 3.1. *For fixed dimension* $2n+1$ *there exists a contact Hamiltonian* $(K_{\mathrm{univ}}, \Delta_{\mathrm{cyl}})$, *specified in Lemma 8.7, such that the following holds. For any almost contact manifold* (M, ξ) *as in Theorem 1.1 there exists an almost contact structure* ξ' *on* M, *which is homotopic to* ξ *relative to* A *through almost contact structures, and a finite collection of disjoint balls* $B_i \subset M \setminus A$ *for* $i=1, ..., L$, *with piecewise smooth boundary such that*

- ξ' *is a genuine contact structure on* $M \setminus \bigcup_{i=1}^{L} \operatorname{Int} B_i$;
- *the contact shells* $\xi'|_{B_i}$ *are equivalent to* $(B_{K_{\mathrm{univ}}}, \eta_{K_{\mathrm{univ}}})$ *for* $i=1, ..., L$.

Remark 3.2. If (B_K, η_K) is dominated by $(B_{K_{\mathrm{univ}}}, \eta_{K_{\mathrm{univ}}})$, then in the statement of Proposition 3.1 we can take K in place of K_{univ}. In particular by Lemma 4.7, *in the* 3-*dimensional case* we can take $K_{\mathrm{univ}}: [-1, 1] \to \mathbb{R}$ to be *any somewhere negative function.* Our proof in higher dimension is not constructive, and we do not know an effective criterion which would allow one to verify whether a particular function K_{univ} satisfies Proposition 3.1. Of course, it is easy to construct a 1-parameter family of Hamiltonians K^{ε} so that any Hamiltonian K is less than K^{ε} for sufficiently small $\varepsilon > 0$ (see Example 3.5). We can then take $K_{\mathrm{univ}} = K^{\varepsilon}$ for sufficiently small ε. It would be interesting to find such a general criterion for which Hamiltonians can be taken as K_{univ}.

3.2. Overtwisted discs and filling of universal holes

Proposition 3.10, which we formulate in this section and prove in §5.2.1, will combine with Proposition 3.1 to prove Theorem 1.1 in §3.3.

A smooth function $k\colon \mathbb{R}_{\geqslant 0}\to\mathbb{R}$ is called *special* if $k(1)>0$ and

$$ak\left(\frac{u}{a}\right) < k(u) \quad \text{for all } a>1 \text{ and } u\geqslant 0. \tag{6}$$

This implies that $k(0)<0$, and hence $k(u)$ has a zero in $(0,1)$. By differentiating (6) with respect to a, we conclude that

$$k(u)-uk'(u)<0 \quad \text{for all } u\geqslant 0, \tag{7}$$

which means that the y-intercept of all tangent lines to the graph of k are negative.

We call a function $K\colon \Delta_{\mathrm{cyl}}\to\mathbb{R}$ *spherically symmetric* if it depends only on the coordinates (u,z), where $u=\sum_{i=1}^{n-1} u_i$. By a slight abuse of notation, we will write $K(u,z)$ rather than $K=\widetilde{K}(u,z)$ for some function $\widetilde{K}\colon[0,1]\times[-1,1]\to\mathbb{R}$.

Definition 3.3. A spherically symmetric contact Hamiltonian $K\colon\Delta_{\mathrm{cyl}}\to\mathbb{R}$ satisfying $K|_{\partial\Delta_{\mathrm{cyl}}}>0$ is called *special* if for some $z_D\in(-1,1)$ and some special $k\colon\mathbb{R}_{\geqslant 0}\to\mathbb{R}$ the following conditions hold for all $u\in[0,1]$:

(SH1) one has $k(u)\leqslant K(u,z)$ and *equality holds* if $z\in\mathcal{O}p\{z_D\}$;

(SH2) the function $K(u,\cdot)\colon[-1,z_D]\to\mathbb{R}$ is non-increasing;

(SH3) $K(u,z)\leqslant K(u,-1)=K(u,1)$.

When $n=1$, where $\Delta_{\mathrm{cyl}}=[-1,1]$, condition (SH1) can be replaced by $K(z_D)<0$.

Remark 3.4. The definition of a special Hamiltonian was picked so that the proofs in §5 would work, in particular the proof of Lemma 5.4, and the main conditions are (SH1) and (SH2). Condition (SH3) is put in strictly for notational convenience for when we do connect sums in §5.1 of contact Hamiltonian shells.

As the following example shows, special contact Hamiltonians exist and furthermore, for any particular contact Hamiltonian $(K',\Delta_{\mathrm{cyl}})$ that is positive on $\partial\Delta_{\mathrm{cyl}}\times S^1$, there is a special contact Hamiltonian $K\colon\Delta_{\mathrm{cyl}}\to\mathbb{R}$ such that $K<K'$.

Example 3.5. For positive constants a, b, and λ with $b<1$ and $\lambda>a/(1-b)$, define the special piecewise-smooth function

$$k(u)=\begin{cases}\lambda(u-b)-a, & \text{if } u\geqslant b,\\ -a, & \text{if } u\leqslant b,\end{cases}$$

and the special piecewise smooth contact Hamiltonian

$$K(u,z)=\max\{k(u),k(|z|)\}.$$

By a perturbation of K near its singular set, we may construct a smooth special contact Hamiltonian $\widetilde{K}$ that is C^0-close to K, though smoothness of K will not be needed in the proof.

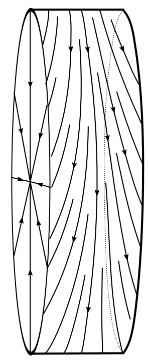

Figure 3.1. A 2-dimensional overtwisted disc $(D_{\mathrm{ot}}, \eta_{\mathrm{ot}})$ with its characteristic foliation.

Let $K\colon \Delta_{\mathrm{cyl}} \to \mathbb{R}$ be a special contact Hamiltonian and define (D_K, η_K) to be the contact germ on the $2n$-dimensional disc

$$D_K := \{(x, v, t) \in \partial B_K : z(x) \in [-1, z_D]\} \subset (B_K, \eta_K), \tag{8}$$

where z_D is the constant in Definition 3.3. Notice that D_K inherits the south pole of the corresponding circle model and the coorientation of ∂B_K as a boundary.

Definition 3.6. Let K_{univ} be as in Proposition 3.1. An *overtwisted disc* $(D_{\mathrm{ot}}, \eta_{\mathrm{ot}})$ is a $2n$-dimensional disc with a germ of a contact structure such that there is a contactomorphism

$$(D_{\mathrm{ot}}, \eta_{\mathrm{ot}}) \cong (D_K, \eta_K),$$

where K is some special contact Hamiltonian with $K < K_{\mathrm{univ}}$. A contact manifold (M^{2n+1}, ξ) is *overtwisted* if it admits a contact embedding $(D_{\mathrm{ot}}, \eta_{\mathrm{ot}}) \to (M, \xi)$ of some overtwisted disc.

Example 3.7. In the 3-dimensional case, it follows from Lemma 4.7 that the disc

$$D_K := \{(z, v, t) \in \partial B_K : z \in [-1, z_D]\} \subset (B_K, \eta_K)$$

is overtwisted in the sense of Definition 3.6 for *any* special contact Hamiltonian, i.e a somewhere negative function on the interval $[-1, 1]$, positive near the end-points ± 1.

Remark 3.8. The definition of the overtwisted disc (D_K, η_K) depends on the choice of a special Hamiltonian $K < K_{\mathrm{univ}}$, and the germs η_K need not be contactomorphic when we vary K. However, as Corollary 1.4 shows, for any two special Hamiltonians $K, K' < K_{\mathrm{univ}}$ any neighborhood of (D_K, η_K) contains $(D_{K'}, \eta_{K'})$.

As the following proposition shows, any overtwisted contact manifold contains infinitely many disjoint overtwisted discs.

Proposition 3.9. *Every neighborhood of an overtwisted disc in a contact manifold contains a foliation by overtwisted discs.*

We prove Proposition 3.9 at the end of §4.2.

Given a special contact Hamiltonian $K: \Delta_{\rm cyl} \to \mathbb{R}$, the contact germ (D_K, η_K) has the following remarkable property, which we will prove in §5.2.2. Let (B, ξ) be a $(2n+1)$-dimensional contact ball with piecewise smooth boundary such that $(D_K, \eta_K) \subset (\partial B, \xi)$, where the coorientation of D_K coincides with the outward coorientation of ∂B.

Proposition 3.10. *Let K_0 and K, with $K_0 \geqslant K$, be two contact Hamiltonians where K is special. Then the contact shell $(B_{K_0} \# B, \eta_{K_0} \# \xi)$, given by performing a boundary connected sum at the north pole of B_{K_0} and the south pole of $D_K \subset \partial B$, is equivalent to a genuine contact structure.*

3.3. Proof of Theorem 1.1

Choose a ball $B \subset M \setminus A$ with piecewise smooth boundary and deform the almost contact structure ξ to make it a contact structure on B with an overtwisted disc $(D_{\rm ot}, \eta_{\rm ot}) \subset (\partial B, \xi)$ on its boundary. This can be done since any two almost contact structures on the ball are homotopic if we do not require the homotopy to be fixed on ∂B.

Using Proposition 3.1 we deform the almost contact structure ξ relative to $A \cup B$ to an almost contact structure ξ on M, which is genuine in the complement of finitely many disjoint balls $B_1, ..., B_N \subset M \setminus (A \cup B)$, where each $(B_i, \xi|_{B_i})$ is isomorphic to $(B_{K_{\rm univ}}, \eta_{K_{\rm univ}})$ as almost contact structures.

According to Proposition 3.9 we can pick disjoint balls $B'_i \subset {\rm Int}\, B$, $i = 1, ..., N$, each with an overtwisted disc on their boundary $(D^i_{\rm ot}, \eta^i_{\rm ot}) \subset (\partial B'_i, \xi)$. As we will describe in §5.1, we can perform an ambient boundary connected sum $B_i \# B'_i \subset M \setminus A$ such that the sets $B_i \# B'_i$ are disjoint for $i = 1, ..., N$ and there are isomorphisms of almost contact structures

$$(B_i \# B'_i, \xi|_{B_i \# B'_i}) \cong (B_i \# B'_i, \xi|_{B_i} \# \xi|_{B'_i}) \cong (B_K \# B'_i, \eta_K \# \xi|_{B'_i}).$$

Now, for $i = 1, ..., L$, by definition we have $(D^i_{\rm ot}, \eta^i_{\rm ot}) = (D_{K_i}, \eta_{K_i})$ for special contact Hamiltonians K_i such that $K_i < K_{\rm univ}$. Therefore we can apply Proposition 3.10 to homotope $\xi|_{B_i \# B'_i}$ relative to the boundary to a genuine contact structure on $B_i \# B'_i$ for each $i = 1, ..., N$. The result will be a contact structure on M that is homotopic relative to A to the original almost contact structure.

3.4. Fibered structures

To prove the parametric version of Theorem 1.1, we need to discuss the parametric form of the introduced above notions. The parameter space, always denoted by T, will be assumed to be a compact manifold of dimension q, possibly with boundary, and we will use the letter τ for points in T.

A family of (almost) contact structures $\{\xi^\tau\}_{\tau\in T}$ on a manifold M can be equivalently viewed as a fiberwise, or as we also say *fibered* (almost) contact structure ${}^T\xi$ on the total space of the trivial fibration ${}^TM:=T\times M\to T$, which on each fiber $M^\tau:=\tau\times M$ coincides with ξ^τ.

A *fibered contact shell* $({}^TM, {}^T\xi)$ is a fibered almost contact structure that is genuine on $\mathcal{O}p\,\partial({}^TM)$, by which we mean that (M^τ,ξ^τ) is genuine for all $\tau\in\mathcal{O}p\,\partial T$ and $(\mathcal{O}p\,\partial M^\tau,\xi^\tau)$ is genuine for all $\tau\in T$. An *equivalence* between fibered contact shells

$$G: ({}^{T_1}B_1, {}^{T_1}\xi_1) \longrightarrow ({}^{T_2}B_2, {}^{T_2}\xi_2)$$

is a diffeomorphism covering a diffeomorphism $g: T_1\to T_2$ such that $G^*({}^{T_2}\xi_2)$ and ${}^{T_1}\xi_1$ are homotopic relative to $\mathcal{O}p\,\partial({}^{T_1}B_1)$ through fibered almost contact structures on ${}^{T_1}B_1$. In particular this requires $G: (B_1^\tau,\xi_1^\tau)\to(B_2^{g(\tau)},\xi_2^{g(\tau)})$ to be an equivalence of contact shells for all $\tau\in T_1$ and to be a contactomorphism when $\tau\in\mathcal{O}p\,\partial T_1$.

Given fibered contact shells ${}^{T_\pm}\zeta_\pm=({}^{T_\pm}B_\pm, {}^{T_\pm}\xi_\pm)$, we say that ${}^{T_+}\zeta_+$ *dominates* ${}^{T_-}\zeta_-$ if there is a third fibered contact shell $\zeta=({}^TB, {}^T\xi)$ such that there are

- a fibered equivalence $G: {}^T\zeta\to{}^{T_+}\zeta_+$;
- a fiberwise embedding $H: {}^{T_-}B_-\to{}^TB$ covering an embedding $h: T_-\to T$ such that $H^*({}^T\zeta)={}^{T_-}\zeta_-$ and ${}^T\xi$ is genuine on ${}^TB\setminus H(\mathrm{Int}^{T_-}B_-)$.

We will refer to the embedding $G\circ H: ({}^{T_-}B_-, {}^{T_-}\xi_-)\to({}^{T_+}B_+, {}^{T_+}\xi_+)$ as a *subordination map.*

Finally we note that the boundary connected-sum construction can be performed in the fibered set-up to define a fibered connected sum

$$({}^TB_1\#{}^TB_2, {}^T\xi_1\#{}^T\xi_2) \quad \text{with fibers } (B_1^\tau\#B_2^\tau, \xi_1^\tau\#\xi_2^\tau),$$

provided that we are given a family of boundary points $p_1^\tau\in\partial B_1^\tau$ and $p_2^\tau\in\partial B_2^\tau$ as in the non-parametric case.

3.5. Parametric contact structures with universal holes

Given a special contact Hamiltonian $K: \Delta_{\mathrm{cyl}}\to\mathbb{R}$, we define a function $E: \Delta_{\mathrm{cyl}}\to\mathbb{R}$ by the formula $E(u,z):=K(u,1)$. By assumption, we have $K\leqslant E$ on Δ_{cyl}. We further define a family of contact Hamiltonians $K^{(s)}: \Delta_{\mathrm{cyl}} \to \mathbb{R}$ by

$$K^{(s)} := sK + (1-s)E \quad \text{for } s \in [0,1]. \tag{9}$$

Given a disc $T := D^q \subset \mathbb{R}^q$, pick a bump function $\delta: T \to [0,1]$ with support in the interior of T and consider the family of contact Hamiltonians $K^{(\delta(\tau))}: \Delta_{\mathrm{cyl}} \to \mathbb{R}$ parameterized by $\tau \in T$ and the fibered circle model shell over T,

$$({}^T B_K, {}^T \eta_K), \quad \text{where } {}^T B_K = \bigcup_{\tau \in T} \{\tau\} \times B_{K^{\delta(\tau)}} \tag{10}$$

and the fiber over $\tau \in T$ is given by $(B_{K^{(\delta(\tau))}}, \eta_{K^{(\delta(\tau))}})$.

Recall Proposition 3.1 and its contact Hamiltonian $K_{\mathrm{univ}}: \Delta_{\mathrm{cyl}} \to \mathbb{R}$. The next proposition, which we prove in §8.2, is the parametric generalization of Proposition 3.1 and says that any fibered almost contact structure is equivalent to a fibered almost contact structure that is genuine away from holes modeled on $({}^T B_{K_{\mathrm{univ}}}, {}^T \eta_{K_{\mathrm{univ}}})$.

Proposition 3.11. *Let $T = D^q$ and $A \subset M$ be a closed subset. Every fibered almost contact structure ${}^T\xi_0$ on ${}^T M = T \times M$ that is genuine on $(T \times \mathcal{O}p\, A) \cup (\mathcal{O}p\, \partial T \times M)$ is homotopic relative to $(T \times A) \cup (\partial T \times M)$ through fibered almost contact structures on ${}^T M$ to some structure ${}^T\xi$ with the following property*:

There is a collection of disjoint embedded fibered shells ${}^{T_i}B_i \subset {}^T(M \setminus A)$ over (not necessarily disjoint) q-dimensional discs $T_i \subset T$ for $i = 1, ..., L$ such that

(i) *the fibers of ${}^T\xi$ are genuine contact structures away from $\bigcup_{i=1}^L \operatorname{Int} {}^{T_i}B_i$;*

(ii) *the fibered contact shells $({}^{T_i}B_i, {}^{T_i}\xi)$ and $({}^{T_i}B_{K_{\mathrm{univ}}}, {}^{T_i}\eta_{K_{\mathrm{univ}}})$ are equivalent.*
Furthermore for every $C \subset \{1, ..., L\}$ the intersection $\bigcap_{i \in C} T_i$ is either empty or a disc.

Recall the setting of Proposition 3.10: (B, ξ) is a $(2n+1)$-dimensional contact ball for which there is a special contact Hamiltonian $K: \Delta_{\mathrm{cyl}} \to \mathbb{R}$ such that $(D_K, \eta_K) \subset (\partial B, \xi)$, where the coorientation of D_K coincides with the outward coorientation of ∂B. The following proposition, which we prove in §5.2.2, is the parametric generalization of Proposition 3.10, where $({}^T B, {}^T \xi)$ is the fibered contact structure $T \times (B, \xi)$.

Proposition 3.12. *Let $(K_0, \Delta_{\mathrm{cyl}})$ be a contact Hamiltonian and consider the fibered contact shell*

$$({}^T B_{K_0} \# {}^T B, {}^T \eta_{K_0} \# {}^T \xi)$$

given by performing a boundary connected sum on each fiber over $\tau \in T$ at the north pole of $B_{K_0^{(\delta(\tau))}}$ and the south pole of $D_K \subset \partial B$. If $K \leqslant K_0$ is special, then $({}^T B_{K_0} \# {}^T B, {}^T \eta_{K_0} \# {}^T \xi)$ is fibered equivalent to a genuine fibered contact structure.

3.6. Proof of Theorem 1.2 and Corollary 1.4

Theorem 1.2 is an immediate corollary of the following theorem, which is a fibered version of Theorem 1.1. In particular, for each $q \geqslant 0$, we see that

$$j_*: \pi_q(\mathfrak{Cont}_{\mathrm{ot}}(M; A, \xi_0, \phi)) \longrightarrow \pi_q(\mathfrak{cont}_{\mathrm{ot}}(M; A, \xi_0, \phi))$$

is an isomorphism by applying the following theorem in the cases of D^q and D^{q+1}.

THEOREM 3.13. *Let $T = D^q$ and $A \subset M$ be a closed subset such that $M \setminus A$ is connected, and let ${}^T\xi$ be a fibered almost contact structure on ${}^TM$ which is genuine on $(T \times \mathcal{O}p\, A) \cup (\partial T \times M)$. If there exists a fixed overtwisted disc $(D_{\mathrm{ot}}, \eta_{\mathrm{ot}}) \subset M \setminus A$ such that for all $\tau \in T$ the inclusion $(D_{\mathrm{ot}}, \eta_{\mathrm{ot}}) \subset (M \setminus A, \xi^\tau)$ is a contact embedding, then ${}^T\xi$ is homotopic to a fibered genuine contact structure through fibered almost contact structures fixed on $(T \times (A \cup D_{\mathrm{ot}})) \cup (\partial T \times M)$.*

Proof. By assumption there is a piecewise smooth disc $D_{\mathrm{ot}} \subset M \setminus A$ such that all almost contact structures ξ^τ, for $\tau \in T$, are genuine on $\mathcal{O}p\, D_{\mathrm{ot}}$ and restrict to D_{ot} as η_{ot}. As $(D_{\mathrm{ot}}, \eta_{\mathrm{ot}})$ determines the germ of the contact structure, we may pick a ball $B \subset \mathcal{O}p\, D_{\mathrm{ot}}$ with $D_{\mathrm{ot}} \subset \partial B$ and assume that $({}^TB, {}^T\xi) = T \times (B, \xi)$.

By applying Proposition 3.11, we may assume that there is a collection of disjoint fibered balls ${}^{T_i}B_i \subset M \setminus (A \cup B)$ over a collection of discs $T_i \subset T$ for $i = 1, ..., L$ such that

(i) ${}^T\xi$ is genuine away from $\bigcup_{i=1}^L \mathrm{Int}({}^{T_i}B_i)$;

(ii) the fibered shells $({}^{T_i}B_i, {}^{T_i}\xi)$ and $({}^{T_i}B_{K_{\mathrm{univ}}}, {}^{T_i}\eta_{K_{\mathrm{univ}}})$ are equivalent.

Apply Proposition 3.9 to get L disjoint balls $B_i' \subset \mathrm{Int}(B \setminus (D_{\mathrm{ot}} \cup A))$ with an overtwisted disc $(D^i_{\mathrm{ot}}, \eta_{\mathrm{ot}}) \subset (\partial B_i', \xi)$ in each of them.

It follows from Lemma 9.1, proven in §9 below, that for each j we can find a parametric family of embedded paths ${}^{T_j}\gamma_j$ connecting ${}^{T_j}B_j$ to ${}^{T_j}B_j'$ in $T \times (M \setminus A \cup D_{\mathrm{ot}})$. Moreover, using Gromov's parametric h-principle for transverse paths, see [31], we may assume that the constructed paths are transverse.

As we explain in §5.1, with these parametric paths we can form disjoint parametric ambient boundary connected sums ${}^{T_j}C_j \subset {}^{T_j}(M \setminus (A \cup D_{\mathrm{ot}}))$ for each $j = 1, ..., L$, between the fibered shells ${}^{T_j}B_j$ and ${}^{T_j}B_j'$. Furthermore, by §5.1 and property (ii) above we have isomorphisms of fibered almost contact structures

$$({}^{T_i}C_i, {}^{T_i}\xi) \cong ({}^{T_i}B_{K_{\mathrm{univ}}} \# {}^{T_i}B_i', {}^{T_i}\eta_{K_{\mathrm{univ}}} \# {}^{T_i}\xi).$$

Applying Proposition 3.12 inductively for $j = 1, ..., L$, we deform ${}^T\xi$ on these connected sums relative to their boundary to get a fibered genuine contact structure on ${}^TM$. □

Proof of Corollary 1.4. By an isotopy of f we can arrange that the complement $M\setminus f(N)$ is overtwisted and the closure $f(N)$ is compact. Then, slightly reducing, if necessary, the manifold M, we may assume that it is non-compact and overtwisted at infinity. Let us exhaust N by compact subsets: $N=\bigcup_{j=1}^{\infty} C_j$, such that $C_j \Subset \operatorname{Int} C_{j+1}$ and $V\setminus C_j$ is connected for all j. Set $C_0:=\varnothing$. The result follows by induction from the following claim:

Suppose we are given an embedding $f^{j-1}: N \to M$ which is contact on $\mathcal{O}p\, C_{j-1}$ and a homotopy of bundle isomorphisms $\Phi_t^{j-1}: TN \to TM$ covering f^{j-1} such that the following property P^{j-1} *is satisfied*:

(P^{j-1}) *The homotopy Φ_t^{j-1} is contact on $T(N)|_{\mathcal{O}p\, C_{j-1}}$ for all $t\in[0,1]$, Φ_0^{j-1} is contact everywhere, and $\Phi_1^{j-1}=df^{j-1}$.*
Then there exists a pair (f^j, Φ_t^j) which satisfies P^j and is such that f^{j-1} and f^j are isotopic via an isotopy fixed on C_{j-1}.

Let $\{\xi_t\}_{t\in[0,1]}$ be a family of almost contact structures on M such that $\xi_t=(\Phi_t^{j-1})_*\zeta$ on $f^{j-1}(C_j)$ and $\xi_t=\xi$ outside $f^{j-1}(C_{j+1})$. We note that $\xi_0=\xi$ on $f^{j-1}(C_j)$, and $\xi_t=\xi$ on $f^{j-1}(C_{j-1})$ for all $t\in[0,1]$. Theorem 1.2 allows us to construct a compactly supported homotopy $\tilde{\xi}_t$ of *genuine* contact structures on M, $t\in[0,1]$, connecting $\tilde{\xi}_0=\xi$ and a contact structure $\tilde{\xi}_1$ which coincides with ξ_1 on $f^{j-1}(C_j)$. Moreover, this can be done to ensure the existence of a homotopy $\Psi_t: TM \to TM$ of bundle isomorphisms such that $\Psi_0=\mathrm{Id}$, $\Psi_t^*\tilde{\xi}_t=\xi_t$, and $\Psi_t|_{f^{j-1}(C_{j-1})}=\mathrm{Id}$, $t\in[0,1]$. Then Gray's theorem [27] provides us with a compactly supported diffeotopy $\phi_t: M \to M$, $t\in[0,1]$, such that $\phi_0=\mathrm{Id}$, $\phi_t^*\xi=\tilde{\xi}_t$, and $\phi_t|_{f^{j-1}(C_{j-1})}=\mathrm{Id}$. Set $f^j:=\phi_1\circ f^{j-1}$ and $\Phi_t^j:=d\phi_t\circ\Psi_t^*\circ\Phi^{j-1}$, $t\in[0,1]$. Then $\Phi_1^j=df^j$, $(\Phi_t^j)^*\xi=(\Phi_t^{j-1})^*\circ(\Psi_t)\circ(d\phi_t)^*\xi=(\Phi_t^{j-1})^*\circ\Psi_t^*\tilde{\xi}_t=(\Phi_t^{j-1})^*\xi_t$. Hence, $(\Phi_t^j)^*\xi|_{C_j}=\zeta$ for all $t\in[0,1]$. We also have $(\Psi_0^j)^*\xi=\zeta$ everywhere. Thus, the pair (f^j, Φ_t^j) satisfies P^j, and the claim follows by induction. □

4. Domination and conjugation for Hamiltonian contact shells

Recall the notation (K,Δ) for a contact Hamiltonian K on a star-shaped domain $\Delta\subset(\mathbb{R}^{2n-1},\xi_{\mathrm{st}})$ such that $K|_{\partial\Delta\times S^1}>0$ as in (1).

In this section we will develop two properties of Hamiltonian contact shells that make them well-suited for the purposes of this paper. Namely in §4.1 we show that a natural partial order $(K,\Delta)\leqslant(K',\Delta')$ is compatible with domination of contact shells, and in §4.2 we show that the action of $\mathrm{Cont}(\Delta)$ on a contact Hamiltonian (K,Δ) by conjugation preserves the equivalence class of the associated contact shell.

A simple, but very important observation is then made in §4.3, where we show how conjugation can be used to make some contact Hamiltonians (K,Δ) much smaller with respect to the partial order. For instance, in the 3-dimensional case where $\Delta\subset\mathbb{R}$ is an interval, we prove that up to conjugation $K\colon\Delta\to\mathbb{R}$ is a minimal element for the partial order if K is somewhere negative. In higher dimensions, the existence of a minimal element up to conjugation is unknown, but the weaker Propositions 4.8 and 4.9 hold in general and they suffice for our purposes.

4.1. A partial order on contact Hamiltonians with domains

Let us introduce a partial order on contact Hamiltonians with domains, where

$$(K,\Delta)\leqslant(K',\Delta')$$

is defined to mean $\Delta\subset\Delta'$ together with

$$K(x,t)\leqslant K'(x,t)\quad\text{for all } x\in\Delta \text{ and} \tag{11}$$

$$0<K'(x,t)\quad\text{for all } x\in\Delta'\setminus\Delta. \tag{12}$$

This partial order is compatible with domination of contact shells.

LEMMA 4.1. *If* $(K,\Delta)\leqslant(K',\Delta')$, *then* (B_K,η_K) *is dominated by* $(B_{K'},\eta_{K'})$. *More specifically, given a contact shell* $(B_{K,C},\eta_{K,\varrho})$, *there is a shell* $(B_{K',C'},\eta_{K',\varrho'})$ *such that the inclusion*

$$(B_{K,C},\eta_{K,\varrho})\subset(B_{K',C'},\eta_{K',\varrho'})$$

is a subordination map.

Proof. If $C'\geqslant C$, then by (11) we have $(B_{K,C},\eta_{K,\varrho})\subset(B_{K',C'},\eta_{K',\varrho'})$ and it will be an embedding of almost contact structures whenever

$$\varrho'=\varrho\quad\text{on } \mathcal{O}p\,B_{K,C}\subset B_{K',C'}.$$

If we pick the extension so that

$$\partial_v\varrho'_{(x,t)}(v)>0\quad\text{on } \mathcal{O}p\{(x,v,t):x\in\Delta, v\geqslant K(x,t)+C\}\cup\mathcal{O}p\{(x,v,t):x\in\Delta'\setminus\operatorname{Int}\Delta\},$$

which is possible on the latter region by (12), it follows that $\eta_{K',\varrho'}$ is contact on

$$\mathcal{O}p(B_{K',C'}\setminus\operatorname{Int}B_{K,C}),$$

and hence the inclusion is a subordination map. □

4.2. Conjugation of contact Hamiltonians

Given a contact manifold (M,α) and a contact Hamiltonian $K\colon M\times S^1\to\mathbb{R}$, let $\{\phi_K^t\}_{t\in[0,1]}$ be the unique contact isotopy with $\phi_K^0=\mathbb{1}$ and

$$\alpha(\partial_t\phi_K^t(x))=K(\phi_K^t(x),t).$$

For a contactomorphism $\Phi\colon(M,\alpha)\to(M',\alpha')$, define the push-forward Hamiltonian

$$\Phi_*K\colon M'\times S^1\longrightarrow\mathbb{R}\quad\text{by } (\Phi_*K)(\Phi(x),t)=c_\Phi(x)\,K(x,t), \tag{13}$$

where $c_\Phi\colon M\to\mathbb{R}_{>0}$ satisfies $\Phi^*\alpha'=c_\Phi\alpha$. One can verify that

$$\{\Phi\phi_K^t\Phi^{-1}\}_{t\in[0,1]}=\{\phi_{\Phi_*K}^t\}_{t\in[0,1]},$$

so Φ_* corresponds to conjugation by Φ.

