Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 15, nº 2 (1988), p. 309-340
<http://www.numdam.org/item?id=ASNSP_1988_4_15_2_309_0>

Existence and Estimates of Green's Function for Degenerate Elliptic Equations

S. CHANILLO - R.L. WHEEDEN

1. - Introduction

In this paper, we study the Green function for equations $\mathbf{L}u = 0$ in a bounded open set Ω in $\mathbb{R}^n$, $n > 2$, in case $\mathbf{L}$ has divergence form

$$\mathbf{L} = -\sum_{i,j} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$$

and the coefficient matrix $A = (a_{ij})$ satisfies

$$w(x)|\xi|^2 \leq \langle A\ \xi, \xi \rangle \leq v(x)|\xi|^2, \ \ \xi \in \mathbb{R}^n. \tag{1.1}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the usual dot product in $\mathbb{R}^n$ and w and v are nonnegative functions which will be further stipulated.

More specifically, we show that a Green function exists and derive interior estimates for its size. By "Green function for Ω with pole y" we mean a function $G(x, y) = G_y(x)$, $x, y \in \Omega$, which solves $\mathbf{L}G_y = \delta_y$ in the weak sense, i.e.

$$\int_\Omega \langle A\nabla G_y, \nabla\varphi \rangle \ dx = \varphi(y), \quad \text{if} \quad \varphi \in \mathrm{Lip}_0(\Omega),$$

where $\mathrm{Lip}_0(\Omega)$ denotes the class of Lipschitz continuous functions supported in Ω. Moreover, G_y vanishes on $\partial\Omega$ in the sense that it is the limit, in an appropriate norm, of functions supported in Ω. It is also possible to represent the solution u of

$$\mathbf{L}u = f \text{ in } \Omega, \text{ with } u = 0 \text{ on } \partial\Omega,$$

in terms of a potential of f which has G as its kernel. This representation will be discussed below.

Research partly supported by NSF Grant DMS 86-01119 for the first author and DMS 85-03329 for the second one.

Pervenuto alla Redazione il 17 Febbraio 1987.

In order to state our main result, we need to introduce some notation. We shall assume throughout that $w \in A_2$, i.e. that

$$\left(\frac{1}{|B|}\int_B w\right)\left(\frac{1}{|B|}\int_B w^{-1}\right) \le c \quad \text{for all balls} \quad B \subset \mathbb{R}^n,$$

although this condition can be somewhat relaxed as indicated at the end of the paper, and that v satisfies the doubling condition: $v(2B) \le cv(B)$, where $v(B) = \int_B v(x)dx$ and $2B$ denotes the ball with the same center as B which is twice as large. We write $v \in D^\infty$ for such v. The assumption that $w \in A_2$ ensures that $w \in D^\infty$. We write $B_h(x)$ for the ball with center x and radius h, and assume that v and w are related by

$$\frac{s}{h}\left[\frac{v(B_s(x))}{v(B_h(x))}\right]^{\frac{1}{q}} \le c\left[\frac{w(B_s(x))}{w(B_h(x))}\right]^{\frac{1}{2}}, \quad 0 < s < h, \ x \in \mathbb{R}^n, \tag{1.2}$$

for some $q > 2$. We shall consistently use the notation $\sigma = q/2$, so that $\sigma > 1$, and we set $s_0 = 2\sigma/(\sigma + 1)$. Thus, $1 < s_0 < 2$. In the classical strongly elliptic case (v and w identically equal to positive constants), the value of q is $2n/(n-2)$ so that $\sigma = n/(n-2)$ and $s_0 = n/(n-1)$.

For $1 \le p \le \infty$, we let

$$L^p_v = \left\{ f : \|f\|_{L^p_v} = \left(\int_\Omega |f(x)|^p v(x)dx\right)^{\frac{1}{p}} < \infty \right\},$$

and we write simply L^p in the case of Lebesgue measure.

Similarly, L^1_{loc} stands for the class of functions which are locally integrable with respect to Lebesgue measure. We use the notation $X = X_{t,s}$ for the Banach space which is the closure of $\text{Lip}_0(\Omega)$ with respect to the norm

$$\|f\|_{L^t_v} + \|\nabla f\|_{L^s_w}.$$

If $1 < s < \infty$, define s' by $\frac{1}{s} + \frac{1}{s'} = 1$.

We can now state our main result. Since we consider only interior estimates, it will be convenient to think of Ω as contained in a large ball and derive the existence and estimates of the Green function for an open ball B when the pole lies in the middle half of B.

THEOREM (1.3). *Suppose that $w \in A_2$, $v \in D^\infty$ and (1.2) holds. Let A be a symmetric matrix which satisfies (1.1) and let $B = B_R(x_0)$ be a ball. For almost every $y \in \frac{1}{2}B$, there is a nonnegative function $G(x,y)$, $x \in B$, which satisfies*

(i) *$G \in X_{t,s}$ for $t < \sigma$ and $s < 2\sigma/(\sigma+1)$, and the sizes of the norms are uniform in y; thus,*

$$\sup_{y \in \frac{1}{2}B} \left[\int_B G(x,y)^t \; v(x)\mathrm{d}x + \int_B |\nabla_x G(x,y)|^s w(x)\mathrm{d}x \right] < \infty$$

for such t and s;

(ii) $\int_B < A\nabla_x G(x,y),\ \nabla\varphi(x) > \mathrm{d}x = \varphi(y),\ \ \varphi \in \mathrm{Lip}_0(B)$,

if $\left(\frac{v}{w}\right)^{s'} w \in L^1_{\mathrm{loc}}$ for some $s' > 2\sigma/(\sigma-1)$ (i.e., $s < 2\sigma/(\sigma+1)$);

(iii) $$\operatorname*{ess\ sup}_{x: r/2 < |x-y| < r} G(x,y) \le c \int_r^R \left[\frac{v(B_t(y))}{w(B_t(y))}\right]^{\frac{1}{p}\frac{\sigma}{\sigma-1}} \frac{t^2}{w(B_t(y))}\frac{\mathrm{d}t}{t}$$

if $0 < r < R/2$ and $0 < p < \sigma$, with c independent of B, y and r;

(iv) $$\operatorname*{ess\ inf}_{x: r/2 < |x-y| < r} G(x,y) \ge c \int_r^R \exp\left(-c_1 \left[\frac{v(B_t(y))}{w(B_t(y))}\right]^{\frac{1}{2}}\right) \frac{t^2}{v(B_t(y))}\frac{\mathrm{d}t}{t}$$

if $0 < r < R/4$, with c and c_1 independent of B, y and r;

(v) *$G \in L^t(B)$ for $t < \max\{n/(n-2),\ \sigma\}$.*

The size of the norm in part (v) may depend on y.

The assumption that $\left(\frac{v}{w}\right)^{s'} w \in L^1(B)$ is made only in part (ii) and is not needed in the other parts of the theorem. This assumption guarantees that the integral in (ii) converges since

$$\int_B | < A\nabla G, \nabla\varphi > | \le \|\nabla\varphi\|_{L^\infty} \int_B |\nabla G| v \quad \text{(by (1.1))}$$

$$\le \|\nabla\varphi\|_{L^\infty}\ \|\nabla G\|_{L^s_w} \left(\int_B \left(\frac{v}{w}\right)^{s'} w\right)^{\frac{1}{s'}}.$$

Of course, $\nabla G \in L^s_w$ by (i). There are alternates for (ii) which do not require the assumption that $\left(\frac{v}{w}\right)^{s'} w \in L^1(B)$. In order to state these, we need to introduce some more notation.

There are two Hilbert spaces H_0 and H naturally associated with the differential operator $\mathbf{L}$. The properties of these spaces will be discussed in §2.

Here, we mention only that H_0 consists of the elements of H which vanish in an appropriate sense at ∂B, and that the inner product $a_0(u,\varphi)$ on H_0 satisfies

$$a_0(u,\varphi) = \int_B < A\nabla u, \nabla\varphi >$$

if $u,\varphi \in \mathrm{Lip}_0(B)$. Furthermore, $a_0(u,\varphi)$ can be defined for $u,\varphi \in H$, and there are then associated functions $\tilde{u},\tilde{\varphi} \in L^2_v(B)$ (even $L^{2\sigma}_v(B)$) such that $\nabla\tilde{u}, \nabla\tilde{\varphi} \in L^2_w(B)$ and

$$a_0(u,\varphi) = \int_B < A\nabla\tilde{u}, \nabla\tilde{\varphi} > .$$

An argument based on the Lax-Milgram theorem shows (see §6) that, if $f/v \in L^{(2\sigma)'}_v(B)$ and the assumptions of Theorem (1.3) hold, then it is possible to solve the problem

$$\mathbf{L}u = f \text{ in } B, \text{ with } u = 0 \text{ on } \partial B, \tag{1.4}$$

in the sense that $\exists\ u \in H_0$ with

$$a_0(u,\varphi) = \int_\Omega f\tilde{\varphi}, \qquad \varphi \in H_0. \tag{1.5}$$

We shall refer to u as the Lax-Milgram solution of (1.4).

Similarly, if F is a vector with $|F|/w \in L^2_w(B)$, it is possible to solve

$$-\mathbf{L}u = \text{ div } F \text{ in } B, \text{ with } u = 0 \text{ on } \partial B, \tag{1.6}$$

in the sense that $\exists\ u \in H_0$ with

$$a_0(u,\varphi) = \int_B < F, \nabla\tilde{\varphi} >, \qquad \varphi \in H_0. \tag{1.7}$$

We shall refer to u as the Lax-Milgram solution of (1.6).

The following result gives representations of these solutions in terms of G, without assuming that $\left(\frac{v}{w}\right)^{s'} w \in L^1_{\mathrm{loc}}$ for some $s' > 2\sigma/(\sigma-1)$.

THEOREM (1.8). *Let v, w and A satisfy the hypothesis of Theorem (1.3). If $f/v \in L^{t'}_v(B)$ for some $t < \sigma$ and u is the Lax-Milgram solution of (1.4), then*

$$\tilde{u}(y) = \int_B f(x)\ G(x,y)\mathrm{d}x \quad \textit{for a.e.} \quad y \in \frac{1}{2}B.$$

Furthermore, if $|F|/w \in L^{s'}_w(B)$ *for some* $s < 2\sigma/(\sigma+1)$ *and* u *is the Lax-Milgram solution of* (1.6), *then*

$$\tilde{u}(y) = \int_B < F(x),\ \nabla G(x,y) > \mathrm{d}x \quad \text{for a.e.} \quad y \in \frac{1}{2}B.$$

The proofs of the theorems rely partly on adapting the methods in [7] for the strongly elliptic case. We also need some facts from [2], including a mean-value inequality and Harnack's inequality, as well as Sobolev's inequality

$$(1.9)\quad \left(\frac{1}{v(B)}\int_B |f|^q v\mathrm{d}x\right)^{\frac{1}{q}} \le c|B|^{\frac{1}{n}} \left(\frac{1}{w(B)}\int_B |\nabla f|^2 w\mathrm{d}x\right)^{\frac{1}{2}},\ f \in \mathrm{Lip}_0(B),$$

with c independent of f and B. The value of q in (1.9) is the same as in (1.2), and the fact that (1.9) is valid, if $w \in A_2$, $v \in D_\infty$ and (1.2) holds, is proved in [1]. Some of the required inequalities from [2], together with other background facts, are given in §2. In §§3-4, estimates for an approximate Green function are derived. In §5, Theorem (1.3) is proved except for the uniform nature of the estimates in part (i); this uniformity is proved together with Theorem (1.8) in §6. In the case of equal weights (by which we mean the case when v is at most a constant multiple of w), our results are contained in [3] and [4]. The classical strongly elliptic case is also treated in [9].

We now state a version of the Wiener test, i.e. a criterion which gives a condition for a point of $\partial\Omega$ to be a regular point. A proof can be obtained by modifying the arguments in [5] or [6], p. 206. In order to state the result, we need a few more definitions. First, we say that $v \in A_p$, $1 \le p < \infty$, if for all balls B

$$\left(\frac{1}{|B|}\int_B v\right)\left(\frac{1}{|B|}\int_B v^{-1/(p-1)}\right)^{p-1} \le c, \quad \text{when} \quad 1 < p < \infty,$$
$$\frac{1}{|B|}\int_B v \le c \ \operatorname*{ess\,inf}_B v, \quad \text{when} \quad p = 1,$$

with c independent of B. We say that $v \in A_\infty$ if $v \in A_p$ for some p.

