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EXISTENCE AND EXPONENTIAL DECAY IN NONLINEAR THERMOELASTICITY

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1. INTRODUCTION

It is well known that in the absence of dissipation, smooth solution of nonlinear elastic materials develop singularities in finite time, while for thermoelastic materials the conduction of the heat equation provides dissipation that competes with the destabilizing effect of nonlinearity in the elastic response. The level of subtlety of this dissipation depends on the boundary condition that the displacement and the thermal difference are supporting. Slemrod [1] showed the global existence of smooth solution for small data when the boundary is either traction-free and at a constant temperature or rigidly clamped and thermally insulated. A similar result was obtained by Zheng [2]. These boundary conditions get a simpler damping mechanism because they imply additional boundary conditions for u and the thermal difference θ, that is, if an end is clamped then the displacement u and the thermal difference θ satisfy $u_{xx} = 0$ and $\theta_{xxx} = 0$ there respectively. So we can make additional partial integrations which led to the desire a priori L^2-estimate.

In case of Dirichlet boundary condition for which the boundary is rigidly clamped and held at a constant temperature we lost the value of u_{xx} in that point and instead of it we get $u_{xx} + \alpha\theta_x = 0$. So this case leads ill behaved boundary terms and it is not possible to apply directly the multiplicative techniques to secure global estimate. Recently Racke and Shibata [3] proved Global existence of a smooth solution for these boundary conditions. To do this the authors showed the algebraic decay of the energy for the linear equation by studying the spectral properties of the stationary linearized problem. The rate of decay depends on higher regularity of the initial data and therefore the global existence result depends on the initial data to be small in $H^m(0, L)$ with m large. One of the authors of this paper proved in [4] (see also the work of Kim [5]) that the solution of the linearized thermoelastic system decays exponentially as time goes to infinity. This fact allows us to get simpler existence result for the corresponding nonlinear equation as was shown in [6] for small data (u_0, u_1) in $H^3(0, L) \times H^2(0, L)$.

The system in question is written as follows

$$u_{tt} - [S(u_x, \theta)]_x = 0, \qquad \text{in }]0, L[\times]0, \infty[\tag{1.1}$$

$$(\theta + \tau_0)[N(u_x, \theta)]_t - [Q(u_x, \theta_x, \theta)]_x = 0, \qquad \text{in }]0, L[\times]0, \infty[\tag{1.2}$$

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with the initial datas given by

$$u(x,0) = u_0(x); \qquad u_t(x,0) = u_1(x); \qquad \theta(x,0) = \theta_0(x) \tag{1.3}$$

and boundary conditions

$$u(0,t) = u(L,t) = \theta(0,t) = \theta(L,t) = 0. \tag{1.4}$$

We are denoting by u the displacement, by $\theta = T_a - \tau_0$ the thermal difference, where T_a is the absolute temperature and τ_0 is the reference temperature which we will assume to be constant. Finally, by S we will denote the stress tensor of Piola–Kirchoff, N stands for the specific entropy and by Q the heat flux.

We would like to remark that the dissipation given by the thermal difference is not strong enough to prevent development of singularities. The work of Hrusa and Messauodi showed that for a special class of nonlinear thermoelastic materials which occupy the whole line, there are smooth initial data for which the solution will develop singularities in finite time.

The main result of this paper is to improve the work in [6] by taking initial data (u_0, u_1) small in $H^2(0,L) \times H^1(0,L)$-norm. This fact allows us to choose large data (u_0, u_1) in the $H^3(0,L) \times H^2(0,L)$-norm. The approach we use here is different from others, we explore the dissipative properties to construct a Liapunov functional whose derivative is negative proportional to itself and we look for estimates of the nonlinear terms in functions of the dissipative terms associated to the thermoelastic system. The fact together with the local existence result (see [1]) give the estimate we need to get the global existence of smooth solutions.