In this paper we will primarily be concerned with contactomorphisms $\Phi\colon\Delta\to\Delta'$ between star-shaped domains in $(\mathbb{R}^{2n-1},\xi_{\mathrm{st}})$, where $c_\Phi\colon\Delta\to\mathbb{R}_{>0}$ is defined by

$$\Phi^*\lambda_{\mathrm{st}}=c_\Phi\,\lambda_{\mathrm{st}}.$$

It is clear that if (K,Δ) satisfies (1), then (Φ_*K,Δ') does as well. As the next lemma shows the push-forward operation induces an equivalence of contact shells.

LEMMA 4.2. *A contactomorphism between two star-shaped domains $\Phi\colon\Delta\to\Delta'$ in $(\mathbb{R}^{2n-1},\xi_{\mathrm{st}})$ induces an equivalence of the contact shells*

$$\hat\Phi\colon(B_K,\eta_K)\longrightarrow(B_{\Phi_*K},\eta_{\Phi_*K})$$

defined by (K,Δ) and (Φ_*K,Δ').

Proof. For a given model $(B_{K,C},\eta_{K,\varrho})$ we will build a model $(B_{\Phi_*K,\widetilde C},\eta_{\Phi_*K,\tilde\varrho})$ such that the two models are isomorphic as almost contact structures.

For $\widetilde C+\min_{\Delta'\times S^1}\Phi_*K>0$, pick a family of diffeomorphisms for $(x,t)\in\Delta\times S^1$,

$$\phi_{(x,t)}\colon[0,K(x,t)+C]\longrightarrow[0,c_\Phi(x)K(x,t)+\widetilde C]$$

and define a smooth family of functions for $(x,t)\in\Delta\times S^1$,

$$\tilde\varrho_{(\Phi(x),t)}\colon[0,c_\Phi(x)K(x,t)+\widetilde C]\longrightarrow\mathbb{R}\quad\text{by } \tilde\varrho_{(\Phi(x),t)}(v)=c_\Phi(x)\varrho_{(x,t)}(\phi^{-1}_{(x,t)}(v)).$$

One sees that $\tilde\varrho$ satisfies the conditions in (3) to define $(B_{\Phi_*K,\widetilde C},\eta_{\Phi_*K,\tilde\varrho})$ provided

$$\phi_{(x,t)}(v)=c_\Phi(x)(v-C)+\widetilde C\quad\text{on } \mathcal{O}p\{(x,v,t):v=K(x,t)+C\}.$$

It follows by construction that the diffeomorphism

$$\hat\Phi\colon(B_{K,C},\eta_{K,\varrho})\longrightarrow(B_{\Phi_*K,\widetilde C},\eta_{\Phi_*K,\tilde\varrho})\quad\text{defined by } \hat\Phi(x,v,t)=(\Phi(x),\phi_{(x,t)}(v),t)$$

is an isomorphism of almost contact structures. ◻

4.2.1. Foliations of overtwisted discs

For a first example of this push-forward procedure, we will prove Proposition 3.9 as a corollary of Lemma 4.2 above and Lemma 4.3 below. For $\delta\in\mathcal{O}p\{1\}$ observe that the contactomorphism $C_\delta:\mathbb{R}^{2n-1}\to\mathbb{R}^{2n-1}$ given by

$$C_\delta(u_1,...,u_{n-1},\phi_1,...,\phi_{n-1},z)=\left(\frac{u_1}{\delta},...,\frac{u_{n-1}}{\delta},\phi_1,...,\phi_{n-1},\frac{z}{\delta}\right)$$

satisfies $C_\delta(\Delta_\delta)=\Delta_{\mathrm{cyl}}$, where $\Delta_\delta:=\{(x,z):u\leqslant\delta \text{ and } |z|\leqslant\delta\}$.

LEMMA 4.3. *Let $K:\Delta_{\mathrm{cyl}}\to\mathbb{R}$ be a special contact Hamiltonian and define*

$$K_\delta:\Delta_\delta\longrightarrow\mathbb{R}\quad \textit{by}\ K_\delta:=K+(\delta-1).$$

If $\delta<1$ is sufficiently close to 1, then $\widetilde{K}_\delta:=(C_\delta)_*K_\delta:\Delta_{\mathrm{cyl}}\to\mathbb{R}$ is also a special contact Hamiltonian.

Proof. Let $k:\mathbb{R}_{\geqslant 0}\to\mathbb{R}$ be the special function for K and let

$$\tilde{k}_\delta(u):=\frac{k(\delta u)}{\delta}+\frac{\delta-1}{\delta}.$$

Computing for $a>1$ and $\delta<1$, we get

$$a\tilde{k}_\delta\left(\frac{u}{a}\right)-\tilde{k}_\delta(u)<(a-1)\frac{\delta-1}{\delta}<0,$$

so $\tilde{k}_\delta$ is special, provided δ is close enough to 1 so that $\tilde{k}_\delta(1)>0$. Since

$$\widetilde{K}_\delta:=(C_\delta)_*K_\delta=\frac{K\circ C_\delta^{-1}}{\delta}+\frac{\delta-1}{\delta},$$

one can now see that it is a special Hamiltonian for $\tilde{z}_D=z_D/\delta$ and $\tilde{k}_\delta$. □

Proof of Proposition 3.9. Consider an overtwisted disc (D_K,η_K) defined by a special contact Hamiltonian $K:\Delta_{\mathrm{cyl}}\to\mathbb{R}$. For $\delta\in[1-\varepsilon,1]$, let $\Delta_\delta=\{(x,z):u\leqslant\delta \text{ and } |z|\leqslant\delta\}$ and consider the family of contact Hamiltonians

$$K_\delta:\Delta_\delta\longrightarrow\mathbb{R},\quad \text{where } K_\delta:=K+(\delta-1).$$

Observe that any neighborhood of $(\partial B_K,\eta_K)$ contains a foliation

$$\{(\partial B_{K_\delta},\eta_{K_\delta})\}_{\delta\in[1-\varepsilon,1]},$$

provided $\varepsilon>0$ is small enough.

Furthermore, when $\varepsilon>0$ is sufficiently small, Lemmas 4.2 and 4.3 give us a family of special contact Hamiltonians $\{\widetilde{K}_\delta:\Delta_{\mathrm{cyl}}\to\mathbb{R}\}_{\delta\in[1-\varepsilon,1]}$ such that $\widetilde{K}_\delta<K_{\mathrm{univ}}$ together with contactomorphisms

$$(\partial B_{K_\delta},\eta_{K_\delta})\cong(\partial B_{\widetilde{K}_\delta},\eta_{\widetilde{K}_\delta}).$$

Therefore every neighborhood of (D_K,η_K) contains a foliation $\{(D_{K_\delta},\eta_{K_\delta})\}_{\delta\in[1-\varepsilon,1]}$ of overtwisted discs. □

4.2.2. Embeddings of contact Hamiltonian shells

As a second application of the push-forward procedure, we have the following lemma about embeddings of contact Hamiltonian shells.

LEMMA 4.4. *Let $(B_{K,C},\eta_{K,\varrho})$ be a contact shell structure for (K,Δ). For any other (K',Δ') there exists a contact shell structure $(B_{K',C'},\eta_{K',\varrho'})$ together with an embedding of almost contact structures*

$$(B_{K,C},\eta_{K,\varrho})\longrightarrow(B_{K',C'},\eta_{K',\varrho'}).$$

If $\Delta\subset\operatorname{Int}\Delta'$, then the embedding can be taken to be an inclusion map.

Proof. Since Δ' is star-shaped, there is a contactomorphism $\Phi\in\operatorname{Cont}_0^c(\mathbb{R}^{2n-1})$ such that $\Delta\subset\operatorname{Int}\Phi(\Delta')$, and therefore, by Lemma 4.2, we may without loss of generality assume that $\Delta\subset\operatorname{Int}\Delta'$.

Given the contact shell structure $(B_{K,C},\eta_{K,\varrho})$, pick any contact shell $(B_{K',C'},\eta_{K',\varrho'})$ subject to the additional conditions that

$$K'(x,t)+C'>K(x,t)+C\quad\text{for all }(x,t)\in\Delta\times S^1 \tag{14}$$

and the smooth family of functions $\varrho'_{(x,t)}:\mathbb{R}_{\geqslant 0}\to\mathbb{R}$ for $(x,t)\in\Delta'\times S^1$ satisfies

$$\varrho'=\varrho\quad\text{on }\mathcal{O}p\,B_{K,C}\subset B_{K',C'}, \tag{15}$$

where the latter is always possible since $\Delta\subset\operatorname{Int}\Delta'$. By (14) we have an inclusion

$$(B_{K,C},\eta_{K,\varrho})\subset(B_{K',C'},\eta_{K',\varrho'}), \tag{16}$$

and by (15) it is an embedding of almost contact structures. □

Remark 4.5. If the inclusion (16) was a subordination map, then

$$\partial_v\varrho'_{(x,t)}(v)>0\quad\text{on }\mathcal{O}p\{(x,v,t):x\in\Delta\text{ and }v\geqslant K(x,t)+C\},$$

which, together with (14) and (15), imply $K'(x,t)>K(x,t)$ for all $x\in\Delta$, since

$$K'(x,t)-K(x,t)=\varrho'_{(x,t)}(K'(x,t)+C')-\varrho'_{(x,t)}(K(x,t)+C)>0.$$

A similar argument shows why assuming $\Delta\subset\Delta'$ is not sufficient, since the conditions (14), (15), and $\partial_v\varrho'>0$ on $\partial\Delta'$ imply $K'(x,t)>K(x,t)$ for all $x\in\partial\Delta\cap\partial\Delta'$.

4.2.3. Changing the contactomorphism type of the domain

Recall that star-shaped domains $\Delta \subset (\mathbb{R}^{2n-1}, \xi_{\mathrm{st}})$ are the ones for which the contact vector field

$$Z = z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}$$

is transverse to $\partial\Delta$, and we denote the *flow* of a vector field X by X^t. While not all star-shaped domains are contactomorphic, up to mutual domination of contact shells (B_K, η_K), the choice of domain does not matter.

LEMMA 4.6. *For any contact Hamiltonian (K, Δ) and star-shaped domain Δ' there is a contact Hamiltonian (K', Δ') such that (B_K, η_K) dominates $(B_{K'}, \eta_{K'})$.*

Proof. For any neighborhood $U \supset \partial\Delta$ there is a contactomorphism $\Phi \in \mathrm{Cont}_0^c(\mathbb{R}^{2n-1})$ such that $\Phi(\Delta') \subset \Delta$ and $\Phi(\partial\Delta') \subset U$. To see this first note that, without loss of generality, we may assume that $\Delta' \subset \Delta$ by replacing Δ' by $Z^{-N}(\Delta')$ for some sufficiently large N. After this reduction, the required contactomorphism is given by $\widetilde{Z}^T$ for T sufficiently large, where $\widetilde{Z}$ is a contact vector field with $\mathrm{supp}(\widetilde{Z}) \subset \mathrm{Int}\,\Delta$ and $\widetilde{Z} = Z$ on $\mathcal{O}p\, Z^{-\varepsilon}(\Delta)$, with $\Delta \setminus Z^{-\varepsilon}(\Delta) \subset U$.

Now pick $U \supset \partial\Delta$ to be such that $K|_{U\times S^1} > 0$, take the constructed contactomorphism Φ above, and consider the contact Hamiltonian $K' = \Phi_*^{-1}(K|_{\Phi(\Delta')})$ on Δ'. It follows from Lemmas 4.1 and 4.2 that (B_K, η_K) dominates $(B_{K'}, \eta_{K'})$. □

4.3. Domination up to conjugation

If we want to prove that the contact shell (B_K, η_K) is dominated by the shell $(B_{K'}, \eta_{K'})$ then Lemmas 4.1 and 4.2 instruct us to care about the partial order from §4.1 up to conjugation. In particular, it is enough to find a contact embedding $\Phi: \Delta \to \Delta'$ such that $(\Phi_* K, \Phi(\Delta)) \leqslant (K', \Delta')$ to prove that (B_K, η_K) is dominated by $(B_{K'}, \eta_{K'})$.

4.3.1. Minimal elements up to conjugation in the 3-dimensional case

In the 3-dimensional case where $\Delta \subset \mathbb{R}$ is always a closed interval, up to conjugation, any somewhere negative Hamiltonian (K, Δ) is minimal with respect to the partial order from §4.1.

LEMMA 4.7. *Suppose that (K, Δ) is somewhere negative, with $\Delta = [-1, 1]$. For any other contact Hamiltonian $(\widetilde{K}, \Delta)$ there is a contactomorphism $\Phi \in \mathrm{Cont}_0(\Delta)$ such that $(\Phi_* K, \Delta) \leqslant (\widetilde{K}, \Delta)$, and hence (B_K, η_K) is dominated by $(B_{\widetilde{K}}, \eta_{\widetilde{K}})$.*

Proof. Without loss of generality, assume $K(0)<0$. Pick $\varepsilon>0$ and $\delta>0$ so that

$$K(z)<-\varepsilon \text{ if } |z|\in[0,\delta] \quad \text{and} \quad \widetilde{K}(z)>\varepsilon \text{ if } |z|\in[1-\delta,1].$$

For $0<\sigma\ll 1$, pick a diffeomorphism $\Phi:[-1,1]\to[-1,1]$ such that it linearly maps

$$[-\sigma,\sigma] \text{ onto } [-1+2\sigma,1-2\sigma] \quad \text{and} \quad \pm[2\sigma,1] \text{ onto } \pm[1-\sigma,1].$$

Since $(\Phi_*K)(\Phi(z))=\Phi'(z)K(z)$, we can pick σ sufficiently small so that

$$\begin{aligned} &(\Phi_*K)(z)<-\frac{1-2\sigma}{\sigma}\delta<\widetilde{K}(z), && \text{if } |z|\in[0,1-2\sigma],\\ &(\Phi_*K)(z)<0<\widetilde{K}(z), && \text{if } |z|\in[1-2\sigma,1-\sigma],\\ &(\Phi_*K)(z)\leqslant\frac{\sigma}{1-2\sigma}\max K<\delta<\widetilde{K}(z), && \text{if } |z|\in[1-\sigma,1], \end{aligned}$$

and hence get that $\Phi_*K<\widetilde{K}$. □

As a consequence of this lemma, the 3-dimensional case simplifies by making §8 unnecessary and allows us to give an effective description of an overtwisted disc. It seems unlikely to us (though we do not have a proof) that the generalization of Lemma 4.7 holds when $\dim(\Delta_{\mathrm{cyl}})\geqslant 3$. The immediate obstacle to adapting the proof is essentially that $\mathcal{O}p\,\partial\Delta_{\mathrm{cyl}}$ is not a star-shaped domain in higher dimensions, while for $\Delta_{\mathrm{cyl}}=[-1,1]$ we get two intervals which are star-shaped.

4.3.2. Remnants of the 3-dimensional case

Proposition 4.8 and its corollary Proposition 4.9 below, represent the remnants of the 3-dimensional Lemma 4.7 that survive in higher dimensions.

Proposition 4.9 essentially says that, up to conjugation, the only part of (K,Δ) that is relevant for the partial order is $K|_{\{K\geqslant 0\}}$, whereas, for instance, $\min K$ is irrelevant if $K<0$ somewhere. It will play a key role in §8, where we prove the existence of universal contact shells.

Given a domain $\Delta\subset\mathbb{R}^{2n-1}_{\mathrm{st}}$, let

$$F_+(\Delta):=\{K\in C^0(\Delta):\operatorname{supp}(K)\subset\operatorname{Int}\Delta,\ K\geqslant 0,\ \text{and } K\neq 0\}$$

and consider the action of $\mathfrak{D}_0(\Delta):=\mathrm{Cont}_0^c(\operatorname{Int}\Delta)$ on $F_+(\Delta)$ given by

$$\Phi_*K:=(c_\Phi\cdot K)\circ\Phi^{-1} \quad \text{for } K\in F_+(\Delta) \text{ and } \Phi\in\mathfrak{D}_0(\Delta),$$

i.e. the push-forward operation from (13).

PROPOSITION 4.8. *If* $\Delta\subset(\mathbb{R}^{2n-1},\xi_{\mathrm{st}})$ *is star-shaped, then for any two* $K,H\in F_+(\Delta)$ *there is a contactomorphism* $\Phi\in\mathfrak{D}_0(\Delta)$ *such that* $\Phi_*K\geqslant H$.

Proof. Without loss of generality, assume that Δ is star-shaped with respect to the radial vector field Z, and that $K(0)>0$. Pick a sufficiently small neighborhood $U\ni 0$ so that, for some $T>0$,

$$\inf_U K>0,\quad \operatorname{supp}(H)\subset Z^T(U)\subset\operatorname{Int}\Delta,\quad\text{and}\quad e^T\inf_U K>\max H,$$

where $Z^t\colon\mathbb{R}^{2n-1}\to\mathbb{R}^{2n-1}$ is the flow of Z and satisfies $(Z^t)^*\lambda_{\mathrm{st}}=e^t\lambda_{\mathrm{st}}$. Let $\widetilde{Z}$ be another contact vector field supported in $\operatorname{Int}\Delta$ and equal to Z on $Z^T(U)$. It follows that the contactomorphism $\Phi:=\widetilde{Z}^T\in\mathfrak{D}_0(\Delta)$ satisfies $\Phi_*K\geqslant H$, since

$$(\Phi_*K)(x)=(c_\Phi\cdot K)(\Phi^{-1}(x))\geqslant e^T\inf_U K\geqslant H(x)\quad\text{if } x\in\operatorname{supp}(H)$$

and $\Phi_*K\geqslant 0$ otherwise. □

Note that Proposition 4.8 shows that, on the conjugacy classes of elements of the positive cone $\mathcal{C}:=\{f\in\mathfrak{D}_0:f\geqslant\mathrm{Id}\text{ and }f\neq\mathrm{Id}\}$, the partial order from [17] is trivial and it would be interesting to understand for which contact manifolds the analog of Proposition 4.8 holds. As pointed out to us by L. Polterovich, a non-trivial bi-invariant metric on Cont_0^c compatible with the notion of order on Cont_0^c from [17] provides an obstruction to Proposition 4.8. For instance Sandon's metric [45] shows that Proposition 4.8 does not hold for $D_R^{2n}\times S^1$ with contact form $dz+\sum_{i=1}^n u_i\,d\phi_i$, where D_R^{2n} is a $2n$-disc of a sufficiently large radius R.

As an application of Proposition 4.8, we show in this next proposition that condition (11) in the definition of the partial order $(K,\Delta)\leqslant(K',\Delta')$ from §4.1 can be weakened so that there is still domination of the contact shells.

PROPOSITION 4.9. *Consider contact Hamiltonians* $K_i\colon\Delta\to\mathbb{R}$ *defining contact shells* (B_{K_i},η_{K_i}) *for* $i=1,2$. *If there is a star-shaped domain* $\tilde\Delta\subset\operatorname{Int}\Delta$ *such that*

$$K_0\leqslant K_1\ \text{ on } \mathcal{O}p(\Delta\setminus\operatorname{Int}\tilde\Delta),\quad 0\leqslant K_1\ \text{ on } \mathcal{O}p\,\partial\tilde\Delta,\quad\text{and}\quad K_0\leqslant 0\ \text{ on } \mathcal{O}p\,\tilde\Delta,$$

with $K_0|_{\operatorname{Int}\tilde\Delta}\not\equiv 0$, *then the contact shell* (B_{K_0},η_{K_0}) *is dominated by* (B_{K_1},η_{K_1}).

Proof. The assumptions ensure that we can pick contact Hamiltonians $\widetilde{K}_i\colon\Delta\to\mathbb{R}$ defining contact shells $(B_{\widetilde{K}_i},\eta_{\widetilde{K}_i})$, for $i=1,2$, so that

(i) $K_0\leqslant\widetilde{K}_0$ and $\widetilde{K}_1\leqslant K_1$;

(ii) $\widetilde{K}_0\leqslant\widetilde{K}_1$ on $\Delta\setminus\tilde\Delta$;

(iii) $-\widetilde{K}_i|_{\tilde\Delta}\in F_+(\tilde\Delta)$ for $i=1,2$.

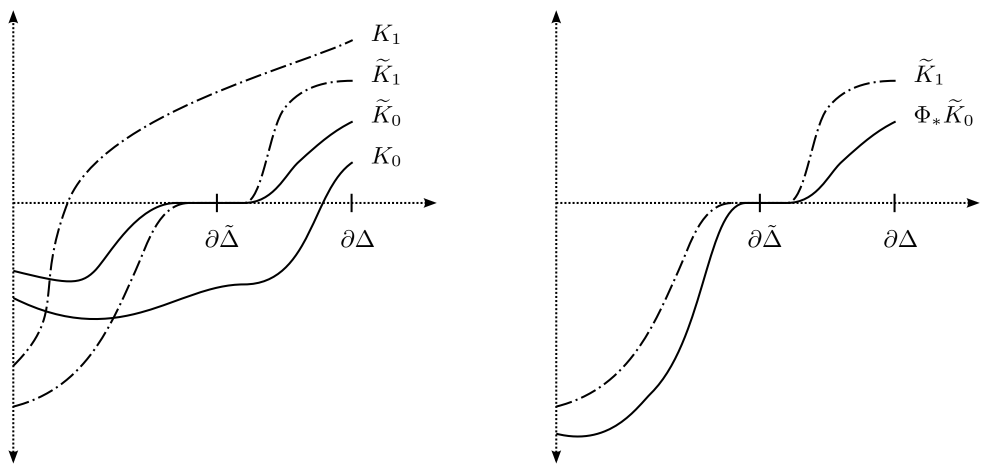

Figure 4.1. Schematic representation of the proof of Proposition 4.9.

By item (i) and Lemma 4.1 it suffices to show that $(B_{\widetilde{K}_0}, \eta_{\widetilde{K}_0})$ is dominated by $(B_{\widetilde{K}_1}, \eta_{\widetilde{K}_1})$.

Applying Proposition 4.8 to item (iii) gives a $\Phi \in \mathrm{Cont}_0^c(\mathrm{Int}\,\tilde{\Delta})$ such that

$$\Phi_*(\widetilde{K}_0|_{\tilde{\Delta}}) \leqslant \widetilde{K}_1|_{\tilde{\Delta}}.$$

Together with item (ii), this means that $\Phi_*\widetilde{K}_0 \leqslant \widetilde{K}_1$, where we think of $\Phi \in \mathrm{Cont}_0^c(\mathrm{Int}\,\Delta)$, and therefore $(B_{\widetilde{K}_0}, \eta_{\widetilde{K}_0})$ is dominated by $(B_{\widetilde{K}_1}, \eta_{\widetilde{K}_1})$ by Lemmas 4.2 and 4.1. □

We also have the following parametric version of Proposition 4.9.

PROPOSITION 4.10. *Assume that $\Delta \subset \mathbb{R}^{2n-1}$ is a star-shaped domain. Let $\Delta' \subset \Delta$ be a smooth star-shaped subdomain and let $K^\tau: \Delta \to \mathbb{R}$, $\tau \in T$, be a family of time-independent functions satisfying $K^\tau|_{\Delta \setminus \mathrm{Int}\,\Delta'} > 0$. Suppose that $K^\tau > 0$ for τ in a closed subset $A \subset T$. Then, for any $\delta > 0$, there exists a family $\widetilde{K}^\tau$ such that*

- *$\widetilde{K}^\tau = K^\tau$ on $\Delta \setminus \mathrm{Int}\,\Delta'$ and $\widetilde{K}^\tau > -\delta$, $\tau \in T$;*
- *$\widetilde{K}^\tau = K^\tau$ for $\tau \in A$;*
- *there exists a family of subordination maps $h^\tau: \eta_{\widetilde{K}^\tau} \to \eta_{K^\tau}$ which are identity maps for $\tau \in A$.*

5. Filling of the universal circle models

In this section we prove Propositions 3.10 and 3.12. Here we set

$$\Delta = \Delta_{\mathrm{cyl}} = \{(x, z) : u \leqslant 1 \text{ and } |z| \leqslant 1\} \subset (\mathbb{R}^{2n-1}, \xi_{\mathrm{st}}), \quad \text{where } u = u_1 + \ldots + u_{n-1}.$$

All contact Hamiltonians (K,Δ) will be assumed time indepedent and spherically symmetric, i.e. functions $K(u,z)$ of only the u and z variables.

The contactomorphism of $(\mathbb{R}^{2n-1},\xi_{\mathrm{st}})$ that is translation in the z-coordinate will be

$$Z_\tau\colon \mathbb{R}^{2n-1}\longrightarrow\mathbb{R}^{2n-1},\quad \text{where } Z_\tau(q,z)=(q,z+\tau),$$

using coordinates $(q,z)\in\mathbb{R}^{2n-2}\times\mathbb{R}$.

5.1. Boundary connected sum

5.1.1. Abstract boundary connected sum

Consider $\mathbb{R}^{2n}$ with polar coordinates $(u_1,\varphi_1,...,u_{n-1},\varphi_{n-1},v,t)$ equipped with the radial Liouville form and vector field

$$\theta:=\sum_{i=1}^{n-1}u_i\,d\varphi_i+v\,dt\quad\text{and}\quad L:=\sum_{i=1}^{n-1}u_i\frac{\partial}{\partial u_i}+v\frac{\partial}{\partial v},$$

and denote by $L^t\colon\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ the Liouville flow.

A *gluing disc* for a contact shell (W,ζ) is a smooth embedding $\iota\colon D\to\partial W$, where $D\subset\mathbb{R}^{2n}$ is a compact domain, star-shaped with respect to L, and with piecewise smooth boundary such that $\iota^*\alpha=\theta$ for a choice of a contact form α for ζ in $\mathcal{O}p\,\partial W$. Note that this implies that $\iota(0)\in\partial W$ is a gluing place in the sense of §2.2 and that the Reeb vector field R_α is transverse to $\iota(D)$.

Given contact shells $(W_\pm^{2n+1},\zeta_\pm)$, with gluing discs $\iota_\pm\colon D\to\partial W_\pm$ such that ι_+ preserves and ι_- reverses orientation, the Reeb flows define contact embeddings

$$\begin{aligned}\Phi_+\colon D\times(-\varepsilon,0]&\longrightarrow\mathcal{O}p\,\iota_+(D)\quad\text{with } \Phi_+^*\alpha_+=dz+\theta,\\ \Phi_-\colon D\times[0,\varepsilon)&\longrightarrow\mathcal{O}p\,\iota_-(D)\quad\text{with } \Phi_-^*\alpha_-=dz+\theta,\end{aligned}\tag{17}$$

such that $\Phi_\pm|_{D\times\{0\}}=\iota_\pm$. For $\ell>0$ consider a smooth function $\beta\colon[-\ell,\ell]\to\mathbb{R}_{\geqslant0}$ such that $\beta(z)=0$ for z near $\pm\ell$ and let $D(z):=L^{-\beta(z)}(D)$. Define the *abstract boundary connected sum* to be the almost contact manifold

$$(W_+\#_TW_-,\zeta_+\#_T\zeta_-):=((W_+,\zeta_+)\cup(T,\ker(dz+\theta))\cup(W_-,\zeta_-))/\sim,\tag{18}$$

where

$$T=\{(p,z)\in\mathbb{R}^{2n}\times[-\ell,\ell]:p\in D(z)\}\subset\mathbb{R}^{2n+1}\tag{19}$$

and one identifies

$$\Phi_+(p,0)\sim(p,-\ell)\in T\quad\text{and}\quad\Phi_-(p,0)\sim(p,\ell)\in T.$$

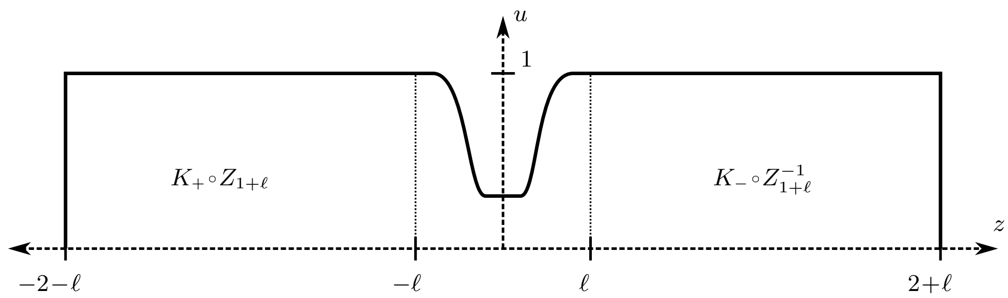

Figure 5.1. The domain of the Hamiltonian $K_+\#_\beta K_-:\Delta\#_{\beta,\ell}\Delta\to\mathbb{R}$.

5.1.2. Abstract connected sum of S^1-model contact shells

Consider a Hamiltonian contact shell $(B_{K,C},\eta_{K,\varrho})$ associated with a contact Hamiltonian (K,Δ). There are canonical gluing discs

$$D_\pm=\{(q,v,t):u\leqslant 1 \text{ and } v\leqslant K(u,\pm 1)\}\subset\mathbb{R}^{2n}$$

with maps $\iota_\pm: D_\pm\to(\partial B_{K,C},\eta_{K,\varrho})$,

$$\iota_\pm(q,v,t)=(q,\pm 1,\varrho^{-1}_{(q,\pm 1)}(v),t)\in\mathbb{R}^{2n-1}\times\mathbb{R}^2,$$

where $\iota_\pm(0,0)=(0,\pm 1,0)$ are the north and south poles of B_K.

