

EXISTENCE AND COMPARISON THEOREMS
FOR DIFFERENTIAL EQUATIONS IN BANACH SPACES

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In our recent paper [3] we have studied the existence of maximal and minimal solutions to the IVP in a Banach space X .

$$(1) \quad u' = f(t, u), \quad u(0) = u_0,$$

where f maps $I \times K$ into X , with $I = [0, a] \subset \mathbb{R}$ and $K \subset X$ a cone. The essential hypotheses have been that f is quasimonotone with respect to K and that K and X have some natural properties. If such extremal solutions exist then it is trivial to prove the usual comparison theorems known from the finite-dimensional case. For example, if u is the minimal solution of (1) on some interval $[a, b] \subset I$ and if $w: [a, b] \rightarrow K$ satisfies $w' \geq f(t, w)$ and $w(0) \geq u_0$ then $w(t) \geq u(t)$ on $[a, b]$.

In the present paper we shall establish existence and comparison theorems for (1) without the hypothesis that f be quasimonotone, but under conditions which have been considered in case $X = \mathbb{R}^n$ in the classical paper of M. Müller [10] in 1926. This is not the first attempt to extend Müller's results to infinite dimensions, since recently P. Volkmann [12] tried to do this. We shall improve the existing results considerably.

I. NOTATIONS AND DEFINITIONS

X will always be a real Banach space. $K \subset X$ will always denote a cone, i.e. a closed convex subset such that $\lambda K \subset K$ for every $\lambda \geq 0$ and $K \cap (-K) = \{0\}$. We let $B_r(x_0)$ be the closed ball with center x_0 and radius r . For bounded subsets $B \subset X$ the ball measure of noncompactness $\beta(B)$ is defined as

$$\beta(B) = \inf\{r > 0: B \text{ can be covered by finitely many } B_r(x_i)\}.$$

Properties of β can be found in §2 of [2],[6].

X^* will be the dual of X and we let $K^* = \{x^* \in X^*: x^*(x) \geq 0 \text{ for all } x \in K\}$. By means of K a partial order \leq is defined as $x \leq y$ iff $y-x \in K$.

A sequence $(e_i) \subset X$ is said to be a (Schauder-) base for X if every $x \in X$ has an unique representation $x = \sum_{i \geq 1} x_i e_i$, the series being convergent in norm. Then there exist uniquely determined $e_i^* \in X^*$ such that $e_i^*(e_j) = \delta_{ij}$ and therefore $x_i = e_i^*(x)$. The cone $K = \left\{ \sum_{i \geq 1} x_i e_i: x_i \geq 0 \text{ for all } i \geq 1 \right\}$ will be called the standard cone (w.r. to (e_i)).

Let us also recall that K is said to be normal if there exists a $\delta > 0$ such that $x, y \in K \cap \{|z| = 1\}$ implies $|x+y| \geq \delta$. An equivalent condition is that there exists an $M > 0$ such that $0 \leq x \leq y$ implies $|x| \leq M|y|$ (take $\delta = \frac{1}{M}$).

II. MÜLLER'S THEOREM

For convenience let us state and prove Müller's Satz 13 in [10]. although our proof is only a little bit shorter than the original one.

Proposition 1. Let $I = [0, a]$; $v, w: I \rightarrow \mathbb{R}^n$ differentiable and $v(t) \leq w(t)$ in I , where \leq is w.r. to $K = \{x \in \mathbb{R}^n: x_i \geq 0 \text{ for all } i\}$. Let $D = \{(t, z): t \in I \text{ and } v(t) \leq z \leq w(t)\}$ and $f: D \rightarrow \mathbb{R}^n$ continuous. Suppose that hypothesis

$$(H) \quad \begin{cases} v'_i(t) \leq f_i(t, z) \text{ whenever } v(t) \leq z \leq w(t) \text{ and } z_i = v_i(t) \\ w'_i(t) \geq f_i(t, z) \text{ whenever } v(t) \leq z \leq w(t) \text{ and } z_i = w_i(t) \end{cases} \quad (i = 1, \dots, n).$$

holds and that $v(0) \leq u_0 \leq w(0)$. Then (1) has a solution on I .