2. EXISTENCE AND ASYMPTOTIC BEHAVIOUR

In this section we will assume that the functions S, N, Q are in C^4 satisfying the following hypotheses

$$\begin{aligned} &\frac{\partial S}{\partial u_x}(0,0) = 1; \qquad \frac{\partial S}{\partial \theta}(0,0) \neq 0; \qquad \frac{\partial N}{\partial u_x}(0,0) \neq 0; \\ &\frac{\partial N}{\partial \theta}(0,0) > 0; \qquad \frac{\partial Q}{\partial u_x}(0,0,0) = 0; \qquad \frac{\partial Q}{\partial \theta}(0,0,0) = 0. \end{aligned} \tag{2.1}$$

To simplify notations we will introduce

$$-\frac{\partial S}{\partial \theta}(0,0) =: \alpha; \qquad \frac{\partial Q/\partial \theta_x}{(\theta + \tau_0)(\partial N/\partial \theta)}(0,0,0) = k > 0; \qquad \frac{\partial N/\partial u_x}{\partial N/\partial \theta}(0,0,0) = \beta.$$

Where the product $\alpha\beta > 0$. For the initial data we will impose

$$u_0 \in H^3(0,L); \quad u_1 \in H^2(0,L); \quad u_2 \in H^1(0,L); \quad \theta_1 \in H^2(0,L); \quad \theta_0 \in H^3(0,L). \tag{2.2}$$

By u_2 and θ_1 we are denoting

$$u_2 =: [S(u_x, \theta)]_x|_{t=0} \tag{2.3}$$

$$\theta_1 =: \frac{\partial N/\partial u_x}{\partial N/\partial \theta} u_{1,x}|_{t=0} + \left.\frac{Q(u_x, \theta_x, \theta)_x}{(\theta + \tau_0)(\partial N/\partial \theta)}\right|_{t=0} \tag{2.4}$$

satisfying the compatibility conditions

$$u_0 = u_1 = u_2 = \theta_1 = \theta_0 = 0 \qquad \text{at } x = L,\ x = 0. \tag{2.5}$$

With this hypotheses we can show that there exists only one local solution for system (1.1)–(1.4) (see [7]), defined in the maximal interval $[0, T_m[$. So, to get global smooth solution we will show that

$$\|\theta(\cdot, t)\|_{H^3(0,L)} + \|u(\cdot, t)\|_{H^3(0,L)} \le c; \qquad \forall t \ge 0.$$

To do this we will regard system (1.1)–(1.4) as

$$u_{tt} - u_{xx} + \alpha\theta_x = F, \qquad \text{in }]\theta, L[\times]0, T_m[\tag{2.6}$$

$$\theta_t - k\theta_{xx} + \beta u_{xt} = G, \qquad \text{in }]0, L(\times]0, T_m[. \tag{2.7}$$

Where

$$F = \left\{\frac{\partial S}{\partial u_x}(u_x, \theta) - 1\right\}u_{xx} + \left\{\frac{\partial S}{\partial \theta}(u_x, \theta) + \alpha\right\}\theta_x$$

$$G = \left\{\frac{\partial Q/\partial \theta_x}{(\theta + \tau_0)(\partial N/\partial \theta)} - k\right\}\theta_{xx} - \left\{\frac{\partial N/\partial u_x}{\partial N/\partial \theta} - \beta\right\}u_{xt}$$
$$+ \frac{\partial Q/\partial u_x}{(\theta + \tau_0)(\partial N/\partial \theta)}u_{xx} + \frac{\partial Q/\partial \theta}{(\theta + \tau_0)(\partial N/\partial \theta)}\theta_x.$$

For simplicity we will put

$$\eta_1 = \frac{\partial S}{\partial u_x}(u_x, \theta) - 1; \qquad \eta_2 = \frac{\partial S}{\partial \theta}(u_x, \theta) + \alpha;$$

$$W_1 = \frac{\partial Q/\partial \theta_x}{(\theta + \tau_0)(\partial N/\partial \theta)} - k; \qquad W_2 = \frac{\partial N/\partial u_x}{\partial N/\partial \theta} - \beta;$$

$$W_3 = \frac{\partial Q/\partial u_x}{(\theta + \tau_0)(\partial N/\partial \theta)}; \qquad W_4 = \frac{\partial Q/\partial \theta}{(\theta + \tau_0)(\partial N/\partial \theta)}.$$