For two contact Hamiltonians $K_\pm:\Delta\to\mathbb{R}$, assume that $E(u)=K_\pm(u,\pm 1)$ is well defined. For any $\ell>0$ and smooth function $\beta:[-\ell,\ell]\to\mathbb{R}_{\geqslant 0}$ such that $\beta=0$ near $z=\pm\ell$, define the domain

$$\Delta\#_{\beta,\ell}\Delta:=Z^{-1}_{1+\ell}(\Delta)\cup T_{\beta,\ell}\cup Z_{1+\ell}(\Delta)\subset\mathbb{R}^{2n-1}, \tag{20}$$

where

$$T_{\beta,\ell}:=\{(q,z):u\leqslant e^{-\beta(z)} \text{ and } |z|\leqslant\ell\}\subset\mathbb{R}^{2n-1} \tag{21}$$

and define the contact Hamiltonian $K_+\#_\beta K_-:\Delta\#_{\beta,\ell}\Delta\to\mathbb{R}$ by

$$(K_+\#_\beta K_-)(u,z)=\begin{cases}(K_+\circ Z_{1+\ell})(u,z), & \text{on } Z^{-1}_{1+\ell}(\Delta),\\ e^{-\beta(z)}E(u), & \text{for } (q,z)\in T_{\beta,\ell},\\ (K_-\circ Z^{-1}_{1+\ell})(u,z), & \text{on } Z_{1+\ell}(\Delta).\end{cases}$$

Going forward, we will drop β from the notation when $\beta\equiv 0$.

It follows from Example 5.8 below that $\Delta\#_{\beta,\ell}\Delta$ is star-shaped since it is contactomorphic to $\Delta\#_\ell\Delta$, which is star-shaped with respect to $Z=\partial/\partial z+L$, and hence $(K_+\#_\beta K_-,\Delta\#_{\beta,\ell}\Delta)$ defines an S^1-model contact shell

$$(B_{K_+\#_\beta K_-},\eta_{K_+\#_\beta K_-})$$

as in §2.3. It is straightforward to check that we have the following lemma.

Lemma 5.1. *The contact shell* $(B_{K_+\#_\beta K_-}, \eta_{K_+\#_\beta K_-})$ *is equivalent to the abstract connected sum* $(B_{K_+}\#_T B_{K_-}, \eta_{K_+}\#_T\eta_{K_-})$ *with tube*

$$T=\{(p,z)\in\mathbb{R}^{2n}\times[-\ell,\ell]: u\leqslant e^{-\beta(z)} \text{ and } v\leqslant e^{-\beta(z)}E(u)\}\subset\mathbb{R}^{2n+1},$$

where the connected sum is done at the north pole of B_{K_+} *and the south pole of* B_{K_-}.

5.1.3. Ambient boundary connected sum

Suppose in an almost contact manifold (W^{2n+1},ξ) there are disjoint codimension-0 submanifolds $W_\pm\subset\operatorname{Int} W$ with piecewise smooth boundary such that ξ is a genuine contact structure in $\mathcal{O}p\,\partial W_\pm$. Assume that the contact shells $(W_\pm,\xi)$ are equipped with gluing discs $\iota_\pm: D\to\partial W_\pm$, where $\iota_\pm^*\alpha=\theta$ for a contact form α for ξ such that ι_+ preserves and ι_- reverses orientation.

For a smooth embedding $\gamma:[0,1]\to\operatorname{Int} W$ such that

- $\gamma(0)=\iota_+(0)$, $\gamma(1)=\iota_-(0)$, and $\gamma(t)\notin W_+\cup W_-$ otherwise;
- ξ is a genuine contact structure on $\mathcal{O}p\,\Gamma$, where $\Gamma:=\gamma([0,1])$;
- γ is transverse to ξ;

we can think of $(W_+\cup\mathcal{O}p\,\Gamma\cup W_-,\xi)$ as an ambient boundary connected sum of the shells $(W_\pm,\xi)$. This is made precise with the following lemma.

Lemma 5.2. *Every neighborhood of* $(W_+\cup\mathcal{O}p\,\Gamma\cup W_-,\xi)$ *contains the image of an almost contact embedding of an abstract connected sum* $(W_+\#_T W_-, \xi\#_T\xi)$.

Proof. The gluing discs $\iota_\pm: D\to\partial W_\pm$ extend to Darboux embeddings

$$\Phi_\pm: D\times(\mp\ell-\varepsilon,\mp\ell+\varepsilon)\longrightarrow\mathcal{O}p\,\iota_\pm(D)\quad\text{with } \Phi_\pm^*\alpha=dz+\theta \text{ and } \Phi_\pm|_{D\times\{\mp\ell\}}=\iota_\pm,$$

and moreover one can ensure that $\Phi_+^{-1}(\Gamma)=\{0\}\times[-\ell,-\ell+\varepsilon)$ and $\Phi_-^{-1}(\Gamma)=\{0\}\times(\ell-\varepsilon,\ell]$.

By the neighborhood theorem for transverse curves in a contact manifold, for $N>0$ sufficiently large the embeddings $\Phi_\pm$ can be extended (after possibly decreasing ε) to a contact embedding

$$\Phi: (D\times(-\ell-\varepsilon,\ell+\varepsilon))\cup(L^{-N}(D)\times[-\ell,\ell])\cup(D\times(\ell-\varepsilon,\ell+\varepsilon))\longrightarrow\operatorname{Int} W,$$

whose image is contained in $\mathcal{O}p(\iota_+(D)\cup\Gamma\cup\iota_-(D))$ and such that $\Phi(\{0\}\times[-\ell,\ell])=\Gamma$. Picking $\beta:[-\ell,\ell]\to\mathbb{R}_{\geqslant 0}$ such that the tube

$$T=\{(p,z)\in\mathbb{R}^{2n}\times[-\ell,\ell]: p\in L^{-\beta(z)}(D)\}$$

is contained in the domain of Φ, we can now use Φ to define the required contact embedding $(W_+\#_T W_-,\xi\#_T\xi)\to(W,\xi)$. □

5.2. Filling a connected sum of a shell with a neighborhood of an overtwisted disc

For the rest of this section, fix a special Hamiltonian (K,Δ). For $\varepsilon'>0$, define

$$K'=K-\varepsilon' \quad \text{and} \quad \Delta'=\{(q,z): u\leqslant 1-\varepsilon' \text{ and } |z|\leqslant 1-\varepsilon'\},$$

and assume $\varepsilon'>0$ is small enough so that $K'|_{\partial\Delta'}>0$.

The goal of this subsection is the proof of Proposition 3.10 and its parametric version Proposition 3.12. All the connected sums as in §5.1.1 and §5.1.2 will be done with a fixed choice of function $\beta: [-\ell,\ell]\to\mathbb{R}_{\geqslant 0}$, which we will suppress from the notation. In particular, we will be considering abstract connected sums such as

$$(B_K\#B_K,\eta_K\#\eta_K) \quad \text{and} \quad (B_{K\#K},\eta_{K\#K}),$$

where we will always use the north-pole gluing place on the first factor and the south-pole gluing place on the second factor. We will also freely use Lemma 5.1 to identify such connected sums.

By Lemma 4.1 we can arrange the inclusion

$$(B_{K',C},\eta_{K',\varrho})\longhookrightarrow(B_{K,C},\eta_{K,\varrho})$$

to be a subordination map, so that we have a $(2n+1)$-dimensional contact annulus

$$(\mathbf{A},\xi_{\mathbf{A}}):=(B_{K,C}\setminus \operatorname{Int} B_{K',C},\ker\eta_{K,\varrho}|_{\mathbf{A}}).$$

Define the contact ball $(\mathbf{B},\xi_{\mathbf{B}})\subset(\mathbf{A},\xi_{\mathbf{A}})$ given by

$$\mathbf{B}:=\{(x,v,t)\in\mathbf{A}: z(x)\in[-1,z_D]\}, \tag{22}$$

and by design the $2n$-dimensional disc $(D_K,\eta_K)\subset(\partial\mathbf{B},\xi_{\mathbf{B}})$ appears with the correct coorientation.

5.2.1. Non-parametric version

To prove Proposition 3.10 it will suffice to show that the contact shell $(B_K\#\mathbf{B},\eta_K\#\xi_{\mathbf{B}})$, defined as a subset of $(B_K\#B_K,\eta_K\#\eta_K)$, is equivalent to a genuine contact structure. Letting $\iota:\Delta\to\Delta\#\Delta$ be the inclusion into the right-hand factor, we will prove in Lemma 5.4 (i) below that there is a family of contact embeddings

$$\Theta_\sigma:\Delta\longrightarrow\operatorname{Int}(\Delta\#\Delta), \quad \text{for } \sigma\in[0,1] \text{ with } \Theta_0=\iota, \tag{23}$$

such that $\Theta_\sigma=\iota$ in $\mathcal{O}p\{x\in\Delta: z(x)\in[z_D,1]\}$ for all $\sigma\in[0,1]$ and $\Theta:=\Theta_1$ satisfies

$$(\Theta_*K',\Theta(\Delta'))<(K\#K,\Delta\#\Delta).$$

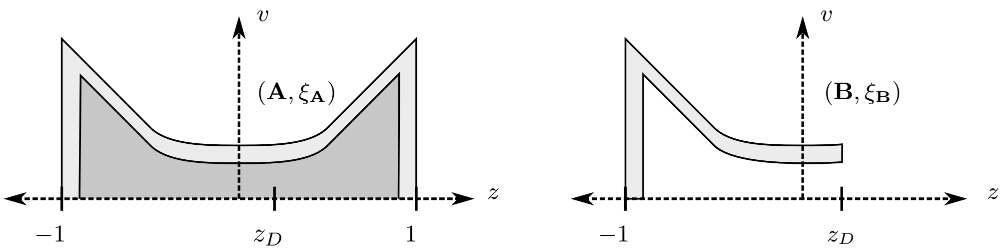

Figure 5.2. On the left: The union of the grey regions is B_K, the dark grey region is $B_{K'}$, and the light grey region is $(\mathbf{A}, \xi_{\mathbf{A}})$. On the right: The contact ball $(\mathbf{B}, \xi_{\mathbf{B}}) \subset B_K$ obtained from $(\mathbf{A}, \xi_{\mathbf{A}})$.

Proof of Proposition 3.10. It suffices to prove that $(B_K \# \mathbf{B}, \eta_K \# \xi_{\mathbf{B}})$ is equivalent to a genuine contact structure, since it is dominated by $(B_{K_0} \# B, \eta_{K_0} \# \xi)$ if we pick $\varepsilon' > 0$ sufficiently small in the definition of $(\mathbf{B}, \xi_{\mathbf{B}})$.

By Lemmas 4.1, 4.2, and 4.4 we can pick a family of contact shell structures on $(B_K \# B_K, \eta_{K \# K, \hat{\varrho}_\sigma})$ such that there is a family of contact shell embeddings

$$\widehat{\Theta}_\sigma \colon (B_{K'}, \eta_{K'}) \longrightarrow (B_K \# B_K, \eta_{K \# K, \hat{\varrho}_\sigma}) \tag{24}$$

with $\widehat{\Theta}_1$ a subordination map. We can arrange that $\eta_{K \# K, \hat{\varrho}_0} = \eta_K \# \eta_K$ and for all $\sigma \in [0, 1]$ to have

$$\eta_{K \# K, \hat{\varrho}_\sigma} = \eta_K \# \eta_K \quad \text{on } \mathcal{O}p\, \hat{\imath}\{x \in \Delta : z(x) \in [z_D, 1]\},$$

where $\hat{\imath} \colon B_K \to B_K \# B_K$ is the inclusion into the right-hand factor.

We can pick an isotopy $\{\Psi_\sigma\}_{\sigma \in [0,1]}$ of $B_K \# B_K$ based at the identity and supported away from the boundary such that

(i) $\Psi_\sigma \circ \hat{\imath} = \widehat{\Theta}_\sigma \colon B_{K'} \to B_K \# B_K$;

(ii) $\Psi_\sigma = \mathrm{Id}$ on $\mathcal{O}p\, \hat{\imath}\{???: z \in [z_D, 1]\}$;

(iii) $\Psi_1(B_K \# \mathbf{A}) = (B_K \# B_K) \setminus \operatorname{Int} \widehat{\Theta}(B_{K'})$.

Observe that a point in $\mathcal{O}p\, \partial(B_K \# \mathbf{B})$ is one of the following regions:

(i) $\mathcal{O}p\, \partial(B_K \# B_K)$, where $\Psi_\sigma = \mathrm{Id}$ and $\eta_K \# \eta_{\mathbf{B}} = \eta_K \# \eta_K = \eta_{K \# K, \hat{\varrho}_\sigma}$;

(ii) $\mathcal{O}p\, \hat{\imath}(\{???: z = z_D\})$, where $\Psi_\sigma = \mathrm{Id}$ and $\eta_K \# \eta_{\mathbf{B}} = \eta_K \# \eta_K = \eta_{K \# K, \hat{\varrho}_\sigma}$;

(iii) $\mathcal{O}p\, \hat{\imath}(\partial B_{K'})$, where $\eta_K \# \eta_{\mathbf{B}} = \hat{\imath}_* \eta_{K'} = \hat{\imath}_* \widehat{\Theta}_\sigma^* (\eta_{K \# K, \hat{\varrho}_\sigma}) = \Psi_\sigma^* (\eta_{K \# K, \hat{\varrho}_\sigma})$.

This shows that $\xi_\sigma := \Psi_\sigma^*(\eta_{K \# K, \hat{\varrho}_\sigma})$ is a family of equivalent contact shells on $B_K \# \mathbf{B}$ with $\xi_0 = \eta_K \# \eta_{\mathbf{B}}$. We know that $\eta_{K \# K, \hat{\varrho}_1}$ is a contact structure away from $\operatorname{Int} \widehat{\Theta}(B_{K'})$, since $\widehat{\Theta}_1$ is a subordination map, and therefore ξ_1 is a genuine contact structure on $B_K \# \mathbf{B}$. □

5.2.2. Parametric version

Recall the family of contact Hamiltonians $K^{(s)} = sK + (1-s)E$ for $s \in [0, 1]$ from (9), where $E(u) = K(u, \pm 1)$ and $E(u) \geqslant K(u, z)$. Let $(\mathbf{B}, \xi)$ be the contact ball from (22)

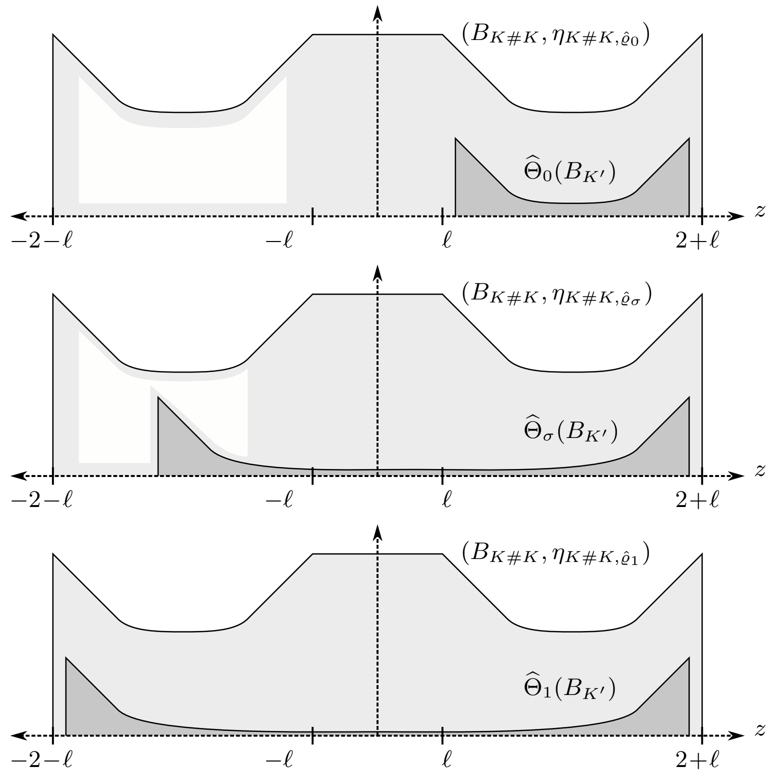

Figure 5.3. Images of the almost contact embeddings $\widehat{\Theta}_\sigma: B_{K'} \to B_{K\#K}$ in dark grey. The white regions denote where outside of $\widehat{\Theta}_\sigma(B_{K'})$ the almost contact structure $\eta_{K\#K,\hat{\varrho}_\sigma}$ is not genuine.

and let $({}^I B_K \# {}^I \mathbf{B}, {}^I\zeta)$ be the family of contact shells fibered over $I=[0,1]$ with fibers $(B_{K^{(s)}} \# \mathbf{B}, \zeta^s)$ for $s \in [0,1]$, where

$$\zeta^s = \eta_{K^{(s)}} \# \xi_{\mathbf{B}} \quad \text{and} \quad (B_{K^{(s)}} \# \mathbf{B}, \eta_{K^{(s)}} \# \xi) \subset (B_{K^{(s)}} \# B_K, \eta_{K^{(s)}} \# \eta_K).$$

We may assume that ζ^s is a genuine contact structure when $s \in \mathcal{O}p\{0\}$, as $K^{(0)} = E$ is positive.

Let us first prove the following proposition similar to Proposition 3.12.

PROPOSITION 5.3. *The fibered family of contact shells* ${}^I\zeta$ *is homotopic relative to*

$$\mathcal{O}p\{s : s=0\} \cup \bigcup_{s\in[0,1]} \mathcal{O}p\, \partial(B_{K^{(s)}} \# \mathbf{B}) \subset {}^I B_K \# {}^I \mathbf{B}$$

through fibered families of contact shells on ${}^I B_K \# {}^I \mathbf{B}$ *to a fibered family of genuine contact structures.*

Proof. Inspecting the proof of Proposition 3.10 shows that it can be done parametrically. In particular, we can get a family of contact shell embeddings

$$\widehat{\Theta}^s_\sigma\colon (B_{K'}, \eta_{K'}) \longrightarrow (B_{K^{(s)}} \# B_K, \eta_{K^{(s)} \# K, \hat{\varrho}^s_\sigma})$$

and associated isotopies $\{\Psi^s_\sigma\}_{\sigma\in[0,1]}$ of $B_{K^{(s)}}\#B_K$, which lead to contact shell structures

$$\hat{\zeta}^s_\sigma := (\Psi^s_\sigma)^*(\eta_{K^{(s)}\#K,\hat{\varrho}^s_\sigma}) \quad \text{on } B_{K^{(s)}}\#\mathbf{B}$$

that define a family of fibered contact shells ${}^I\hat{\zeta}_\sigma$ on ${}^IB_K\#{}^I\mathbf{B}$. It follows from the second part of Lemma 5.4 that

$$((\Theta_\sigma)_*K', \Theta_\sigma(\Delta')) < (K^{(s)}\#K, \Delta\#\Delta) \quad \text{if } s\in \mathcal{O}p\{0\}, \tag{25}$$

and therefore we can arrange for $\widehat{\Theta}^s_\sigma$ to be a subordination map when $s\in\mathcal{O}p\{0\}$.

With this set-up the proof of Proposition 3.10 shows that we can ensure that the family of fibered contact shells ${}^I\hat{\zeta}_\sigma$ is such that ${}^I\hat{\zeta}_0={}^I\zeta$ as well as

(i) $\hat{\zeta}^s_\sigma=\zeta^s$ on $\mathcal{O}p\,\partial(B_{K^{(s)}}\#\mathbf{B})$ for all s and σ;

(ii) $\hat{\zeta}^s_1$ is a genuine contact structure for all s;

(iii) $\hat{\zeta}^s_\sigma$ is a genuine contact structure for all $(\sigma,s)\in[0,1]\times[0,3a]$ for some $a>0$.

Pick any smooth function

$$f\colon [0,1]\times[0,1]\longrightarrow[0,1], \quad \text{with } f(\sigma,s)=\begin{cases} 0, & \text{if } \sigma=0,\\ 0, & \text{if } s\in[0,a],\\ 1, & \text{if } s\in[2a,1] \text{ and } \sigma=1,\end{cases}$$

and define the family of contact shells $\zeta^s_\sigma:=\hat{\zeta}^s_{f(\sigma,s)}$ on $B_{K^{(s)}}\#\mathbf{B}$, which represents a homotopy of fibered families of contact shells

$$\{{}^I\zeta_\sigma\}_{\sigma\in[0,1]} \quad \text{on} \quad {}^IB_K\#{}^I\mathbf{B}.$$

It follows from item (i) and the fact $f(\sigma,s)=0$ if $s\in[0,a]$, that this homotopy is relative to the appropriate set. Observe that ζ^s_1 is a genuine contact structure for all $s\in[0,1]$, since either $s\leqslant 3a$ and $\zeta^s_1:=\hat{\zeta}^s_{f(1,s)}$ is genuine by item (iii), or $s\geqslant 2a$ and $\zeta^s_1:=\hat{\zeta}^s_{f(1,s)}=\hat{\zeta}^s_1$ is genuine by item (ii). Therefore we have the desired homotopy between ${}^I\zeta={}^I\zeta_0$ and a fibered family of genuine contact structures ${}^I\zeta_1$. □

Proof of Proposition 3.12. Recall that $({}^TB_{K_0}\#{}^TB, {}^T\eta_{K_0}\#{}^T\xi)$ is the fibered contact shell, which at the point $\tau\in T=D^q$ is given by

$$(B_{K_0^{(\delta(\tau))}}\#B, \eta_{K_0^{(\delta(\tau))}}\#\xi),$$

where $\delta: T \to [0,1]$ is a bump function that vanishes near the boundary. It suffices to prove that $({}^T B_K \# {}^T \mathbf{B}, {}^T \eta_K \# {}^T \xi_{\mathbf{B}})$ is fibered equivalent to a fibered contact structure over T, since it is dominated by $({}^T B_{K_0} \# {}^T B, {}^T \eta_{K_0} \# {}^T \xi)$ if we pick $\varepsilon' > 0$ sufficiently small in the definition of $(\mathbf{B}, \xi_{\mathbf{B}})$.

In the notation of Proposition 5.3, we have the identification

$$\zeta^{\delta(\tau)} = \eta_{K(\delta(\tau))} \# \xi_{\mathbf{B}} \quad \text{as contact shell structures on } B_{K(\delta(\tau))} \# \mathbf{B},$$

and a fibered contact structure on $({}^T B_K \# {}^T \mathbf{B}, {}^T \zeta_1)$ with contact structure

$$\zeta_1^{\delta(\tau)} \text{ on the fiber } B_{K(\delta(\tau))} \# \mathbf{B}.$$

Since $\delta(\tau) = 0$ if $\tau \in \mathcal{O}p\, \partial T$, the homotopy constructed in Proposition 5.3, when used fiberwise, gives a homotopy between ${}^T \eta_K \# {}^T \xi_{\mathbf{B}}$ and ${}^T \zeta_1$ showing that they are fibered equivalent. □

5.3. Main lemma

Consider the connected sums $(K \#_\beta K, \Delta \#_{\beta,\ell} \Delta)$ and $(E \#_\beta K, \Delta \#_{\beta,\ell} \Delta)$ as in §5.1.2. The main goal of this section will be to prove the following lemma, which we will break up into two sublemmas below.

LEMMA 5.4. *There is a family of contact embeddings for* $\sigma \in [0,1]$,

$$\Theta_\sigma : \Delta \longrightarrow \Delta \#_{\beta,\ell} \Delta \quad \textit{with } \Theta_\sigma = Z_{1+\ell} \textit{ on } \mathcal{O}p\{x \in \Delta : z(x) \in [z_D, 1]\}$$

based at $\Theta_0 := Z_{1+\ell}$ *such that*

(i) $((\Theta_1)_* K', \Theta_1(\Delta')) < (K \#_\beta K, \Delta \#_{\beta,\ell} \Delta)$;

(ii) $((\Theta_\sigma)_* K', \Theta_\sigma(\Delta')) < (E \#_\beta K, \Delta \#_{\beta,\ell} \Delta)$ *for all* $\sigma \in [0,1]$.

Proof. It follows from Lemma 5.7 that it suffices to prove this lemma when $\beta \equiv 0$ and this special case is proved in Lemma 5.9. □

Let us remark that in §5.2.1 we only used the first part of Lemma 5.4, while in §5.2.2 we used both parts.

Remark 5.5. In the 3-dimensional case, where $\Delta = [-1,1]$, this lemma essentially follows from Lemma 4.7.

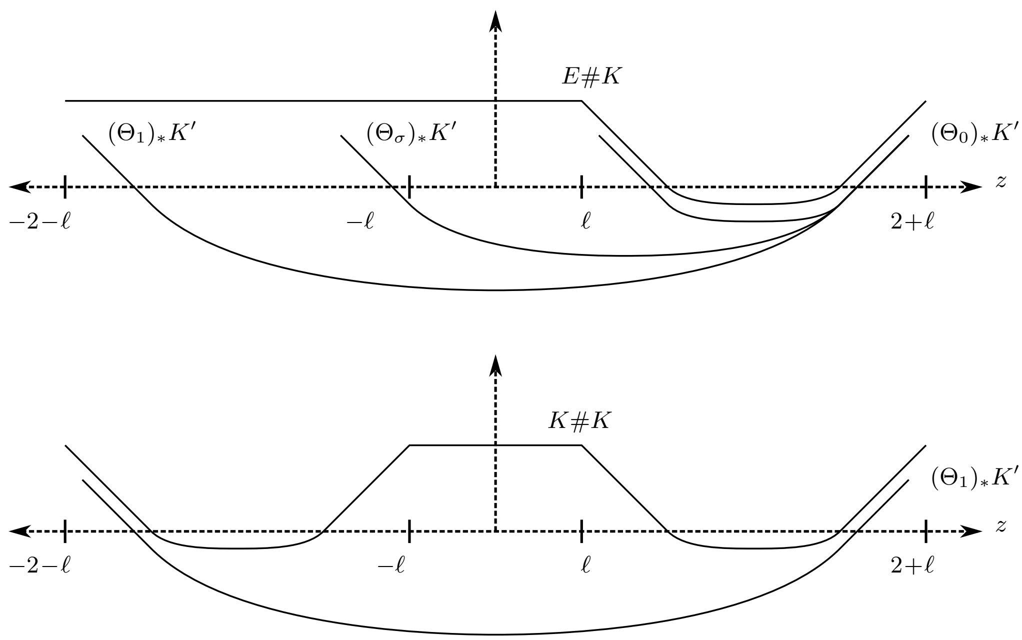

Figure 5.4. Schematic representation of Lemma 5.4, where we have that $\Delta=\{z:|z|\leqslant 1\}\subset\mathbb{R}$. Here $K'(z)=K(z)-\varepsilon'$ is restricted to $\Delta'=\{z:|z|\leqslant 1-\varepsilon'\}$.

5.3.1. Transverse scaling and simplifying the neck region

Transverse scaling. An orientation-preserving diffeomorphism $h:\mathbb{R}\to\mathbb{R}$ defines a contactomorphism Φ_h of $(\mathbb{R}^{2n-1},\xi_{\mathrm{st}})$ by

$$\Phi_h(u_i,\varphi_i,z)=(h'(z)u_i,\varphi_i,h(z)),$$

where $\Phi_h^{-1}=\Phi_{h^{-1}}$. By (13) we have

$$(\Phi_h)_*H(u,z)=h'(h^{-1}(z))H\left(\frac{u}{h'(h^{-1}(z))},h^{-1}(z)\right) \tag{26}$$

for a contact Hamiltonian $H(u,z):\mathbb{R}^{2n-1}\to\mathbb{R}$.

Example 5.6. For our purposes Φ should be thought of as a way to manipulate the z-variable at the cost of a scaling factor on the u-variable, in particular we have a contactomorphism Φ_h between domains in $(\mathbb{R}^{2n-1},\xi_{\mathrm{st}})$ given by

$$\Phi_h:\{(q,z):u\leqslant f(z) \text{ and } z\in[a,b]\}\longrightarrow\{(q,z):u\leqslant(h'\cdot f)(h^{-1}(z)) \text{ and } z\in[h(a),h(b)]\},$$

where $f:\mathbb{R}\to\mathbb{R}_{>0}$.

This contactomorphism allows us to reduce the proof of Lemma 5.4 to when $\beta\equiv 0$.

LEMMA 5.7. *For every connected sum* $(K\#_\beta K, \Delta\#_{\beta,\ell}\Delta)$, *if* $\ell'>\ell$ *is sufficiently large, then there is a contact embedding*

$$\Phi\colon \Delta\#_{\ell'}\Delta \longrightarrow \Delta\#_{\beta,\ell}\Delta, \quad \textit{with } \Phi=Z_{\pm(\ell-\ell')} \textit{ on } \mathcal{O}p\{x\in\Delta\#_{\ell'}\Delta : \pm z(x)\geqslant \ell'\},$$

such that $(\Phi_*(K\#K), \Phi(\Delta\#_{\ell'}\Delta))\leqslant(K\#_\beta K, \Delta\#_{\beta,\ell}\Delta)$.