Next, for any open bounded set Ω, the Hilbert spaces $H(\Omega)$ and $H_0(\Omega)$ can be defined as before. As noted in [2], a simple argument based on the Lax-Milgram theorem shows that if $\psi \in H(\Omega)$, the Dirichlet problem $\mathbf{L}u = 0$ in Ω, with $u = \psi$ on $\partial\Omega$, can be solved in the sense that $\exists\ u \in H(\Omega)$ with $a_0(u,\varphi) = 0$ for all $\varphi \in H_0(\Omega)$ and $u - \psi \in H_0(\Omega)$.

We can also give a meaning to solving $\mathbf{L}u = 0$ in Ω with $u = \psi$ on $\partial\Omega$ in case ψ is a function which is defined and continuous only on $\partial\Omega$. This can be done by choosing a sequence $\{p_k\}$ of polynomials which converge uniformly to ψ on $\partial\Omega$ and solving $\mathbf{L}u_k = 0$ in Ω with $u_k - p_k \in H_0(\Omega)$. It can be shown from the weak maximum principle (§2) that $\sup_{\Omega}|\tilde{u}_k| \leq \sup_{\partial\Omega}|p_k|$. From this inequality and Caccioppoli's inequality, we can show that $\{u_k\}$ converges to a limit u in $H(\Omega')$ for any Ω' with closure in Ω and that $a_0(u,\varphi) = 0$ for $\varphi \in \mathrm{Lip}_0(\Omega)$. We say that a point $x \in \partial\Omega$ is a *regular point* if

$$\lim_{r\to 0} \operatorname*{ess\,sup}_{y\in B_r(x)\cap\Omega} |\tilde{u}(y) - \psi(x)| = 0$$

whenever u is a solution, in the sense described above, of $\mathbf{L}u = 0$ in Ω with $u = \psi$ on $\partial\Omega$, ψ continuous on $\partial\Omega$.

Finally, if B is a fixed open ball containing $\overline{\Omega}$ and $E \subset B$, define the *capacity* of E by

$$\operatorname{cap} E = \inf \left\{ \int_B < A\nabla\varphi, \nabla\varphi > \middle| \varphi \geq 1 \text{ on } E,\ \varphi \in \mathrm{Lip}_0(B) \right\}.$$

The Wiener test can now be stated as follows.

THEOREM (1.10). *Let A be symmetric and satisfy* (1.1) *for a pair of weights v, w for which $v \in A_\infty$, $w \in A_2$ and* (1.2) *holds. Let Ω be a bounded open set and $x \in \partial\Omega$. There is a positive constant c_1 such that if*

$$\int_0^\varepsilon \exp\left(-c_1\left[\frac{v(B_t(x))}{w(B_t(x))}\right]^{\frac{1}{2}}\right) \frac{t^2}{v(B_t(x))} \operatorname{cap}(B_t(x)\backslash\Omega) \frac{dt}{t} = +\infty$$

for some $\varepsilon > 0$, then x is a regular point.

In passing, we note that if $x \in \partial\Omega$ and the complement of Ω contains a truncated cone with vertex x, then

$$\operatorname{cap}(B_t(x)\backslash\Omega) \geq c\ \frac{w(B_t(x))}{t^2},$$

as is easy to see by Sobolev's inequality. Thus, in this case, x is a regular point if

$$\int_0^\varepsilon \exp\left(-c_1\left[\frac{v(B_t(x))}{w(B_t(x))}\right]^{\frac{1}{2}}\right) \frac{w(B_t(x))}{v(B_t(x))} \frac{dt}{t} = +\infty.$$

2. - Preliminaries

As in [2], for a bounded open set Ω, let

$$a_0(u,\varphi) = \int_\Omega < A\nabla u, \nabla\varphi >, \qquad u,\varphi \in \mathrm{Lip}(\overline{\Omega}),$$

where Lip($\overline{\Omega}$) denotes the class of functions which are Lipschitz continuous in the closure of Ω. By the degeneracy condition (1.1),

$$\int_\Omega |\nabla u|^2 w \leq a_0(u,u) \leq \int_\Omega |\nabla u|^2 v. \tag{2.1}$$

It follows that $a_0(u,\varphi)$ is an inner product on $\mathrm{Lip}_0(\Omega)$, and therefore that $a_0(u,u)^{\frac{1}{2}}$ is a norm on $\mathrm{Lip}_0(\Omega)$. In particular, $a_0(u,\varphi) \leq a_0(u,u)^{\frac{1}{2}}\ a_0(\varphi,\varphi)^{\frac{1}{2}}$. Note also that since A is symmetric, $|<Ax,y>| \leq <Ax,x>^{\frac{1}{2}} <Ay,y>^{\frac{1}{2}}$.

We denote the completion of $\mathrm{Lip}_0(\Omega)$ with respect to this norm by $H_0 = H_0(\Omega)$. An element u of H_0 is thus an equivalence class of Cauchy sequences $\{u_k\}$, $u_k \in \mathrm{Lip}_0(\Omega)$.

If $u, \varphi \in H_0$ with $u = \{u_k\}$ and $\varphi = \{\varphi_k\}$, $u_k, \varphi_k \in \mathrm{Lip}_0(\Omega)$, it is easy to see that $a_0(u_k,\varphi_k)$ converges, and we define

$$a_0(u,\varphi) = \lim_{k\to\infty} a_0(u_k,\varphi_k).$$

It follows that $\|u\|_0 = a_0(u,u)^{\frac{1}{2}}$ is a norm on H_0.

We now show that it is possible to associate with each $\varphi \in H_0$ a unique pair $(\tilde{\varphi}, \nabla\tilde{\varphi})$ so that if $\varphi = \{\varphi_k\}$ then $\varphi_k \to \tilde{\varphi}$ in $L^2_v(\Omega)$ (even in $L^{2\sigma}_v(\Omega)$) and $\nabla\varphi_k \to \nabla\tilde{\varphi}$ in $L^2_w(\Omega)$. We shall refer to $(\tilde{\varphi}, \nabla\tilde{\varphi})$ as the pair of functions associated with φ. To see this, note that since $\varphi_k \in \mathrm{Lip}_0(\Omega)$, φ_k can be extended to a function in $\mathrm{Lip}_0(\mathbb{R}^n)$ by setting $\varphi_k = 0$ outside Ω. In particular, if B_R is any ball containing Ω, for this new $\varphi_k \in \mathrm{Lip}_0(B_R)$, and by Sobolev's inequality (1.9),

$$\begin{aligned}
\left(\frac{1}{v(B_R)}\int_{B_R} |\varphi_k - \varphi_j|^2 v\right)^{\frac{1}{2}} &\leq c\ R\left(\frac{1}{w(B_R)}\int_{B_R} |\nabla(\varphi_k - \varphi_j)|^2 w\right)^{\frac{1}{2}} \\
&= c\ R\left(\frac{1}{w(B_R)}\int_{\Omega} |\nabla(\varphi_k - \varphi_j)|^2 w\right)^{\frac{1}{2}} \\
&\leq c_{w,B_R}\ \|\varphi_k - \varphi_j\|_0 \quad \text{(by (2.1))}.
\end{aligned}$$

Thus, $\{\varphi_k\}$ is a Cauchy sequence in $L^2_v(\Omega)$. Also, by the last inequality, $\{\nabla\varphi_k\}$ is a Cauchy sequence in $L^2_w(\Omega)$. Let $\tilde{\varphi}$ and $\nabla\tilde{\varphi}$ denote the limits, respectively, and observe that these are independent of the particular sequence $\{\varphi_k\}$ representing φ. Of course, if $\varphi \in \mathrm{Lip}_0(\Omega)$, then $\tilde{\varphi} = \varphi$ and $\nabla\tilde{\varphi} = \nabla\varphi$.

Since $w^{-1} \in L^1(\Omega)$, it is easy to see that $\nabla\tilde{\varphi}$ is the distributional gradient of $\tilde{\varphi}$; in fact, by Schwarz's inequality, it follows that $\varphi_k \to \tilde{\varphi}$ in $L^1(\Omega)$ and $\nabla\varphi_k \to \nabla\tilde{\varphi}$ in $L^1(\Omega)$ (since also $v^{-1} \in L^1(\Omega)$), and therefore if $\psi \in \mathrm{Lip}(\Omega)$,

$$\int \tilde{\varphi}\ \nabla\psi = \lim_{k\to\infty}\int \varphi_k\ \nabla\psi = -\lim_{k\to\infty}\int \nabla\varphi_k\ \psi = -\int \nabla\tilde{\varphi}\ \psi.$$

We will also have to consider the Hilbert space $H = H(\Omega)$ which is the completion of $\mathrm{Lip}(\overline{\Omega})$ under the inner product

$$a(u,\varphi) = a_0(u,\varphi) + \int_\Omega u\varphi v, \qquad u,\varphi \in \mathrm{Lip}(\overline{\Omega}). \tag{2.2}$$

Facts about H are given in [2]. In particular, $H_0 \subset H$ continuously by Sobolev's inequality, and if $u \in H$, $u = \{u_k\}$, $u_k \in \mathrm{Lip}(\overline{\Omega})$, then u_k converges in L^2_v to a function $\tilde{u}$ and ∇u_k converges in L^2_w to a vector $\nabla\tilde{u}$. Furthermore, if $\varphi \in H$, $\varphi = \{\varphi_k\}$, the limits $a(u,\varphi) = \lim\ a(u_k,\varphi_k)$ and $a_0(u,\varphi) = \lim\ a_0(u_k,\varphi_k)$ exist and satisfy

$$a(u,\varphi) = a_0(u,\varphi) + \int_\Omega \tilde{u}\tilde{\varphi}v,$$

and $a(u,\varphi)$ is an inner product on H.

It will be useful to have a representation for $a_0(u,\varphi)$ in terms of $\nabla\tilde{u}$ and $\nabla\tilde{\varphi}$. This is given in the following lemma.

LEMMA (2.3). *Let $u,\varphi \in H$ and let $\nabla\tilde{u}$ and $\nabla\tilde{\varphi}$ be the associated gradients, respectively. If $u = \{u_k\}$ and $\varphi = \{\varphi_k\}$, then*

$$< A\nabla u_k, \nabla\varphi_k > \rightarrow < A\nabla\tilde{u}, \nabla\tilde{\varphi} > \ \text{ in } L^1(\Omega).$$

In particular,

$$a_0(u,\varphi) = \int_\Omega < A\nabla\tilde{u},\nabla\tilde{\varphi} > \ \text{ and } \ a(u,\varphi) = \int_\Omega < A\nabla\tilde{u},\nabla\tilde{\varphi} > + \int_\Omega \tilde{u}\tilde{\varphi}v.$$

PROOF. Let $h_k = < A\nabla u_k, \nabla\varphi_k >$. It is easy to see that $\{h_k\}$ is a Cauchy sequence in $L^1(\Omega)$ by using the inequality

$$| < Ax, y > | \leq \ < Ax, x >^{\frac{1}{2}} < Ay, y >^{\frac{1}{2}} .$$

Thus, $h_k \to h$ in $L^1(\Omega)$ for some h. Consequently, $\int_\Omega h_k \to \int_\Omega h$. Since also $\int_\Omega h_k = a_0(u_k,\varphi_k) \to a_0(u,\varphi)$, we obtain $a_0(u,\varphi) = \int_\Omega h$. The lemma will therefore be proved if we show that $h = \ < A\nabla\tilde{u},\nabla\tilde{\varphi} >$ a.e. Since $h_k \to h$ in L^1 and both $\nabla u_k \to \nabla\tilde{u}$ and $\nabla\varphi_k \to \nabla\tilde{\varphi}$ in L^2_w, this follows easily by selecting subsequences which converge pointwise a.e.

An element $u \in H$ is called a solution of $\mathrm{L}u = 0$ if $a_0(u,\varphi) = 0$ for all $\varphi \in H_0$. Also, if $u \in H$, we say $u \geq 0$ in Ω if u can be represented by a sequence $\{u_k\}$, $u_k \in \mathrm{Lip}(\overline{\Omega})$, with $u_k \geq 0$ in Ω. If $u \geq 0$ in Ω, then clearly $\tilde{u} \geq 0$ a.e. in Ω. We will need the following two facts from [2] about solutions.