Proof: Consider $P: I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $P_i(t, x) = \max\{v_i(t), \min\{z_i, w_i(t)\}\}$. Then $f(t, P(t, x))$ defines a continuous extension of f to $I \times \mathbb{R}^n$ which is also bounded since f is bounded on D . Therefore, $u' = f(t, P(t, u))$ has a solution u on I with $u(0) = u_0$. Let us show that u is in D and therefore a solution of (1). For $\epsilon > 0$ and $e = (1, \dots, 1)$ consider $w_\epsilon(t) = w(t) + \epsilon(1+t)e$ and $v_\epsilon(t) = v(t) - \epsilon(1+t)e$. We have $v_\epsilon(0) < u_0 < w_\epsilon(0)$. Suppose that $t_0 \in (0, a]$ is such that $v_\epsilon(t) < u(t) < w_\epsilon(t)$ in $[0, t_0)$ but for example $u_j(t_0) = w_{\epsilon j}(t_0)$. Then we have $v(t_0) \leq P(t_0, u(t_0)) \leq w(t_0)$ and $P_j(t_0, u(t_0)) = w_{\epsilon j}(t_0)$, hence, $w'_j(t_0) \geq f_j(t_0, P(t_0, u(t_0))) = u'_j(t_0)$, i.e. $u'_j(t_0) < w'_{\epsilon j}(t_0)$, contradicting $u_j(t) < w_{\epsilon j}(t)$ in $t < t_0$. Therefore, $v_\epsilon(t) < u(t) < w_\epsilon(t)$ in I . Now, $\epsilon \rightarrow 0$ yields $(t, u(t)) \in D$ on I .

q.e.d.

Remarks. From the proof it is obvious that v, w need not be differentiable, since inequalities with Dini-derivatives instead of derivatives are sufficient. The proof also shows that, in case $v(t) < w(t)$ on I , $v(t) < u(t) < w(t)$ provided that the first inequalities in (H) are strict and $v(0) < u(0) < w(0)$.

III. EXTENSIONS OF BANACH SPACES WITH A BASE

The infinite dimensional extensions of Proposition 1 are most natural in Banach spaces X with a base (e_i) , since we still have "coordinates" there. In this section K will always be the standard cone w.r. to (e_i) . We also use the projections $P_n: X \rightarrow \text{span}\{e_1, \dots, e_n\}$, defined by $P_n(\sum_{i \geq 1} x_i e_i) = \sum_{i \leq n} x_i e_i$, and the remainders $R_n x = x - P_n x$. Note that $x \in K$ implies $P_n x \in K$ and $R_n x \in K$, and $P_n x \rightarrow x$ for every $x \in X$, the convergence being uniform on compact sets. We need the following

Proposition 2. Let $I = [0, a] \subset \mathbb{R}$ and $v, w: I \rightarrow X$ continuous and such that $v(t) \leq w(t)$ on I . Suppose that K is normal. Then $D = \{(t, z): t \in I \text{ and } v(t) \leq z \leq w(t)\}$ is compact.

Proof: Evidently D is closed, and bounded since K is normal. Thus, we only have to prove that $R_n z_j \rightarrow 0$ as $n \rightarrow \infty$, uniformly with respect to j , whenever (t_j, z_j) is a sequence in D . Since v and w are continuous on I , we have $R_n v(t) \rightarrow 0$ and $R_n w(t) \rightarrow 0$ as $n \rightarrow \infty$, uniformly on I . Since K is normal, we therefore obtain

$$|R_n z_j| \leq M |R_n (w(t_j) - v(t_j))| + |R_n w(t_j)| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

uniformly in j .

q.e.d.