Proof. Pick a constant

$$0<C<\frac{\min E}{\max E}\leqslant 1 \tag{27}$$

and a diffeomorphism $h\colon[-\ell',\ell']\to[-\ell,\ell]$ with $h'(z)=1$ on $z\in\mathcal{O}p\{\pm\ell\}$ and

$$h'(h^{-1}(z))\leqslant Ce^{-\beta(z)}, \tag{28}$$

which is possible provided

$$\ell'>\frac{1}{2C}\int_{-\ell}^{\ell} e^{\beta(z)}\,dz.$$

Extend h by translation to get a diffeomorphism $h\colon\mathbb{R}\to\mathbb{R}$ and consider the associated contactomorphism $\Phi_h\colon(\mathbb{R}^{2n-1},\xi)\to(\mathbb{R}^{2n-1},\xi)$ from (26). This is the desired contact embedding, for by (28) we have

$$\Phi_h(\Delta\#_{\ell'}\Delta)=\{(q,z): u\leqslant h'(h^{-1}(z)) \text{ and } z\in[-2-\ell,2+\ell]\}\subset\Delta\#_{\beta,\ell}\Delta.$$

To check the order on the Hamiltonians, it suffices to check on $\Phi_h(T_{\ell'})$, where we have

$$(\Phi_h)_*E(u,z)=h'(h^{-1}(z))E\left(\frac{u}{h'(h^{-1}(z))}\right)<e^{-\beta(z)}E(u)=(K\#_\beta K)(u,z),$$

by (27) and (28). □

5.3.2. The twist contactomorphism and a special case of Lemma 5.4

We will use the transverse scaling contactomorphisms Φ_h together with the following contactomorphism.

Twist contactomorphism. For $g\in C^\infty(\mathbb{R})$ and $z_0\in\mathbb{R}$, define

$$\Psi_{g,z_0}(u_i,\varphi_i,z):=\left(\frac{u_i}{1+g(z)u},\varphi_i-\int_{z_0}^{z} g(s)\,ds, z\right),$$

which is a contactomorphism between the subsets of $(\mathbb{R}^{2n-1},\xi_{\mathrm{st}})$,

$$\Psi_{g,z_0}\colon\{(q,z): 1+g(z)u>0\}\longrightarrow\{(q,z): 1-g(z)u>0\},$$

where $\Psi_{g,z_0}^{-1}=\Psi_{-g,z_0}$. By (13), we have

$$(\Psi_{g,z_0})_*H(u,z)=(1-g(z)u)H\left(\frac{u}{1-g(z)u},z\right) \tag{29}$$

for a contact Hamiltonian $H(u,z)\colon\mathbb{R}^{2n-1}\to\mathbb{R}$.

Example 5.8. For our purposes, Ψ should be thought of as a way to manipulate the u-variable at the cost of a rotation in the angular coordinates, in particular we have a contactomorphism between domains in $(\mathbb{R}^{2n-1},\xi_{\mathrm{st}})$,

$$\Psi_g\colon\{(q,z):u\leqslant f_2(z)\}\longrightarrow\{(q,z):u\leqslant f_1(z)\},$$

where $f_j\colon\mathbb{R}\to\mathbb{R}_{>0}$ and

$$g(z)=\frac{1}{f_1(z)}-\frac{1}{f_2(z)}.$$

Composing twist and scaling. Fix an orientation-preserving diffeomorphism $h\colon\mathbb{R}\to\mathbb{R}$ and define

$$g(z):=1-\frac{1}{h'(h^{-1}(z))}.$$

It follows from Examples 5.6 and 5.8 that

$$\Gamma_{h,z_0}:=\Psi_{g,z_0}\circ\Phi_h\colon\{(q,z):u\leqslant 1,z\in[a,b]\}\longrightarrow\{(q,z):u\leqslant 1\text{ and }z\in[h(a),h(b)]\} \tag{30}$$

is a contactomorphism of these domains in $(\mathbb{R}^{2n-1},\xi_{\mathrm{st}})$. So Γ_{h,z_0} lets us change the z-length of a region without changing the u-width, albeit still at the cost of a rotation in the angular coordinates.

A computation shows that

$$\Gamma_{h,z_0}(u_i,\varphi_i,z)=\left(\frac{h'(z)u_i}{1+(h'(z)-1)u},\varphi_i-\int_{z_0}^{z}\left(1-\frac{1}{h'(h^{-1}(z))}\right)ds,h(z)\right)$$

so if $h(z)=z+\tau$ for $z\in A\subset\mathbb{R}$ and $z_0\in h(A)$, then Γ_{h,z_0} is just a translation

$$\Gamma_{h,z_0}=Z_\tau\quad\text{on }\{(q,z):z\in A\}\subset\mathbb{R}^{2n-1}. \tag{31}$$

If we define

$$\tilde{h}(u,z):=h'(h^{-1}(z))-(h'(h^{-1}(z))-1)u, \tag{32}$$

then for a contact Hamiltonian $H(u,z)\colon\mathbb{R}^{2n-1}\to\mathbb{R}$ we have that

$$(\Gamma_{h,z_0})_*H(u,z)=\tilde{h}(u,z)\,H\left(\frac{u}{\tilde{h}(u,z)},h^{-1}(z)\right). \tag{33}$$

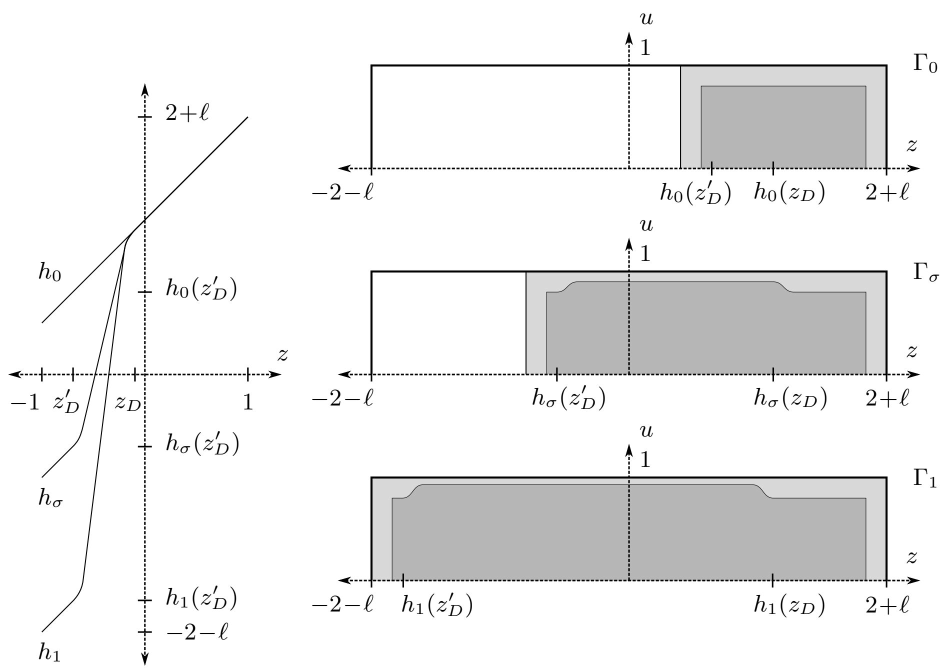

Figure 5.5. The family of diffeomorphisms h_σ and embeddings $\Gamma_\sigma\colon\Delta\to\Delta\#_\ell\Delta$. The union of the grey regions denotes the image $\Gamma_\sigma(\Delta)$ while the dark grey regions denotes the image $\Gamma_\sigma(\Delta')$ for $\Delta'=\{(q,z):u\leqslant 1-\varepsilon' \text{ and } |z|\leqslant 1-\varepsilon'\}$.

Proving Lemma 5.4 when $\beta\equiv 0$**.** Assume now that our special contact Hamiltonian K is special with respect to the function $k\colon\mathbb{R}_{\geqslant 0}\to\mathbb{R}$ and the point $z_D\in(-1,1)$ as in Definition 3.3. Note that Definition 3.3 implies that

$$k(u)\leqslant K(u,z)\leqslant E(u), \tag{34}$$

where $E(u):=K(u,\pm 1)$, and we can pick $z'_D<z_D$ so that

$$K(u,z)=k(u) \quad \text{when } z\in\mathcal{O}p[z'_D,z_D]. \tag{35}$$

Pick a family of diffeomorphisms $h_\sigma\colon\mathbb{R}\to\mathbb{R}$ for $\sigma\in[0,1]$ such that

$$h_\sigma(z)=\begin{cases} z+(1-2\sigma)(1+\ell), & \text{for } z\in\mathcal{O}p(-\infty,z'_D],\\ h'_\sigma(z)\geqslant 1, & \text{for } z\in[z'_D,z_D],\\ z+1+\ell, & \text{for } z\in\mathcal{O}p[z_D,\infty). \end{cases} \tag{36}$$

Recall the contactomorphism Γ_{h,z_0} from (30) and define the contact embeddings

$$\Gamma_\sigma:=\Gamma_{h_\sigma,2+\ell}\colon\Delta\longrightarrow\Delta\#_\ell\Delta \quad \text{for } s\in[0,1]. \tag{37}$$

By (31), we see that $\Gamma_0 = Z_{1+\ell}$ and on $\mathcal{O}p\{(q,z): z\in[z_D,1]\}$ we have $\Gamma_\sigma = Z_{1+\ell}$ for all $\sigma\in[0,1]$. With this family of contactomorphisms we can prove Lemma 5.4 with the simplifying assumption that $\beta\equiv 0$.

LEMMA 5.9. *The family of contact embeddings $\Gamma_\sigma:\Delta\to\Delta\#_\ell\Delta$ for $\sigma\in[0,1]$ satisfies the following conditions*:

(i) $(\Gamma_\sigma)_*K\leqslant E\#K$ *on* $\Gamma_\sigma(\Delta)$ *for all* $\sigma\in[0,1]$;

(ii) $(\Gamma_1)_*K\leqslant K\#K$ *on* $\Gamma_1(\Delta)$.

Proof. By (33) we have

$$(\Gamma_\sigma)_*K(u,z)=\tilde h_\sigma(u,z)K\left(\frac{u}{\tilde h_\sigma(u,z)}, h_\sigma^{-1}(z)\right),$$

where, recalling from (32),

$$\tilde h_\sigma(u,z) := h'_\sigma(h_\sigma^{-1}(z))-(h'_\sigma(h_\sigma^{-1}(z))-1)u\geqslant 1,$$

where the inequality follows from $h'_\sigma(h_\sigma^{-1}(z))\geqslant 1$ and $u\leqslant 1$. For $z\in h_\sigma([z'_D,z_D])$ we have

$$(\Gamma_\sigma)_*K(u,z)=(\Gamma_\sigma)_*k(u,z)=\tilde h_\sigma(u,z)k\left(\frac{u}{\tilde h_\sigma(u,z)}\right)\leqslant k(u),$$

where the first equality follows since here $K(u,z)=k(u)$ by (SH1) in Definition 3.3 and the last inequality follows from the definition (6) of $k:\mathbb{R}_{\geqslant 0}\to\mathbb{R}$ being special. Therefore we have

$$(\Gamma_\sigma)_*K(u,z)\begin{cases} =K(u,h_\sigma^{-1}(z)), & \text{if } z\in\mathcal{O}p\,h_\sigma([-1,z'_D]),\\ \leqslant k(u), & \text{if } z\in h_\sigma([z'_D,z_D]),\\ =K(u,z-(1+\ell)), & \text{if } z\in\mathcal{O}p[z_D+1+\ell,2+\ell],\end{cases} \tag{38}$$

since h_σ is just translations on the ends.

To verify (i), since

$$(E\#K)(u,z)=\begin{cases} E(u), & \text{if } z\in[-2-\ell,\ell],\\ K(u,z-(1+\ell)), & \text{if } z\in[\ell,2+\ell],\end{cases}$$

it follows from (38) and the inequality (34) that it suffices to check that

$$K(u,h_\sigma^{-1}(z))\leqslant K(u,z-(1+\ell))\quad\text{when } z\in[\ell,h_\sigma(z'_D)].$$

Since $h_\sigma^{-1}(z)=z-(1-2\sigma)(1+\ell)$ here, this is equivalent to

$$K(u,z+2\sigma(1+\ell))\leqslant K(u,z)\quad\text{when } z\in[-1,z'_D-2\sigma(1+\ell)],$$

and this latter condition follows from (SH2) in Definition 3.3.

To verify (ii), using (36) we see that (38) at $\sigma=1$ becomes

$$(\Gamma_1)_*K(u,z)\begin{cases} =K(u,z+(1+\ell)), & \text{if } z\in\mathcal{O}p[-2-\ell,z'_D-1-\ell],\\ \leqslant k(u), & \text{if } z\in[z'_D-1-\ell,z_D+1+\ell],\\ =K(u,z-(1+\ell)), & \text{if } z\in\mathcal{O}p[z_D+1+\ell,2+\ell],\end{cases}$$

while, by definition,

$$(K\#K)(u,z)=\begin{cases} K(u,z+(1+\ell)), & \text{if } z\in[-2-\ell,-\ell],\\ E(u), & \text{if } z\in[-\ell,\ell],\\ K(u,z-(1+\ell)), & \text{if } z\in[\ell,2+\ell],\end{cases}$$

so (ii) follows from (34). □

6. Contact structures with holes

The goal of this section is Proposition 6.2 and its parametric version Proposition 7.6, which are the first steps in proving Propositions 3.1 and 3.11.

6.1. Semi-contact structures

Let Σ be a closed $2n$-dimensional manifold. A *semi-contact structure on an annulus* $C=\Sigma\times[a,b]$ is a smooth family $\{\zeta_s\}_{s\in[a,b]}$ such that ζ_s is a germ of a contact structure along the slice $\Sigma_s:=\Sigma\times\{s\}$. If $\{\alpha_s\}_{s\in[a,b]}$ is a smooth family of 1-forms with $\zeta_s=\ker\alpha_s$ on $\mathcal{O}p\,\Sigma_s$, then one gets an almost contact structure (λ,ω) on C, where

$$\lambda(x,s)=\alpha_s(x,s)\quad\text{and}\quad\omega(x,s)=d\alpha_s(x,s).$$

It follows that every semi-contact structure on C defines an almost contact structure on C that equals ζ_s on $TC|_{\Sigma_s}$.

Given a contact structure ξ on $\Sigma\times\mathbb{R}$ and a smooth family of functions $\psi_s:\Sigma\to\mathbb{R}$ for $s\in[a,b]$, if we pick $\Psi_s:\mathcal{O}p\,\Sigma_s\to\mathcal{O}p(\operatorname{graph}\psi_s)\subset\Sigma\times\mathbb{R}$ to be a smooth family of diffeomorphisms such that $\Psi_s|_{\Sigma_s}=\mathrm{Id}\times\psi_s$, then we can define a semi-contact structure on $\Sigma\times[a,b]$ by $\zeta_s:=\Psi_s^*\xi$. Any semi-contact structure of this form will be said to be of *immersion type*.

Remark 6.1. The term is motivated by the fact that on the boundary of each domain $\Sigma^{[a',b']}:=\Sigma\times[a',b']$ for $a\leqslant a'<b'\leqslant b$ the structure $\zeta|_{\partial\Sigma^{[a',b']}}$ is induced from the genuine contact structure ξ by an immersion $\partial\Sigma^{[a',b']}\to\Sigma\times\mathbb{R}$. Of course, this is an immersion of a very special type, which maps the boundary components $\Sigma\times\{a'\}$ and $\Sigma\times\{b'\}$ onto intersecting graphical hypersurfaces.

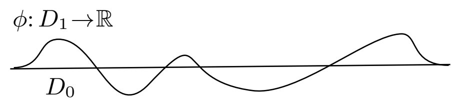

Figure 6.1. A typical regular contact saucer.

6.2. Saucers

A *saucer* is a domain $B \subset D \times \mathbb{R}$, where D is a $2n$-disc possibly with a piecewise smooth boundary, of the form

$$B = \{(w,v) \in D \times \mathbb{R} : f_-(w) \leqslant v \leqslant f_+(w)\},$$

where $f_\pm: D \to \mathbb{R}$ are smooth functions such that $f_- < f_+$ on $\operatorname{Int} D$ and whose ∞-jets coincide along ∂D. Observe that every saucer comes with a family of discs

$$D_s = \{(w,v) \in D \times \mathbb{R} : v = (1-s) f_-(w) + s f_+(w)\} \quad \text{for } s \in [0,1],$$

such that the interiors $\operatorname{Int} D_s$ foliate $\operatorname{Int} B$ and the family of discs D_s coincide with their ∞-jets along their common boundary $S = \partial D_s$, which is called the *border* of the saucer B.

A *semi-contact structure on a saucer* B is a family $\{\zeta_s\}_{s \in [0,1]}$ of germs of contact structures along the discs D_s for $s \in [0,1]$, which coincide as germs along the border S. As in §6.1, a semi-contact structure on a saucer B defines an almost contact structure ξ on B. Furthermore, (B, ξ) is a contact shell since ζ_0 and ζ_1 are germs of contact structures on D_0 and D_1 and the family ζ_s coincide along the border of B.

6.3. Regular semi-contact saucers

In $(\mathbb{R}^{2n+1}, \xi_{\mathrm{st}}^{2n+1})$, where $\xi_{\mathrm{st}}^{2n+1} = \{\lambda_{\mathrm{st}}^{2n-1} + v\, dt = 0\}$, $v := -y_n$ and $t := x_n$, define the hyperplane $\Pi := \{(w,v) \in \mathbb{R}^{2n} \times \mathbb{R} : v = 0\}$. Observe that the characteristic foliation on $\Pi \subset (\mathbb{R}^{2n+1}, \xi_{\mathrm{st}}^{2n+1})$ is formed by the fibers of the projection $\pi: \Pi \to \mathbb{R}^{2n-1}$ given by $\pi(x,t) = x$ for $x \in \mathbb{R}^{2n-1}$.

Let $D \subset \Pi$ be a $2n$-disc and let $\phi: D \to \mathbb{R}$ be a smooth function such that $\phi > 0$ on $(\operatorname{Int} D) \cap \mathcal{O}p\, \partial D$ and whose ∞-jet vanishes on ∂D. Let $F: D \to \mathbb{R}$ be a function, compactly supported in $\operatorname{Int} D$, such that $\phi + F$ is positive on $\operatorname{Int} D$. Define the saucer

$$B := \{(w,v) \in D \times \mathbb{R} : 0 \leqslant v \leqslant \phi(w) + F(w)\}.$$

Note that, up to a canonical diffeomorphism, the saucer B is independent of the choice of the function F. There is a natural family of diffeomorphisms between $D_s \subset B$ and the graphs

$$\Gamma_{s\phi} := \{(w,v) \in D \times \mathbb{R} : v = s\phi(w)\} \subset \mathbb{R}^{2n+1},$$

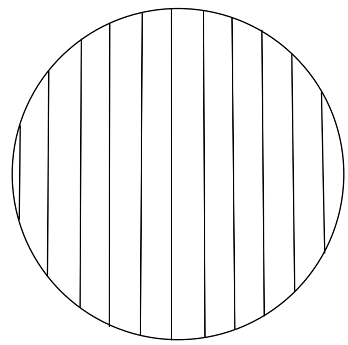

Figure 6.2. A regular foliation on the disc.

whose ∞-jets coincide along the border.

Define $\sigma_\phi = \{\zeta_s\}_{s\in[0,1]}$ to be the semi-contact structure on B, where ζ_s is the pull-back of the germ of the contact structure on $\Gamma_{s\phi} \subset (\mathbb{R}^{2n+1}, \xi_{\text{st}}^{2n+1})$. We see that ϕ defines the contact shell (B, σ_ϕ) up to diffeomorphisms of the domain.

Parameterize B with coordinates $(w,s) \in D \times [0,1]$ so that $D_{s_0} = \{s = s_0\} \subset B$, and consider the map

$$\Phi: B \longrightarrow \mathbb{R}^{2n+1}, \quad \text{where } \Phi(w,s) = (w, s\phi(w)).$$

If ϕ is positive everywhere on $\operatorname{Int} D$, then Φ is an embedding, and hence σ_ϕ is a genuine contact structure since it can be identified with $\Phi^*\xi_{\text{st}}^{2n+1}$. Similarly, for $2n$-discs $D' \subset D$ and associated semi-contact structures $\sigma_{\phi'}$ and σ_ϕ, a contact shell $\sigma_{\phi'}$ is dominated by a shell σ_ϕ if $\phi' \leqslant \phi|_{D'}$ and $\phi|_{\operatorname{Int} D \setminus D'} > 0$.

An embedded $2n$-disc $D \subset \Pi$ is called *regular* if

- the characteristic foliation $\mathcal{F}$ on $D \subset (\mathbb{R}^{2n+1}, \xi_{\text{st}}^{2n+1})$ is diffeomorphic to the characteristic foliation on the standard round disc in Π;
- the ball $\Delta := D/\mathcal{F}$ with its induced contact structure is star-shaped.

An embedded $2n$-disc $D \subset (M^{2n+1}, \xi)$ in a contact manifold is *regular* if the contact germ of ξ on D is contactomorphic to the contact germ of a regular disc in Π. A semi-contact saucer is *regular* if it is equivalent to a semi-contact saucer of the form (B, σ_ϕ) defined over a regular $2n$-disc $D \subset \Pi$.

In §7 we will prove the following proposition.

Proposition 6.2. *Let M be a $(2n+1)$-manifold, $A \subset M$ be a closed subset, and ξ_0 be an almost contact structure on M that is genuine on $\mathcal{O}p\, A \subset M$. There exist a finite number of embedded saucers $B_i \subset M$ for $i = 1, ..., N$ such that ξ_0 is homotopic relative to A to an almost contact structure ξ_1 which is genuine on $M \setminus \bigcup_{i=1}^N B_i$ and whose restriction to each saucer B_i is semi-contact and regular.*

6.4. Fibered saucers

Slightly stretching the definition of a fibered shell, we will allow $(2n+1)$-dimensional discs B^τ for $\tau\in\partial T$ to degenerate into $2n$-dimensional discs, as in the following definition of fibered saucers. A domain ${}^TB\subset T\times D\times\mathbb{R}$ is called a *fibered saucer* if $T=D^q$ and it has the form

$$ {}^TB=\{(\tau,x,v)\in T\times D\times\mathbb{R}: f_-(\tau,x)\leqslant v\leqslant f_+(\tau,x)\}, $$

where $f_\pm: T\times D\to\mathbb{R}$ are two C^∞-functions such that $f_-(\tau,x)<f_+(\tau,x)$ for all $(\tau,x)\in \operatorname{Int}(T\times D)$ and $f_\pm$ coincide along $\partial(T\times D)$ together with their ∞-jet. Every fibered saucer comes with a family of discs

$$ D^\tau_s=\{(\tau,x,v): x\in D \text{ and } v=(1-s)f_-(\tau,x)+sf_+(\tau,x)\}, $$

where for fixed $\tau\in T$ the discs $\{D^\tau_s\}_{s\in[0,1]}$ coincide with their ∞-jets along their common boundary $S^\tau=\partial D^\tau_s$. We call the union ${}^TS:=\bigcup_{\tau\in T}S^\tau$ the *border* of the fibered saucer ${}^TB$.

A fibered semi-contact structure ${}^T\xi$ on a fibered saucer B is a family ζ^τ_s of germs of contact structures along discs D^τ_s for $s\in[0,1]$ and $\tau\in T$, which coincide along the border ${}^TS$. A fibered semi-contact structure defines a fibered almost contact structure on ${}^TB$. In particular, any fibered semi-contact structure on a fibered saucer ${}^TB$ defines a fibered contact shell.

A fibered semi-contact structure on a fibered saucer B is called *regular* if the saucer (B^τ,ξ^τ) is regular for each $\tau\in\operatorname{Int}T$. More precisely, a fibered semi-contact saucer ${}^T\zeta=({}^TB,{}^T\xi)$ is regular if there exist a regular $2n$-ball $D\subset\Pi$ and a C^∞-function

$$ \phi: {}^TD=\bigcup_{\tau\in T}\{\tau\}\times D\longrightarrow\mathbb{R} $$

such that

– ϕ vanishes with its ∞-jet along $\partial({}^TD)$, and $\phi>0$ on $\mathcal{O}p\,\partial({}^TD)\cap\operatorname{Int}{}^TD$;

– for each $s\in[0,1]$ the contact structure ζ_s is induced by an embedding onto a neighborhood of the graph $\{(\tau,x,v):=s\phi(\tau,x),\tau\in T,\text{ and } x\in D^\tau\}\subset\mathbb{R}^{2n+1}_{\mathrm{st}}$;

– the disc D is regular.

Thus a fibered regular semi-contact saucer is determined by the function ϕ, and we will denote it by ${}^T\sigma_\phi$.

6.5. Interval model

Proposition 6.2 says that any contact shell dominates a collection of regular semi-contact saucers. So the next step towards proving Proposition 3.1 will be to relate regular semi-contact structures and circle model contact shells and this will be the goal of the remainder of the section.

We will start by introducing one more model contact shell, which we call an *interval model*, and it will help us interpolate between regular semi-contact saucers and circle models shells.

Recall the standard contact $(\mathbb{R}^{2n-1}, \xi_{\mathrm{st}})$ with ξ_{st} given by the contact form

$$\lambda_{\mathrm{st}} = dz + \sum_{i=1}^{n-1} u_i \, d\varphi_i.$$

In this section the notation (v, t) stands for canonical coordinates on the cotangent bundle T^*I.

For a compact star-shaped domain $\Delta \subset \mathbb{R}^{2n-1}$ and a contact Hamiltonian

$$K\colon \Delta \times S^1 \longrightarrow \mathbb{R} \quad \text{such that } K|_{\partial\Delta\times S^1} > 0 \text{ and } K|_{\Delta\times\{0\}} > 0, \tag{39}$$

we will build a contact shell structure, similar to the circle model, on a piecewise smooth $(2n+1)$-dimensional ball

$$(B_K^I, \eta_K^I) \subset \Delta \times T^*I,$$

which we will refer to as the *interval model contact shell* for K.

For any constant $C > -\min K$, define the domain

$$B_{K,C}^I := \{(x, v, t) \in \Delta \times T^*I : 0 \leqslant v \leqslant K(x, t) + C\},$$

which is a piecewise smooth $(2n+1)$-dimensional ball in $\mathbb{R}^{2n-1} \times T^*I$, whose diffeomorphism type is independent of the choice of C. Denote the boundary by

$$\Sigma_{K,C}^I = \partial B_{K,C}^I = \Sigma_{0,K,C}^I \cup \Sigma_{1,K,C}^I \cup \Sigma_{2,K,C}^I,$$

where

$$\begin{aligned}
\Sigma_{0,K,C}^I &= \{(x, v, t) : v = 0\} \subset \Delta \times T^*I, \\
\Sigma_{1,K,C}^I &= \{(x, v, t) : v = K(x, t) + C\} \subset \Delta \times T^*I, \\
\Sigma_{2,K,C}^I &= \{(x, v, t) : 0 \leqslant v \leqslant K(x, t) + C \text{ and } (x, t) \in \partial(\Delta \times I)\} \subset \Delta \times T^*I.
\end{aligned}$$

Now pick a smooth family of functions

$$\varrho_{(x,t)}\colon \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R} \quad \text{for } (x, t) \in \Delta \times I \tag{40}$$

such that

(i) $\varrho_{(x,t)}(v) = v$ when $v \in \mathcal{O}p\{0\}$;
(ii) $\varrho_{(x,t)}(v) = v - C$ for $(x, v, t) \in \mathcal{O}p\{v \geqslant K(x, t) + C\}$;
(iii) $\partial_v \varrho_{(x,t)}(v) > 0$ for $(x, t) \in \mathcal{O}p\, \partial(\Delta \times I)$,

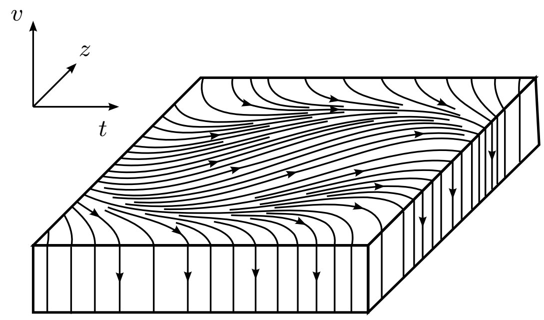

Figure 6.3. The interval model with its characteristic distribution.

which is possible by (39), and consider the distribution on $\Delta\times T^*I$,

$$\ker\alpha_\varrho \quad \text{for the 1-form } \alpha_\varrho=\lambda_{\rm st}+\varrho\, dt.$$

We now have the following lemma, whose proof is analogous to Lemma 2.1.

Lemma 6.3. *The almost contact structure given by α_ϱ defines a contact shell*

$$(B^I_{K,C},\eta^I_{K,\varrho})$$

that is independent of the choice of ϱ and C, up to equivalence. If $K>0$, then the contact germ (Σ^I_K,η^I_K) extends canonically to a contact structure on B^I_K.

Similarly, we also have a direct description of the contact germ (Σ^I_K,η^I_K) without the shell given by gluing together the contact germs on the hypersurfaces

$$\begin{aligned}
\widetilde{\Sigma}^I_{0,K} &= \{(x,v,t): v=0\}\subset\Delta\times T^*I,\\
\widetilde{\Sigma}^I_{1,K} &= \{(x,v,t): v=K(x,t)\}\subset\Delta\times T^*I,\\
\widetilde{\Sigma}^I_{2,K} &= \{(x,v,t): 0\leqslant v\leqslant K(x,t) \text{ and } (x,t)\in\partial(\Delta\times I)\}\subset\Delta\times T^*I,
\end{aligned}$$

to form a contact germ on $\widetilde{\Sigma}^I_K:=\widetilde{\Sigma}^I_{0,K}\cup\widetilde{\Sigma}^I_{1,K}\cup\widetilde{\Sigma}^I_{2,K}$.

Lemma 6.4. *The contact germs on Σ^I_K and $\widetilde{\Sigma}^I_K$ are contactomorphic.*

The proof is completely analogous to Lemma 2.3. Note one important distinction compared to the circle model: the contact germ on $\widetilde{\Sigma}^I_K$ is defined by a global immersion of the sphere into $\Delta\times T^*I$ (piecewise smooth and topologically embedded at the non-smooth points). This property allows us to use the interval model as a bridge between regular contact saucers and the circle model.