To facilitate our analysis let us introduce the linear system

$$U_{tt} - U_{xx} + \alpha\psi_x = \mathfrak{F}, \qquad \text{in }]0, L[\times]0, T_m[\tag{2.8}$$

$$\psi_t - k\psi_{xx} + \beta U_{xt} = \mathcal{G}, \qquad \text{in }]0, L[\times]0, T_m[\tag{2.9}$$

$$U(x, 0) = U_0, \qquad U_t(x, 0) = U_1, \qquad \psi(x, 0) = \psi_0,$$

$$U(0, t) = U(L, t) = \psi(0, t) = \psi(L, t) = 0.$$

From now on and without loss of generality we will assume that α and β are positive real numbers. First we will study the asymptotic behaviour of the linearized equation (2.8)–(2.9). To do this we define the following functionals

$$E_1(t; U, \psi) = \frac{1}{2}\int_0^L \left\{|U_t|^2 + |U_x|^2 + \frac{\alpha}{\beta}|\psi|^2\right\} dx;$$

$$E_2(t; U, \psi) = \frac{1}{2}\int_0^L \left\{|U_{tt}|^2 + |U_{xt}|^2 + \frac{\alpha}{\beta}|\psi_t|^2\right\} dx$$

$$E_3(t; U, \psi) = \frac{1}{2}\int_0^L \left\{|U_{xt}|^2 + |U_{xx}|^2 + \frac{\alpha}{\beta}|\psi_x|^2\right\} dx.$$

Let us multiply equation (2.8) by U_t and (2.9) by $(\alpha/\beta)\psi$ and summing the product result we have

$$\frac{d}{dt}E_1(t; U, \psi) = -k\frac{\alpha}{\beta}\int_0^L |\psi_x|^2\,dx + \int_0^L \left\{\mathfrak{F}U_t + \frac{\alpha}{\beta}\mathcal{G}\psi\right\}dx.$$

Assuming regular data, and since U_t and ψ_t have the same boundary conditions, we get

$$\frac{d}{dt}E_2(t; U, \psi) = -k\frac{\alpha}{\beta}\int_0^L |\psi_{xt}|^2\,dx + \int_0^L \left\{\mathfrak{F}_t U_{tt} + \frac{\alpha}{\beta}\mathcal{G}_t\psi_t\right\}dx. \tag{2.10}$$

To get the above identities we use essentially the fact that U_t and ψ_t have the same boundary conditions as U and ψ. But this is not the case for U_x and ψ_x. It is in this point that the typical difficulty for boundary conditions of type Dirichlet-Dirichlet appears. Let us see in detail this fact. Multiplying equation (2.8) by $-U_{xxt}$ and (2.9) by $-(\alpha/\beta)\psi_{xx}$ and summing up the product result we have

$$\frac{d}{dt}E_3(t; U,\psi) = -k\frac{\alpha}{\beta}\int_0^L \psi_{xx}^2\,dx + \alpha\psi_x(x,t)U_{xt}(x,t)_{x=0}^{x=L}$$
$$-\int_0^L \left\{\mathfrak{F}U_{xxt} + \frac{\alpha}{\beta}\mathcal{G}\psi_{xx}\right\}dx. \tag{2.11}$$

The derivative of E_3 has a pointwise term involving second order derivatives. Which is not possible to bound using directly the Sobolev inequalities. To overcome this fact we will use the following lemma.