These facts are valid assuming (1.1) with $w \in A_2$, $v \in D^\infty$ and (1.2). First, a solution u in $H(2B)$ satisfies the mean-value inequality

$$\text{(2.4)} \qquad \operatorname*{ess\,sup}_B |\tilde{u}| \le c_p \left[\frac{v(B)}{w(B)}\right]^{\frac{1}{p}\left(\frac{\sigma}{\sigma-1}\right)} \left(\frac{1}{v(B)}\int_{2B} |\tilde{u}|^p v\right)^{1/p}, \quad 0 < p < \infty.$$

Moreover, if also $u \ge 0$, it satisfies Harnack's inequality

$$\text{(2.5)} \qquad \operatorname*{ess\,sup}_B \tilde{u} \le \exp\left(c_1 \left[\frac{v(B)}{w(B)}\right]^{\frac{1}{2}}\right) \operatorname*{ess\,inf}_B \tilde{u}.$$

The next two lemmas may be viewed as maximum (or minimum) principles. The first one is an adaptation of a similar result in [11] or [8].

LEMMA (2.6) (Weak Maximum Principle). *Let u be a supersolution in $H(\Omega)$, i.e., $u \in H(\Omega)$ and $a_0(u,\varphi) \ge 0$ if $\varphi \in \mathrm{Lip}_0(\Omega)$, $\varphi \ge 0$. Let $u = \{u_k\}$, $u_k \in \mathrm{Lip}(\overline{\Omega})$, and assume that $u_k \ge 0$ in some neighborhood (depending on k) of $\partial\Omega$. Then $\tilde{u} \ge 0$ a.e. in Ω.*

PROOF. Consider $u_k^- = -\min\{u_k, 0\}$. Note that $u_k^- \in \mathrm{Lip}_0(\Omega)$ since $u_k \ge 0$ near $\partial\Omega$. Since $\{u_k\}$ is bounded in H it is easy to see that $\{u_k^-\}$ is bounded in H_0. We may then select a subsequence $u_{k_j}^-$ which converges weakly in H_0 to $\psi \in H_0$. Thus,

$$\lim\ a_0(u_{k_j}, u_{k_j}^-) = \lim\ a_0(u, u_{k_j}^-) = a_0(u, \psi) \ge 0$$

since u is a supersolution. Thus,

$$\lim \int < A\nabla u_{k_j},\ \nabla u_{k_j}^- > \ \ge 0,$$

i.e.

$$\lim \int < A\nabla u_{k_j}^-,\ \nabla u_{k_j}^- > \ \le 0.$$

Therefore, $\|\nabla u_{k_j}^-\|_{L^2_w} \to 0$. Extending $u_{k_j}^-$ to a large ball containing Ω ($u_{k_j}^-$ has support in Ω) and applying Sobolev's inequality, we see $\|u_{k_j}^-\|_{L^2_v} \to 0$. But $u_{k_j} \to \tilde{u}$ in L^2_v, so $u_{k_j}^- \to (\tilde{u})^-$ in L^2_v. Thus $(\tilde{u})^- = 0$ a.e. in Ω and the proof is complete.

LEMMA (2.7). *Let B_1, B_2 and B_3 be balls with a common center and radii r_1, r_2, r_3, respectively, satisfying $r_1 < r_2 < r_3$. If $\varphi \in H(B_3)$ and $\tilde{\varphi} \le \ell$ a.e. in $B_3 \backslash B_1$, then given $L > \ell$, there exists $\varphi_k \in \mathrm{Lip}(\overline{B}_2)$ such that $\varphi_k \to \varphi$ in $H(B_2)$ and $\varphi_k \le L$ in some neighborhood of ∂B_2. Moreover, if u is a solution in $H(B_2)$, $u = \{u_k\}$, and if $u_k \le \varphi_k$ near ∂B_2 for these φ_k, then $\tilde{u} \le L$ a.e. in B_2.*

PROOF. The second statement follows from the first by applying the weak maximum principle to the solution $L-u$ in B_2: since $L-u_k \geq L-\varphi_k \geq 0$ near ∂B_2, we obtain $L-\tilde{u} \geq 0$ a.e. in B_2.

To prove the first statement, note that since $\varphi \in H(B_3)$, $\exists\ h_k \in \mathrm{Lip}(\overline{B}_3)$ with $h_k \to \varphi$ in $H(B_3)$. Thus, $h_k \to \tilde{\varphi}$ in $L^2_v(B_3)$, and by using a subsequence, we may assume that $h_k \to \tilde{\varphi}$ a.e. in B_3. By hypothesis, $\tilde{\varphi} \leq \ell$ a.e. in $B_3 \backslash B_1$, so by Egorov's theorem, given $L > \ell$ and $\delta > 0$, $\exists\ E \subset B_3\backslash B_1$ and k_0 such that $|(B_3\backslash B_1)\backslash E| < \delta$ and $h_k \leq L$ on E if $k \geq k_0$. Make a smooth partition of unity $\chi_1+\chi_2+\chi_3 = 1$ on B_3 such that χ_2 is supported on $B\backslash B_1$, where B is a ball with the same center as B_1 and radius r satisfying $r_2 < r < r_3$, and $\chi_1 \equiv 1$ on B_1 and is supported on a slight enlargement of B_1. Then $h_k = h_k\chi_1 + h_k\chi_2 + h_k\chi_3$, and we define $\varphi_k = h_k\chi_1 + (h_k\chi_{2\wedge}L)$. Clearly, $\varphi_k \in \mathrm{Lip}(\overline{B}_2)$ and $\varphi_k \leq L$ near ∂B_2. It remains to show that $\varphi_k \to \varphi$ in $H(B_2)$. We will do this by showing (see (2.2))

$$\|\varphi_k - h_k\|^2_{L^2_v(B_2)} + \int_{B_2} < A\nabla(\varphi_k - h_k),\ \nabla(\varphi_k - h_k) > \to 0. \tag{2.8}$$

We have

$$\|\varphi_k - h_k\|_{L^2_v(B_2)} = \|(h_k\chi_{2\wedge}L) - h_k\chi_2\|_{L^2_v(B_2\backslash B_1)}.$$

Since $h_k \to \tilde{\varphi}$ in $L^2_v(B_3)$, also $h_k\chi_2 \to \tilde{\varphi}\chi_2$ in this norm; moreover, $h_k\chi_{2\wedge}L \to \tilde{\varphi}\chi_{2\wedge}L$ in this norm since

$$|(\alpha_\wedge L) - (\beta_\wedge L)| \leq |\alpha - \beta| \text{ for any } \alpha, \beta.$$

Since $\tilde{\varphi} \leq \ell$ on $B_2\backslash B_1$, it follows that both $h_k\chi_2$ and $h_k\chi_{2\wedge}L$ converge in $L^2_v(B_2\backslash B_1)$ to the same limit, and so $\|\varphi_k - h_k\|_{L^2_v(B_2)} \to 0$.

The second term in (2.8) equals

$$\begin{aligned}
&\int_{B_2\backslash B_1} < A\nabla([h_k\chi_{2\wedge}L] - h_k\chi_2),\ \nabla([h_k\chi_{2\wedge}L] - h_k\chi_2) > \\
&\quad = \int_{(B_2\backslash B_1)\cap\{h_k\chi_2 > L\}} < A\nabla(h_k\chi_2),\ \nabla(h_k\chi_2) > \\
&\quad \leq \int_{(B_2\backslash B_1)\backslash E} < A\nabla(h_k\chi_2),\ \nabla(h_k\chi_2) >,
\end{aligned}$$

if $k \geq k_0$, since then $h_k \leq L$ on E. The integrand in the last integral is nonnegative and equals

$$
\begin{aligned}
&< h_k A\nabla\chi_2 + \chi_2 A\nabla h_k,\ h_k\nabla\chi_2 + \chi_2\nabla h_k > \\
&\qquad = h_k^2 < A\nabla\chi_2, \nabla\chi_2 > +2\chi_2 h_k < A\nabla\chi_2, \nabla h_k > \\
&\qquad + \chi_2^2 < A\nabla h_k, \nabla h_k > \\
&\qquad \le h_k^2 < A\nabla\chi_2, \nabla\chi_2 > +2\chi_2|h_k| < A\nabla\chi_2, \nabla\chi_2 >^{\frac{1}{2}} < A\nabla h_k, \nabla h_k >^{\frac{1}{2}} \\
&\qquad + \chi_2^2 < A\nabla h_k, \nabla h_k > \\
&\qquad \le h_k^2|\nabla\chi_2|^2 v + 2\chi_2|h_k|\ |\nabla\chi_2| v^{\frac{1}{2}} < A\nabla h_k, \nabla h_k >^{\frac{1}{2}} \\
&\qquad + \chi_2^2 < A\nabla h_k, \nabla h_k > .
\end{aligned}
$$

Thus, using Schwarz's inequality and the fact that $0 \le \chi_2 \le 1$, we see that the last integral is bounded by

$$
c\|\nabla\chi_2\|_{L^\infty}^2 \int_{(B_2\backslash B_1)\backslash E} h_k^2 v + \int_{(B_2\backslash B_1)\backslash E} < A\nabla h_k, \nabla h_k > .
$$

Since h_k converges in L_v^2, $h_k^2 v$ converges in L^1. Furthermore, by lemma (2.3), $< A\nabla h_k, \nabla h_k >$ converges in L^1. Thus, since the domain of integration in both integrals above has small measure $(< \delta)$, the integrals are small uniformly in k for large k, and the lemma follows.

We shall need the following compactness result.

LEMMA (2.9). *Let Ω be a bounded open set and $w \in A_2$. Let Σ be a ball in Ω and $\{f_j\}$ be a sequence of functions each supported in Ω with*

$$
\|f_j\|_{L_w^2(\Omega\backslash\Sigma)} + \|\nabla f_j\|_{L_w^2(\Omega\backslash\Sigma)} \le c < \infty
$$

uniformly in j. Then $\exists$ a subsequence $\{f_{j_k}\}$ such that if Σ^* is any enlargement of Σ, $\{f_{j_k}\}$ converges in $L_w^2(\Omega\backslash\Sigma^*)$.

PROOF. Let $\eta(x)$ be a smooth function supported in $|x| < 1$ with $\int \eta = 1$, and let $\eta_t(x) = t^{-n}\eta(x/t)$ for $t > 0$. If $g(x)$ is defined in $\mathbb{R}^n$, let

$$
g^t(x) = \int g(y)\ \eta_t(x-y)\mathrm{d}y.
$$

Note that the definition of $g^t(x)$ only involves values of $g(y)$ with $y \in B_t(x)$. Also, if Σ^* and Ω^* are neighborhoods of $\overline{\Sigma}$ and $\overline{\Omega}$, resp., and if $x \in \overline{\Omega}\backslash\Sigma^*$ and t is small, then $B_t(x) \subset \Omega^*\backslash\Sigma$. Thus, for such x and t,

$$
|g^t(x)| \le ct^{-n}\int_{B_t(x)} |g| \le ct^{-n}\|g\|_{L_w^2(\Omega^*\backslash\Sigma)} \left(\int_{\Omega^*\backslash\Sigma} w^{-1}\right)^{\frac{1}{2}}
$$

and

$$|\nabla g^t(x)| \le ct^{-n-1} \int_{B_t(x)} |g| \le ct^{-n-1} \; \|g\|_{L^2_w(\Omega^*\backslash\Sigma)} \left(\int_{\Omega^*\backslash\Sigma} w^{-1} \right)^{\frac{1}{2}} .$$

Moreover,

$$\begin{aligned} |g^t(x) - g(x)| &= \left| \int [g(y) - g(x)] \; \eta_t(x-y)\mathrm{d}y \right| \\ &\le ct^{-n} \int_{B_t(x)} |g(y) - g(x)|\mathrm{d}y. \end{aligned}$$

By Lemma (1.4) of [3], this is bounded by

$$c \int_{B_t(x)} \frac{|\nabla g(y)|}{|x-y|^{n-1}} \,\mathrm{d}y. \tag{2.10}$$

If $x \in \Omega\backslash\Sigma^*$ and t is small, we may rewrite (2.10) after multiplying and dividing by t as

$$ct \left(\frac{1}{t} \int_{|x-y|<t} \frac{|\nabla g(y)|\chi_{\Omega^*\backslash\Sigma}(y)}{|x-y|^{n-1}} \,\mathrm{d}y \right),$$

which is at most $ct \; M(|\nabla g|\chi_{\Omega^*\backslash\Sigma})(x)$, where M denotes the Hardy-Littlewood maximal operator. Since $w \in A_2$, it then follows from [10] that for small t,

$$\|g^t - g\|_{L^2_w(\Omega\backslash\Sigma^*)} \le ct \; \|\nabla g\|_{L^2_w(\Omega^*\backslash\Sigma)}.$$

Since the f_j are supported in Ω, we may think of them as defined on all of $\mathbb{R}^n$ by just setting $f_j = 0$ outside Ω. Then

$$\|f_j\|_{L^2_w(\Omega^*\backslash\Sigma)} + \|\nabla f_j\|_{L^2_w(\Omega^*\backslash\Sigma)} = \|f_j\|_{L^2_w(\Omega\backslash\Sigma)} + \|\nabla f_j\|_{L^2_w(\Omega\backslash\Sigma)} \le c.$$

Hence, by above, $\exists$ c independent of j such that if $x \in \overline{\Omega}\backslash\Sigma^*$ and t is small,

$$|f_j^t(x)| \le ct^{-n}, \;\; |\nabla f_j^t(x)| \le ct^{-n-1} \text{ and } \|f_j^t - f_j\|_{L^2_w(\Omega\backslash\Sigma^*)} \le ct. \tag{2.11}$$

The rest of the proof follows from (2.11) by applying Ascoli's theorem to the first two inequalities in (2.11) for each fixed t. We shall not give the details but instead refer to [6], p. 167.