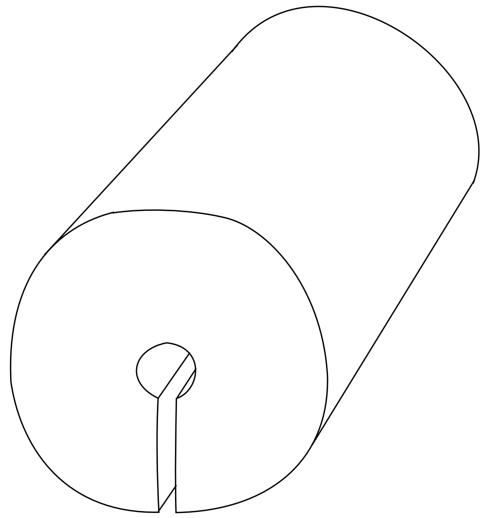

Figure 6.4. The keyhole model inside B_K.

6.6. Relations between the model contact shells

We will now establish some domination relations between our three models.

PROPOSITION 6.5. *For star-shaped domains* $\Delta' \subset \operatorname{Int}\Delta$, *let* $K:\Delta\times S^1 \to \mathbb{R}$ *be such that* $K|_{\Delta\times\{0\}}>0$ *and* $K|_{\Delta\setminus\Delta'\times S^1}>0$. *For* $K':=K|_{\Delta'\times S^1}$, *the interval model contact shell* (B^I_K, η^I_K) *dominates the circle model contact shell* $(B_{K'}, \eta_{K'})$.

Proof. Fix $C>-\min K$ and ϱ as in (40) that defines contact shell models

$$(B_K, \eta_{K,\varrho}), \quad (B_{K'}, \eta_{K',\varrho}), \quad \text{and} \quad (B^I_K, \eta^I_{K,\varrho}).$$

Take any $\varepsilon>0$ such that $K|_{\Delta\times[-\varepsilon,\varepsilon]}>\varepsilon$ and consider the domain

$$\begin{aligned} B^\varepsilon_K &:= B^I_K \setminus (\{(x,v,t) : v \leqslant \varepsilon\} \cup \{(x,v,t) : t \in [-\varepsilon, \varepsilon]\}) \\ &= \{(x,v,t) \in \Delta\times T^*I : \varepsilon \leqslant v \leqslant K(x,t)+C \text{ and } \varepsilon \leqslant t \leqslant 1-\varepsilon\}. \end{aligned} \tag{41}$$

Note that $\eta^I_{K,\varrho}$ restricted to B^ε_K defines a contact shell $(B^\varepsilon_K, \eta^\varepsilon_{K,\varrho})$ that we will call the *keyhole* model, and it follows from (41) that $(B^\varepsilon_K, \eta^\varepsilon_{K,\varrho})$ is dominated by $(B^I_K, \eta^I_{K,\varrho})$. It remains to show for sufficiently small ε that the shell $(B^\varepsilon_K, \eta^\varepsilon_{K,\varrho})$ dominates $(B_{K'}, \eta_{K',\varrho})$.

Note that $(B^\varepsilon_K, \eta^\varepsilon_{K,\varrho})$ can be cut out of $(B_K, \eta_{K,\varrho})$ by the same inequalities as in (41), where (v,t) are viewed as coordinates $v=r^2$ and $t=\phi/2\pi$ on $\mathbb{R}^2$, rather than on T^*I. This embedding is shown in Figure 6.4 and explains the term *keyhole*.

For standard coordinates $(q,p)\in\mathbb{R}^2$, where $q=\sqrt{v}\cos(2\pi t)$ and $p=\sqrt{v}\sin(2\pi t)$, and by the assumptions on ϱ in (40), the 1-form on $\Delta\times\mathbb{R}^2$ defining $\eta_{K,\varrho}$ can be written as

$$\alpha_\varrho=\lambda_{\rm st}+\frac{\varrho(v)}{2\pi v}(q\,dp-p\,dq),$$

and on $\Delta\times\mathcal{O}p\{(q,0)\in\mathbb{R}^2:q\geqslant-2\delta\}$ is a genuine contact form for some $\delta>0$.

Pick a smooth function $k:\Delta\to[-\delta,\infty)$ such that both $k(x)=-\delta$ on $\mathcal{O}p\,\partial\Delta$ and $k(x)=K(x,0)$ on $\mathcal{O}p\,\Delta'$, and

$$\Gamma_k:=\{(x,q,0)\in\Delta\times\mathbb{R}^2:-2\delta\leqslant q\leqslant k(x)\}\subset B_K.$$

Consider a smooth isotopy $\{\psi_s\}_{s\in[0,1]}$ of $\Delta\times\{(q,p):q\geqslant-2\delta \text{ and } p=0\}$ of the form

$$\psi_s(x,q)=(x,g_s(x,q)),$$

supported away from ∂B_K, and such that

$$\psi_1(\Gamma_k)=\{(x,q,0)\in\Delta\times\mathbb{R}^2:-2\delta\leqslant q\leqslant-\delta\}\subset B_K.$$

Since this isotopy preserves $\alpha_\varrho|_{\Delta\times\{(q,p):p=0\}}=\lambda_{\rm st}$, it follows from a Moser-method argument (cf. [21, Theorem 2.6.13]) that ψ_s can be extended to a contact isotopy Ψ_s of B_K supported in $\Delta\times\mathcal{O}p\{(q,0)\in\mathbb{R}^2:q\geqslant-2\delta\}$.

If ε is small enough, then the contactomorphism Ψ_1 satisfies $\Psi_1(B_{K'})\subset B_K^\varepsilon$, and hence the keyhole model shell $(B_K^\varepsilon,\eta_{K,\varrho}^\varepsilon)$ dominates the circle model shell $(B_{K'},\eta_{K',\varrho})$. □

We also have the following parametric version of Proposition 6.5, whose proof is analogous.

Proposition 6.6. *Let $K^\tau:\Delta\times S^1\to\mathbb{R}$ be a family of contact Hamiltonians such that $K^\tau|_{\Delta\times\{0\}}>0$ and $K^\tau|_{\partial\Delta\times S^1}>0$. If $\Delta'\subset\operatorname{Int}\Delta$ is a star-shaped domain and $K'^\tau:=K^\tau|_{\Delta'\times S^1}$, then the fibered shell ${}^T\eta^I_{{}^TK}$ dominates ${}^T\eta_{{}^TK'}$.*

The next proposition relates our saucer models from the previous section with the interval models discussed here.

Proposition 6.7. *Let $\zeta=(B,\xi)$ be a regular semi-contact saucer viewed as a shell. Then ζ dominates an interval model η^I_K for some $K:\Delta\times I\to\mathbb{R}$.*

To prove this we will need the following two Lemmas 6.8 and 6.9.

LEMMA 6.8. *Let Δ be a compact contact manifold with boundary with a fixed contact form. Let $h, g: \Delta \to \Delta$ be contactomorphisms that are the time-1 maps of isotopies generated by contact Hamiltonians $H, G: \Delta \times I \to \mathbb{R}$ that vanish with their ∞-jet on $\partial\Delta$. If $h = g$ on $\mathcal{O}p\,\partial\Delta$, then h can be generated as the time-1 map of a Hamiltonian $\widetilde{H}: \Delta \times I \to \mathbb{R}$, where $\widetilde{H} = G$ on $\mathcal{O}p\,\partial(\Delta \times I)$.*

Proof. Denote by h_t and g_t the contact diffeotopies generated by H and G. Pick a contact diffeotopy such that $\hat{h}_t = h_t$ on $\partial\Delta$ as ∞-jets and

$$\hat{h}_t = \begin{cases} g_t, & \text{if } t \in [0, \varepsilon], \\ h_t, & \text{if } t \in [2\varepsilon, 1-2\varepsilon], \\ g_t \circ g_1^{-1} \circ h_1, & \text{if } t \in [1-\varepsilon, 1]. \end{cases}$$

Observe that $\widehat{H} = G$ when $t \in [0, \varepsilon] \cup [1-\varepsilon, 1]$ if $\hat{h}_t$ is generated by $\widehat{H}: \Delta \times I \to \mathbb{R}$. Hence, without loss of generality, we may assume that $H = G$ when $t \in \mathcal{O}p\,\partial I$.

Since h_t and g_t are C^∞-small on $\mathcal{O}p\,\partial\Delta$, we can pick an isotopy

$$\psi^s: \Delta \times I \longrightarrow \Delta \times I \quad \text{for } s \in [0,1], \text{ with } \psi^0 = \mathrm{Id},$$

supported in $\Delta \times \operatorname{Int} I$ and such that for $x \in \mathcal{O}p\,\partial\Delta$ we have

$$\psi^s(\,\cdot\,, t) \text{ is a contactomorphism} \quad \text{and} \quad \psi^1(h_t(x), t) = (g_t(x), t).$$

Applying the Gray–Moser argument parametrically in t builds a contact isotopy $\tilde{\psi}_t$ such that $\tilde{\psi}_t = \mathrm{Id}$ when $t \in \mathcal{O}p\,\partial I$ and $\tilde{\psi}_1(h_t(x)) = g_t(x)$ when $x \in \mathcal{O}p\,\partial\Delta$. Defining $\tilde{h}_t := \tilde{\psi}_t \circ h_t$ and $\widetilde{H}$ to be its generating contact Hamiltonian gives the result. □

For the following lemma let $\Pi := \{(w, t, v) \in \mathbb{R}^{2n-1} \times T^*\mathbb{R} : v = 0\} \subset (\mathbb{R}^{2n+1}, \xi_{\mathrm{st}}^{2n+1})$.

LEMMA 6.9. *For a star-shaped domain $\Delta \subset (\mathbb{R}^{2n-1}, \xi_{\mathrm{st}})$ consider the disc*

$$D = \{(w, t) \in \Delta \times \mathbb{R} : h_-(w) \leqslant t \leqslant h_+(w)\} \subset \Pi,$$

where $h_\pm: \Delta \to \mathbb{R}$ are C^∞-functions such that $h_- < h_+$. If (B, σ_ϕ) is a contact saucer defined over D, then it is equivalent to a contact saucer $(\widetilde{B}, \sigma_{\tilde{\phi}})$ defined over

$$\widetilde{D} = \Delta \times [0,1].$$

Proof. We may assume that

$$B = \{(w, t, v) : 0 \leqslant v \leqslant \Phi(w, t) \text{ and } (w, t) \in D\},$$

where the function Φ is positive on $\operatorname{Int} D$ and coincides with ϕ on $\mathcal{O}p\,\partial D$, and hence $\partial B = D\cup\Gamma$, where

$$\Gamma := \{(w,t,v) : v = \Phi(w,t) \text{ and } (w,t)\in D\}.$$

Given a point $w\in\Delta$, consider a leaf ℓ_w through the point $(w,h_-(w))\in D$ of the characteristic foliation $\mathcal{F}_\phi$ on the graph $\Gamma_\phi := \{(w,t,v) : v=\phi(w,t) \text{ and } (w,t)\in D\}$, and let $(w', h_+(w'))\in D$ be the other end of the leaf ℓ_w. This rule $f_\phi(w) := w'$ defines the holonomy contactomorphism $f_\phi\colon\Delta\to\Delta$.

Consider the diffeomorphism $G\colon D\to\widetilde{D} := \Delta\times[0,1]$ defined by the formula

$$G(w,t) = \left(w, \frac{t-h_-(w)}{h_+(w)-h_-(w)}\right).$$

Since $\lambda_{\mathrm{st}}+v\,dt$ restricted to D and $\widetilde{D}$ is λ_{st} and is preserved by G, it follows that G extends to a contactomorphism $G\colon\mathcal{O}p\,D\to\mathcal{O}p\,\widetilde{D}$. The diffeomorphism G moves the graph of the function $\phi|_{\mathcal{O}p\,\partial D}$ onto a graph of some function $\tilde{\phi}\colon\mathcal{O}p\,\partial\widetilde{D}\to\mathbb{R}$ whose ∞-jet vanishes on $\partial\widetilde{D}$. Pick any smooth extension $\tilde{\phi}\colon\widetilde{D}\to\mathbb{R}$.

The characteristic foliation $\mathcal{F}_{\tilde{\phi}}$ on the graph $\Gamma_{\tilde{\phi}} = \{(w,t,v) : v=\tilde{\phi}(w,t) \text{ and } (w,t)\in\widetilde{D}\}$ is represented by $\partial/\partial t - X_{\tilde{\phi}_t}$, where $X_{\tilde{\phi}_t}$ is the contact vector field on Δ for $\tilde{\phi}_t\colon\Delta\to\mathbb{R}$ thought of as a contact Hamiltonian. It follows that the holonomy contactomorphism $f_{\tilde{\phi}}\colon\Delta\to\Delta$, defined similarly to f_ϕ, coincides with the time-1 map of the contact isotopy of Δ defined by $-\tilde{\phi}\colon\Delta\times[0,1]\to\mathbb{R}$ thought of as a time-dependent contact Hamiltonian. According to Lemma 6.8, we can modify $\tilde{\phi}$, keeping it fixed over $\mathcal{O}p\,\partial D$, to make the holonomy contactomorphism $f_{\tilde{\phi}}$ equal to f_ϕ.

Since the holonomy maps $f_{\tilde{\phi}}$ and f_ϕ are equal, it follows that there is a diffeomorphism $F\colon\Gamma_\phi\to\Gamma_{\tilde{\phi}}$ equal to G on $\mathcal{O}p\,\partial\Gamma_\phi$, mapping the characteristic foliation $\mathcal{F}_\phi$ to the characteristic foliation $\mathcal{F}_{\tilde{\phi}}$, and with the form

$$F(w,t,\phi(v,t)) = (f(w,t), \tilde{\phi}(f(w,t))) \quad \text{for } (v,t)\in D,$$

for some diffeomorphism $f\colon D\to\widetilde{D}$. It follows that F extends to a contactomorphism of neighborhoods $F\colon\mathcal{O}p\,\Gamma_\phi\to\mathcal{O}p\,\Gamma_{\tilde{\phi}}$.

Let $(\widetilde{B}, \sigma_{\tilde{\phi}})$ be a contact saucer over $\widetilde{D}$, where

$$\widetilde{B} = \{(w,t,v) : 0\leqslant v\leqslant\tilde{\Phi}(w,t) \text{ and } (w,t)\in\widetilde{D}\}$$

for some function $\tilde{\Phi}\colon\widetilde{D}\to\mathbb{R}$ that coincides with $\tilde{\phi}$ on $\mathcal{O}p\,\partial\widetilde{D}$ and is positive on $\operatorname{Int}\widetilde{D}$. Note that $\partial\widetilde{B} = \widetilde{D}\cup\widetilde{\Gamma}$, where $\widetilde{\Gamma} := \{(w,t,v) : v=\tilde{\Phi}(w,t) \text{ and } (w,t)\in\widetilde{D}\}$. Let us define a diffeomorphism $H\colon\partial B\to\partial\widetilde{B}$ so that

$$H|_{\mathcal{O}p\,D} = G \quad\text{and}\quad H|_\Gamma(w,t,\Phi(v,t)) = (f(w,t), \tilde{\Phi}(f(v,t))).$$

This diffeomorphism matches the traces of contact structures on the boundaries ∂B and $\partial\widetilde{B}$ of the saucers, and hence extends to a contactomorphism between $\mathcal{O}p\,\partial B$ and $\mathcal{O}p\,\partial\widetilde{B}$ which can be further extended to an equivalence between the saucers (B,σ_ϕ) and $(\widetilde{B},\sigma_{\tilde{\phi}})$. □

Proof of Proposition 6.7. By the definition of regular semi-contact saucer from §6.2, we may assume that $\zeta=(B,\sigma_\phi)$, where B is defined over a regular domain

$$D\subset\Pi:=\{(w,v)\in\mathbb{R}^{2n}\times\mathbb{R}:v=0\},$$

and σ_ϕ is given by a family of contact structures ζ_s on neighborhoods of graphs

$$D_s:=\{(w,v)\in D\times\mathbb{R}:v=s\phi(w)\},$$

where $\phi\colon D\to\mathbb{R}$ is a C^∞-function supported in D which is positive on $\mathcal{O}p(\partial D)\cap\operatorname{Int}D$.

By the regularity assumption on D, the projection $\pi\colon D\to\mathbb{R}^{2n-1}$ is equivalent to the linear projection of the round ball and the image $\Delta=\pi(D)$ is contactomorphic to a star-shaped domain in $\mathbb{R}^{2n-1}$. Choose a slightly smaller star-shaped ball $\Delta'\subset\operatorname{Int}\Delta$ such that $\phi|_{\operatorname{Int}D\setminus\operatorname{Int}D'}>0$, where $D':=\pi^{-1}(\Delta')\cap D$. Note that the characteristic foliation on D' is *not* a regular foliation, rather it is diffeomorphic to the product foliation of the $2n-1$ disc and the interval. There exist functions $h_\pm\colon\Delta'\to\mathbb{R}$, $h_-<h_+$, such that

$$D'=\{(w,t):h_-(w)\leqslant t\leqslant h_+(w)\text{ and }w\in\Delta'\}\subset\Pi.$$

Choose a function $\phi'\colon D'\to\mathbb{R}$ that defines an immersion type semi-contact saucer $(B',\sigma_{\phi'})$ over D' such that $\phi'\leqslant\phi|_{D'}$ and hence $(B',\sigma_{\phi'})$ is dominated by (B,σ_ϕ).

Hence, we can apply Lemma 6.9 and find a function $\tilde{\phi}\colon\widetilde{D}:=\Delta'\times[0,1]\to\mathbb{R}$ which is positive near $\partial\widetilde{D}$ and such that the corresponding saucer $(\widetilde{B},\sigma_{\tilde{\phi}})$ over $\widetilde{D}$ is equivalent to $(B',\sigma_{\phi'})$.

Let us rescale the saucer $(\widetilde{B},\sigma_{\tilde{\phi}})$ by an affine contactomorphism of $\mathbb{R}^{2n+1}_{\mathrm{st}}$

$$(z,u,\varphi,t,v)\longmapsto\big((1+\delta)^2z,(1+\delta)^2u,\varphi,(1+\delta)t-\tfrac{1}{2}\delta,(1+\delta)v\big)$$

for $\delta>0$, to an equivalent saucer $(\widehat{B},\sigma_{\hat{\phi}})$ over the domain $\widehat{D}:=\hat{\Delta}\times\big[-\tfrac{1}{2}\delta,1+\tfrac{1}{2}\delta\big]$. The notation φ stands for the tuple $(\varphi_1,...,\varphi_{n-1})$ of angular coordinates. Note that $\Delta'\subset\operatorname{Int}\hat{\Delta}$ and we may choose δ sufficiently small so that $\hat{\phi}|_{\operatorname{Int}\widehat{D}\setminus\operatorname{Int}\widetilde{D}}>0$.

Then the restriction $K:=\hat{\phi}|_{\widetilde{D}}$ of the function $\hat{\phi}$ to the domain $\widetilde{D}=\Delta'\times[0,1]$ defines an interval model shell (B^I_K,η^I_K). It is dominated by the saucer $(\widehat{B},\sigma_{\hat{\phi}})$, which is equivalent to $(B',\sigma_{\phi'})$, which is in turn dominated by (B,σ_ϕ). □

Similarly, one can prove the following parametric version of Proposition 6.7.

Proposition 6.10. *Let* ${}^T\zeta=({}^TB,{}^T\xi)$ *be a fibered regular semi-contact saucer. Then* ${}^T\zeta$ *dominates a fibered interval model* ${}^T\eta^I_K$ *for some* $K:({}^T\Delta)\times I\to\mathbb{R}$.

Remark 6.11. Let us point out that the shells (B^τ,ξ^τ) degenerate when τ approaches ∂T. Hence, the subordination map $({}^TB^I_K,{}^T\eta^I_K)\to({}^TB,{}^T\xi)$ has to cover an embedding $T\to\operatorname{Int}T$.

The next proposition is the main result in this section.

Proposition 6.12. *If* $(B,\xi)=\sigma_\phi$ *is a regular semi-contact saucer viewed as a shell, then there is a time-independent contact Hamiltonian* $K:\Delta\to\mathbb{R}$ *such that* (B,ξ) *dominates the circle model contact shell* (B_K,η_K).

Proof. We first use Proposition 6.7 to find an interval model $\eta^I_{\widetilde{K}}$ dominated by ζ for some $\widetilde{K}:D\times I\to\mathbb{R}$. Then we apply Proposition 6.5 to get a circle model contact shell $(B_{K'},\eta_{K'})$ dominated by $(B^I_{\widetilde{K}},\eta^I_{\widetilde{K}})$. Finally, choosing a time-independent contact Hamiltonian $K<K'$ and applying Lemma 4.1, we get the required circle model contact shell (B_K,η_K) dominated by (B,ξ). □

Similarly, the parametric versions Propositions 6.10 and 6.6 prove the following result.

Proposition 6.13. *Let* $({}^TB,{}^T\xi)$ *be a fibered regular semi-contact saucer. Then there exists a family of time-independent contact Hamiltonians* $K^\tau:\Delta\to\mathbb{R}$ *for* $\tau\in T$ *which satisfies* $K^\tau>0$ *for* $\tau\in\partial T$, *such that* $({}^TB,{}^T\xi)$ *dominates the corresponding fibered circle model contact shell* $({}^TB_K,{}^T\eta_K)$.

7. Reduction to saucers

7.1. Construction of contact structures in the complement of saucers

The goal of this section is to prove the Proposition 6.2. The starting point of the proof is Gromov's h-principle for contact structures on open manifolds, which we will now formulate. Given a $(2n+1)$-dimensional manifold M, possibly with boundary, a closed subset $A\subset M$, and a contact structure ξ_0 on $\mathcal{O}p\,A\subset M$ define $\mathfrak{Cont}(M;A,\xi_0)$ to be the space of contact structures on M that coincide with ξ_0 on $\mathcal{O}p\,A$ and $\mathfrak{cont}(M;A,\xi_0)$ to be the space of *almost* contact structures that agree with ξ_0 on $\mathcal{O}p\,A$. Let $j:\mathfrak{Cont}(M;A,\xi_0)\to\mathfrak{cont}(M;A,\xi_0)$ be the inclusion map. We say that the pair (M,A) is *relatively open* if for any point $x\in M\setminus A$ either there exists a path in $M\setminus A$ connecting x with a boundary point of M or a proper path $\gamma:[0,\infty)\to M\setminus A$ with $\gamma(0)=x$.

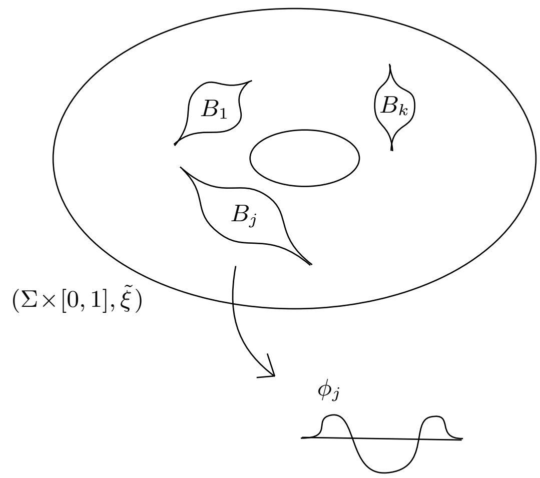

Figure 7.1. A representation of the statement of Lemma 7.2.

THEOREM 7.1. (Gromov [28], [31]) *Let M be a $(2n+1)$-manifold, $A\subset M$ a closed subset, and ξ_0 a contact structure on $\mathcal{O}p\,A\subset M$. Suppose that (M,A) is relatively open. Then the inclusion*

$$j\colon \mathfrak{Cont}(M;A,\xi_0)\longrightarrow \mathfrak{cont}(M;A,\xi_0)$$

is a homotopy equivalence.

As we will see, Proposition 6.2 follows from the following special case.

LEMMA 7.2. *For a closed manifold Σ, any semi-contact structure $\xi=\{\zeta_s\}_{s\in[0,1]}$ on the annulus $C=\Sigma\times[0,1]$ is homotopic relative to $\mathcal{O}p\,\partial C$ to an almost contact structure $\tilde{\xi}$ which is a genuine contact structure in the complement of finitely many saucers $B_1,...,B_k\subset C$ and such that the restriction $\tilde{\xi}|_{B_j}$, $j=1,...,k$, is semi-contact and regular.*

Proof of Proposition 6.2. Choose an embedded annulus $C=S^{2n}\times[0,1]\subset M\setminus A$ and first use the existence part of Gromov's h-principle (Theorem 7.1) to deform ξ_0 relative to A to an almost contact structure which is genuine on $M\setminus C$. Next we use the 1-parametric part of Theorem 7.1, applied to the family of neighborhoods of spheres $S^{2n}\times t$ for $t\in[0,1]$, to make the almost contact structure semi-contact on C. Finally, we use Lemma 7.2 to complete the proof. □

We will need two lemmas in order to prove Lemma 7.2.

Observe that we are free to partition $[0,1]=\bigcup_{i=0}^{N}[a_i,a_{i+1}]$, where $a_i<a_j$ for $i<j$ and prove the lemma for the restriction of the semi-contact structure to each $\Sigma\times[a_i,a_{i+1}]$, which we will do multiple times in the proof.

LEMMA 7.3. *For a closed manifold* Σ *and a semi-contact structure* $\{\zeta_s\}_{s\in[0,1]}$ *on* $\Sigma\times[0,1]$ *there exists* $N>0$ *such that the restriction of* $\{\zeta_s\}_{s\in[0,1]}$ *to* $\Sigma\times[a_i,a_{i+1}]$ *for* $a_i:=a+i/N$ *is of immersion type for each* $i=0,...,N-1$.

Proof. Choose an $\varepsilon>0$ such that the contact structure $\{\zeta_s\}_{s\in[0,1]}$ is defined on $\Sigma\times[s-\varepsilon,s+\varepsilon]$ for each $s\in[a,b]$. We will view $\{\zeta_s\}_{s\in[0,1]}$ as a family of contact structures on $\Sigma\times[-\varepsilon,\varepsilon]$. For each s_0 and $\sigma>0$ sufficiently small, a Darboux–Moser-type argument implies there is an isotopy $\phi_s^{s_0}:\Sigma\times[-\sigma,\sigma]\to\Sigma\times[-\varepsilon,\varepsilon]$ such that $\phi_{s_0}^{s_0}=\mathrm{Id}$ and $(\phi_{s_0}^s)^*\zeta_s=\zeta_{s_0}$ for $s\in[s_0,s_0+\sigma]$. Moreover by shrinking σ if necessary, we can ensure that the hypersurfaces $\phi_{s_0}^s(\Sigma\times\{0\})$ are graphical in $\Sigma\times[-\varepsilon,\varepsilon]$. Hence for any $s_0\in[a,b]$ the restriction of ζ_s to $[s_0,s_0+\sigma]$ is of immersion type and therefore choosing $N>1/\sigma$ we get the required partition of $\Sigma\times[a,b]$ into the annuli of immersion type. $\square$

LEMMA 7.4. *Let* $\{\xi_s\}_{s\in[0,1]}$ *be a semi-contact structure on* $\Sigma\times[0,1]$. *Then, after partitioning,* $\{\xi_s\}_{s\in[0,1]}$ *is equivalent to a semi-contact structure* $\{\zeta_s\}_{s\in[0,1]}$ *of immersion type satisfying the following properties: there exists a smooth function* $\psi:\Sigma\to\left[-\frac{1}{2}R,\frac{1}{2}R\right]$ *and a contact structure* μ *on* $\widehat{C}:=\Sigma\times[-R,R]$ *such that*

- *for all* $s\in[0,1]$ *the contact structure* ζ_s *on* $\Sigma\times[s-\delta,s+\delta]$ *equals* $\Psi_s^*\mu$, *where*

$$\Psi_s:\Sigma\times[s-\delta,s+\delta]\longrightarrow\widehat{C}$$

is the embedding $(x,s+t)\mapsto(x,s\psi(x)+t)$, *with* $x\in\Sigma$ *and* $t\in[-\delta,\delta]$;

- *there are closed domains* $V\subset\Sigma$ *and* $\widehat{V}\subset\mathrm{Int}\,V$ *such that* $\psi|_V>0$, *and over* $\Sigma\setminus\mathrm{Int}\,\widehat{V}$ *the contact structure* μ *is transverse to the graph of the function* $s\psi$ *for all* $s\in[0,1]$.

Proof. Using Lemma 7.3 we may assume, by passing to a partition, that the semi-contact structure $\{\xi_s\}_{s\in[0,1]}$ on the annulus C is of immersion type. So there is a contact structure μ on $\widehat{C}:=\Sigma\times[-R,R]$ and a smooth family of embeddings

$$\Psi_s:\Sigma\times[-\delta,\delta]\longrightarrow\widehat{C}\quad\text{for } s\in[0,1]$$

such that $\Psi_s(x,u)=(x,\psi_s(x)+u)$ for $u\in[-\delta,\delta]$, and $\Psi_s^*\mu$ is identified with ξ_s. Let us endow $\Sigma\times[-R,R]$ with the product metric. The partition argument also allows us, furthermore, to assume that $\psi_0=0$ and that the C^1-norm of ψ_1 is arbitrary small. We will impose the appropriate bound on its C^1-norm further down in the proof.

Choose a contact form α for μ, let $\mathfrak{R}$ be its associated Reeb vector field and set

$$k:=\min_{(x,u)\in\widehat{C}}\|\mathfrak{R}(x,u)\|.$$

Define the constant $\beta\in\left(0,\frac{1}{4}\pi\right]$ to be

$$\beta:=\min_{(x,u)\in\widehat{C}}\mathrm{angle}(\mu_{x,u},\mathfrak{R}(x,u))$$

and choose $\sigma>0$ small enough to ensure that

$$|\text{angle}(\mu_{x,u},\mu_{x,0})|\leqslant\tfrac{1}{32}\beta$$

for all $x\in\Sigma$ and $|u|\leqslant\sigma$.