LEMMA 2.1. Let us take $(v_0, v_1, f) \in H_0^1(0,L) \cap H^2(0,L) \times H_0^1(0,L) \times H^1(0,T; L^2(0,L))$ and let v be the solution of

$$v_{tt} - v_{xx} = f(x,t) \quad \text{in }]0,L[\times]0,T[$$
$$v(x,0) = v_0(x); \quad v_t(x,0) = v_1(x) \quad \text{in }]0,L[\tag{2.12}$$
$$v(0,t) = v(L,t) = 0 \quad \text{on }]0,T[$$

then the following identity holds

$$\frac{L}{4}[v_x^2(L,t) + v_x^2(0,t)] = \frac{d}{dt}\int_0^L \left(x - \frac{L}{2}\right)v_t v_x\,dx + \frac{1}{2}\int_0^L (v_x^2 + v_t^2)\,dx$$
$$-\int_0^L \left(x - \frac{L}{2}\right) f v_x\,dx.$$

Proof. Multiplying (2.12) by $(x - L/2)v_x$ and integrating over $[0,L]$ we have

$$\int_0^L \left(x - \frac{L}{2}\right)v_{tt}v_x\,dx - \int_0^L \left(x - \frac{L}{2}\right)v_{xx}v_x\,dx = \int_0^L \left(x - \frac{L}{2}\right)f v_x\,dx. \tag{2.13}$$

Since $v_t(0, t) = v_t(L, t) = 0$, direct calculations yields

$$\int_0^L \left(x - \frac{L}{2}\right) v_{tt} v_x \, dx = \frac{d}{dt} \int_0^L \left(x - \frac{L}{2}\right) v_t v_x \, dx - \int_0^L \left(x - \frac{L}{2}\right) v_t v_{xt} \, dx$$

$$= \frac{d}{dt} \int_0^L \left(x - \frac{L}{2}\right) v_t v_x \, dx + \frac{1}{2} \int_0^L v_t^2 \, dx. \tag{2.14}$$

On the other hand

$$\int_0^L \left(x - \frac{L}{2}\right) v_{xx} v_x \, dx = \frac{1}{2} \int_0^L \left(x - \frac{L}{2}\right) (v_x^2)_x \, dx$$

$$= \frac{L}{4} [v_x^2(L, t) + v_x^2(0, t)] - \frac{1}{2} \int_0^L v_x^2 \, dx. \tag{2.15}$$

From (2.13) to (2.15) our result follows. ■

Motivated in Lemma 2.1 we introduce the following functional

$$E_4(t) = -\int_0^L \left(x - \frac{L}{2}\right) U_{xt} U_{tt} \, dx.$$

Using equation (2.8) and Lemma 2.1 we easily get

$$\frac{d}{dt} E_4(t) = -\frac{L}{4} \{ |U_{xt}(0, t)|^2 + |U_{xt}(L, t)|^2 \} + \frac{1}{2} \int_0^L \{ |U_{xt}|^2 + |U_{tt}|^2 \} \, dx$$

$$+ \alpha \int_0^L \left(x - \frac{L}{2}\right) \psi_{xt} U_{xt} \, dx - \int_0^L \left(x - \frac{L}{2}\right) \mathfrak{F}_t U_{xt} \, dx. \tag{2.16}$$

Finally, we define the following functions

$$E_5(t) = \int_0^L U_{xt} \psi \, dx; \qquad E_6(t) = \int_0^L U_{xt} U_x \, dx,$$

$$\mathfrak{N}(t; U, \psi) = \int_0^L \{ |U_x|^2 + |U_t|^2 + |U_{xx}|^2 + |U_{xt}|^2 + |\psi|^2 + |\psi_t|^2 + |\psi_x|^2 \} \, dx$$

$$\mathfrak{M}(t; U, \psi) = \int_0^L \{ |U_{xx}|^2 + |U_{xt}|^2 + |\psi_x|^2 + |\psi_{xt}|^2 + |\psi_{xx}|^2 \} \, dx.$$

We will prove in the following lemma that there exists a linear combination of the functions E_i ($i = 1, \ldots, 6$), we will denote by K, that is

$$K(t; u, \psi) = \kappa_1 E_1 + \kappa_2 E_2 + \frac{10k}{\alpha} E_3 + \frac{\beta}{6} E_4 + E_5 + \frac{\beta}{2} E_6,$$

which is a Liapunov functional. This is shown more precisely in the following lemma.