COROLLARY (2.12). *The previous lemma holds as stated if we replace the hypothesis that the f_j have support in Ω by the hypothesis that $\exists$ f_j^k supported in Ω such that $f_j^k \to f_j$ and $\nabla f_j^k \to \nabla f_j$ in $L^2_w(\Omega\backslash\Sigma)$ as $k \to \infty$.*

This follows easily by applying the previous lemma to the sequence $\{f_j^{k_j}\}$, where for a given j we choose k_j so that the $L^2_w(\Omega\backslash\Sigma)$ norm of both $f_j - f_j^{k_j}$ and $\nabla f_j - \nabla f_j^{k_j}$ tend to 0.

REMARK (2.13). (i) Lemma (2.9) has an analogue for L^p_w norms, $1 \le p < \infty$, if $w \in A_p$. Similarly, there are versions for $L^p_w(\Omega)$ norms, i.e. for all of Ω without deleting Σ^*. In case $1 < p < \infty$, the proof is the same as when $p = 2$ except that L^2_w norms are replaced by L^p_w norms. In case $p = 1$, a small change is needed when estimating (2.10). Instead of majorizing (2.10) by the maximal function, we simply integrate over $\Omega\backslash\Sigma^*$ and use Fubini's theorem to obtain

$$\|g^t - g\|_{L^1_w(\Omega\backslash\Sigma^*)} \le ct \int_{\Omega^*\backslash\Sigma} |\nabla g(y)| \left\{ \frac{1}{t} \int_{\Omega\cap B_t(y)} \frac{w(x)}{|x-y|^{n-1}}\, dx \right\} dy \tag{2.14}$$
$$\le ct\, \|\nabla g\|_{L^1_w(\Omega^*\backslash\Sigma)},$$

since the part of the integrand in curly brackets is at most $cM(w)(y) \le cw(y)$ due to $w \in A_1$.

(ii) We note that other variants of Lemma (2.9) can be obtained by altering its proof. For example, the conclusion holds for $L^1_w(\Omega\backslash\Sigma^*)$ instead of $L^2_w(\Omega\backslash\Sigma^*)$ if we still assume that the f_j are supported in Ω and that $\{f_j\}$ and $\{\nabla f_j\}$ are bounded in $L^2_w(\Omega\backslash\Sigma)$ but replace the hypothesis that $w \in A_2$ by the weaker hypothesis that w^{-1} is integrable over a neighborhood of $\overline{\Omega}$ and

$$\lim_{t\to 0} \int_\Omega \left(\int_{\Omega\cap B_t(y)} \frac{w(x)}{|x-y|^{n-1}}\, dx \right)^2 w^{-1}(y)dy = 0. \tag{2.15}$$

This last condition is valid if $w \in A_2$ since then $w^{-1} \in A_2$ and the integral in (2.15) is at most

$$\int [tM(w\chi_\Omega)(y)]^2 w(y)^{-1} dy \le ct^2 \int_\Omega w(y)dy \quad \text{(by [10])},$$

which tends to 0 with t. The only real change in the proof of the lemma comes in estimating $\|f_j^t - f_j\|_{L^1_w(\Omega\backslash\Sigma^*)}$, which by the first inequality in (2.14) is at most

$$c \int_{\Omega\backslash\Sigma} |\nabla f_j(y)| \left(\int_{\Omega\cap B_t(y)} \frac{w(x)}{|x-y|^{n-1}}\, dx \right) dy.$$

By Schwarz's inequality, this is bounded by

$$c\, \|\nabla f_j\|_{L^2_w(\Omega\backslash\Sigma)} \left(\int_\Omega \left(\int_{\Omega\cap B_t(y)} \frac{w(x)}{|x-y|^{n-1}}\, dx \right)^2 w^{-1}(y)dy \right)^{\frac{1}{2}},$$

and the result follows as before if (2.15) holds.

3. - Estimates for $\tilde{G}^\rho$

Fix $y \in \Omega$ and $\rho > 0$ with $B_\rho = B_\rho(y) \subset \Omega$. For $\varphi \in H_0$, let $\tilde{\varphi}$ be the associated function in L^2_v. The mapping

$$\varphi \to \frac{1}{v(B_\rho)} \int_{B_\rho} \tilde{\varphi} v$$

is a continuous linear functional on H_0: see, e.g. the argument in §2 showing how to associate $\tilde{\varphi}$ with φ. Since the bilinear form $a_0(u, \varphi)$ is continuous and coercive on H_0, the Lax-Milgram theorem implies there is a unique $G^\rho \in H_0$ such that

$$a_0(G^\rho, \varphi) = \frac{1}{v(B_\rho)} \int_{B_\rho} \tilde{\varphi} v, \qquad \varphi \in H_0. \tag{3.1}$$

$\tilde{G}^\rho$ will be called the approximate Green function for Ω with pole y.

We claim that $G^\rho \geq 0$ as an element of H_0, i.e. that $\exists\ G^\rho_k \in \mathrm{Lip}_0(\Omega)$ with $G^\rho_k \geq 0$ and $G^\rho_k \to G^\rho$ in H_0. This of course implies that $\tilde{G}^\rho \geq 0$ a.e. since $G^\rho_k \to \tilde{G}^\rho$ in L^2_v and thus $\exists$ a subsequence $G^\rho_{k_j} \to \tilde{G}^\rho$ a.e. To prove the claim, let $G^\rho = \{G^\rho_k\}$, $G^\rho_k \in \mathrm{Lip}_0(\Omega)$. Note that $\{|G^\rho_k|\}$ is bounded in H_0: in fact,

$$a_0(|G^\rho_k|,\ |G^\rho_k|) = a_0(G^\rho_k,\ G^\rho_k)$$

since $\nabla |G^\rho_k| = (\mathrm{sign}\ G^\rho_k)\ \nabla G^\rho_k$ where $G^\rho_k \neq 0$. Hence, a subsequence $|G^\rho_{k_j}| \rightharpoonup h$ (weak convergence in H_0). Thus,

$$\begin{aligned} a_0(G^\rho,\ h - G^\rho) &= \lim\ a_0(G^\rho,\ |G^\rho_{k_j}| - G^\rho_{k_j}) \\ &= \lim\ \frac{1}{v(B_\rho)} \int_{B_\rho} (|G^\rho_{k_j}| - G^\rho_{k_j}) v \quad \text{by (3.1).} \end{aligned}$$

Since the last expression is clearly nonnegative,

$$a_0(G^\rho, G^\rho) \leq a_0(G^\rho, h), \text{ so that } a_0(G^\rho, G^\rho) = c\ a_0(G^\rho, h)$$

for some c with $0 < c \leq 1$. We have

$$\begin{aligned} 0 \leq \|G^\rho - c|G^\rho_{k_j}|\ \|_0 &= a_0(G^\rho - c|G^\rho_{k_j}|,\ G^\rho - c|G^\rho_{k_j}|) \\ &= a_0(G^\rho, G^\rho) - 2c a_0(G^\rho,\ |G^\rho_{k_j}|) + c^2 a_0(|G^\rho_{k_j}|,\ |G^\rho_{k_j}|) \\ &= a_0(G^\rho, G^\rho) - 2c a_0(G^\rho,\ |G^\rho_{k_j}|) + c^2 a_0(G^\rho_{k_j}, G^\rho_{k_j}). \end{aligned}$$

The right side converges to

$$a_0(G^\rho, G^\rho) - 2c a_0(G^\rho, h) + c^2 a_0(G^\rho, G^\rho) = (c^2 - 1) a_0(G^\rho, G^\rho) \leq 0.$$

Hence $\|G^\rho - c|G^\rho_{k_j}|\,\|_0 \to 0$. This shows that G^ρ is the limit in H_0 of $c|G^\rho_{k_j}| \geq 0$, and so establishes the claim. It follows incidentally that $c = 1$, since by above, $c|G^\rho_{k_j}| \rightharpoonup \tilde{G}^\rho$ in L^2_v, while also $G^\rho_k \rightharpoonup \tilde{G}^\rho$ in L^2_v: hence, $c|\tilde{G}^\rho| = \tilde{G}^\rho$ a.e. and $c = 1$. Thus, the argument above shows that if $G^\rho = \{G^\rho_k\}$, then also $G^\rho = \{|G^\rho_{k_j}|\}$ for some subsequence.

Consider now the case $\Omega = B = B_R(x_0)$. By above, we may assume that $G^\rho_k \geq 0$. For $t > 0$, define

$$\varphi_k = \left[\frac{1}{t} - \frac{1}{G^\rho_k}\right]^+ = \left[\frac{1}{t} - \frac{1}{G^\rho_k}\right] \chi_{\{G^\rho_k > t\}}.$$

Then

$$\nabla\varphi_k = \frac{\nabla G^\rho_k}{(G^\rho_k)^2}\, \chi_{\{G^\rho_k > t\}}$$

and $\varphi_k \in \mathrm{Lip}_0(B)$. We have

$$\begin{aligned}\|\varphi_k\|_0^2 &= \int_B < A\nabla\varphi_k, \nabla\varphi_k > = \int_{\{G^\rho_k > t\}} < A\nabla G^\rho_k,\ \nabla G^\rho_k > \frac{1}{(G^\rho_k)^4} \\ &\leq \frac{1}{t^4}\int_B < A\nabla G^\rho_k,\ \nabla G^\rho_k > = \frac{1}{t^4}\|G^\rho_k\|_0^2.\end{aligned}$$

Since this is bounded in k, there is a subsequence $\varphi_{k_j} \rightharpoonup \varphi$ in H_0. Then

$$\lim\ a_0(G^\rho_{k_j},\ \varphi_{k_j}) = \lim\ a_0(G^\rho, \varphi_{k_j}) = a_0(G^\rho, \varphi),$$

where the second equality follows from weak convergence and the first follows from the strong convergence of $G^\rho_{k_j}$ to G^ρ in H_0 and the boundedness of $\|\varphi_k\|_0$. The middle term equals

$$\lim\ \frac{1}{v(B_\rho)}\int_{B_\rho} \varphi_{k_j} v \leq \frac{1}{t}, \text{ since } \tilde{\varphi}_{k_j} = \varphi_{k_j} \leq \frac{1}{t}.$$

Therefore,

$$\begin{aligned}&\lim \int < A\nabla G^\rho_{k_j},\ \nabla\varphi_{k_j} > \leq \frac{1}{t}, \text{ or} \\ &\lim \int < A\nabla G^\rho_{k_j},\ \frac{\nabla G^\rho_{k_j}}{(G^\rho_{k_j})^2}\,\chi_{\{G^\rho_{k_j} > t\}} > \leq \frac{1}{t}.\end{aligned}$$