By means of Proposition 2 we can prove

Theorem 1. Let X be a Banach space with base (e_i) , K the standard cone w.r. to (e_i) and K be normal. Let $v, w: I \rightarrow X$ be differentiable, $v(t) \leq w(t)$ on I and $D = \{(t, z): t \in J \text{ and } v(t) \leq z \leq w(t)\}$. Let $f: D \rightarrow X$ be continuous and suppose that condition (H) from Proposition 1 holds, now for every $i \geq 1$. Suppose also that $(0, u_0) \in D$. Then (1) has a solution on I .

Proof: Consider the IVP

$$(1_n) \quad z' = f_n(t, z) = P_n f(t, z + R_n v(t)), \quad z(0) = P_n u_0$$

in $D_n = \{(t, z): t \in I \text{ and } P_n v(t) \leq z \leq P_n w(t)\}$. Evidently, f_n is continuous, and f_n is bounded since f , being continuous on the compact set D , is bounded. Furthermore, f_n satisfies hypothesis (H) of Proposition 1. Therefore, (1_n) has a solution z_n on I . Since $|z'_n(t)| \leq c$ for some $c > 0$, every $n \geq 1$ and every $t \in I$, (z_n) is equicontinuous. In order to be ready for the application of the Ascoli/Arzelà Lemma, we have to prove that $(z_n(t))$ is relatively compact, for every $t \in I$. Let $\psi(t) = \beta(\{z_n(t): n \geq 1\})$. Since $\beta(B) = 0$ iff B is relatively compact, we have to show $\psi(t) \equiv 0$. Obviously, $\psi(0) = 0$. Since X is separable, Theorem 1 of H. Mönch and (G. v. Harten [9] implies $\psi'(t) \leq \beta(\{z'_n(t): n \geq 1\})$ a.e. on I . By Proposition 7.2 in [2] and (1_n) , this implies

$$\psi'(t) \leq \lambda \beta(f(t, \{z_n(t) + R_n v(t) : n \geq 1\})) = 0 \text{ a.e., with } \lambda = \sup_n |P_n|.$$

Therefore, $\psi(t) \equiv 0$ on I and we are allowed to assume that $z_n(t) \rightarrow u(t)$ uniformly on I . Evidently, $v(t) \leq u(t) \leq w(t)$ and u is a solution of (1), since also $R_n v(t) \rightarrow 0$ uniformly on I .

q.e.d.

Remarks. (i) Obviously, it is again enough that v, w be continuous and that the inequalities in (H) are satisfied for Dini-derivatives.

(ii) Theorem 1 has been announced by Volkman [12] for the special cases (σ_0) and $\ell_p (1 \leq p < \infty)$ and has been proved for ℓ_p . The idea of that proof has been to show that (H) implies the boundary condition

$$(2) \lim_{\lambda \rightarrow 0} \text{dist}((t, x) + \lambda(1, f(t, x)), D) = 0 \text{ for } (t, x) \in D \text{ and } 0 \leq t < a,$$

which then allows the application of known existence results for differential equations on closed sets; see e.g. §4 of [2],[6]. This approach is very circumstantial and the true nature of the problem is hidden by tedious calculations in a special norm. If one is glad that with condition (H) one has found another example where (2) is satisfied then let us note that (2) is a trivial consequence in all cones covered by Theorem 1. In fact, what we have done for $(0, u_0)$ can be proved for every initial value $(s, x) \in D$ with $s < a$, and therefore the corresponding initial value problem has a solution u on $[s, a]$; since $u(s+\lambda) = x + \lambda f(s, x) + o(\lambda) \in D$, (2) is then obvious.

In case K is not normal, we are not sure whether D is compact or even only bounded and therefore we have to assume more than mere continuity about f .