LEMMA 2.2. There exist positive constants κ_i ($i = 1, 2$) and c_0, c_1 such that the derivative of $K(t; U, \psi)$ defined above satisfies

$$\frac{d}{dt}K(t; U, \psi) \le -\frac{k^2}{\beta}\int_0^L \{|\psi_x|^2 + |\psi_{xt}|^2 + |\psi_{xx}|^2\}\,dx - \frac{\beta}{8}\int_0^L \{|U_{xt}|^2 + |U_{xx}|^2\}\,dx$$
$$- \frac{\beta L}{48}\{|U_{xt}(0, t)|^2 + |U_{xt}(L, t)|^2\} + R(t; U, \psi), \tag{2.17}$$

where

$$R(t; U, \psi) = \kappa_1 \int_0^L \left\{\mathfrak{F} U_t + \frac{\alpha}{\beta}\mathcal{G}\psi\right\} dx + \kappa_2 \int_0^L \left\{\mathfrak{F}_t U_{tt} + \frac{\alpha}{\beta}\mathcal{G}_t \psi_t\right\} dx$$
$$- \frac{10k}{\alpha}\int_0^L \left\{\mathfrak{F} U_{xxt} + \frac{\alpha}{\beta}\mathcal{G}\psi_{xx}\right\} dx - \frac{\beta}{6}\int_0^L \left(x - \frac{L}{2}\right)\mathfrak{F}_t U_{xt}\,dx$$
$$- \int_0^L \{\mathfrak{F}\psi_x - \mathcal{G}H_{xt}\}\,dx - \frac{\beta}{2}\int_0^L \mathfrak{F} U_{xx}\,dx + \frac{\beta}{4}\int_0^L |\mathfrak{F}|^2\,dx,$$

and

$$c_0\mathfrak{N}(t; U, \psi) \le K(t, U, \psi) \le c_1\mathfrak{N}(t; U, \psi). \tag{2.18}$$

Proof. Using (2.8)-(2.9) we obtain that

$$\frac{d}{dt}E_5(t) = -\beta\int_0^L |U_{xt}|^2\,dx + \alpha\int_0^L |\psi_x|^2\,dx + k\int_0^L \psi_{xx}U_{xt}\,dx$$
$$- \int_0^L \psi_x U_{xx}\,dx - \int_0^L \{\mathfrak{F}\psi_x - \mathcal{G}U_{xt}\}\,dx, \tag{2.19}$$

similarly, using (2.8) we get

$$\frac{d}{dt}E_6(t) = \int_0^L |U_{xt}|^2\,dx - \int_0^L |U_{xx}|^2\,dx + \alpha\int_0^L U_{xx}\psi_x\,dx - \int_0^L \mathfrak{F} U_{xx}\,dx. \tag{2.20}$$

From (2.19) and (2.20) we easily obtain

$$\frac{d}{dt}\left\{E_5(t) + \frac{\beta}{2}E_6(t)\right\} = -\frac{\beta}{2}\int_0^L |U_{xt}|^2\,dx - \frac{\beta}{2}\int_0^L |U_{xx}|^2\,dx + \frac{\alpha\beta}{2}\int_0^L U_{xx}\psi_s\,dx$$
$$+ \alpha\int_0^L |\psi_x|^2\,dx + k\int_0^L \psi_{xx}U_{xt}\,dx - \int_0^L U_{xx}\psi_x\,dx$$
$$- \frac{\beta}{2}\int_0^L \mathfrak{F} U_{xx}\,dx - \int_0^L \{\mathfrak{F}\psi_x - \mathcal{G}U_{xt}\}\,dx. \tag{2.21}$$

On the other hand using (2.8) it is not difficult to see that

$$\int_0^L |U_{tt}|^2\,dx \le 3\int_0^L \{|U_{xx}|^2 + \alpha^2|\psi_x|^2 + |\mathfrak{F}|^2\}\,dx.$$