By the degeneracy condition,

$$\limsup \int_{\{G^\rho_{k_j} > t\}} \frac{|\nabla G^\rho_{k_j}|^2}{(G^\rho_{k_j})^2}\, w \leq \frac{1}{t}.$$

If we define $\psi_k = [\log\ G_k^\rho - \log\ t]\chi_{\{G_k^\rho > t\}} = [\log\ G_k^\rho - \log\ t]^+$, then $\nabla\psi_k = (\frac{\nabla G_k^\rho}{G_k^\rho})\chi_{\{G_k^\rho > t\}}$ and $\psi_k \in \mathrm{Lip}_0(B)$. From the estimate above,

$$\lim\ \sup \int |\nabla\psi_{k_j}|^2 w \leq \frac{1}{t},$$

so that by Sobolev's inequality (integration is over $B = B_R(x_0)$),

$$\lim\ \sup \left[\frac{1}{v(B)} \int_{B\cap\{G_{k_j}^\rho > t\}} (\log(G_{k_j}^\rho/t))^q\ v\right]^{\frac{2}{q}} \leq c\ \frac{R^2}{w(B)}\ \frac{1}{t}.$$

Restricting the integration to $\{G_{k_j}^\rho > 2t\}$ gives

$$(\log\ 2)^2 \lim\ \sup \left[\frac{v(G_{k_j}^\rho > 2t)}{v(B)}\right]^{\frac{2}{q}} \leq c\ \frac{R^2}{w(B)}\ \frac{1}{t}.$$

We may replace $2t$ by t without affecting the form of this inequality. Since $G_k^\rho \to \tilde{G}^\rho$ in L_v^2, by using a further subsequence if necessary, we may assume $G_{k_j}^\rho \to \tilde{G}^\rho$ pointwise a.e. Thus, $\chi_{\{\tilde{G}^\rho > t\}} \leq \lim\ \inf\ \chi_{\{G_{k_j}^\rho > t\}}$ a.e., and by Fatou's lemma, $v(\tilde{G}^\rho > t) \leq \lim\ \inf\ v(G_{k_j}^\rho > t)$. Therefore,

$$v(\tilde{G}^\rho > t) \leq c\ \left[\frac{R^2}{w(B)}\right]^\sigma \frac{1}{t^\sigma}\ v(B),\ \sigma = \frac{q}{2}. \tag{3.2}$$

Here, $\tilde{G}^\rho$ is the approximate Green function for $B = B_R(x_0)$. Note that the constant c in (3.2) is independent of ρ and y.

Consider now the special case $R = r$, $x_0 = y$, and look at values of x in the annulus $r/2 < |x - y| < 3r/4$. Fix $\rho < r/4$. Then $B_{r/4}(x) \subset B_r \backslash B_\rho$. Note G^ρ is a solution in $B_r \backslash B_\rho$ since if $\varphi \in H_0(B_r \backslash B_\rho)$ then $a_0(G^\rho, \varphi) = 0$: this is because of (3.1) since φ is the limit with respect to $\|\cdot\|_0$ of functions supported outside B_ρ. Hence, by the mean-value property (2.4) and the doubling condition $(v(B_{r/4}(x)) \approx v(B_r),\ w(B_{r/4}(x)) \approx w(B_r))$,

$$\operatorname*{ess\ sup}_{B_{r/8}(x)} \tilde{G}^\rho \leq c\ \left[\frac{v(B_r)}{w(B_r)}\right]^{\frac{1}{p}\frac{\sigma}{\sigma-1}} \left[\frac{1}{v(B_r)} \int_{B_{r/4}(x)} (\tilde{G}^\rho)^p v\right]^{\frac{1}{p}}.$$

To estimate the integral, we use (3.2) with $R = r$ and $x_0 = y$ together with the obvious inequality $v(B_{r/4}(x); \tilde{G}^\rho > t) \leq cv(B_r)$ to obtain

$$\int_{B_{r/4}(x)} (\tilde{G}^\rho)^p v \leq c\ v(B_r)\ \left[\frac{r^2}{w(B_r)}\right]^p,\ \text{provided } 1 < p < \sigma.$$

Substituting this estimate above and recalling that x is any point in $r/2 < |x-y| < 3r/4$ gives

$$(3.3)\qquad \operatorname*{ess\,sup}_{x:r/2<|x-y|<3r/4} \tilde{G}^\rho(x) \le c \left[\frac{v(B_r)}{w(B_r)}\right]^{\frac{1}{p}\frac{\sigma}{\sigma-1}} \frac{r^2}{w(B_r)}, \text{ if } p<\sigma,\ \rho<r/4.$$

Here, $\tilde{G}^\rho$ is the approximate Green function for $B_r = B_r(y)$ with pole y.

We wish to show that (3.3) holds without the restriction $|x-y| < 3r/4$ on the left, i.e. that

$$(3.4)\qquad \operatorname*{ess\,sup}_{B_r\setminus B_{r/2}} \tilde{G}^\rho \le c \left[\frac{v(B_r)}{w(B_r)}\right]^{\frac{1}{p}\frac{\sigma}{\sigma-1}} \frac{r^2}{w(B_r)}, \text{ if } p<\sigma,\ \rho<r/4.$$

This will clearly follow from (3.3) by replacing r by $4r/3$ and using doubling provided we show that $\tilde{G}^\rho$ increases when we enlarge domains, i.e. that if G^ρ and $G^{*\rho}$ are defined for Ω and Ω^* resp., then $\Omega \subset \Omega^*$ implies $\tilde{G}^\rho \le \tilde{G}^{*\rho}$ in Ω. This follows from the weak maximum principle applied to Ω and $G^{*\rho} - G^\rho$ if we verify the appropriate hypotheses. Note that $G^{*\rho} - G^\rho \in H(\Omega)$ (not $H_0(\Omega)$) and $G_k^{*\rho} - G_k^\rho = G_k^{*\rho} - 0 \ge 0$ near $\partial\Omega$ for each k; moreover, as it is easily seen from (3.1), $G^{*\rho} - G^\rho$ is a solution in Ω.

LEMMA (3.5). *Let $B = B_R(x_0)$ and $y \in \frac{1}{2}B$. If $\tilde{G}^\rho$ is the approximate Green function for B with pole y and if $0 < r < R/2$, then*

$$\operatorname*{ess\,sup}_{x:r/2<|x-y|<r} \tilde{G}^\rho(x) \le c \int_r^R \left[\frac{v(B_t(y))}{w(B_t(y))}\right]^{\frac{1}{p}\frac{\sigma}{\sigma-1}} \frac{t^2}{w(B_t(y))}\frac{dt}{t},\ \rho<\frac{r}{4},$$

for $0<p<\sigma$. The constant c is independent of R, x_0, y, r and ρ.

PROOF. We first claim that it is enough to prove the lemma in case $y = x_0$. To see this, first note that if $y \in \frac{1}{2}B$ then $B \subset B_{2R}(y) = \Sigma$, so that $\tilde{G}^\rho \le \tilde{G}^\rho_\Sigma$ in B, where $\tilde{G}^\rho_\Sigma$ denotes the approximate Green function for Σ with pole y. Hence, if we knew the estimate for balls centered at y, we would have

$$\begin{aligned}\operatorname*{ess\,sup}_{x:r/2<|x-y|<r} \tilde{G}^\rho(x) &\le \operatorname*{ess\,sup}_{x:r/2<|x-y|<r} \tilde{G}^\rho_\Sigma(x)\\ &\le c\int_r^{2R} \left[\frac{v(B_t(y))}{w(B_t(y))}\right]^{\frac{1}{p}\frac{\sigma}{\sigma-1}} \frac{t^2}{w(B_t(y))}\frac{dt}{t}\\ &= c\Big(\int_r^R + \int_R^{2R}\Big) \le c\int_r^R\end{aligned}$$

since by doubling $\int_R^{2R} \approx \int_{R/2}^R \le \int_r^R$.

Thus, we may consider only balls centered at y. For $s > 0$, let $B_s = B_s(y)$ and let $\tilde{G}_s^\rho$ denote the approximate Green function for B_s with pole y. Our goal is to prove that if $r < \frac{R}{2}$, $\rho < \frac{r}{4}$ and $p < \sigma$, then

$$\operatorname*{ess\,sup}_{B_r \backslash B_{r/2}} \tilde{G}_R^\rho \leq d \int_r^R \left[\frac{v(B_t)}{w(B_t)}\right]^{\frac{1}{p}\frac{\sigma}{\sigma-1}} \frac{t^2}{w(B_t)} \frac{dt}{t}. \tag{3.6}$$

Pick $m = 1, 2, \cdots$ with $r\left(\frac{3}{2}\right)^{m-1} < R \leq r\left(\frac{3}{2}\right)^m$. In $B_r \backslash B_{r/2}$,

$$\tilde{G}_R^\rho \leq \tilde{G}^\rho_{r\left(\frac{3}{2}\right)^m} = \tilde{G}_r^\rho + \sum_{j=1}^m \left[\tilde{G}^\rho_{r\left(\frac{3}{2}\right)^j} - \tilde{G}^\rho_{r\left(\frac{3}{2}\right)^{j-1}}\right]. \tag{3.7}$$

We now estimate the size of each term on the right in (3.7). For the first term, from (3.4),

$$\operatorname*{ess\,sup}_{B_r \backslash B_{r/2}} \tilde{G}_r^\rho \leq c \left[\frac{v(B_r)}{w(B_r)}\right]^{\frac{1}{p}\frac{\sigma}{\sigma-1}} \frac{r^2}{w(B_r)}.$$

To estimate the remaining terms, we claim that if $\rho < \frac{s}{4}$ then

$$\operatorname*{ess\,sup}_{B_s} \left[\tilde{G}^\rho_{\frac{3}{2}s} - \tilde{G}^\rho_s\right] \leq c \left[\frac{v(B_s)}{w(B_s)}\right]^{\frac{1}{p}\frac{\sigma}{\sigma-1}} \frac{s^2}{w(B_s)}. \tag{3.8}$$

If so, it follows from (3.7) and doubling that

$$\begin{aligned}\operatorname*{ess\,sup}_{B_r \backslash B_{r/2}} \tilde{G}_R^\rho &\leq c \sum_{j=0}^m \left[\frac{v(B_{(3/2)^j r})}{w(B_{(3/2)^j r})}\right]^{\frac{1}{p}\frac{\sigma}{\sigma-1}} \frac{\left((3/2)^j r\right)^2}{w\left(B_{(3/2)^j r}\right)} \\ &\leq c \int_r^R \left[\frac{v(B_t)}{w(B_t)}\right]^{\frac{1}{p}\frac{\sigma}{\sigma-1}} \frac{t^2}{w(B_t)} \frac{dt}{t}, \quad r < \frac{R}{2},\end{aligned}$$

which proves (3.6).

The proof of (3.8) uses Lemma (2.7) with φ there taken to be $G^\rho_{3s/2}$, $u = G^\rho_{3s/2} - G^\rho_s$, B_1, B_2, B_3 taken to be $B_{3s/4}$, B_s, $B_{3s/2}$ resp. and

$$\ell = c \left[\frac{v(B_s)}{w(B_s)}\right]^{\frac{1}{p}\frac{\sigma}{\sigma-1}} \frac{s^2}{w(B_s)}.$$

Note that by (3.4) with $r = 3s/2$ and doubling, we have $\tilde{\varphi} \leq \ell$ a.e. in $B_{3s/2} \backslash B_{3s/4}$. Choose $\{\varphi_k\}$ as in Lemma (2.7) for $L = 2\ell$. Note that u is a solution in B_s and $u = \{u_k\}$, $u_k = \varphi_k - \psi_k$ where $G_s^\rho = \{\psi_k\}$ with $\psi_k \geq 0$ in B_s. Thus $u_k \leq \varphi_k$ in B_s, so by Lemma (2.7), $\tilde{u} \leq 2\ell$ a.e. in B_s, which proves (3.8).