Theorem 2. Let everything be as in Theorem 1, but instead of the normality of K assume that f satisfies the estimate $\beta(f(t,B)) \leq \omega(t,\beta(B))$ for every bounded $B \subset X$ such that $(t,B) \subset D$, where $\omega: I \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous and such that the initial value problem $\rho' = \lambda\omega(t,\rho)$, $\rho(0) = 0$ admits the trivial solution $\rho(t) \equiv 0$ only ($\lambda = \sup_n |P_n|$). Then (1) has a local solution.

Proof: Since $\{(t, z + R_n v(t)) : (t, z) \in D_n\}$ is compact, f_n is bounded and therefore (1_n) has a solution z_n on I . Since f is bounded near $(0, u_0)$, there exists an interval $[0, b] \subset I$ such that (z_n) is equicontinuous there, for n sufficiently large, say $n \geq n_0$. Then $\psi(t) = \beta(\{z_n(t) : n \geq n_0\})$ satisfies $\psi(0) = 0$ and

$$\psi'(t) \leq \lambda \beta(f(t, \{z_n(t) + R_n v(t) : n \geq n_0\})) \leq \lambda \omega(t, \psi(t)) \text{ a.e. on } [0, b].$$

Therefore $\psi(t) \equiv 0$ and this implies that there is a uniformly convergent subsequence (z_{n_k}) the limit u of which is a solution of (1) on $[0, b]$.

q.e.d.

Clearly, (1) has a solution on I if, in addition to the hypotheses of Theorem 2, we have that either D is bounded or f satisfies one of the usual growth conditions, e.g. $|f(t, x)| \leq M(1 + |x|)$ for all $(t, x) \in D$ and some constant $M \geq 0$.

Naturally, the question arises for which bases (e_i) the standard cone K is normal. Since this is a nontrivial problem, our next section deals with it.

IV. NORMAL CONES

Let X be a Banach space. A set $B \subset X$ is called reproducing if $X = B - B$. It is a classical result of M. Krein [5] that a cone $K \subset X$ is normal iff K^* is reproducing. The dual statement is also true, i.e. K^* is normal iff K is reproducing; see e.g. T. Ando [1] or H. H. Schaefer [11].

Now, let us consider the special case where K is the standard cone w.r. to the base (e_i) of X . Recall that (e_i) is said to be unconditional if for every $x \in X$ the series $\sum_{i \geq 1} e_i^*(x)e_i$ is unconditionally convergent. Otherwise (e_i) is called conditional. Note also that a series $\sum_{i \geq 1} x_i e_i$ is unconditionally convergent iff it is subseries convergent, i.e. iff $\sum_k x_{i_k} e_{i_k}$ converges for every increasing sequence (i_k) ; see e.g. p. 20 of I. Marti [8]. Let us prove

Proposition 3. Let X be a Banach space with the base (e_i) and let K be the standard cone w.r. to (e_i) . Then (e_i) is unconditional iff K is reproducing and normal.

Proof: (i) Let (e_i) be unconditional. Then $x = \sum_{i \geq 1} x_i e_i = u - v$ with $u = \sum_{i \in \mu} x_i e_i \in K$ and $v = - \sum_{i \in \nu} x_i e_i \in K$, where $\mu = \{i: x_i > 0\}$ and $\nu = \{i: x_i < 0\}$. Therefore K is reproducing. Similarly, for $x^* \in X^*$ we have $x^*(x) = \sum_{i \geq 1} x_i x^*(e_i) = u^*(x) - v^*(x)$ with $u^*, v^* \in K^*$ defined by

$$u^*(x) = \sum_{i \geq 1} x_i \max\{x^*(e_i), 0\} \quad \text{and} \quad v^*(x) = - \sum_{i \geq 1} x_i \{\min x^*(e_i), 0\}.$$

Thus, K^* is reproducing and therefore K is normal.