From (2.16) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E_4(t) \leq -\frac{L}{4}\{|U_{xt}(0,t)|^2 + |U_{xt}(L,t)|^2\} + \frac{3}{2}\int_0^L \{|U_{xt}|^2 + |U_{xx}|^2\}\,\mathrm{d}x$$
$$+ \frac{3\alpha^2}{2}\int_0^L |\psi_x|^2\,\mathrm{d}x + \alpha\int_0^L \left(x - \frac{L}{2}\right)\psi_{xt}U_{xt}\,\mathrm{d}x$$
$$- \int_0^L \left(x - \frac{L}{2}\right)\mathfrak{F}_t U_{xt}\,\mathrm{d}x + \frac{3}{2}\int_0^L |\mathfrak{F}|^2\,\mathrm{d}x. \tag{2.22}$$

Relation (2.21) together with (2.22) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\beta}{6}E_4(t) + E_5(t) + \frac{\beta}{2}E_6(t)\right\}$$
$$\leq -\frac{\beta L}{24}\{|U_{xt}(0,t)|^2 + |U_{xt}(L,t)|^2\} - \frac{\beta}{4}\int_0^L \{|U_{xt}|^2 + |U_{xx}|^2\}\,\mathrm{d}x$$
$$+ \frac{\alpha^2\beta}{4}\int_0^L |\psi_x|^2\,\mathrm{d}x + \frac{L\alpha\beta}{12}\int_0^L \psi_{xt}U_{xt}\,\mathrm{d}x + \left(\frac{\alpha\beta}{2} - 1\right)\int_0^L U_{xx}\psi_x\,\mathrm{d}x$$
$$+ \alpha\int_0^L |\psi_x|^2\,\mathrm{d}x + k\int_0^L \psi_{xx}U_{xt}\,\mathrm{d}x - \int_0^L \{\mathfrak{F}\psi_x - \mathcal{G}U_{xt}\}\,\mathrm{d}x$$
$$- \frac{\beta}{6}\int_0^L \left(x - \frac{L}{2}\right)\mathfrak{F}_t U_{xt}\,\mathrm{d}x + \frac{\beta}{4}\int_0^L |\mathfrak{F}|^2\,\mathrm{d}x - \frac{\beta}{2}\int_0^L \mathfrak{F}U_{xx}\,\mathrm{d}x. \tag{2.23}$$

Note that

$$\frac{L\alpha\beta}{12}\int_0^L |\psi_{xt}U_{xt}|\,\mathrm{d}x + k\int_0^L |\psi_{xx}U_{xt}|\,\mathrm{d}x$$
$$\leq \frac{\beta}{8}\int_0^L |U_{xt}|^2\,\mathrm{d}x + \frac{L^2\alpha^2\beta}{36}\int_0^L |\psi_{xt}|^2\,\mathrm{d}x + \frac{4k^2}{\beta}\int_0^L |\psi_{xx}|^2\,\mathrm{d}x \tag{2.24}$$

and

$$\left(\frac{\alpha\beta}{2} - 1\right)\int_0^L U_{xx}\psi_x\,\mathrm{d}x \leq \frac{\beta}{8}\int_0^L |U_{xx}|^2\,\mathrm{d}x + \frac{2}{\beta}\left(\frac{\alpha\beta}{2} - 1\right)^2\int_0^L |\psi_x|^2\,\mathrm{d}x. \tag{2.25}$$

Substitution of (2.24) and (2.25) into (2.23) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\beta}{6}E_4(t) + E_5(t) + \frac{\beta}{2}E_6(t)\right\}$$
$$\leq -\frac{\beta L}{24}\{|U_{xt}(0,t)|^2 + |U_{xt}(L,t)|^2\} - \frac{\beta}{8}\int_0^L \{|U_{xt}|^2 + |U_{xx}|^2\}\,\mathrm{d}x + c_2\int_0^L |\psi_x|^2\,\mathrm{d}x$$
$$+ \frac{4k^2}{\beta}\int_0^L |\psi_{xx}|^2\,\mathrm{d}x + \frac{L^2\alpha^2\beta}{36}\int_0^L |\psi_{xt}|^2\,\mathrm{d}x - \int_0^L \{\mathfrak{F}\psi_x - \mathcal{G}U_{xt}\}\,\mathrm{d}x$$
$$- \frac{\beta}{6}\int_0^L \left(x - \frac{L}{2}\right)\mathfrak{F}_t U_{xt}\,\mathrm{d}x + \frac{\beta}{4}\int_0^L |\mathfrak{F}|^2\,\mathrm{d}x - \frac{\beta}{2}\int_0^L \mathfrak{F}U_{xx}\,\mathrm{d}x. \tag{2.26}$$