Define the subsets

$$\begin{aligned}\widehat{V}'&:=\{x\in\Sigma:\text{the angle between }\mu_{x,0}\text{ and }T_{x,0}\Sigma\text{ in }T_{x,0}\widehat{C}\text{ is at most }\tfrac{1}{4}\beta\},\\ \widehat{V}&:=\{x\in\Sigma:\text{the angle between }\mu_{x,0}\text{ and }T_{x,0}\Sigma\text{ in }T_{x,0}\widehat{C}\text{ is at most }\tfrac{1}{6}\beta\},\\ \widehat{V}''&:=\{x\in\Sigma:\text{the angle between }\mu_{x,0}\text{ and }T_{x,0}\Sigma\text{ in }T_{x,0}\widehat{C}\text{ is at most }\tfrac{1}{8}\beta\},\end{aligned}$$

and noting that $\widehat{V}''\subset\operatorname{Int}\widehat{V}\subset\widehat{V}\subset\operatorname{Int}\widehat{V}'$ define the constant

$$d:=\min\{\text{dist}(\widehat{V},\Sigma\setminus\widehat{V}'),\text{dist}(\widehat{V}'',\Sigma\setminus\widehat{V})\}.$$

For all $(x,u)\in\widehat{V}'\times[-\sigma,\sigma]$, note that $\mathfrak{R}(x,u)$ is transverse to $T_{x,u}\Sigma=T_{x,u}(\Sigma\times\{u\})$, since

$$\text{angle}(\mathfrak{R}(x,u),T_{x,u}\Sigma)\geqslant\text{angle}(\mathfrak{R}(x,u),\mu_{x,u})-\text{angle}(\mu_{x,u},T_{x,u}\Sigma)\geqslant\tfrac{1}{2}\beta. \qquad (42)$$

In particular we have $du(\mathfrak{R}(x,0))\neq0$ for all $x\in\widehat{V}'$, and we will write $\widehat{V}'$ as the disjoint union $\widehat{V}'=\widehat{V}'_+\cup\widehat{V}'_-$, where $\widehat{V}'_\pm=\{x\in\widehat{V}':\pm du(\mathfrak{R}(x,0))>0\}$.

Pick a smooth function $G:\Sigma\to[-1,1]$ such that $G\equiv\pm1$ on $\widehat{V}'_\pm$ and define the following smooth function on $\widehat{C}=\Sigma\times[-R,R]$:

$$H:\widehat{C}\longrightarrow\mathbb{R},\quad\text{where }H(x,u)=\theta(u)G(x),$$

where $\theta:[-R,R]\to[0,1]$ is a cutoff function such that $\theta(u)=1$ when $|u|\leqslant\sigma$ and $\theta(u)=0$ when $|u|$ is close to R. Let $h_t:\widehat{C}\to\widehat{C}$ be the contact isotopy generated by the contact Hamiltonian H. Pick $\varepsilon>0$ sufficiently small so that $\varepsilon k\sin\left(\tfrac{1}{2}\beta\right)<\sigma$ and for all $t\in[0,\varepsilon]$ we have

- $\|h_t\|_{C^0}<\tfrac{1}{2}\min\{d,\sigma\}$;
- dh_t rotates every hyperplane by an angle less than $\tfrac{1}{32}\beta$.

Recall that the function $\psi_1:\Sigma\to\mathbb{R}$ entering the definition of the semi-contact annulus can be chosen arbitrarily C^1-small. In particular we will assume that the graph of ψ_1 is in the narrow band

$$\Gamma_1:=\{(x,u)\in\widehat{C}:u=\psi_1(x)\}\subset\left\{(x,u):u\in\left[-\tfrac{1}{2}\varepsilon k\sin\left(\tfrac{1}{2}\beta\right),\tfrac{1}{2}\sigma\right]\right\},$$

and the angle between $T_z\Gamma_1$ and the horizontal plane $T_z\Sigma$ is always less than $\tfrac{1}{32}\beta$ for all $z\in\Gamma_1$. We claim that with these bounds the hypersurface

$$\widetilde{\Gamma}_1:=h_\varepsilon(\Gamma_1)=h_\varepsilon(\{(x,u)\in\widehat{C}:u=\psi_1(x)\})$$

is graphical, meaning that $\widetilde{\Gamma}_1=\{(x,u)\in\widehat{C}:u=\tilde{\psi}_1(x)\}$ for a function $\tilde{\psi}_1:\Sigma\to\mathbb{R}$, and furthermore satisfies the following two conditions:

(i) $0<\frac{1}{2}k\varepsilon\sin\left(\frac{1}{2}\beta\right)\leqslant\tilde{\psi}_1|_{\widehat{V}}\leqslant\sigma$;

(ii) for every $x\in\Sigma\setminus\operatorname{Int}\widehat{V}$, at $z=(x,\tilde{\psi}_1(x))\in\widetilde{\Gamma}_1$, we have $\operatorname{angle}(T_z\widetilde{\Gamma}_1,\mu_z)>\frac{1}{32}\beta$.

To see that $\widetilde{\Gamma}_1$ is graphical, just note that the C^1-norm of ψ_1, together with the fact that dh_ε rotates all hyperplanes by no more than $\frac{1}{32}\beta$, implies that each plane tangent to $\widetilde{\Gamma}_1$ forms an angle less that $\frac{1}{16}\beta\leqslant\frac{1}{16}\pi$ with the horizontal plane $T_z\Sigma$.

To prove (i) and (ii), let us write the contact vector field $X_H(x,u)$ for H as

$$X_H(x,u)=Y(x,u)+v(x,u)\frac{\partial}{\partial u},\quad \text{where } Y(x,u)\in T_{x,u}(\Sigma\times\{u\}),$$

recalling that h_t is the flow for X_H. By design, the contact vector field X_H satisfies

$$X_H(x,u)=\pm\mathfrak{R}(x,u)\quad \text{if } (x,u)\in\widehat{V}'_\pm\times[-\sigma,\sigma],$$

and therefore, using (42), we have the lower bound

$$v(x,u)\geqslant k\sin\left(\tfrac{1}{2}\beta\right)>0\quad \text{for all } (x,u)\in\widehat{V}'\times[-\sigma,\sigma]. \tag{43}$$

Noting that $\Gamma_1\subset\Sigma\times\left[-\frac{1}{2}\varepsilon k\sin\left(\frac{1}{2}\beta\right),\frac{1}{2}\sigma\right]$, to prove (i) and (ii) let us assume that

$$(x,u)\in\Sigma\times\left[-\tfrac{1}{2}\varepsilon k\sin\left(\tfrac{1}{2}\beta\right),\tfrac{1}{2}\sigma\right].$$

- If $h_\varepsilon(x,u)\in\widehat{V}\times[-R,R]$ then, by the C^0-bound on h_ε, we know that $x\in\widehat{V}'$. So

$$h_\varepsilon(x,u)\in\widehat{V}\times\left[\tfrac{1}{2}\varepsilon k\sin\left(\tfrac{1}{2}\beta\right),\sigma\right],$$

where the lower bound follows from (43) and the upper bound comes from the C^0-bound on h_ε, and this proves (i).

- If $h_\varepsilon(x,u)\in(\Sigma\setminus\operatorname{Int}\widehat{V})\times[-R,R]$ then, by the C^0-bound on h_ε, then we know that $x\in\Sigma\setminus\widehat{V}''$ and $z=(x',u')=h_\varepsilon(x,u)\in(\Sigma\setminus\operatorname{Int}\widehat{V})\times[-\sigma,\sigma]$. Therefore we have

$$\begin{aligned}\operatorname{angle}(\mu_z,T_z\widetilde{\Gamma}_1)&\geqslant\operatorname{angle}(\mu_{x',0},T_{x',0}\Sigma)-\operatorname{angle}(\mu_z,\mu_{x',0})-\operatorname{angle}(T_z\tilde{\Gamma}_1,T_{x',0}\Sigma)\\&>\tfrac{1}{8}\beta-\tfrac{1}{32}\beta-\tfrac{1}{16}\beta=\tfrac{1}{32}\beta,\end{aligned}$$

and this proves (ii).

The proof of (ii) generalizes to show that

$$\operatorname{angle}(T_z\widetilde{\Gamma}_s,\mu_z)>\tfrac{1}{32}\beta\quad \text{at } z=(x,s\tilde{\psi}_1(x))\in\widetilde{\Gamma}_s:=\{(x,u):u=s\tilde{\psi}_1(x)\}$$

for each $x\in\Sigma\setminus\operatorname{Int}\widehat{V}$ and $s\in[0,1]$.

We now set $V:=\{x:\tilde{\psi}_1(x)\geqslant\frac{1}{4}k\varepsilon\sin\left(\frac{1}{2}\beta\right)\}$, so that $\widehat{V}\subset\operatorname{Int}V$. Note that the family of contact structures ζ_s on $\Sigma\times\{s\}\subset\Sigma\times[0,1]$ induced by μ on the neighborhoods of graphs $\widetilde{\Gamma}_s=\{(x,u):u=s\tilde{\psi}(x)\}$ defines a semi-contact structure $\{\zeta_s\}_{s\in[0,1]}$ which is equivalent to $\{\xi_s\}_{s\in[0,1]}$. This concludes the proof of Lemma 7.4. □

Proof of Lemma 7.2. According to Lemma 7.4 we may assume that the semi-contact structure $\{\zeta_s\}_{s\in[0,1]}$ on $\Sigma\times[0,1]$ is of immersion type with the following properties. There exist a contact structure μ on $\Sigma\times[-R,R]$ and a function $\psi:\Sigma\to\left[-\frac{1}{2}R,\frac{1}{2}R\right]$ such that

(i) the germ of contact structure ζ_s is induced from μ on a neighborhood of $\Sigma\times\{s\}$ by an embedding $(x,s+t)\mapsto(x,s\psi(x)+t)\in\Sigma\times[-R,R]$ for all $x\in\Sigma$ and $t\in[-\delta,\delta]$;

(ii) there are closed domains $V\subset\Sigma$ and $\widehat{V}\subset\operatorname{Int}V$ such that $\psi|_V>0$, and the contact structure μ is transverse to graphs of functions $s\psi$ over $\widehat{W}:=\Sigma\setminus\operatorname{Int}\widehat{V}$ for all $s\in[0,1]$.

We will keep the notation ψ for the restriction of ψ to $\widehat{W}$. Note that ψ can be presented as the difference $\psi=\psi_+-\psi_-$ of two positive functions $\psi_\pm\in C^\infty(\widehat{W})$ such that the graphs of the functions $s\psi_\pm$ are transverse to μ.

Let $\{U_i\}_{i=1}^N$ be a finite covering of $W:=\Sigma\setminus\operatorname{Int}V$ by interiors of balls with smooth boundaries and such that $\bigcup_{i=1}^N\overline{U}_i\subset\widehat{W}$.

Let $\{\lambda_i^\pm:\Sigma\to[0,1]\}_{i=1}^N$ be two partitions of unity on W subordinate to the covering $\{U_i\}_{i=1}^N$, with $\sum_{i=1}^N\lambda_i^\pm|_W=1$, such that $\operatorname{Support}(\lambda_i^-)\Subset\operatorname{Support}(\lambda_i^+)$, $i=1,...,N$, and

$$\sum_{i=1}^N\lambda_i^\pm|_{\widehat{W}}\leqslant 1.$$

For $0\leqslant k\leqslant N$ define

$$L_k:=\sum_{i=1}^k\lambda_i^+,$$

noting that $L_N|_W=1$ and $L_N|_{\widehat{V}}=0$ shows $\widehat{V}\subset U\subset\Sigma\setminus W$, where $U:=\{x\in\Sigma:L_N(x)<1\}$.

For $1\leqslant i\leqslant N$ define the functions

$$\psi_i^\pm:=\psi^\pm\lambda_i^\pm:\Sigma\longrightarrow\mathbb{R}\quad\text{and}\quad\psi_i:=\psi_i^+-\psi_i^-:\Sigma\longrightarrow\mathbb{R}$$

and for $0\leqslant k\leqslant N$ the functions

$$\Psi_k^\pm:=\sum_{i=1}^k\psi_i^\pm\quad\text{and}\quad\Psi_k:=\Psi_k^+-\Psi_k^-=\sum_{i=1}^k\psi_i.$$

One can further ensure that the graphs of the functions $\Psi_k^\pm$ are transverse to μ. Let $\Gamma(\Psi_k):=\{(x,u):u=\Psi_k(x)\text{ and }x\in\Sigma\}\subset\widehat{C}=\Sigma\times[-R,R]$ be the graph of Ψ_k and likewise $\Gamma(L_k):=\{(x,u):u=L_k(x)\text{ and }x\in\Sigma\}\subset C=\Sigma\times[-1,1]$ be the graph of L_k. Set

$$\Gamma_L:=\bigcup_{k=0}^N\Gamma(L_k)$$

and consider the map $p:\Gamma_L\to\Sigma\times[-R,R]$,

$$p(x,u)=(x,\psi^+(x)u-\Psi_k^-(x))\quad\text{for }(x,u)\in\Gamma(L_k).$$

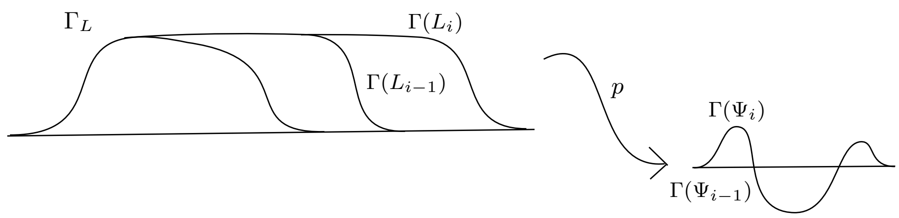

Figure 7.2. The set Γ_L used in the proof of the lemma. The region below a bump function is a saucer, so a partition of unity decomposes a general region into small saucers. Each saucer comes equipped with a map to $\widehat{C}$, making it a contact shell.

This map is well defined because if $(x,u)\in\Gamma(L_i)\cap\Gamma(L_j)$ for $0\leqslant i<j\leqslant N$ then $\Psi_i^-(x)=\Psi_j^-(x)$. Indeed, $L_i(x)=L_j(x)$ implies that $\psi_l^+(x)=0$ for all $i<l\leqslant j$, and hence $\psi_l^-(x)=0$ for $i<l\leqslant j$ because $\mathrm{Support}(\psi_l^-)\subset\mathrm{Support}(\psi_l^+)$. Note that $p(\Gamma_L)=\bigcup_{i=0}^N\Gamma(\Psi_i)$. The map p extends to an immersion

$$P\colon \mathcal{O}p\,\Gamma_L \longrightarrow \Sigma\times[-R,R];$$

see Figure 7.2.

The complement $C\setminus\Gamma_L$ is the union $\mathrm{Int}\,\Omega\cup\mathrm{Int}\,B_1\cup...\cup\mathrm{Int}\,B_N$, where

$$\Omega:=\{(x,u):L_N(x)\leqslant u\leqslant 1 \text{ and } x\in U\}$$

and B_i are the saucers bounded by the graphs $\Gamma(L_{i-1})$ and $\Gamma(L_i)$ over the balls $\overline{U}_i$ for $i=1,...N$. On Ω we can extend the immersion P to a diffeomorphism

$$\Omega\longrightarrow\{(x,u):\Psi_N(x)\leqslant u\leqslant\psi_1(x) \text{ and } x\in U\}$$

that is fiberwise linear with respect to the projection to Σ, so it remains to extend the induced contact structure $P^*\mu$ on $\mathcal{O}p\,\Gamma_L\cup\Omega$ as a regular semi-contact structure on the saucers. The following lemma is obvious.

LEMMA 7.5. *Let Σ be a hypersurface in a $(2n+1)$-dimensional contact manifold transversal to the contact structure, and $f\colon D^{2n}\to\Sigma$ be a smooth embedding of the unit $2n$-ball. Then there exists $\varepsilon>0$ such that the disc $f(D^{2n}_\varepsilon)$ is regular.*

It follows that the covering by balls U_i^+ can be chosen to ensure that the discs $(\overline{U}_i^+,\zeta_0)$ are regular. If the function ψ is sufficiently C^1-small, then for each $i=1,...,N$ the graphs $\Gamma_i^-\subset\Gamma(\Psi_{i-1})$ and $\Gamma_i^+\subset\Gamma(\Psi_i)$ of the functions $\Psi_{i-1}|_{\overline{U}_i^+}$ and $\Psi_i|_{\overline{U}_i^+}$, respectively, are regular as well. Hence there is a contactomorphism g_i between a neighborhood $O_i\supset\Gamma_i^-$ and a neighborhood of a disc in $\Pi=\{(w,v)\in\mathbb{R}^{2n}\times\mathbb{R}:v=0\}$. Again, if ψ is sufficiently small, then the neighborhood O_i contains the disc Γ_i^+ and, moreover, $g_i(\Gamma_i^+)$ is transverse to the vector field $\partial/\partial y_n$, which ensures the regularity of the contact saucer B_i, $i=1, ..., N$.

Finally, it remains to observe that the required C^1-smallness of the function ψ can be achieved by passing to a partition. This concludes the proof of Lemma 7.2. □

7.2. Contact structures in the complement of saucers. Parametric version

We prove in this section the following parametric version of Proposition 6.2.

PROPOSITION 7.6. *Let M be a $(2n+1)$-manifold and $A\subset M$ be a closed set, so that $M\setminus A$ is connected. Let ${}^T\xi_0$ be a fibered almost contact structure on ${}^TM$ which is genuine on $(T\times \mathcal{O}p\,A)\cup(\partial T\times M)\subset T\times M$. Then there exist a finite number of (possibly overlapping) discs $T_i\subset T$ and disjoint embedded fibered saucers ${}^{T_i}B_i\subset {}^TM$, $i=1, ..., N$, such that ${}^T\xi_0$ is homotopic relative to $(T\times A)\cup(\partial T\times M)$ to a fibered almost contact structure ${}^T\xi_1$ which is genuine on ${}^TM\setminus\bigcup_{i=1}^N B_i$ and whose restriction to each fibered saucer ${}^{T_i}B_i$ is semi-contact and regular. Moreover, we can choose the discs in such a way that any non-empty intersection $T_{i_1}\cap...\cap T_{i_k}$, $1\leqslant i_1<...<i_k\leqslant N$, is again a disc with piecewise smooth boundary.*

As in the non-parametric case, the following lemma is the main part of the proof.

LEMMA 7.7. *Any fibered semi-contact structure ${}^T\xi=\{\zeta_s^\tau\}_{s\in[0,1],\tau\in T}$ on the fibered annulus ${}^TC:=T\times\Sigma\times[0,1]$ is homotopic relative to $(T\times\mathcal{O}p\,\partial C)\cup(\partial T\times C)$ to a fibered almost contact structure ${}^T\tilde{\xi}$ which is a genuine contact structure in the complement of finitely many fibered saucers ${}^{T_1}B_1, ..., {}^{T_k}B_k\subset {}^TC$, and such that the restriction ${}^{T_j}\tilde{\xi}|_{T_j B_j}$ for each $j=1, ..., k$ is semi-contact and regular. Moreover, we can choose the discs $T_i\subset T$ in such a way that any non-empty intersection $T_{i_1}\cap...\cap T_{i_p}$, $1\leqslant i_1<...<i_p\leqslant k$, is again a disc with piecewise smooth boundary.*

First, note that Lemma 7.3 has the following parametric analogue.

LEMMA 7.8. *Given a fibered semi-contact structure ${}^T\xi$ on $T\times\Sigma\times[0,1]$ there exists $N>0$ such that the restriction of ${}^T\xi$ to $T\times\Sigma\times[a_i, a_{i+1}]$, $a_i:=a+(b-a)/N$, is of immersion type for each $i=0, ..., N-1$.*

Proof of Lemma 7.7. By Lemma 7.8, we may assume that the fibered semi-contact structure ${}^T\xi$ on the annulus ${}^TC=T\times C$ is of immersion type, i.e. there exist a fibered contact structure ${}^T\mu=\{\mu^\tau\}_{\tau\in T}$ on ${}^T\widehat{C}:=T\times\Sigma\times[-R,R]$ and a family of functions $\psi_s^\tau:\Sigma\to[-r,r]$ for $r<R$, $s\in[0,1]$ and $\tau\in T$, such that the contact structure ζ_s^τ on a neighborhood of Σ_s^τ is induced from μ^τ by an embedding of this neighborhood onto a neighborhood of the graph of the function ψ_s^τ.

Let $T'\subset \operatorname{Int} T''$ and $T''\subset \operatorname{Int} T$ be two slightly smaller compact parameter spaces such that the semi-contact structure μ^τ is contact for $\tau\in T\setminus \operatorname{Int} T'$, i.e. $\psi^\tau_s(x)>\psi^\tau_{s'}(x)$ for any $x\in\Sigma$, $\tau\in T\setminus\operatorname{Int} T'$ and $s>s'$.

Similarly to the non-parametric case (see Lemma 7.4), we can reduce to the case when the following property holds: $\psi^\tau_0=0$, $\psi^\tau_s=s\psi^\tau$, and there exist domains $\widehat{W}, W\subset T\times\Sigma$, $\operatorname{Int}\widehat{W}\Supset W$, such that $\psi^\tau(x)>0$ for $(x,\tau)\in T'\times\Sigma\setminus\operatorname{Int} W$ and the contact structures μ^τ transverse to the graphs Γ^τ_s of the functions $s\psi^\tau$ over $V^\tau:=(\Sigma\setminus\operatorname{Int} W)\cap\{\tau\}\times\Sigma$.

Denote by ψ the function $\psi(\tau,x,s):=\psi^\tau(x,s)$ for $(\tau,x,s)\in W$, and present ψ as the difference of two positive functions, $\psi=\psi^+-\psi^-$.

Let us choose a finite covering of W denoted $\{U_i\}_{i=1}^N$, such that $U_i=\operatorname{Int} T_i\times\operatorname{Int}\Delta_i^\pm$, where $\Delta_i\subset\Sigma$ and $T_i\subset T''$ are balls with smooth boundaries, $i=1,...,N$, and we have $\bigcup_{i=1}^N \overline{U}_i\subset\widehat{W}$. Here we choose the discs $T_i\subset T$ in such a way that any non-empty intersection $T_{i_1}\cap...\cap T_{i_k}$, $1\leqslant i_1<...<i_k\leqslant L$, is again a disc with piecewise smooth boundary. More geometric constraints on the coverings will be imposed below.

Let $\{\lambda_i^\pm\}_{i=1}^N$ be two partitions of unity over $\widehat{W}$ subordinated to $\{U_i\}_{i=1}^N$ so that $\operatorname{Support}(\lambda_i^-)\Subset\operatorname{Support}(\lambda_i^+)$ for $I=1\ ...,N$,

$$\sum_{i=1}^N \lambda_i^\pm|_W=1 \quad\text{and}\quad \sum_{i=1}^N \lambda_i^\pm|_{\widehat{W}}\leqslant 1.$$

Let

$$\psi_i^\pm:=\psi^\pm\lambda_i^\pm \ \text{ and }\ \psi_i:=\psi_i^+-\psi_i^-,\quad i=1,...,N.$$

Set, for $k=1,...,N$,

$$\Psi_k^\pm:=\sum_{i=1}^k\psi_i^\pm,\quad \Psi_k:=\Psi_k^+-\Psi_k^-=\sum_{i=1}^k\psi_i\quad\text{and}\quad \Phi_k:=\sum_{i=1}^k\lambda_i^+.$$

Note that $L_N|_W=1$ and $L_N|_{\widehat{V}}=0$, so $V\subset U:=\{x:L_N(x)<1\}\subset T\times\Sigma\setminus W$.

In ${}^T C=T\times\Sigma\times[0,1]$ we let $\Gamma(L_k)$ be the graph of the function L_k, and in ${}^T\widehat{C}=T\times\Sigma\times[-R,R]$ we let $\Gamma(\Psi_k)$ be the graph of the function Ψ_k. Set $\Gamma_L=\bigcup_{i=1}^N\Gamma(L_i)\subset {}^T C$.

Consider the map $p:\Gamma_L\to T\times\Sigma\times[-R,R]$ given by the formula

$$p(\tau,x,s)=(\tau,x,\psi^+(\tau,x)s-\Psi_i^-(\tau,x))\quad\text{for } (\tau,x,s)\in\Gamma(L_i).$$

This map is well defined because if $(\tau,x,s)\in\Gamma(L_i)\cap\Gamma(L_j)$ for $0\leqslant i<j\leqslant N$ then $\Psi_i^-(\tau,x)=\Psi_j^-(\tau,x)$. Indeed, $\Phi_i(\tau,x)=\Phi_j(\tau,x)$ implies that $\psi_l^+(\tau,x)=0$ for all $i<l\leqslant j$, and hence $\psi_l^-(\tau,x)=0$ for $i<l\leqslant j$ since $\operatorname{Support}(\psi_l^-)\subset\operatorname{Support}(\psi_l^+)$. Note that $p(\Gamma_L)=\bigcup_{i=1}^N\Gamma(\Psi_i)$. The map p extends to an immersion $P:\mathcal{O}p\,\Gamma_L\to T\times\Sigma\times[-R,R]$.

The complement ${}^TC\setminus\Gamma_L$ is a union of the interior of the domain

$$\Omega := \{(x,s) : L_N(\tau,x) \leqslant s \leqslant 1, \tau \in T, \text{ and } x \in U\}$$

and interiors of fibered saucers ${}^{T_i}B_i$ bounded by the graphs $\Gamma(L_{i-1})$ and $\Gamma(L_i)$ over the balls $\overline{U}_i$ for $i=1,\dots N$. We can extend the immersion P to Ω as a fiberwise linear, with respect to the projection to $T\times\Sigma$, diffeomorphism

$$\Omega \longrightarrow \{(\tau,x,u) : \Psi_N(\tau,x) \leqslant u \leqslant \psi(\tau,x) \text{ and } (\tau,x) \in U\},$$

so it remains to extend the induced contact structure $P^*({}^T\mu)$ on $\mathcal{O}p\,\Gamma_L\cup\Omega$ as a fibered regular semi-contact structure to the fibered saucers.

It follows from Lemma 7.5 that the covering by balls U_i can be chosen to ensure that for each $i=1,\dots,N$ and $\tau\in T_i^+$ the disc $(\{\tau\}\times\Delta_i^+,\zeta_0^\tau)$ is regular. If the functions ψ_1 and ψ_0 are sufficiently C^1-close, then for each $i=1,\dots,N$ the graphs $\Gamma_i^-\subset\Gamma(\Psi_{i-1})$ and $\Gamma_i^+\subset\Gamma(\Psi_i)$ of the functions $\Psi_{i-1}|_{\overline{U}_i^+}$ and $\Psi_i|_{\overline{U}_i^+}$, respectively, are fibered over T_i^+ by regular discs as well. Hence, there is a fibered-over-T_i^+ contactomorphism g_i between a neighborhood $O_i\supset\Gamma_i^-$ and a neighborhood of a fibered disc in $T_i\times\Pi=\{(w,v)\in\mathbb{R}^{2n}\times\mathbb{R}:v=0\}$. Again, if ψ is sufficiently close, then the neighborhood O_i contains the disc Γ_i^+ and, moreover, $g_i(\Gamma_i^+)$ is transverse to the vector field $\partial/\partial y_n$, which ensures the regularity of the fibered contact saucer ${}^{T_i}B_i$, $i=1,\dots,N$.

Finally, it remains to observe that the required smallness of the function ψ can be achieved by passing to a partition. □

Proof of Proposition 7.6. Assume $T=D^q$. Choose an embedded annulus $C=S^{2n}\times[0,1]\subset M\setminus A$ and first use the q-parametric part of Gromov's h-principle (Theorem 7.1) to deform ${}^T\xi_0$ relative to $(T\times A)\cup(\partial T\times M)$ to a fibered almost contact structure which is a genuine fibered contact structure on $T\times M\setminus C$. Next, with use of the $(q+1)$-parametric part of Theorem 7.1 applied to the family of neighborhoods of spheres $\{\tau\}\times S^{2n}\times\{t\}$, with $\tau\in T$ and $t\in[0,1]$, we make the fibered almost contact structure semi-contact on ${}^TC=T\times C$. Finally we use Lemma 7.7 to complete the proof. For general T we triangulate it and inductively over skeleta apply the previous proof to each simplex. □

8. Reduction to a universal model

In this section we prove Proposition 3.1.

8.1. Equivariant coverings

The key step in the proof of Proposition 3.1 is the following result.

Proposition 8.1. *For a fixed dimension, there is a finite list of saucers*

$$\{(B_p,\zeta_p)\}_{p=1}^L$$

with the following property: for any circle model contact shell (B_K,η_K) *defined by a time-independent contact Hamiltonian* $K:\Delta\to\mathbb{R}$, *there exist finitely many disjoint balls* $B_i\subset B_K$, *for* $i=1,...,q$, *such that the contact shell* (B_K,η_K) *is homotopic relative to* $\mathcal{O}p\,\partial B_K$ *to an almost contact structure* ξ *that is genuinely contact on* $B_K\setminus\bigcup_{i=1}^q B_i$ *and each contact shell* $\xi|_{B_i}$ *is equivalent to one of the saucers* (B_p,ζ_p) *for* $p=1,...,L$.

Remark 8.2. The proof of Proposition 8.1 follows roughly the same scheme as the proof of Proposition 6.2, but uses the idea of equivariant coverings in a crucial way. The basic idea can be seen in the following trivial observation about real functions. Consider the piecewise constant function $\phi:\mathbb{R}\to\mathbb{R}$ which is equal to 1 on $[0,1)\cup[2,3)$, equal to -3 on $[1,2)$, and 0 elsewhere. Let the group $\mathbb{Z}$ act on $\mathbb{R}$ by translation: $j\in\mathbb{Z}$ being identified with the map $x\mapsto x+j$. Then the function $\sum_{j=1}^k\phi\circ j$ is equal to 1 on $[0,1)\cup[k+2,k+3)$ and it is strictly negative on $[1,k+2)$.