(ii) Let K be reproducing and normal. For $x \in X$ we have $x = u - v = \sum_{i \geq 1} u_i e_i - \sum_{i \geq 1} v_i e_i$ and $u, v \in K$. Since K is normal we have $\left| \sum_{k=m+1}^{m+p} u_k e_k \right| \leq M \left| \sum_{i=m+1}^{m+p} u_i e_i \right| \rightarrow 0$ as $m \rightarrow \infty$, uniformly with respect to $p \geq 1$. Therefore $x = \sum_{i \geq 1} x_i e_i = \sum_{i \geq 1} (u_i - v_i) e_i$ is sub-series convergent, hence (e_i) is unconditional. q.e.d.

Remark. If (e_i) is unconditional then $\|x\| = \sup \left\{ \left| \sum_{i \in \mu} \gamma_i x_i e_i \right| : (\gamma_i) \in S, \mu \in \Sigma \right\}$, where S is the unit ball of ℓ^∞ and Σ is the set of all finite subsets of \mathbb{N} , defines a norm equivalent to $|\cdot|$, and evidently $\|x\| \leq \|y\|$ if $0 \leq x \leq y$; see e.g. Marti [8, p. 38] or I. Lindenstrauss/L. Tzafriri [7, p. 16]. Therefore the normality of K is evident without appeal to Krein's theorem.

Examples. (i) (e_i) , defined by $e_{ij} = \delta_{ij}$, is an unconditional base for c_0 , the space of all sequences convergent to zero with $|x| = \max_{i \geq 1} |x_i|$, and for ℓ^p ($1 \leq p < \infty$). Therefore the standard cone K is normal.

(ii) $(\tilde{e}_i, \tilde{e}_i^*)$, defined by $\tilde{e}_i = \sum_{j \leq i} e_j$ and $\tilde{e}_i^* = e_i - e_{i+1}$, is conditional for c_0 , consider for example $x = \sum_{i \geq 1} \frac{(-1)^i}{i} e_i$. However, K is normal with constant $M = 1$. In fact, if $0 \leq x \leq y$, i.e. $0 \leq x_i - x_{i+1} \leq y_i - y_{i+1}$ for all $i \geq 1$, then (x_i) and (y_i) are decreasing and $|x| = x_1 \leq y_1 = |y|$. Thus, K may be normal although (e_i) is conditional.

(iii) Haar's system, see e.g. [8, p. 49], is unconditional for $L^p(0,1)$ with $1 < p < \infty$; cp. e.g. [7, p. 145].

(iv) Every orthonormal base of a Hilbert space is unconditional; in particular, the trigonometric system $e_0 = (2\pi)^{-\frac{1}{2}}$, $e_{2k-1}(t) = \pi^{-\frac{1}{2}} \sin(kt)$, $e_{2k}(t) = \pi^{-\frac{1}{2}} \cos(kt)$ is unconditional for $L^2(0, 2\pi)$.

(v) The trigonometric system is only conditional for $L^p(0, 2\pi)$ with $1 < p < \infty$ and $p \neq 2$; see e.g. [8, p. 51]. With this example we have found a standard cone which is not normal. In fact, the coefficient-functionals are given by $e_i^* = e_i$, considered as elements of $X^* = L^q(0, 2\pi)$, with $p^{-1} + q^{-1} = 1$. Therefore $K_p^* = K_q$, the standard cone w.r. to (e_i) in $L^q(0, 2\pi)$. Hence, if K_p is normal then K_p is not reproducing, by Proposition 3; therefore $K_p^* = K_q$ is not normal.

V. AN EXTENSION TO $l^\infty(\Omega)$

Let Ω be an arbitrary set and $l^\infty(\Omega)$ the space of all bounded functions $x: \Omega \rightarrow \mathbb{R}$ with norm $|x| = \sup_{\omega \in \Omega} |x_\omega|$. We let $K = \{x: x_\omega \geq 0 \text{ for all } \omega \in \Omega\}$, and the elements $x \vee y$ and $x \wedge y$ be defined by $(x \vee y)_\omega = \max(x_\omega, y_\omega)$ and $(x \wedge y)_\omega = \min(x_\omega, y_\omega)$.