COROLLARY (3.9). *With the same notation and hypothesis as in Lemma* (3.5), *for a.e.* $y \in \frac{1}{2}B$ *there is a constant* c *independent of* r *and* ρ *(but depending on* y, R, w, v*) such that*

$$\operatorname*{ess\,sup}_{x:\frac{r}{2}<|x-y|<r} \tilde{G}^{\rho}(x) \le c \ \min\{r^{2-n},\ r^{-n/\sigma}\}, \qquad \rho < \frac{r}{4},\ n > 2.$$

PROOF. This will follow from Lemma (3.5). The integral in the conclusion is at most

$$\left[\sup_{t\le R} \frac{v(B_t(y))}{w(B_t(y))}\right]^{\frac{1}{p}\frac{\sigma}{\sigma-1}} \left[\sup_{t\le R} \frac{t^n}{w(B_t(y))}\right] \int_r^R \frac{t^2}{t^n}\frac{dt}{t}$$

$$= c_{y.R.w.v} \int_r^R t^{1-n}dt \le c_{y.R.w.v}\ r^{2-n},$$

with $c_{y.R.w.v}$ finite for a.e. y. Similarly, since $\frac{t^2}{w(B_t(y))} \le c\ v(B_t(y))^{-\frac{1}{\sigma}}$ for $t < R$ by (1.2), the integral in the conclusion of Lemma (3.5) is also majorized by

$$\left[\sup_{t\le R} \frac{v(B_t(y))}{w(B_t(y))}\right]^{\frac{1}{p}\frac{\sigma}{\sigma-1}} \left[\sup_{t\le R} \frac{t^n}{v(B_t(y))}\right]^{\frac{1}{\sigma}} \int_r^R t^{-\frac{n}{\sigma}}\frac{dt}{t}$$

$$= c_{y.R.w.v}\ r^{-\frac{n}{\sigma}}.$$

4. - Estimates for $\nabla\tilde{G}^{\rho}$

The goal of this section is to obtain an estimate for $\|\nabla\tilde{G}^{\rho}\|_{L^s_w(B)}$ which is uniform in ρ. We shall prove

LEMMA (4.1). *Let* $B = B_R(x_0)$ *and* $\tilde{G}^{\rho}$ *denote the approximate Green function for* B *with pole* y. *There is a number* s_0, *with* $1 < s_0 < 2$, *such that* $\int_B |\nabla\tilde{G}^{\rho}|^s\ w$ *is bounded uniformly in* ρ *for each* $s < s_0$ *and a.e.* $y \in \frac{1}{2}B$. *The bound depends on* s, y, B, w *and* v.

As we shall see, the value of s_0 is $2\sigma/(\sigma+1)$.

The first step in proving the lemma is the following Caccioppoli-type estimate.

LEMMA (4.2). *Let* B *and* $\tilde{G}^{\rho}$ *be as above, and let* $B_r = B_r(y)$ *for* $r \le \frac{R}{2}$. *For* $y \in \frac{1}{2}B$ *and* $\rho < \frac{r}{2}$,

$$\int_{B\setminus B_r} < A\nabla\tilde{G}^{\rho},\ \nabla\tilde{G}^{\rho} > \le \frac{c}{r^2} \int_{B_r\setminus B_{r/2}} (\tilde{G}^{\rho})^2 v$$

with c independent of r, ρ and y.

PROOF. The proof is very similar to that of Lemma (3.1) in [2]. Pick $\eta(x)$ so that $\eta = 1$ outside B_r, $\eta = 0$ in $B_{r/2}$ and $|\nabla\eta| \leq c/r$. The function $\varphi_k = G_k^\rho \eta^2$ belongs to $\mathrm{Lip}_0(B)$ and the argument of Lemma (3.1) of [2], with u and β there taken to be G^ρ and 1 resp., yields (cf. (3.7) of [2])

$$\begin{aligned}\int < A\nabla G_k^\rho, \nabla G_k^\rho > \eta^2 &\leq 2\left[\int < A\nabla\eta,\ \nabla\eta > (G_k^\rho)^2 + |\delta_k|\right] \\ &\leq 2\left[\int |\nabla\eta|^2 (G_k^\rho)^2 v + |\delta_k|\right]\end{aligned}$$

with $\delta_k \to 0$. One small change is needed in the argument in [2]: namely, since G^ρ is not a subsolution, we must justify (3.3) of [2]. However, if $\{\varphi_{k_j}\}$ is a subsequence which converges weakly in H_0 to φ, then, as usual,

$$\begin{aligned}\lim\ a_0(G_{k_j}^\rho, \varphi_{k_j}) = a_0(G^\rho, \varphi) &= \lim\ a_0(G^\rho, \varphi_{k_j}) \\ &= 0, \text{ since } \varphi_{k_j} = 0 \text{ on } B_\rho, \rho < \frac{r}{2}\ .\end{aligned}$$

This serves as a replacement for (3.3) of [2].

From the properties of η,

$$\int_{B\setminus B_r} < A\nabla G_k^\rho,\ \nabla G_k^\rho > \leq c\left[\frac{1}{r^2}\int_{B_r\setminus B_{r/2}} (G_k^\rho)^2 v + |\delta_k|\right].$$

The lemma now follows by letting $k \to \infty$ and applying Lemma (2.3).

LEMMA (4.3). *Let B and $\tilde{G}^\rho$ be as in Lemma* (4.1), *and let $B_r = B_r(y)$ for $r < \frac{R}{2}$. For a.e. $y \in \frac{1}{2}B$,*

$$\int_{B\setminus B_r} |\nabla\tilde{G}^\rho|^2 w \leq c\ r^{-\frac{n}{\sigma}}$$

with c dependent on y, R, w and v but independent of ρ and r.

PROOF. We consider first the case $\rho < \frac{r}{4}$. Combining Lemma (4.2), (1.1) and Corollary (3.9), we see that for a.e. $y \in \frac{1}{2}B$,

$$\begin{aligned}\int_{B\setminus B_r} |\nabla\tilde{G}^\rho|^2 w &\leq \frac{c}{r^2}\ \min\{r^{2-n}, r^{-n/\sigma}\}^2\ v(B_r) \\ &= cr^{-n/\sigma}\min\{r^{2-n+n/\sigma}, r^{-(2-n+n/\sigma)}\}\frac{v(B_r)}{r^n} \\ &\leq cr^{-n/\sigma}\frac{v(B_r)}{r^n}\end{aligned}$$

with $c = c_{y.B.w.v}$ independent of r and ρ. Since

$$\frac{v(B_r)}{r^n} \leq \sup_{t\leq R}\left[\frac{v(B_t(y))}{t^n}\right] = c_{y.R.v} < \infty$$

for a.e. y, we obtain the desired estimate.

In case $\rho > \frac{r}{4}$, write

$$a_0(G^\rho, G_k^\rho) = \frac{1}{v(B_\rho)}\int_{B_\rho} G_k^\rho \; v \leq \left(\frac{1}{v(B_\rho)}\int_{B_\rho}(G_k^\rho)^q v\right)^{\frac{1}{q}}.$$

Enlarge the domain of integration in the last integral to B, recall that G_k^ρ is supported in B, and apply Sobolev's inequality to obtain

$$\begin{aligned} a_0(G^\rho, G_k^\rho) &\leq cR\frac{v(B)^{1/q}}{v(B_\rho)^{1/q}}\left(\frac{1}{w(B)}\int_B |\nabla G_k^\rho|^2 w\right)^{\frac{1}{2}} \\ &= c\frac{1}{v(B_\rho)^{1/q}}\left(\int_B |\nabla G_k^\rho|^2 w\right)^{\frac{1}{2}}, \quad c = c_{B.w.v} \\ &\leq \frac{c}{v(B_\rho)^{1/q}}\left(\int < A\nabla G_k^\rho, \nabla G_k^\rho >\right)^{\frac{1}{2}}. \end{aligned}$$

Since $a_0(G^\rho, \; G_k^\rho) \to a_0(G^\rho, G^\rho)$, Lemma (2.3) gives

$$\int < A\nabla\tilde{G}^\rho, \nabla\tilde{G}^\rho > \leq \frac{c}{v(B_\rho)^{1/q}}\left(\int < A\nabla\tilde{G}^\rho, \nabla\tilde{G}^\rho >\right)^{\frac{1}{2}}.$$

Thus,

$$\begin{aligned} \int < A\nabla\tilde{G}^\rho, \nabla\tilde{G}^\rho > &\leq \frac{c}{v(B_\rho)^{2/q}} = \frac{c}{v(B_\rho)^{1/\sigma}} \\ &\leq \frac{c}{v(B_r)^{1/\sigma}}, \text{ since } \rho > \frac{r}{4}. \end{aligned}$$

Multiplying and dividing on the right by $r^{n/\sigma}$, letting

$$\sup_{r\leq R}\frac{r^n}{v(B_r)} = c_{y.R.v} \; (< \infty \text{ for a.e. } y),$$

and using (1.1), we obtain

$$\int |\nabla\tilde{G}^\rho|^2 w \leq cr^{-\frac{n}{\sigma}}, \qquad c = c_{y.B.w.v}.$$

This completes the proof of Lemma (4.3).

PROOF OF LEMMA (4.1). For any $t > 0$ and $r > 0$,

$$w(\{x : |\nabla \tilde{G}^{\rho}(x)| > t\}) \le w(\{x \in B \backslash B_r : |\nabla \tilde{G}^{\rho}(x)| > t\}) + w(B_r)$$
$$\le \frac{1}{t^2} \int_{B \backslash B_r} |\nabla \tilde{G}^{\rho}|^2 w + w(B_r).$$

For $r \le \frac{R}{2}$, use Lemma (4.3) to estimate the first term on the right, and use

$$w(B_r) \le \left[\sup_{r \le R} \frac{w(B_r(y))}{r^n} \right] r^n$$

to estimate the second one. Thus, for a.e. $y \in \frac{1}{2}B$, $\exists\ c = c_{y.R.w.v}$ such that

$$w(|\nabla \tilde{G}^{\rho}| > t) \le c[\frac{1}{t^2} r^{-n/\sigma} + r^n], \ r \le \frac{R}{2}.$$

Choosing $r = t^{-\frac{2\sigma}{n(\sigma+1)}}$, which is less than $\frac{R}{2}$ if $t > (\frac{R}{2})^{-\frac{n(\sigma+1)}{2\sigma}}$, we get

$$w(|\nabla \tilde{G}^{\rho}| > t) \le ct^{-\frac{2\sigma}{\sigma+1}} = ct^{-s_0}.$$

Since we also have the trivial estimate $w(|\nabla \tilde{G}^{\rho}| > t) \le w(B)$ for all t, Lemma (4.1) follows easily.

5. - Existence of the Green Function

In §4, we showed that $\exists\ s_0,\ 1 < s_0 < 2$, such that $\nabla \tilde{G}^{\rho} \in L^s_w$ uniformly in ρ for $s < s_0$ and a.e. $y \in \frac{1}{2}B$. Also, from (3.2),

$$v(\tilde{G}^{\rho} > t) \le c \ \min\left\{ \left[\frac{R^2}{w(B)} \right]^{\sigma} \frac{1}{t^{\sigma}},\ 1 \right\} v(B)$$

with c independent of ρ, y and t. Thus, $\tilde{G}^{\rho} \in L^t_v$ uniformly in ρ and y for $t < \sigma$. Since G^{ρ}_k is supported in B and $G^{\rho}_k \to \tilde{G}^{\rho}$ in $L^{2\sigma}_v$ and $\nabla G^{\rho}_k \to \nabla \tilde{G}^{\rho}$ in L^2_w, it follows that $\tilde{G}^{\rho}$ belongs to $X = X_{t.s}$ uniformly in ρ for $1 < t < \sigma$, $1 < s < s_0$ and a.e. $y \in \frac{1}{2}B$. Since $t, s > 1$, X is reflexive, so $\exists$ a subsequence $\{\tilde{G}^{\rho_j}\}$ which converges weakly in X to an element $G \in X$. Moreover, by taking sequences of t and s values increasing to σ and s_0 resp. and using a diagonal method, we may choose G independent of t and s for $t < \sigma$, $s < s_0$, i.e. $\exists\ \{\tilde{G}^{\rho_j}\}$ and G such that

$$\tilde{G}^{\rho_j} \rightharpoonup G \text{ (weakly) in } X_{t.s} \text{ for all } t < \sigma,\ s < s_0 \text{ and a.e. } y \in \frac{1}{2}B.$$

We have

$$a_0(G^\rho, \phi) = \frac{1}{v(B_\rho)} \int\limits_{B_\rho} \varphi v, \text{ for } \varphi \in \mathrm{Lip}_0(\Omega).$$

It follows by applying Lemma (2.3) to the left side that

$$\int < A\nabla \tilde{G}^\rho, \nabla\varphi > = \frac{1}{v(B_\rho)} \int\limits_{B_\rho} \varphi v. \tag{5.1}$$

The right side of (5.1) converges to $\varphi(y)$ as $\rho \to 0$. The left side, by (1.1) and Hölder's inequality, is at most

$$\begin{aligned}\int |\nabla \tilde{G}^\rho| \, |\nabla\varphi| v &< \|\nabla\tilde{G}^\rho\|_{L^s_w} \, \|\nabla\varphi\|_{L^{s'}_{(v/w)^{s'}w}} \\ &\leq \|\nabla\tilde{G}^\rho\|_{L^s_w} \, \|\nabla\varphi\|_{L^\infty} \int\limits_B \left(\frac{v}{w}\right)^{s'} w.\end{aligned}$$

Since $(\frac{v}{w})^{s'}w \in L^1(B)$, it follows that the left side of (5.1) defines a continuous linear functional on X for fixed φ, i.e.

$$\ell(f) = \int < A\nabla f, \nabla\varphi >$$

is a continuous linear functional on X. Since $\tilde{G}^{\rho_j} \to G$ in X, we obtain from (5.1) by passing to the limit that

$$\int < A\nabla G, \nabla\varphi > = \varphi(y), \ \ \varphi \in \mathrm{Lip}_0(B), \ \text{ a.e. } y \in \frac{1}{2}B.$$

This proves part (ii) of Theorem (1.3). It also proves part (i) except for the uniformity in y of the sizes of the norms of G and ∇G. Actually, the uniformity is shown for the norm of G but not for the norm of ∇G. In §6, we will establish the uniformity by using an argument that works equally well for G and ∇G.