The key point of this example is two-fold: firstly, that a function which is negative on an arbitrarily large portion of its support can be written as a sum of functions which are negative on a small subset of their support. And secondly, that in fact these functions can be taken to be translations of a single function by a group action.

Consider $\mathbb{R}^{2n+1}$ with the contact structure ξ_{st} given by the form

$$\alpha=dz+\sum_{i=1}^{n-1}(x_i\,dy_i-y_i\,dx_i)-y_n\,dx_n=dz+\sum_{i=1}^{n-1}u_i\,d\varphi_i-y_n\,dx_n.$$

Let $\Pi=\{(z,x,y):y_n=0\}$. In the group of contactomorphisms $\mathrm{Cont}(\mathbb{R}^{2n+1},\xi_{\mathrm{st}})$ consider the $2n$-dimensional lattice Θ generated by the following transformations:

- the translations

$$T_z:(x,y,z)\longmapsto(x,y,z+1),$$
$$T_{x_n}:(x_1,...,x_n,y,z)\longmapsto\big(x_1,...,x_n+\tfrac{1}{2},y,z\big);$$

- the sheers in the (y_j,z) and (x_j,z) planes for each $j=1,...,n-1$,

$$S_{y_j}:(x,y_1,...,y_j,...,y_n,z,t)\longmapsto(x,y_1,...,y_j+1,...,y_n,z+x_j),$$
$$S_{x_j}:(x_1,...,x_j,...,x_n,y,z)\longmapsto(x_1,...,x_j+1,...,x_n,y,z-y_j).$$

Note that Θ preserves Π, we have $S_{y_j} S_{x_j} S_{y_j}^{-1} S_{x_j}^{-1} = T_z^2$, and all other transformations commute. Hence every element of Θ may be written as

$$S_{x_1}^{k_1} \dots S_{x_{n-1}}^{k_{n-1}} S_{y_1}^{l_1} \dots S_{y_{n-1}}^{l_{n-1}} T_{x_n}^{k_n} T_z^{l_n},$$

and from this it follows that Θ acts properly discontinuously on Π, that is for any compact set $Q \subseteq \Pi$, the set

$$S(Q) := \{g \in \Theta : g(Q) \cap Q \neq \varnothing\} \subset \Theta \tag{44}$$

is finite.

For a positive integer N, let C_N be the scaling contactomorphism

$$(x, y, z) \longmapsto (Nx, Ny, N^2 z)$$

and let $\Theta_N = C_N^{-1} \Theta C_N$, that is the group generated by the translations and sheers:

$$T_{j,N} := C_N^{-1} \circ T_j \circ C_N, \quad T_{z,N} := C_N^{-1} \circ T_z \circ C_N, \quad \text{and} \quad S_{j,N} := C_N^{-1} \circ S_j \circ C_N.$$

Say that a compact set Q *generates a Θ_N-equivariant cover of* Π if $\Theta_N \cdot \mathrm{Int}(Q) = \Pi$.

Since $T_{x_n}^2 = T_{x_n,N}^{2N}$, the cyclic group $\Upsilon := \langle T_{x_n}^2 \rangle$ is always a subgroup of Θ_N, in fact a normal subgroup, and we define $\widehat{\Theta}_N$ to be the quotient group Θ_N / Υ. Say that a compact set $Q \subset \Pi$ is *sufficiently small* if $T_{x_n}^2(Q) \cap Q = \varnothing$.

Note that the quotient of $\mathbb{R}^{2n+1}$ by the contactomorphism $T_{x_n}^2$ is the contact manifold

$$\left(\mathbb{R}^{2n-1} \times T^* S^1, \ker\left(dz + \sum_{i=1}^{n-1} (x_i\, dy_i - y_i\, dx_i) + v\, dt \right) \right),$$

where $v = -y_n$ is identified with the fiber coordinate of $T^* S^1$ and the base coordinate $t \in \mathbb{R}/\mathbb{Z}$ is given by the quotient by translation $T_{x_n}^2$. Denote this quotient by

$$\pi: \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^{2n-1} \times T^* S^1.$$

The group $\widehat{\Theta}_N$ can be viewed as a subgroup of the group of contactomorphisms of $\mathbb{R}^{2n-1} \times T^* S^1$ preserving $\widehat{\Pi} = \pi(\Pi) \simeq \mathbb{R}^{2n-1} \times \mathbb{R}/\mathbb{Z}$. Any compactly supported function $\Phi: \Pi \to \mathbb{R}$ defines a function $\sum_{h \in \Upsilon} \Phi \circ h^{-1}$ which is 1-periodic in the x_n-variable, and therefore defines a function $\hat{\Phi}: \widehat{\Pi} \to \mathbb{R}$.

Remark 8.3. (a) If Q generates a Θ-equivariant covering of Π, then $Q_N := C_N^{-1}(Q)$ generates a Θ_N-equivariant covering. For a sufficiently large N the set Q_N is sufficiently small.

(b) Suppose that $a > \frac{1}{2}$. Then the parallelepiped

$$P = \{(x_1, \dots, x_n, y_1, \dots, y_n, z) : |x_j|, |y_j| \leqslant a, 1 \leqslant j \leqslant n-1, 0 < x_n \leqslant a, y_n = 0, |z| \leqslant a\} \subset \Pi$$

generates a Θ-equivariant covering of Π. If $a < 1$, then P is sufficiently small. In particular, there are sufficiently small sets generating equivariant coverings.

(c) If $Q' \subset Q \subset \Pi$ are two compact sets, and Q' generates a Θ_N-equivariant covering of Π, then so does Q.

Let us *fix for the rest of the paper* a regular sufficiently small disc $Q\subset\Pi$ and a smaller disc $Q'\subset\operatorname{Int} Q$ which generates a Θ-equivariant covering of Π. Denote by m the cardinality $|S(Q)|$ of the set $S(Q)$.

We also fix two non-negative C^∞-functions $\phi_+,\phi_-:\Pi\to\mathbb{R}$, which are supported in Q and satisfy the following conditions:

(i) $\phi_+|_{\operatorname{Int} Q}>0$, $\phi_-|_{Q'}>0$, and $\phi_-|_{\mathcal{O}p(\partial Q)}=0$;

(ii) $\max\phi|_{Q'}<-(m+1)\mu$, where $\phi:=\phi_+-\phi_-$, $\mu:=\max\phi$, and m is the cardinality of the set $S(Q)$ defined in (44);

(iii) let $\phi^s=\phi_+-s\phi_-$, $s\in[0,1]$ (so that $\phi^s\geqslant\phi^1=\phi$) and for any finite subset $F\subset\Theta$ let

$$\Phi^s_F := \mu+\sum_{g\in F}\phi^s\circ g^{-1}|_Q,\quad s\in[0,1];$$

then the graph $y_n=\Phi^s_F(q)$, $q\in Q$, with the induced contact structure, is regular.

Remark 8.4. In condition (iii) the elements $g\in F$ with $g(Q)\cap Q=\varnothing$ are irrelevant, so it suffices to verify (iii) only for subsets F of the finite set $S(Q)$. Hence, the condition can always be satisfied by taking ϕ_+ and ϕ_- sufficiently small (e.g. replacing the pair (ϕ_+,ϕ_-) which satisfy (i) and (ii) by $(\varepsilon\phi_+,\varepsilon\phi_-)$ for a sufficiently small $\varepsilon>0$).

Pick a linear ordering of $\Theta=\{g_1,g_2,\dots\}$, and order $\widehat{\Theta}_N$ accordingly; we fix this ordering during the rest of the paper. Define functions $\Pi\to\mathbb{R}$ by the formulas

$$\Phi_k := \mu+\sum_{j=1}^k \phi\circ g_j^{-1}\quad\text{and}\quad \Psi_k=\mu+\sum_{j=1}^k\phi_+\circ g_j^{-1}$$

for $k=0,1,\dots$, and note that $\Phi_0=\Psi_0\equiv\mu$.

Let

$${}^\Phi\Gamma_k^- := \{y_n=\Phi_{k-1}(x_1,\dots,x_n,y_1,\dots,y_{n-1},z)\text{ and }(x_1,\dots,x_n,y_1,\dots,y_{n-1},z)\in g_k(Q)\}$$

be the graph of Φ_{k-1} over the set $g_k(Q)$, and similarly denote by ${}^\Phi\Gamma_k^+$ the graph of Φ_k over $g_k(Q)$. Denote by ${}^\Psi\Gamma_k^+$ the graph of Ψ_k over $g_k(Q)$ and by ${}^\Psi\Gamma_k^-$ the graph of Ψ_{k-1} over $g_k(Q)$. Define B_k to be the saucer

$$\begin{aligned}B_k := \{&\Psi_{k-1}(x_1,\dots,x_n,y_1,\dots,y_{n-1},z)\leqslant y_n\leqslant\Psi_k(x_1,\dots,x_n,y_1,\dots,y_{n-1},z)\\&\text{and }(x_1,\dots,x_n,y_1,\dots,y_{n-1},z)\in g_k(Q)\}\end{aligned}$$

bounded by ${}^\Psi\Gamma_k^-$ and ${}^\Psi\Gamma_k^+$. Similar to the proof of Proposition 6.2, we observe that there is an immersion $\mathcal{O}p\,\partial B_k\to\mathbb{R}^{2n+1}$ which maps ${}^\Psi\Gamma_k^\pm\to{}^\Phi\Gamma_k^\pm$ diffeomorphically. The induced contact structure ζ_k with its canonical semi-contact extension to B_k defines a shell structure on the saucer B_k.

More generally, for $s\in[0,1]$, set

$$\Phi_k^s := \mu + \sum_{j=1}^{k} \phi^u \circ g_j^{-1},$$

so that $\Phi_k^1 = \Phi_k$ and $\Phi_k^0 = \Psi_k$. Define the regular contact saucer (B_k, ζ_k^s) to be the one induced by an immersion of $\mathcal{O}p\, B_k \to \mathbb{R}^{2n+1}$ that maps $\partial B_k = {}^\Psi\Gamma_k^+ \cup {}^\Psi\Gamma_k^-$ diffeomorphically onto the graphs of Φ_k^s and Φ_{k-1}^s over $g_k(Q)$. Regularity is ensured by condition (iii) above.

The above construction builds a countable collection of 1-parameter families of regular contact saucers (B_k, ζ_k^s). However, as the next lemma shows, up to equivalence the number of these 1-parameter families is always bounded by $L=2^m$, the number of subsets of the set $S(Q)$ from (44).

LEMMA 8.5. *Up to equivalence, the above construction builds at most $L=2^m$ 1-parametric families of regular contact saucers (B_k, ζ_k^u), where $m=|S(Q)|$.*

Proof. By the contactomorphism g_k^{-1}, we know that (B_k, ζ_k^s) is equivalent to a saucer whose boundary contact germ is defined by the two graphs over Q,

$$\{(w, y_n) : y_n = (\Phi_{k-1}^s \circ g_k)|_Q\} \quad \text{and} \quad \{(w, y_n) : y_n = (\Phi_k^s \circ g_k)|_Q\}.$$

However, $\Phi_k^s|_Q \circ g_k = \mu + \sum_{j=1}^k (\phi^s \circ (g_j^{-1} g_k))|_Q$, and the number of different sums of this type is bounded above by the number $L=2^m$ of finite subsets of the set $S(Q)$. □

Given a positive N and an element $g\in\Theta_N$ we let $\phi_{g,N} := (\phi/N)\circ C_N \circ g^{-1}$. Notice that by the contactomorphism C_N^{-1} the regular semi-contact saucer which is defined over the domain Q by the functions Φ_{k-1}^s and Φ_k^s is equivalent to the saucer over the domain $C_N^{-1}(Q)$ defined by the functions

$$\Phi_{k-1,N}^s := \frac{\mu}{N} + \sum_{j=1}^{k-1} \phi_{g_j,N}^u \quad \text{and} \quad \Phi_{k,N}^s := \frac{\mu}{N} + \sum_{j=1}^{k} \phi_{g_j,N}^s,$$

where $\phi_{g,N}^s := (\phi^s/N)\circ C_N \circ g^{-1}$.

Consider the function $\hat\phi_{g,N} : \widetilde\Pi \to \mathbb{R}$. We note that $\hat\phi_{g,N} = \hat\phi_{g',N}$ if g and g' are in the same conjugacy class from $\widehat\Theta_N = \Theta_N/\Upsilon$, so that in the notation for $\hat\phi_{g,N}$ we can use $g\in\widehat\Theta_N$.

LEMMA 8.6. *With the above choices of Q, Q', ϕ_+, and ϕ_-, for any bounded open domains U' and $U \Supset U'$ in $\mathbb{R}^{2n-1}$, and any C^∞-function $K: U \to \mathbb{R}$ which is positive on $(U \setminus \overline{U}')$, there exist $N>0$ and a finite subset $\Lambda \subset \widehat\Theta_N$ such that*

$$U' \times S^1 \Subset \bigcup_{g\in\Lambda} g(\operatorname{Int} \pi(Q'_N)) \subset \bigcup_{g\in\Lambda} g(\operatorname{Int}\pi(Q_N)) \Subset U \times S^1$$

and

$$\sum_{g\in\Lambda}\hat{\phi}_{g,N} < \begin{cases} -\dfrac{2\mu}{N}, & \text{on } U'\times S^1,\\ K-\dfrac{\mu}{N}, & \text{on } (U\setminus U')\times S^1.\end{cases}$$

Proof. Suppose that $K{:}\,U\to\mathbb{R}$ is given. Since K is positive on $U\setminus U'$, we may fix some $\varepsilon>0$ with the property that the set $P:=\{(x,y,z)\in U\setminus U'{:}\,K(x,y,z)>\varepsilon\}$ disconnects U' from ∂U. Note that the conclusion of the lemma only becomes stronger if we enlarge the set $U'\Subset U$. With this in mind, we redefine U' to be the interior of the union of all components of $U\setminus P$ which are disjoint from ∂U.

Set $\widehat{Q}_N:=\pi(Q_N)$ and $\widehat{Q}'_N:=\pi(Q'_N)$. For a sufficiently large N there exists a finite set $\Lambda\subset\widehat{\Theta}$ such that $(U'\cup P)\times S^1\Supset\bigcup_{g\in\Lambda}g(\widehat{Q}_N)\supset\bigcup_{g\in\Lambda}g(\widehat{Q}'_N)\Supset U'\times S^1$. Furthermore, suppose that

$$N>(m+1)\mu\varepsilon^{-1}. \tag{45}$$

Then, using (45), we get on $P\times S^1$ that

$$\sum_{g\in\Lambda}\phi_{g,N}<\frac{m\mu}{N}=\frac{(m+1)\mu}{N}-\frac{\mu}{N}<\varepsilon-\frac{\mu}{N}<K-\frac{\mu}{N}.$$

On the other hand, on $U'\times S^1$ we have

$$\sum_{g\in\Lambda}\phi_{g,N}<-\frac{(m+1)\mu}{N}+\frac{(m-1)\mu}{N}<-\frac{2\mu}{N}.$$

Indeed, this holds for given $(x,y,z)\in g(Q'_N)$, because according to inequality (ii) a single negative term $\hat{\phi}_{g,N}(x,y,z)$ is larger in absolute value by at least $2\mu/N$ than the sum of all positive terms (the denominator N appears because of the scaling factor of the function in the definition of $\phi_{g,N}$). □

Proof of Proposition 8.1. Let $U=\operatorname{Int}\Delta$ and $U'\Subset U$ be a star-shaped subset such that $K|_{U\setminus U'}>0$. Lemma 8.6 provides an integer $N>0$ and a finite set $\Lambda\subset\widehat{\Theta}_N$ such that the corresponding function

$$\Phi=\Phi^{S^1,N}=\frac{\mu}{N}+\sum_{g\in\Lambda}\phi_{g,N}{:}\,\Delta\times S^1\longrightarrow\mathbb{R}$$

satisfies $\Phi(w,t)<K(v)$ for $w\in\Delta\setminus U'$ and $\Phi|_{U'\times S^1}<-\mu/N$. According to Proposition 4.9, there exists a contact Hamiltonian $\widetilde{K}$ such that $\eta_{\widetilde{K}}$ is dominated by η_K, where $\widetilde{K}|_{\Delta\setminus U'}=K|_{\Delta\setminus U'}$ and $\widetilde{K}|_{U'}>-\mu/N$. Therefore, $\Phi(w,t)<\widetilde{K}(w)$ for all $(w,t)\in\Delta\times S^1$. The function Φ is equal to $\mu/N>0$ near $\partial\Delta\times S^1$, and hence defines a circular shell model η_Φ which is dominated by $\eta_{\widetilde{K}}$. Hence, it is sufficient to prove the required extension result for η_Φ. We order Λ using the chosen ordering of $\widehat{\Theta}$ and define functions

$$\Phi_k = \frac{\mu}{N}+\sum_{j=1}^{k}\phi_{g_j,N}\colon \Delta\times S^1\to\mathbb{R},\quad k=0,...,|\Lambda|,$$

where $|\Lambda|$ is the cardinality of Λ. We have $\Phi_0=\mu/N$ and $\Phi_{|\Lambda|}=\Phi$. The shells η_{Φ_k} and $\eta_{\Phi_{k-1}}$ differ by one of the regular saucers (B_p,ζ_p), from the finite list provided by Lemma 8.5, while the shell η_{Φ_0} is solid, since $\Phi_0>0$ everywhere. □

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. Proposition 6.2 allows us to assume that ξ is contact outside of a finite collection of disjoint saucers $\{B_i\}_{i=1}^N$, so that the restriction $\xi|_{B_i}$ for each $i=1,...,N$, is a regular semi-contact saucer. Using Proposition 6.12 we replace saucers by circle model shells defined by time-independent contact Hamiltonians. Applying Proposition 8.1 we can further reduce to the case of a contact structure in the complement of saucers from the finite list (B_p,ζ_p), $p=1,...,L$. Using again Proposition 6.12 we replace saucers by circle model shells (B_{K_p},η_{K_p}) defined by time-independent contact Hamiltonians. We may then choose any special Hamiltonian K_{univ} satisfying $K_{\mathrm{univ}}(x)<\min_p K_p(x)$. □

8.2. The standardization of the holes in the parametric case

In this section we prove Proposition 3.11.

Given a special Hamiltonian $K\colon\Delta_{\mathrm{cyl}}\to\mathbb{R}$, we recall the following notation from §3.1:

$$K^{(s)}:=sK+(1-s)E,\ \ s\in[0,1],\quad \text{where } E(u,z):=K(u,1).$$

LEMMA 8.7. *There exist a special Hamiltonian $K_{\mathrm{univ}}\colon\Delta_{\mathrm{cyl}}\to\mathbb{R}$ and a non-increasing function $\theta\colon[0,1]\to[0,1]$ with $\theta(0)=0$ and $\theta(1)=1$, which depend only on the choice of Q, Q', ϕ_+, and ϕ_-, and such that for each $p=1,...,2^m$ there exists a family of subordination maps*

$$\eta_{K_{\mathrm{univ}}^{(\theta(s))}}\longrightarrow\eta_p^s:=(B_p,\zeta_p^s),\quad s\in[0,1],$$

which are solid for $s=0$.

Proof. We note that there exists $\delta>0$ such that the regular saucer (B_p,ζ_p^s) is solid for $s\in[0,\delta]$, i.e. the contact structure on its boundary is extended inside as a genuine contact structure. Proposition 6.13 implies that the family of saucers (B_p,ζ_p^s) dominates a family of circle models $\eta_{\widetilde{K}_p^s}$, where $\widetilde{K}_p^s>0$ for $s\in[0,\delta']$, $p=1,...,2^m$, and some $\delta'<\delta$. We also note that Lemma 4.6 allows us to assume that the domain Δ in the definition of the Hamiltonians $\widetilde{K}^s_p$ coincides with Δ_{cyl}. Choose as K_{univ} any special contact Hamiltonian which satisfies

$$K_{\mathrm{univ}} < \min_{\substack{s\in[0,1]\\ p=1,\dots,L=2^m}} \widetilde{K}^s_p.$$

(See Example 3.5.) There exists $\delta''\in(0,\delta')$ such that $\widetilde{K}^s_p > K^{(0)}_{\mathrm{univ}}$ for all $s\in[0,1]$ and $p=1,\dots,2^m$. Choose a non-decreasing function $\theta\colon[0,1]\to[0,1]$ such that $\theta(s)=0$ for $s\in\left[0,\frac{1}{2}\delta'\right]$ and $\theta(s)=1$ for $s\in[\delta'',1]$. Then $K^{(\theta(s))}_{\mathrm{univ}} < \widetilde{K}^s_p$ for all $s\in[0,1]$ and $p=1,\dots,2^n$. Hence, by Lemma 4.1, one can arrange the inclusion maps $\eta_{K^{(\theta(s))}_{\mathrm{univ}}}\to\eta^s_p$ to be subordinations. □

Remark 8.8. It is not clear if any Hamiltonian K_{univ} satisfying Proposition 3.1 also satisfies Lemma 8.7, or conversely. But once we know that there are two Hamiltonians separately satisfying Proposition 3.1 and Lemma 8.7, we can simply choose K_{univ} to be less than both of them, and this Hamiltonian will suffice for both.

Let $T=D^q\subset\mathbb{R}^q$ be the unit disc. Choose any decreasing C^∞-function $\theta\colon[0,1]\to[0,1]$, which is equal to 1 on $\left[0,\frac{1}{3}\right]$ and to 0 on $\left[\frac{2}{3},1\right]$.

PROPOSITION 8.9. *There is a universal finite list of families of saucers* (B^s_p,ζ^s_p), $p=1,\dots,L$ *and* $s\in[0,1]$, *where* L *depends only on dimension* n, *with the following property. Let* $K^\tau\colon\Delta\to\mathbb{R}$, $\tau\in T$, *be a family of time-independent contact Hamiltonians parameterized by the unit disc* $T=D^q\subset\mathbb{R}^q$, *and such that* $K^\tau(x)>0$ *for* $(\tau,x)\in\partial(T\times\Delta)$. *Let* $({}^TB,{}^T\eta)$, *where* ${}^TB=T\times B$, *be the fibered circular shell defined by this family. Denote by* ${}^T\eta_p$ *the shell corresponding to the family of saucers* $\eta_p^{(\theta(\|\tau\|))}$. *Then there exist finitely many balls* $B_i\subset B$, $i=1,\dots,N$, *with piecewise smooth boundary, such that the fibered contact shell* ${}^T\eta$ *(viewed as a fibered almost contact structure on* ${}^TB$*) is homotopic relative to* $\mathcal{O}p\,\partial(T\times B)$ *to a fibered almost contact structure* ${}^T\xi$, *which is genuinely contact on* $T\times(B\setminus\bigcup_{i=1}^N B_i)$, *and such that each fibered contact shell* ${}^T\xi|_{B_i}$ *is equivalent to one of the fibered saucer shells* ${}^T\eta_p$, $p=1,\dots,L$.

Proof. First, we can choose $\overline{K}^s\colon\Delta\to\mathbb{R}$, $s\in[0,1]$, so that $\overline{K}^{\|\tau\|}\leqslant K^\tau$ everywhere, $\overline{K}^1>0$, and $\overline{K}^s|_{\partial\Delta}>0$ for all $s\in[0,1]$. Therefore, it suffices to prove the proposition for the family $\overline{K}^{\|\tau\|}$. We may also assume that $\overline{K}^0(x)\leqslant\overline{K}^s(x)$ for any $x\in\Delta$ and $s\in[0,1]$.

Let $U=\operatorname{Int}\Delta$ and $U'\Subset U$ be a star-shaped subset such that $\overline{K}^{\|\tau\|}\big|_{U\setminus U'}>0$ for all $\tau\in T$. We also choose $\delta>0$ so that $\overline{K}^s>0$ for all $s\in[1-\delta,1]$. Lemma 8.6 applied to $\overline{K}^0$ provides an integer $N>0$ and a finite set $\Lambda=\{g_1,\dots,g_k\}\subset\widehat{\Theta}_N$ such that the function

$$\Phi=\Phi^{S^1,N}=\frac{\mu}{N}+\sum_{g\in\Lambda}\phi_{g,N}\colon\Delta\times S^1\longrightarrow\mathbb{R}$$

satisfies $\Phi(w,t)<\overline{K}^s(w)$ for $w\in\Delta\setminus U'$ and $\Phi|_{U'\times S^1}<-\mu/N$. Choosing N large enough, we may also arrange that

$$\min_{\|\tau\|\geqslant 1-\delta}\overline{K}^\tau>\Psi=\Psi^{S^1,N}=\frac{\mu}{N}+\sum_{g\in\Lambda}\phi_{+,g,N}=\frac{\mu}{N}+\frac{1}{N}\sum_{g\in\Lambda}(\phi_+\circ C_N\circ g^{-1}).$$

According to Proposition 4.10, there exists a family of functions $\widetilde{K}^s$, $s\in[0,1]$, such that

- $\widetilde{K}^s=\overline{K}^s$ on $\Delta\setminus U'$ and $\widetilde{K}^s>-\mu/N$ for $s\in[0,1]$;
- $\widetilde{K}^s=\overline{K}^s$ for $s\in[1-\delta,1]$;
- there exists a family of subordination maps $h^s:\eta_{\overline{K}^s}\to\eta_{\widetilde{K}^s}$ which are identity maps for $s\in[1-\delta,1]$.

In particular, $\Phi(w,t)<\widetilde{K}^s(w)$ for all $w\in\Delta$, $t\in S^1$, and $s\in[0,1]$.

Recall the notation $\phi^s=\phi_+-s\phi_-$, $\phi^s_{g_j,N}=(\phi^s/N)\circ C_N\circ g_i^{-1}$ from §8.1. Choose a diffeomorphism $f:[0,1]\to[0,1]$ such that $f\big(1-\tfrac{1}{2}\delta\big)=\tfrac{2}{3}$ and $f(1-\delta)=\tfrac{1}{3}$. Then the function $\tilde{\theta}:=\theta\circ f$ satisfies $\tilde{\theta}|_{[0,1-\delta]}=1$ and $\tilde{\theta}|_{[1-\delta/2,1]}=0$.

We define the families of functions, for $s\in[0,1]$ and $i=1,\dots k$,

$$\Phi^s_i:=\frac{\mu}{N}+\sum_{j=1}^{i}(\phi^{\tilde{\theta}(s)}_{g_j,N}):\Delta\times S^1\longrightarrow\mathbb{R},$$

so we have $\Phi^s_0=1/N$ for $s\in[0,1]$, $\Phi^s_{|\Lambda|}=\Phi$ for $s\leqslant 1-\delta$, and $\Phi^1_{|\Lambda|}=\Psi$. Here $|\Lambda|$ is the cardinality of Λ.

The function $\Phi^s:=\Phi^s_k$ for each $s\in[0,1]$ is equal to $1/N>0$ near $\partial\Delta\times S^1$, and it satisfies the inequality $\Phi^s<\widetilde{K}^s$. Indeed, for $s\in[0,1-\delta]$, we have $\Phi^s=\Phi<\widetilde{K}^s$, and for $s\in[1-\delta,1]$ we have $\Phi^s<\Psi<\overline{K}^u=\widetilde{K}^s$. Therefore, the family of circle model shells $\eta_{\Phi^{\|\tau\|}}$ is dominated by $\eta_{\widetilde{K}^{\|\tau\|}}$, and hence it is sufficient to prove the required extension result for the family $\eta_{\Phi^{\|\tau\|}}$.

The families of model shells $\eta_{\Phi_i^{\|\tau\|}}$ and $\eta_{\Phi_{i-1}^{\|\tau\|}}$, $\tau\in T$, differ by one of the regular saucer families $(B_p,\zeta_p^{\tilde{\theta}(\|\tau\|)})$, $p=1,\dots,L=2^m$, from the finite list provided by Lemma 8.5. The shell $\eta_{\Phi_0^{\|\tau\|}}$ is solid for all $\tau\in T$, since $\Phi_0^{\|\tau\|}>0$ everywhere. Similarly, the saucers $(B_p,\zeta_p^{\|\tau\|})$ for $\tau\in\mathcal{O}p\,\partial T$ are solid for $\tau\in\mathcal{O}p\,\partial T$, because we have $\Phi_j^{\|\tau\|}\geqslant\Phi_{j-1}^{\|\tau\|}$ for all $j=1,\dots,|\Lambda|$. But the fibered saucer corresponding to the family $(B_p,\zeta_p^{\tilde{\theta}(\|\tau\|)})$ is equivalent to ${}^T\eta_p$, $p=1,\dots,L$. □

Proof of Proposition 3.11. Proposition 7.6 allows us to assume that ${}^T\xi_0$ is fibered contact outside of a finite collection of disjoint saucers $\{B_i\}_{i=1}^N$, so that the restriction $\xi_0|_{B_i}$, for each $i=1,\dots,N$, is a fibered regular semi-contact saucer. Applying Proposition 8.9, we further reduce the holes to a finite list of fibered saucers ${}^T\eta_p$, $p=1,\dots,L=2^m$.

Proposition 6.13 allows us to replace each saucer ${}^T\eta_p$, $p=1, ..., L$, by a fibered circle model shell ${}^T\eta_{K_p}$ defined by a family of time-independent contact Hamiltonian $K_p^\tau, \tau\in T$. But then, using Lemma 8.7, we conclude that each circle model shell ${}^T\eta_{K_p}$ dominates the fibered circle model ${}^T\eta_{K_{\mathrm{univ}}}:=\{\eta_{K_{\mathrm{univ}}^{(\theta(\|\tau\|))}}\}_{\tau\in T}$. □

9. Leafwise contact structures

Theorem 1.5 follows from Theorem 1.6 because any leafwise almost contact structure is homotopic to a structure from $\mathfrak{cont}_{\mathrm{ot}}(\mathcal{F}; h_1, ..., h_N)$ for an appropriate choice of embeddings $h_1, ..., h_N$. Hence, it is sufficient to prove Theorem 1.6.