Let X be a closed subspace of $l^\infty(\Omega)$ which is invariant under the operations \wedge and \vee ; for example $X = C(\Omega)$ with $\Omega \subset \mathbb{R}^n$ compact; $X = \sigma$, the space of all convergent sequences (where $\Omega = \mathbb{N}$), etc. The intersection of K and X will be denoted by K again. In such spaces the analogue of Proposition 1 can be proved along the lines of the proof to Proposition 1 since K has inner points and \vee, \wedge are defined. Thus, we have

Theorem 3. Let X be a closed subspace of $\ell^\infty(\Omega)$ invariant under \wedge and \vee ; $K \subset X$ the cone of the nonnegative elements; $I = [0, a] \subset \mathbb{R}$; $v, w: I \rightarrow X$ continuous and $v(t) \leq w(t)$ on I ; $D = \{(t, z) \in I \times X: t \in I \text{ and } v(t) \leq z \leq w(t)\}$; $f: D \rightarrow X$ continuous and $\beta(f(t, B)) \leq \psi(t, \beta(B))$ for all $B \subset X$ such that $\{t\} \times B \subset D$, where $\psi: I \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous, monotone increasing in the second argument and such that $\rho(t) \equiv 0$ is the only solution of $\rho' = \psi(t, \rho)$, $\rho(0) = 0$. Then, problem (1) has a local solution for every u_0 satisfying $v(0) \leq u_0 \leq w(0)$, provided that condition (H), now with w instead of i , holds for all $\omega \in \Omega$.

Proof: We define $P: I \times X \rightarrow X$ by $P(t, z) = v(t) \vee (z \wedge w(t))$. It is easy to verify that

$$|P(t, z) - P(\bar{t}, \bar{z})| \leq |v(t) - v(\bar{t})| + |w(t) - w(\bar{t})| + |z - \bar{z}|. \text{ In}$$

particular, $|P(t, z) - P(t, \bar{z})| \leq |z - \bar{z}|$ and therefore

$\beta(P(t, B)) \leq \beta(B)$ for all $t \in I$ and all bounded $B \subset X$. Thus,

$\tilde{f}(t, z) = f(t, P(t, z))$ is continuous and satisfies $\beta(\tilde{f}(t, B)) \leq \psi(t, \beta(B))$.

Therefore, $u' = \tilde{f}(t, u)$, $u(0) = u_0$ has a local solution u . Now we choose an interior point e of K and obtain that u is a solution of (1) as in the proof of Proposition 1.

q.e.d.

VI. FINAL REMARKS

(i) We have always considered solutions which are differentiable w.r. to the norm under consideration. If we look for "solutions" of countable systems which are only componentwise differentiable then the

analogue to Proposition 1 is easy to establish in case f is continuous w.r. to the product topology, a condition on f much stronger than continuity in norm in many cases. To be more precise, let $v, w: I \rightarrow \mathbb{R}^\infty$ be continuous and such that $v(t) \leq w(t)$ on I , i.e. all components v_i, w_i are continuous and $v_i(t) \leq w_i(t)$ on I for all $i \geq 1$. Let D be as before, $f: D \rightarrow \mathbb{R}^\infty$ continuous, and suppose that (H) holds (for every $i \geq 1$). Then (1) has a "solution" on I for every u_0 such that $v(t) \leq u_0 \leq w(t)$. To see this, we extend f as in the proof of Theorem 3; the extended system has a "solution" u on I , by Corollary 6.1 in [2], and as before we can show that u is a "solution" of the original problem.