We now wish to show that there is a subsequence of $\{\tilde{G}^{\rho_j}\}$ which converges to G *pointwise* a.e. Let $r < \frac{1}{2}R$ and $B_r = B_r(y)$. By Corollary (3.9) and Lemma (4.2), for a.e. $y \in \frac{1}{2}B$,

$$\|\tilde{G}^\rho\|_{L^2_w(B\backslash B_r)} + \|\nabla\tilde{G}^\rho\|_{L^2_w(B\backslash B_r)} \leq c < \infty, \quad \rho < r/4,$$

with c independent of ρ (but depending on r): in fact, the same would be true if the first summand were replaced by $\|\tilde{G}^\rho\|_{L^\infty(B\backslash B_r)}$. Of course, $G^\rho_k \to \tilde{G}^\rho$ in L^2_v, so also in L^2_w, and $\nabla\tilde{G}^\rho_k \to \nabla\tilde{G}^\rho$ in L^2_w. Since G^ρ_k is supported in B, the hypothesis of Corollary (2.12) holds with f_j taken to be $\tilde{G}^{\rho_j}$. It follows that there is a subsequence $\tilde{G}^{\rho_{j_k}}$ which converges in $L^2_w(B\backslash B_r)$. (The subsequence depends on r). We now show that the limit must be G. Given a bounded function φ, let

$$\ell(g) = \int g\varphi w.$$

This defines a continuous linear functional on X since

$$|\ell(g)| \leq \|\varphi\|_{L^\infty}\|g\|_{L^1_w} \leq \{\|\varphi\|_{L^\infty} w(B)^{1/t'}\}\ \|g\|_{L^t_w} \leq c\|g\|_X.$$

Thus,

$$\int \tilde{G}^{\rho_j}\varphi w \to \int G\varphi w.$$

But if $G^\#$ is the limit in $L^2_w(B\backslash B_r)$ of $\tilde{G}^{\rho_{j_k}}$, then

$$\int \tilde{G}^{\rho_{j_k}}\varphi w \to \int G^\#\varphi w \text{ for any bounded } \varphi \text{ supported in } B\backslash B_r.$$

Thus, $\int G\varphi w = \int G^\#\varphi w$ for such φ, and consequently, $G^\# = G$ a.e. in $B\backslash B_r$. Summarizing, we have now shown that $\tilde{G}^{\rho_{j_k}} \to G$ in $L^2_w(B\backslash B_r)$ for a subsequence $\{\rho_{j_k}\}$ which depends on r, $\rho_{j_k} \to 0$. Hence, there is a further subsequence, again denoted $\{\rho_{j_k}\}$, such that $\tilde{G}^{\rho_{j_k}} \to G$ pointwise a.e. in $B\backslash B_r$. Letting $r \to 0$ through a sequence and using repeated subsequences and a diagonal process, we see there is a fixed subsequence $\{\rho_{j_k}\} \to 0$ such that $\tilde{G}^{\rho_{j_k}} \to G$ a.e. in B, as desired.

We now obtain (iii) and (v) of Theorem (1.3) from Lemma (3.5) and Corollary (3.9) by letting $\rho = \rho_{j_k} \to 0$. Note also that $\tilde{G}^{\rho_{j_k}}$ converges to G weakly in X, (strongly) in $L^2_w(B\backslash B_r)$ for any $r > 0$, and pointwise a.e.

Finally, to prove part (iv) of Theorem (1.3), recall from Lemma (4.2) that if $\tilde{G}^\rho$ is the approximate Green function for $B_{2r} = B_{2r}(y)$Theorem (1.3), recall from Lemma (4.2) that if $\tilde{G}^\rho$ is the approximate Green function for $B_{2r} = B_{2r}(y)$ with pole y, then

$$\begin{aligned}\int_{B_{2r}\backslash B_r} < A\nabla\tilde{G}^\rho, \nabla\tilde{G}^\rho > &\leq \frac{c}{r^2}\int_{B_{3r/2}\backslash B_{r/2}} (\tilde{G}^\rho)^2 v, \ \rho < r/2,\\ &\leq \frac{c}{r^2}(\operatorname*{ess\,sup}_{B_{3r/2}\backslash B_{r/2}} \tilde{G}^\rho)^2\ v(B_r).\end{aligned}$$

Since G^ρ is a nonnegative solution in $B_{2r}\backslash B_\rho$, Harnack's inequality (2.5) gives

$$\operatorname*{ess\,sup}_{B_{3r/2}\backslash B_{r/2}} \tilde{G}^\rho \leq e^{c_1\mu(B_r)} \operatorname*{ess\,inf}_{B_{3r/2}\backslash B_{r/2}} \tilde{G}^\rho, \text{ with } \mu(B_r) = [v(B_r)/w(B_r)]^{\frac{1}{2}}.$$

Thus, if $\rho < \frac{r}{2}$,

$$(5.2) \qquad (\operatorname*{ess\,inf}_{B_{3r/2}\backslash B_{r/2}} \tilde{G}^\rho)^2 \geq c\frac{r^2}{v(B_r)}e^{-c_1\mu(B_r)}\int_{B_{2r}\backslash B_r} < A\nabla\tilde{G}^\rho, \nabla\tilde{G}^\rho > .$$

Now pick φ with $\varphi(x) = 1$ in B_r, $\text{supp}(\varphi) \subset B_{2r}$ and $|\nabla\varphi| \le c/r$. If $\rho < r$,

$$1 = \frac{1}{v(B_\rho)} \int_{B_\rho} \varphi v = a_0(G^\rho, \varphi) = \int < A\nabla\tilde{G}^\rho, \nabla\varphi >$$
$$\le \left(\int_{\text{supp}\nabla\varphi} < A\nabla\tilde{G}^\rho, \nabla\tilde{G}^\rho > \right)^{\frac{1}{2}} \left(\int < A\nabla\varphi, \nabla\varphi > \right)^{\frac{1}{2}}$$
$$\le \left(\int_{B_{2r}\setminus B_r} < A\nabla\tilde{G}^\rho, \nabla\tilde{G}^\rho > \right)^{\frac{1}{2}} \left([\frac{c}{r}]^2 v(B_r) \right)^{\frac{1}{2}}$$

since supp $\nabla\varphi \subset B_{2r}\setminus B_r$. Combining this estimate with (5.2) gives

$$\underset{B_{3r/2}\setminus B_{r/2}}{\text{ess inf}}\ \tilde{G}^\rho \ge c\frac{r^2}{v(B_r)} e^{-c_1\mu(B_r)}, \quad \rho < \frac{r}{2}. \tag{5.3}$$

Here, $\tilde{G}^\rho = \tilde{G}^\rho_{2r}$ is the approximate Green function for $B_{2r} = B_{2r}(y)$ with pole y.

The rest of the proof of part (iv) of Theorem (1.3) will be similar to the summing procedure used to prove Lemma (3.5). It follows from (5.3) by using Lemma (2.7) with $\varphi = -G^\rho_{2r}$ (note $G^\rho_{2r} \in H(B_{3r/2})$) that

$$G^\rho_{2r.k} \ge c\frac{r^2}{v(B_r)} e^{-c_1\mu(B_r)} \text{ near } \partial B_r.$$

Thus,

$$G^\rho_{2r.k} - G^\rho_{r.k} \ge c\frac{r^2}{v(B_r)} e^{-c_1\mu(B_r)} \text{ near } \partial B_r.$$

Consequently, by the weak maximum principle,

$$\tilde{G}^\rho_{2r} - \tilde{G}^\rho_r \ge c\frac{r^2}{v(B_r)} e^{-c_1\mu(B_r)} \text{ a.e. in } B_r.$$

Assume now for simplicity that $B = B_R$ has center y. If $r < \frac{R}{2}$, choose a positive integer m with $r2^m \le R < r2^{m+1}$. In B_{2r},

$$\tilde{G}^\rho_R \ge \tilde{G}^\rho_{r2^m} = \tilde{G}^\rho_{2r} + \sum_{j=1}^{m-1} [\tilde{G}^\rho_{r2^{j+1}} - \tilde{G}^\rho_{r2^j}] \ ;$$

if $m = 1$, the sum is missing. Thus, a.e. in $B_r \backslash B_{r/2}$,

$$G_R^\rho \geq c\frac{r^2}{v(B_r)}e^{-c_1\mu(B_r)} + c\sum_{j=1}^{m-1}\frac{(2^j r^2)}{v(B_{2^j r})}e^{-c_1\mu(B_{2^j r})}$$
$$\geq c\int_r^R \frac{t^2}{v(B_t)}e^{-c_1'\mu(B_t)}\frac{\mathrm{d}t}{t}$$

for some $c_1' > 0$ by the doubling condition on the weights. The desired result follows by letting $\rho \to 0$. In case B is not centered at y, note that $y \in \frac{1}{2}B$ implies $B \supset B_{R/2}(y) = B'$, and apply the estimate above to B', $r < \frac{1}{4}R$.

6. - Theorem (1.8)

In this section, we prove Theorem (1.8) and the uniformity for $y \in \frac{1}{2}B$ of the estimates in part (i) of Theorem (1.3).

To prove Theorem (1.8), first note by Hölder's inequality that

$$\ell(\varphi) = \int_B f\tilde{\varphi}\mathrm{d}x$$

defines a continuous linear functional on H_0 if $f/v \in L_v^{(2\sigma)'}(B)$ since

$$\left(\int_B |\tilde{\varphi}|^{2\sigma}v\right)^{1/2\sigma} \leq c_{v,w,B}\left(\int_B |\nabla\tilde{\varphi}|^2 w\right)^{\frac{1}{2}} \quad \text{by (1.9)}$$
$$\leq c_{v,w,B}\|\varphi\|_{H_0} \quad \text{by (1.1)}.$$

Hence, by the Lax-Milgram theorem, there is a unique $u \in H_0$ with $a_0(u, \varphi) = \int_B f\tilde{\varphi}$ (cf. (1.4) and (1.5)). Moreover,

$$\|u\|_{H_0} \leq c_{v,w,B}\ \|f/v\|_{L_v^{(2\sigma)'}(B)}.$$

Taking $\varphi = G^\rho$ (with pole y), we get

$$a_0(u, G^\rho) = \int_B f\tilde{G}^\rho\ \mathrm{d}x. \tag{6.1}$$

The left side equals

$$\frac{1}{v(B_\rho)}\int_{B_\rho}\tilde{u}v \to \tilde{u}(y), \text{ for a.e. } y \in B \text{ as } \rho \to 0.$$

Note that

$$\left|\int_B fg\right| \leq \left(\int_B \left(\frac{|f|}{v}\right)^{t'} v\right)^{\frac{1}{t'}} \left(\int_B |g|^t v\right)^{\frac{1}{t}}$$
$$\leq \left(\int_B \left(\frac{|f|}{v}\right)^{t'} v\right)^{\frac{1}{t'}} \|g\|_X, \quad (X = X_{t,s}).$$

Hence, under the hypothesis of the first part of Theorem (1.8), the map $g \to \int_B fg$ is a continuous linear functional on X for some $t < \sigma$. Since $\tilde{G}^{\rho_j}(x)$ converges weakly in X to $G(x,y)$ if $y \in \frac{1}{2}B$, $t < \sigma$ and $s < s_0$, it follows that the right side of (6.1), with $\rho = \rho_j$, converges to $\int_B f(x)G(x,y)\mathrm{d}x$, $y \in \frac{1}{2}B$. This proves the first part of Theorem (1.8).