We begin with the following lemma, which we already used in §3.6 in the proof of Theorem 3.13.

LEMMA 9.1. *Let U be a connected manifold of dimension $m>1$, T be a compact contractible set, and $T_1, ..., T_k\subset T$ be its compact subsets such that*

($\star$) *any intersection $T_{i_1}\cap...\cap T_{i_p}$ for $1\leqslant i_1<...<i_p\leqslant k$ is either empty or contractible.*

Let B be a closed m-dimensional ball with a given point $p\in\partial B$, $S_j: T_j\times B\to T_j\times U$, $S_j(\tau, x)=(\tau, s_j(\tau, x))$, and $S_\pm: T\times B\to T\times U$, $S_\pm(\tau, x)=(\tau, s_\pm(\tau, x))$, be pairwise disjoint fiberwise smooth embeddings. Then there exists a fiberwise embedding $S: T\times[-1, 1]\to T\times U$ such that

(i) $S(\tau, \pm1)=S_\pm(\tau, p)$, $\tau\in T$;

(ii) $S(T\times[-1, 1])\cap\bigcup_{j=1}^k S_j(T_j\times B)=\varnothing$;

(iii) $S(T\times(-1, 1))\cap(S_-(T\times B)\cup S_+(T\times B))=\varnothing$.

Proof. We will prove the statement by induction on k. When $k=0$ the statement follows from the fact that the space of maps of the contractible set T into the space of pairs of disjoint embeddings of B into U is connected, and hence by a fiberwise isotopy we may assume that the embeddings $s_\pm(\tau, \cdot): B\to U$ are independent of τ, i.e. $s_\pm(\tau, x)=\tilde{s}_\pm(x)$ for all $(\tau, x)\in T\times B$. Then to construct the required embedding it is sufficient to connect the points $\tilde{s}_\pm(p)$ by an embedded arc in U which does not intersect the balls $\tilde{s}_\pm(B)$ in their interior points.

Suppose that the statement is already proven for $k=j$ (and any U). Suppose first that one of the $k=j+1$ sets $T_1, ..., T_k$, say T_k, coincides with T. By a fiberwise isotopy we can make the embedding $s_k(\tau, \cdot): B\to U$ independent of τ, i.e. $s_k(\tau, x)=\tilde{s}_k(x)$ for all $(\tau, x)\in T\times B$. Therefore, the statement reduces to the case of $k-1=j$ sets $T_1, ..., T_j$ and their embeddings into $\widetilde{U}=U\setminus\tilde{s}_k(B)$, which is connected as well.

Consider now the general case. By an argument as above, we may assume that the embeddings $s_+(\tau, \cdot)$ are independent of τ, i.e. $s_+(\tau, x)=\hat{s}_+(x)$ for all $(\tau, x)\in T\times B$.

Let $\widehat{U}:=U\setminus\hat{s}_+(B)$. Suppose that T_k is a proper subset of T. Set $\widehat{T}:=T_k$, $\widehat{T}_i:=T_i\cap T_k$ and $\widehat{S}_i:=S_i|_{\widetilde{T}_i\times B}$, $i=1,...,k-1$, $\widehat{S}_-:=S_-|_{T_k\times B}$, and $\widehat{S}_+:=S_k|_{T_k\times B}$. Note that the sets $\widehat{T}_i$, $i=1,...,k-1$, and $\widehat{T}$ satisfy the condition $(\star)$.

Considering $\widehat{S}_i$ as embeddings into $\widehat{T}_i\times\widehat{U}$, and $\widehat{S}_\pm$ as embeddings into $\widehat{T}\times\widehat{U}$, we may apply the induction hypothesis to construct a fiberwise embedding $\widehat{S}:\widehat{T}\times[-1,1]\to\widehat{T}\times\widehat{U}$ such that

– $\widehat{S}(\tau,\pm1)=S_\pm(\tau,\hat{p})$, $\tau\in\widehat{T}$, where $\hat{p}\in\partial B$ is a point different from p;
– $\widehat{S}(\widehat{T}\times[-1,1])\cap\bigcup_{j=1}^{k-1}\widehat{S}_j(\widehat{T}_j\times B)=\varnothing$;
– $\widehat{S}(\widehat{T}\times(-1,1))\cap(\widehat{S}_-(\widehat{T}\times B)\cup\widehat{S}_+(\widehat{T}\times B))=\varnothing$.

Using the embedding $\widehat{S}$, we can make a fiberwise connected sum of the embeddings $\widehat{S}_\pm$ to construct a fiberwise embedding $\widetilde{S}_-:T\times B\to T\times\widehat{U}$ with the following properties:

- $\widetilde{S}_-(T\times B)\cap\bigcup_{j=1}^{k-1}S_j(T_j\times B)=\varnothing$;
- $\widetilde{S}_-(T\times B)\supset S_-(T\times B)\cup S_k(T_k\times B)$;
- the embeddings $\widetilde{S}_-$ and S_- coincide near $T\times\{p\}\subset T\times\partial B$.

Hence, by applying again the induction hypothesis to the embeddings $\widetilde{S}_-$, S, and $S_j, j=1,...,k-1$, we may construct a fiberwise embedding $S:T\times[-1,1]\to T\times U$ with the required properties. □

Proof of Theorem 1.6. Let T be an m-ball. We need to prove that any map

$$(T,\partial T)\longrightarrow(\mathfrak{cont}_{\mathrm{ot}}(\mathcal{F};h_1,...,h_N),\mathfrak{Cont}_{\mathrm{ot}}(\mathcal{F};h_1,...,h_N))$$

is homotopic relative to ∂T to a map into $\mathfrak{Cont}_{\mathrm{ot}}(\mathcal{F};h_1,...,h_N)$. In other words, let $\xi_\tau\in\mathfrak{cont}_{\mathrm{ot}}(\mathcal{F};h_1,...,h_N)$, $\tau\in T$, be a family of leafwise almost contact structures which are genuine leafwise contact structures for $\tau\in\partial T$. We will construct a homotopy relative to ∂T to a family of genuine leafwise contact structures $\tilde{\xi}_\tau,\tau\in T$.

Consider a foliation $\widehat{\mathcal{F}}$ on $T\times V$ with leaves $\{\tau\}\times L$, where $\tau\in T$ and L is a leaf of $\mathcal{F}$. Let $\hat{h}_j:T\times T_j\times B\to T\times V$ be the embeddings given by

$$\hat{h}_j(\tau,\tau',x)=(\tau,h_j(\tau',x),(\tau,\tau',x))\in T\times T_j\times B,\quad j=1,...,N.$$

Note that the family $\xi_\tau,\tau\in T$, can be viewed as a leafwise almost contact structure Ξ from $\mathfrak{cont}(\widehat{\mathcal{F}};\hat{h}_1,...,\hat{h}_N)$, which is genuine on leaves $\{\tau\}\times L$ for $\tau\in\partial T$. Moreover, we may assume that Ξ is a genuine leafwise contact structure on a neighborhood $U\supset\partial T\times V$ and neighborhoods $U_j\supset\hat{h}_j(T\times T_j\times B)$, $j=1,...,N$.

There exists a triangulation $\mathcal{T}$ of $T\times V$ with the following properties:

- there are compact *subcomplexes* $\widehat{U}$ and $\widehat{U}_j$, $j=1,...,N$, of the triangulation $\mathcal{T}$ such that $\partial T\times V\subset\widehat{U}\subset U$ and $\hat{h}_j(T\times T_j\times B)\subset\widehat{U}_j\subset U_j$, $j=1,...,N$;
- the restriction $\mathcal{T}_0$ of the triangulation $\mathcal{T}$ to $(T\times V)\setminus\mathrm{Int}\big(\widehat{U}\cup\bigcup_{j=1}^N\widehat{U}_j\big)$ is transverse to the foliation $\widehat{\mathcal{F}}$;

- for every top-dimensional simplex σ of $\mathcal{T}_0$ there is a submersion $\pi_\sigma\colon \operatorname{Int}\sigma \to B^{q+m}$ which is a fibration over an open $(q+m)$-ball with the ball fibers, and such that the pre-images $\pi_\sigma^{-1}(s)$, $s\in B^{q+m}$, are intersections of the leaves of $\widetilde{\mathcal{F}}$ with $\operatorname{Int}\sigma$.

Applying Gromov's parametric h-principle for contact structures on open manifolds (see [28] and Theorem 7.1) inductively over skeleta of the triangulation, we can deform Ξ, keeping it fixed on $\widehat{U}\cup\bigcup_{j=1}^N \widehat{U}_j$ to make it a leafwise genuinely contact in a neighborhood of the codimension-1 skeleton of the triangulation $\mathcal{T}_0$.

Our next goal is to further deform Ξ on each top-dimensional simplex σ of the triangulation $\mathcal{T}_0$, keeping it fixed on $\mathcal{O}p\,\partial\sigma$, to make it a leafwise genuine contact structure on σ. Let us choose one of such simplices. There exists a compact subset $\bar\sigma\subset\operatorname{Int}\sigma$ such that the leafwise almost contact structure Ξ is genuine on $\mathcal{O}p(\sigma\setminus\operatorname{Int}\bar\sigma)$ and $\pi_\sigma|_{\bar\sigma}$ is a fibration over a closed $(m+q)$-ball X with fibers diffeomorphic to a closed $(2n+1)$-ball.

Hence, $\Xi|_{\bar\sigma}$ can be viewed as a fibered-over-X almost contact structure on $\bar\sigma$, and applying Proposition 7.6 we can further deform Ξ keeping it fixed on $\mathcal{O}p\,\partial\sigma$, to make it genuine away from a finite number of disjoint domains Z_i fibered over $X_i\subset X$ with piecewise smooth boundary, $i=1,\dots,K$. These domains are not necessarily disjoint but could be chosen arbitrarily small and in such a way that all non-empty intersections $X_{i_1}\cap\dots\cap X_{i_k}$, $1\leqslant i_1<\dots<i_k\leqslant K$, are again balls with piecewise smooth boundaries. Let $Y\subset\bar\sigma$ and $Y_i\subset Z_i$ be subfibrations of the fibrations $\bar\sigma\to X$ and $Z_i\to X_i$, $i=1,\dots,K$, formed by boundaries of the corresponding ball-fibers.

Next, we use Lemma 9.1 to construct for each X_j a fiberwise embedding $S_j\colon X_j\times[0,1]\to\overline{Z}_j\setminus\bigcup_{i\neq j}Z_i$ with $S_j(\tau,0)\in Y_j$ and $S_j(\tau,1)\in Y$. Recall that by assumption every point $(\tau,x)\in Y$ can be connected to a point on the boundary of one of the overtwisted balls $B_{i,\tau,\tau'}:=h_i(\{\tau\}\times\{\tau'\}\times B_i)$, $i=1,\dots,N$ and $\tau\in T$, by an embedded path in the corresponding leaf. This path can be chosen inside an arbitrarily small neighborhood of the codimension-1 skeleton of the triangulation $\mathcal{T}_0$. Hence, if the sets X_j are chosen sufficiently small, we can extend each of the embeddings S_j to a leafwise embedding $\widetilde{S}_j\colon X_j\times[0,2]\to V$ such that

- $S_j(\tau,0)\in Y_j$;
- $S_j(\tau,2)\in h_i(T_i\times\partial B)$ for some $i=i(\sigma,j)$.

Moreover, using Proposition 3.9 to increase the number of embeddings h_i, we can additionally arrange that the map $(\sigma,j)\mapsto i(\sigma,j)$ is injective. Then, successively applying Theorem 3.13 to neighborhoods of $Z_j\cup S_j(X_j\times[0,2])\cup\bigcup_{\tau\in X_j}h_{i(\sigma,j)}(S(\tau,2)\times B)$ for all top-dimensional simplices σ of the triangulation, we deform Ξ to make it a leafwise genuinely contact on these neighborhoods. $\square$

10. The overtwisted contact structures. Discussion

Recall the definition of an overtwisted contact structure from §3.2: a contact structure ξ on a manifold M is called *overtwisted* if there is a contact embedding $(D_{\mathrm{ot}},\xi_{\mathrm{ot}})\to(M,\xi)$; see §3.2. We note that the disc D_{ot} is only *piecewise smooth*. We do not know if it is possible to characterize overtwisted structures in dimension greater than 3 by existence of a *smooth* overtwisted disc.

In the 3-dimensional case a contact structure which is overtwisted in our sense is also overtwisted in the sense of [12]. This should be clear from the picture of the characteristic foliation on the disc D_{ot}; see Figure 3.1. The converse is also true. This can be seen directly by finding a copy of $(D_{\mathrm{ot}},\xi_{\mathrm{ot}})$ in a neighborhood of the traditional overtwisted disc, or indirectly, from the classification theorem from [12]. Indeed, one can first find a contact structure on the ball with standard boundary which contains $(D_{\mathrm{ot}},\xi_{\mathrm{ot}})$ and which is in the standard almost contact class. Then, implanting this ball in an overtwisted contact manifold does not change the isotopy class of this structure.

Overtwisting and plastikstufes

As it was already mentioned in the introduction, an overtwisted contact manifold contains a plastikstufe, see [40]. Let us recall that given a smooth closed $(n-1)$-dimensional manifold Q, the *model plastikstufe with core* Q is the contact germ of the $(n+1)$-dimensional manifold $(\mathcal{P}_Q,\zeta)\subset(\mathbb{R}^3_{\mathrm{ot}}\times T^*Q,\ker(\alpha_{\mathrm{ot}}+\lambda_{T^*Q}))$, where $\mathcal{P}_Q:=D^2_{\mathrm{ot}}\times Q_0$ is the product of an overtwisted disc D^2_{ot} and the zero section.

Corollary 10.1. *Let (M^{2n+1},ξ) be an overtwisted contact manifold. If the complexified tangent bundle $TQ\otimes_{\mathbb{R}}\mathbb{C}$ is trivial, then there is a contact embedding $(\mathcal{P}_Q,\zeta)\to(M^{2n+1},\xi)$ of the contact germ of the model plastikstufe with core Q.*

Proof. The contractibility of $\mathbb{R}^3$ implies the existence of a contact bundle isomorphism $\Phi\colon(T\mathbb{R}^3,\xi_{\mathrm{ot}})\to(T\mathbb{R}^3,\xi_{\mathrm{st}})$ that covers the identity map on $\mathbb{R}^3$, is homotopic through bundle isomorphisms to the identity, and respects the conformal symplectic structures. By extending by the identity, we get a similar contact bundle map

$$\Psi\colon(T\mathbb{R}^3\times T^*Q,\ker(\alpha_{\mathrm{ot}}+\lambda_{T^*Q}))\longrightarrow(T\mathbb{R}^3\times T^*Q,\ker(\alpha_{\mathrm{st}}+\lambda_{T^*Q})).$$

On the other hand, the triviality of the complexified tangent bundle of Q implies that there is a contact bundle homomorphism $\Phi\colon(T\mathbb{R}^3\times T^*Q,\ker(\alpha_{\mathrm{st}}+\lambda_{T^*Q}))\to(T\mathbb{R}^{2n+1},\xi_{\mathrm{st}})$. Combining $\Phi\circ\Psi$ with an inclusion $(\mathbb{R}^{2n+1},\xi_{\mathrm{st}})$ as a Darboux chart into (M,ξ), we get a contact bundle homomorphism $\Psi\colon(T\mathbb{R}^3\times T^*Q,\ker(\alpha_{\mathrm{ot}}+\lambda_{T^*Q}))\to(TM,\xi)$, and hence Corollary 1.4 provides us with a contact embedding $(\mathcal{P}_Q,\zeta)\to(M^{2n+1},\xi)$. □

Changing $\boldsymbol{\Delta}_{\mathbf{cyl}}$

The definition of an overtwisted disc depends on the choice of the special contact Hamiltonian $K_\varepsilon: \Delta_{\mathrm{cyl}} \to \mathbb{R}$, where $\Delta_{\mathrm{cyl}} = D \times [-1,1]$, where D is the unit ball in $\mathbb{R}^{2n-2}$. Suppose that $\widetilde{D} \subset \mathbb{R}^{2n-2}$ is any other star-shaped domain with a piecewise smooth boundary. Let

$$\begin{aligned}
\tilde{\Delta}_{\mathrm{cyl}} &:= \{(q,z) \in \mathbb{R}^{2n-2} \times \mathbb{R} : q \in \widetilde{D} \text{ and } |z| \leqslant 1\},\\
\Delta^-_{\mathrm{cyl}} &:= \Delta_{\mathrm{cyl}} \cap \{(q,z) : z \leqslant 0\},\\
\tilde{\Delta}^-_{\mathrm{cyl}} &:= \tilde{\Delta}_{\mathrm{cyl}} \cap \{(q,z) : z \leqslant 0\}.
\end{aligned}$$

Let C_+ be the space of continuous piecewise-smooth functions $\tilde{\Delta}^-_{\mathrm{cyl}} \to \mathbb{R}$ which are positive on $\mathcal{O}p\, \partial \tilde{\Delta}_{\mathrm{cyl}} \cap \tilde{\Delta}^-_{\mathrm{cyl}}$.

Given two functions $K_\pm \in C_+$ such that $K_- < K_+$, we let

$$\begin{aligned}
U_{K_-,K_+} &= \{(x,v,t) : K_-(x,t) \leqslant v \leqslant K_+(x,t), z(x) \leqslant 0\} \subset (\tilde{\Delta}^-_{\mathrm{cyl}} \times T^*S^1, \ker(\lambda_{\mathrm{st}} + v\, dt)),\\
\Sigma_{K_+} &= \{(x,v,t) : 0 \leqslant v \leqslant K_+(x,t), x \in \partial\Delta, z(x) \leqslant 0\} \subset (\Delta \times \mathbb{R}^2, \ker(\lambda_{\mathrm{st}} + v\, dt)).
\end{aligned}$$

Gluing these pieces together via the natural identification between their common parts, we define $\widehat{U}_{K_-,K_+} := U_{K_-,K_+} \cup \Sigma_{K_+}$.

LEMMA 10.2. *For any $K_+ \in C_+(\tilde{\Delta})$ there exists $K_- \in C_+(\tilde{\Delta})$ such that $K_- < K_+$ and $\widehat{U}_{K_-,K_+}$ is overtwisted.*

Proof. Choosing a representative η of the contact germ along Σ_{K_+}, let U be a neighborhood of $\partial \tilde{\Delta}_{\mathrm{cyl}}$ such that $K|_{U \cap \tilde{\Delta}^-_{\mathrm{cyl}}} > 0$ and the contact structure η is defined on $\{(x,v,t) : x \in U, z(x) \geqslant 0, \text{ and } v \leqslant K_+(x)\}$. There is a contact embedding $\Phi: \Delta^-_{\mathrm{cyl}} := \tilde{\Delta}^-_{\mathrm{cyl}}$ such that $\Phi(\partial\Delta_{\mathrm{cyl}} \cap \Delta^-_{\mathrm{cyl}}) \subset U$ and $\Phi(\Delta^-_{\mathrm{cyl}} \cap \{z=0\}) \subset \{z=0\}$. Indeed, the contact vector field

$$Z = L + z \frac{\partial}{\partial z}, \quad \text{where } L = \sum_{i=1}^{n-1} u_i \frac{\partial}{\partial u_i},$$

is given by the contact Hamiltonian z with respect to the standard contact form

$$\lambda_{\mathrm{st}} = dz + \sum_{i=1}^{n-1} u_i\, d\phi_i.$$

Consider a cut-off function $\sigma: \mathbb{R}^{2n-1} \to \mathbb{R}_+$ which is equal to 1 on $\tilde{\Delta}_{\mathrm{cyl}} \setminus U$ and supported in $\operatorname{Int} \tilde{\Delta}_{\mathrm{cyl}}$, and let $\widetilde{Z}$ denote the contact vector field defined by the contact Hamiltonian $K := z\sigma$. Let us observe that $\widetilde{Z}$ is tangent to the hyperplane $\{z=0\}$ because K vanishes on this hyperplane. Let Z^t and $\widetilde{Z}^t$ be the contact flows generated by Z and $\widetilde{Z}$. Then the formula $\Phi:=\widetilde{Z}^{\widetilde{C}}\circ Z^{-C}$ is the required contact embedding for appropriately chosen positive constants C and $\widetilde{C}$.

For an appropriate choice of a special Hamiltonian $K<K_{\text{univ}}$ we have $\Phi_*K<K_+$. On the other hand, there exists $K_-\in C_+$ such that $\Phi_*K>K_-$. Hence, the overtwisted disc $D_{\text{ot}}=D_K$ embeds into an arbitrarily small neighborhood of $\widehat{U}_{K_-,K_+}$, i.e. $\widehat{U}_{K_-,K_+}$ is overtwisted. □

Wrinkles and overtwisting

Consider the standard contact manifold $(\mathbb{R}^{2n+1},\xi_{\text{st}})$, where

$$\xi_{\text{st}}=\left\{dz+\sum_{i=1}^{n-1}u_i\,d\varphi_i-y_n\,dx_n=0\right\}.$$

Let B denote the unit ball in $\mathbb{R}^{2n+1}$ and $w\colon B\to\mathbb{R}^{2n+1}$ be the *standard wrinkle* (see [16]), i.e. a map given by the formula

$$(v,y_n)\longmapsto(v,y_n^3-3\alpha(r)y_n),\quad \text{where } v\in\mathbb{R}^{2n},$$

$r:=\|v\|^2$, and $\alpha\colon[0,1]\to\mathbb{R}$ is a C^∞-function which is positive on $\left(\frac14,\frac34\right)$, negative on $\left(\frac34,1\right]$, constant near 0, has a negative derivative at $\frac34$, and satisfies the inequality $\alpha(r)\leqslant 1-r^2$.

Let $W:=\{(v,y_n):y_n^2\leqslant\alpha(r)\}$ and $U\subset B$ be a neighborhood of W. Corollary 1.4 allows us to construct a contact embedding of $(U\setminus W,(w^*\xi_{\text{st}})|_{U\setminus W})$ into any overtwisted contact manifold of the same dimension. One can also show, though we do not know a simple proof of this fact, that $(U\setminus W,(w^*\xi_{\text{st}})|_{U\setminus W})$ contains an overtwisted disc. Hence, *a contact structure ξ on a manifold M is overtwisted if and only if there exist a neighborhood $U\supset W$ in B and a contact embedding $(U\setminus W,(w^*\xi_{\text{st}})|_{U\setminus W})\to(M,\xi)$.*

Stabilization of overtwisted contact manifolds

Given a contact manifold (Y,ξ) with a contact form λ, its *stabilization* is the manifold $Y^{\text{stab}}:=Y\times\mathbb{R}^2$ with the contact structure $\xi^{\text{stab}}:=\{\lambda+v\,d\phi=0\}$. It is straightforward to check that, up to a canonical contactomorphism, the contact manifold $(Y^{\text{stab}},\xi^{\text{stab}})$ is independent of the choice of the contact form λ.

After the first version of the current paper was posted on the `arXiv`, R. Casals, E. Murphy, and F. Presas observed that stabilization preserves the following overtwisting property.

Theorem 10.3. ([4]) *The stabilization $(Y^{\text{stab}},\xi^{\text{stab}})$ of every overtwisted contact manifold (Y,ξ) is overtwisted.*

In particular, this implies that an overtwisted contact manifold of dimension $2n+1$ can be equivalently defined as a contact manifold containing a neighborhood of the standard 3-dimensional overtwisted disc stabilized $n-1$ times.

Note that Theorem 10.3 also implies the following result.

COROLLARY 10.4. *For any overtwisted contact manifold* $(M, \{\lambda=0\})$, *the contact manifold* $(M\times T^*S^1, \{\lambda+v\,dt=0\})$ *is overtwisted. Moreover,*(2)

$$M\times T^*_+S^1 := (M\times T^*S^1)\cap\{v>0\}$$

is overtwisted as well.

Proof. $(M\times T^*_+S^1, \{\lambda+v\,dt=0\})$ is contactomorphic to

$$(M\times(\mathbb{R}^2\setminus\{0\}), \{\lambda+x\,dy-y\,dx=0\}).$$

On the other hand, there exists a contact embedding

$$(M\times D^2_R, \{\lambda+x\,dy-y\,dx=0\}) \longrightarrow (M\times(\mathbb{R}^2\setminus\{0\}), \{\lambda+x\,dy-y\,dx=0\}).$$

It can be defined, for instance, by the formula $(w,x,y)\mapsto(\mathfrak{R}^{-2Ry}(w), x+2R, y)$, where $\mathfrak{R}^t$ is the Reeb flow of the contact form λ. But according to Theorem 10.3 the product $(M\times D^2_R, \{\lambda+x\,dy-y\,dx=0\})$ is overtwisted if the radius of the 2-disc D^2_R is sufficiently large, and the claim follows. □

We refer the reader to [4] for further discussion of equivalent definitions of overtwisting.

Overtwisting and (non)-orderability

In [17] a relation $\leqslant$ on the universal cover $\widetilde{\mathfrak{Cont}}(Y,\xi)$ of the identity component of the group of contactomorphisms of (Y,ξ) was introduced. Namely, $f\leqslant g$ for $f,g\in\widetilde{\mathfrak{Cont}}(Y,\xi)$ if there is a path in $\widetilde{\mathfrak{Cont}}(Y,\xi)$ connecting f to g which is generated by a non-negative contact Hamiltonian. This relation is either trivial (e.g. in the case of the standard contact sphere of dimension >1), see [15], and in this case the contact manifold (Y,ξ) is called *non-orderable*, or it is a genuine partial order, e.g. in the case of $\mathbb{R}P^{2n-1}$ (see [26]) or the unit cotangent bundle $UT^*(M)$ of a closed manifold M; see [15] and [10].

Given a contact manifold (Y,ξ) with a fixed contact form λ, consider the contact manifold $(Y\times T^*S^1, \{\lambda+v\,dt=0\})$, where $S^1=\mathbb{R}/\mathbb{Z}$. Let $f_K\in\widetilde{\mathfrak{Cont}}(Y,\xi)$ be generated by

(2) Klaus Niederkrüger conjectured in [41] that this could be an appropriate definition for overtwistedness in higher dimensions.

a time-dependent contact Hamiltonian $K_t: Y\to\mathbb{R}$, which can be assumed 1-periodic in t. We consider the domain

$$V^+(f_K)=\{(x,v,t): v+K_t(x)\geqslant 0 \text{ and } x\in Y\}\subset Y\times T^*S^1.$$

In [17] it was proved that if $f\leqslant g$ then there exists a contact isotopy

$$h_t: Y\times T^*S^1\longrightarrow Y\times T^*S^1 \quad \text{such that } h_0=\mathrm{Id} \text{ and } h_1(V^+(f))\subset V^+(g). \tag{46}$$

However, it is not known whether the converse is true. Thus, it seems natural to introduce a weaker relation: *we say that* $f\lessapprox g$ *if there exists an isotopy* h_t *as in* (46). As $f\leqslant g$ implies $f\lessapprox g$, it follows that if a contact manifold is not orderable then it is not $\lessapprox$-orderable. The converse is not known, but in all cases known to us where orderability has been proved, one can also prove $\lessapprox$-orderability.

It has been a longstanding open question if closed overtwisted contact manifolds are orderable or not (see [8] and [7] for partial results in this direction). We have the following weak answer to this question.

THEOREM 10.5. *Every closed overtwisted contact manifold is not* $\lessapprox$*-orderable.*

Proof. According to Corollary 10.4, for a sufficiently large contact Hamiltonian $K>0$, the domain $V^+(f_K)\subset Y\times T^*S^1$ is overtwisted. Hence Corollary 1.4 allows us to construct a contact isotopy of $V^+(f_{2K})$ into $V^+(f_K)$ inside $Y\times T^*S^1$. This isotopy extends to a global contact isotopy, and hence $f_{2K}\lessapprox f_K$. Since we clearly have $f_K\lessapprox f_{2K}$, it follows that the order $\lessapprox$ is trivial. ☐

Classification of overtwisted contact structures on spheres

We will finish the paper by discussing the classification of overtwisted contact structures on S^{2n+1} explicitly. Almost contact structures on the sphere S^{2n+1} are classified by the homotopy group $\pi_{2n+1}(\mathrm{SO}(2n+2)/U(n+1))$. The following lemma gives this group.(3)

LEMMA 10.6. (Harris [32])

$$\pi_{2n+1}(\mathrm{SO}(2n+2)/U(n+1))=\begin{cases}\mathbb{Z}/n!\mathbb{Z}, & \textit{if } n=4k,\\ \mathbb{Z}, & \textit{if } n=4k+1,\\ \mathbb{Z}/\tfrac{1}{2}n!\mathbb{Z}, & \textit{if } n=4k+2,\\ \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}, & \textit{if } n=4k+3.\end{cases}$$

(3) We thank Søren Galatius for providing this reference.

Thus, Corollary 1.3 implies that on spheres S^{8k+1}, $k>0$, there are exactly $(4k)!$ different overtwisted contact structures, on spheres S^{8k+5}, $k\geqslant 0$, there are $\frac{1}{2}(4k+2)!$ different overtwisted contact structures, while on all other spheres there are infinitely many. In particular, there is a unique overtwisted contact structure on S^5.

It is interesting to note that S^5 has infinitely many *tight*, i.e. non-overtwisted, contact structures. Besides the standard contact structure, these are examples given by Brieskorn spheres (see [46]). In fact, it is shown in [35] that the monoid of contact structures on S^5 (under connected sum) is infinitely generated, as it admits a homomorphism onto $\mathbb{Q}$. The full classification of tight contact structures on any manifold of dimension greater than 3 is an open problem.

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