(ii) Condition (H) has always been formulated in terms of components. It is easy to see that there is an equivalent formulation in terms of functionals from K^* , namely

$$(H) \quad \begin{aligned} v(t) \leq z \leq w(t) \text{ and } x^*(z - v(t)) = 0 \text{ for some } x^* \in K^* &\Rightarrow x^*(v'(t) - f(t, z)) \leq 0 \\ v(t) \leq z \leq w(t) \text{ and } x^*(z - w(t)) = 0 \text{ for some } x^* \in K^* &\Rightarrow x^*(w'(t) - f(t, z)) \geq 0 \end{aligned}$$

This version of the condition allows us to consider also cones K other than the standard cones. However, Proposition 1 may not be valid as has been shown by Volkmann [12]. He has the counter-example $K = \{x \in \mathbb{R}^3: (x_1^1 + x_2^2)^{\frac{1}{2}} \leq x_3\}$, $v(t) \equiv 0$, $w(t) \equiv (2, 0, 2)$ and $f: D \rightarrow \mathbb{R}^3$, defined by $f_1 = f_3 = 0$ and

$$f_2(x) = \begin{cases} x_1 & \text{for } x_1 \in [0, 1] \\ 2 - x_1 & \text{for } x_1 \in [1, 2] \\ 0 & \text{otherwise,} \end{cases}$$

which is Lipschitz and such that the solution through $(1,0,1)$ is $u(t) = (1,t,1)$.

(iii) We have always regarded v and w as a-priori bounds, i.e. we wanted to find solutions in between v and w . Mere comparison theorems, where a solution u is assumed to exist and where one wants to know whether v and w are bounds for u are trivial in case K has nonempty interior K^0 and the relevant inequalities are strict in the sense that $w(0) - u_0 \in K^0$, $u_0 - v(0) \in K^0$, and $x^*(v'(t) - f(t,z)) < 0$, $x^*(w'(t) - f(t,z)) > 0$ (for $x^* \in K^* \setminus \{0\}$) in condition (H). Clearly, in this case we need no β -estimate or any other condition on f which guarantees existence of a solution. In case $K^0 = \emptyset$ no comparison results are known under condition (H). The only thing we can say is that u is in between v and w if (1) has at most one solution and conditions like that in Theorem 1 or Theorem 2, guaranteeing existence of a solution, are satisfied.

(iv) If we are in the situation given by Theorem 1 and if f is T -periodic in time then it is sometimes easy to find T -periodic solutions. For example, suppose that f satisfies the "Lipschitz conditions"

$$(3) \quad |P_n f(t, P_n \underline{z} + R_n v(t)) - P_n f(t, P_n \bar{z} + R_n v(t))| \leq L_n(t) |P_n \underline{z} - P_n \bar{z}|$$

for $n \geq 1$, $L_n \in C(I)$

and for $v(t) \leq \underline{z}, \bar{z} \leq w(t)$. Suppose also that v and w are T -periodic.

Then $u' = f(t, u)$ has a T -periodic solution. In fact, we know that the IVP

$$(4) \quad z' = P_n f(t, z + R_n v(t)), \quad z(0) = x \quad \text{with} \quad P_n v(0) \leq x \leq P_n w(0)$$

has a unique solution $z(t; x)$ in between $P_n v(t)$ and $P_n w(t)$, by Proposition 1 (with $\alpha = T$) and the Lipschitz-condition on $P_n f$.

Define $U_T x = z(T; x)$. Then U_T is a continuous map of the order interval $[P_n v(0), P_n w(0)]$ into itself. By Brouwer's fixed point

theorem, U_T has a fixed point, i.e. (4) has a solution z_n such

that $z_n(0) = z_n(T)$. In the proof to Theorem 1 we have seen that there

is a uniformly convergent subsequence $z_{n_k}(t) \rightarrow u(t)$. Then u is a solution of $u' = f(t, u)$ satisfying $u(0) = u(T)$. Since f is

T -periodic, this implies the existence of a T -periodic solution. We

do not know whether this result is true if f is only continuous. In

the autonomous case, and with constant functions v and w , these results can be used to prove fixed point theorems; see [4].

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