The proof of the second part is similar. In fact, since

$$\left|\int_B < F, \nabla\varphi >\right| \leq \left(\int_B \left(\frac{|F|}{w}\right)^2 w\right)^{\frac{1}{2}} \left(\int_B |\nabla\varphi|^2 w\right)^{\frac{1}{2}}$$
$$\leq \left(\int_B \left(\frac{|F|}{w}\right)^2 w\right)^{\frac{1}{2}} \|\varphi\|_{H_0},$$

there is a unique $u \in H_0$ with

$$a_0(u,\varphi) = \int_B < F, \nabla\tilde{\varphi} > \mathrm{d}x, \qquad \varphi \in H_0.$$

Also,

$$\|u\|_{H_0} \leq \left(\int_B \left(\frac{|F|}{w}\right)^2 w\right)^{\frac{1}{2}}.$$

Taking $\varphi = G^\rho$ and observing that

$$\left|\int_B < F, \nabla g >\right| \leq \left(\int_B \left(\frac{|F|}{w}\right)^{s'} w\right)^{1/s'} \left(\int_B |\nabla g|^s w\right)^{1/s}$$
$$\leq \left(\int_B \left(\frac{F}{w}\right)^{s'} w\right)^{1/s'} \|g\|_X,$$

the result follows as before if $s < 2\sigma/(\sigma+1)$, i.e., $s' > 2\sigma/(\sigma-1)$.

We now show that the sizes of the norms in part (i) of Theorem (1.3) are uniform in y for $y \in \frac{1}{2}B$. The proof uses the representations in Theorem (1.8) and an iteration of the Moser type as in [2]. Since many of the details are like those in the proof of Lemma (3.1) of [2], we shall be brief. Let u be the Lax-Milgram solution of $\mathbf{L}u = f$, $u \in H_0$, $u = \{u_j\}$, $u_j \in \mathrm{Lip}_0(B)$, $\|u_j - u\|_0 \to 0$. Let $t < \sigma$ and

$$k = \left(\int\limits_B \left(\frac{|f|}{v}\right)^{t'} v \right)^{1/t'}.$$

For $\beta \geq 1$ and $k < M < \infty$, define $H_M(\tau) = \tau^\beta - k^\beta$ for $\tau \in [k, M]$ and $H_M(\tau) = M^\beta - k^\beta + \beta M^{\beta-1}(\tau - M)$ for $\tau > M$. Let $\psi_j = u_j^+ + k$, and for fixed M define

$$\varphi_j(x) = G(\psi_j(x)) = \int\limits_k^{\psi_j(x)} H_M'(\tau)^2 \, d\tau.$$

As in [2], $\varphi_j \in \mathrm{Lip}_0(B)$ and $\|\varphi_j\|_0$ is bounded. Hence, there is a subsequence, which we again denote by $\{\varphi_j\}$, which converges weakly in H_0 to φ. Then

$$\lim\ a_0(u_j, \varphi_j) = \lim\ a_0(u, \varphi_j) = a_0(u, \varphi),$$

and since

$$a_0(u_j, \varphi_j) = \int\limits_B < A\nabla u_j, \nabla \varphi_j >$$

and (Lax-Milgram)

$$a_0(u, \varphi_j) = \int\limits_B f\varphi_j,$$

we obtain

$$\int\limits_B < A\nabla u_j, \nabla\varphi_j > = \int\limits_B f\varphi_j + \delta_j, \quad \delta_j \to 0. \tag{6.2}$$

The left side equals

$$\int\limits_B < A\nabla\psi_j, \nabla\psi_j > G'(\psi_j) \geq \int\limits_B |\nabla\psi_j|^2 G'(\psi_j) w.$$

Writing $\varphi_j = G(\varphi_j)$ and using $G(\tau) \leq \tau G'(\tau)$ and $\psi_j \geq k$, we obtain

$$\int\limits_B |\nabla\psi_j|^2 G'(\psi_j) w \leq \frac{1}{k} \int\limits_B |f| \psi_j^2 G'(\psi_j) + |\delta_j|.$$

Since the integrand on the left is $|\nabla(H_M(\psi_j))|^2 w$ and the one on the right is $|f|\ |H'_M(\psi_j)\psi_j|^2$, Sobolev's inequality gives

$$\left(\int_B |H_M(\psi_j)|^{2\sigma} v\right)^{\frac{1}{\sigma}} \le c\left(\frac{1}{k}\int_B |f|\ |H'_M(\psi_j)\psi_j|^2 + |\delta_j|\right)$$
$$\le c\left(\int_B |H'_M(\psi_j)\psi_j|^{2t} v\right)^{1/t} + c|\delta_j|,$$

by Hölder's inequality and the definition of k. As $j \to \infty$, $\psi_j \to \tilde{u}^+ + k$ a.e. for a subsequence. Setting $\psi = \tilde{u}^+ + k$, we obtain

$$\left(\int_B H_M(\psi)|^{2\sigma} v\right)^{1/2\sigma} \le c\left(\int_B |H'_M(\psi)\psi|^{2t} v\right)^{1/2t}$$

Note that $H_M(\tau) \ge (\tau^\beta - k^\beta)\chi_{(k.M)}(\tau)$ and $H'_M(\tau) \le \beta\tau^{\beta-1}$. Thus, letting $M \to \infty$,

$$\left(\int_B (\psi^\beta - k^\beta)^{2\sigma} v\right)^{1/2\sigma} \le c\beta\left(\int_B \psi^{2t\beta} v\right)^{1/2t}$$

Since $\psi \ge k$,

$$k^\beta \le c\left(\int_B \psi^{2t\beta} v\right)^{1/2t} \le c\beta\left(\int_B \psi^{2t\beta} v\right)^{1/2t}, \qquad (\beta \ge 1).$$

Hence, by Minkowski's inequality,

$$\left(\int_B \psi^{2\sigma\beta} v\right)^{1/2\sigma\beta} \le (c\beta)^{1/\beta}\left(\int_B \varphi^{2t\beta} v\right)^{1/2t\beta}. \tag{6.3}$$

When $\beta = 1$ and $t \le \sigma$, the integral on the right of (6.3) is finite, then

$$\left(\int_B \psi^{2t} v\right)^{1/2t} \le c\left(\int_B \psi^{2\sigma} v\right)^{1/2\sigma} \le c\left(\int_B |\tilde{u}|^{2\sigma} v\right)^{1/2\sigma} + ck$$
$$\le c\left(\int_B |\nabla\tilde{u}|^2 w\right)^{\frac{1}{2}} + ck \le c\|u\|_0 + ck$$
$$\le c\|u\|_0 + ck \le ck.$$

Hence, starting with $\beta = 1$ in (6.3) and assuming $\sigma > t$, iteration of (6.3) leads to $\|\psi\|_{L^\infty(B)} \leq ck$. Thus, $\|\tilde{u}^+\|_{L^\infty(B)} \leq ck$. A similar estimate for $\tilde{u}^-$ together with the representation

$$\tilde{u}(y) = \int_B f(x)G(x,y)\mathrm{d}x, \text{ for a.e. } y \in \frac{1}{2}B,$$

given by Theorem (1.8), implies by duality that if $t < \sigma$,

$$\int_B G(x,y)^t v(x)\mathrm{d}x \leq c \text{ independent of } y$$

for a.e. $y \in \frac{1}{2}B$, as desired.

The argument for $\nabla_x G(x,y)$ is similar. In this case, u is the Lax-Milgram solution of $-\mathbf{L}u =$ div F, i.e. $u \in H_0$ and

$$a_0(u,\varphi) = \int < F, \nabla\tilde{\varphi} >$$

if $\varphi \in H_0$. We assume that $s' > 2\sigma/(\sigma - 1)$, (> 2), and $|F|/w \in L_w^{s'}(B)$; then

$$\tilde{u}(y) = \int_B < F(x), \nabla G(x,y) > \mathrm{d}x, \text{ for a.e. } y \in \frac{1}{2}B$$

by Theorem (1.8). Use the same test functions φ_j as before except that now

$$k = \left(\int_B \left(\frac{|F|}{w}\right)^{s'} w \right)^{1/s'}.$$

The analogue of (6.2) is

$$\begin{aligned}\int_B < A\nabla u_j, \nabla\varphi_j > &= \int_B < F, \nabla\varphi_j > + \delta_j \\ &= \int_B < F, G'(\psi_j)\nabla\psi_j > + \delta_j.\end{aligned}$$

Thus, for $\varepsilon > 0$,

$$\int_B |\nabla\psi_j|^2 G'(\psi_j)w \leq \varepsilon \int_B |\nabla\psi_j|^2 G'(\psi_j)w + \frac{1}{\varepsilon}\int_B \left(\frac{|F|}{w}\right)^2 G'(\psi_j)w + |\delta_j|.$$

Taking $\varepsilon = \frac{1}{2}$ and using the fact that $\psi_j \geq k$, we obtain

$$\int_B |\nabla(H_M(\psi_j))|^2 w \leq \frac{4}{k^2} \int_B \left(\frac{|F|}{w}\right)^2 (H'_M(\psi_j)\psi_j)^2 w + 2|\delta_j|$$
$$\leq 4 \left[\int_B (H'_M(\psi_j)\psi_j)^{\frac{2s'}{s'-2}} w \right]^{\frac{s'-2}{s'}} + 2|\delta_j|,$$

by Hölder's inequality with exponents $s'/2$ and $s'/(s'-2)$. Now use Sobolev's inequality and let $j \to \infty$ and $M \to \infty$ as before to obtain

$$\left(\int_B \left(\psi^\beta - k^\beta\right)^{2\sigma} v\right)^{1/2\sigma} \leq c\beta \left(\int_B \psi^{2s'\beta/(s'-2)} w \right)^{(s'-2)/2s'}$$

Thus,

$$\left(\int_B \psi^{2\sigma\beta} v \right)^{1/2\sigma\beta} \leq (c\beta)^{1/\beta} \left(\int_B \psi^{2s'\beta/(s'-2)} w \right)^{(s'-2)/2s'\beta}$$

When $\beta = 1$ and $s'/(s'-2) < \sigma$, the expression on the right is bounded by a multiple of k, and iteration leads to $\|\tilde{u}\|_{L^\infty(B)} \leq ck$. It follows that

$$\int_B |\nabla_x G(x,y)|^s w(x) dx \leq c \text{ independent of } y$$

for a.e. $y \in \frac{1}{2}B$. The condition $s'/(s'-2) < \sigma$ is the same as $s' > 2\sigma/(\sigma-1)$, or $s < 2\sigma/(\sigma+1)$.

REMARK. A careful examination of the proof shows that the assumptions $w \in A_2$ and (1.2) can be replaced by assuming $w \in D^\infty$, Sobolev's inequality (1.9), the analogous Poincaré inequality in [1], $w^{-1} \in L^1_{\text{loc}}$ and (2.15). Of these, the last two guarantee a version of the compactness lemma (2.9) with L^1_w in the conclusion rather than L^2_w (see the last part of remark (2.13)), which means that where we used $\tilde{G}^{\rho_{jk}} \to G^\#$ in $L^2_w(B\backslash B_r)$ earlier, we can use $\tilde{G}^{\rho_{jk}} \to G^\#$ in $L^1_w(B\backslash B_r)$, and this makes the argument work.

We also note that (1.2) is necessary for Sobolev's inequality for doubling weights; see [1].

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