From the Gagliardo–Nirenberg inequality we get

$$\alpha\psi_x U_{xt}|_{x=0}^{x=L} \le 2\alpha c_0\left\{\int_0^L |\psi_x|^2\,\mathrm{d}x\right\}^{1/4}\left\{\int_0^L |\psi_x|^2 + |\psi_{xx}|^2\,\mathrm{d}x\right\}^{1/4}\{|U_{xt}(0,t)|^2 + |U_{xt}(L,t)|^2\}^{1/2}$$

$$\le c_3\int_0^L |\psi_x|^2\,\mathrm{d}x + \frac{k\alpha}{2\beta}\int_0^L |\psi_{xx}|^2\,\mathrm{d}x + \frac{\alpha\beta L}{480k}\{|U_{xt}(0,t)|^2 + |U_{xt}(L,t)|^2\},$$

for a positive constant c_3. Using (2.11) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E_3(t) \le -\frac{k\alpha}{2\beta}\int_0^L |\psi_{xx}|^2\,\mathrm{d}x + c_3\int_0^L |\psi_x|^2\,\mathrm{d}x$$

$$-\int_0^L \left\{\mathcal{F}U_{xxt} + \frac{\alpha}{\beta}\mathcal{G}\psi_{xx}\right\}\mathrm{d}x + \frac{\alpha\beta L}{480k}\{|U_{xt}(0,t)|^2 + |U_{xt}(L,t)|^2\}.$$

Our result follows from (2.26) and the last inequality for κ_1 and κ_2 satisfying

$$\kappa_1 \ge \frac{\beta}{k\alpha}\left(\frac{10kc_3}{\alpha} + c_2\right) + \frac{k}{\alpha} \quad \text{and} \quad \kappa_2 \ge \frac{L^2\alpha\beta^2}{36k} + \frac{k}{\alpha}.$$

Finally, for κ_1 and κ_2 big enough inequality (2.18) holds. The proof is now complete. ∎

To get global solution we will suppose that the initial data satisfy

$$\|u_0\|^2_{H^2(0,L)} + \|u_1\|^2_{H^1(0,L)} + \|\theta_0\|^2_{H^2(0,L)} + \|\theta_1\|^2_{H^1(0,L)} + \|u_2\|^2_{H^2(0,L)} < \varepsilon^2. \qquad (2.27)$$

Since u_1, u_2 and θ_1 satisfy condition (2.2), then there exist a positive constant μ such that

$$\|u_{1,xx}\|^2_{L^2(0,L)} + \|\theta_{1,xx}\|^2_{L^2(0,L)} + \|u_{2,x}\|^2_{L^2(0,L)} < \mu^2, \qquad (2.28)$$

where ε is small (<1) and μ is large (>1). From (2.27) and (2.28) we have

$$\mathfrak{N}(0, u, \theta) < \varepsilon^2; \qquad \mathfrak{N}(0, u_t, \theta_t) < \mu^2. \qquad (2.29)$$

Using the continuity of the solution, it follows

$$\mathfrak{N}(t, u, \theta) + \frac{2\lambda}{c_0}\int_0^t\int_0^L \{|\theta_{xx}|^2 + |\theta_{xt}|^2\}\,\mathrm{d}x\,\mathrm{d}\tau \le c\varepsilon^2 \qquad \forall t \in [0, t_0[, \qquad (2.30)$$

$$\mathfrak{N}(t, u_t, \theta_t) + \frac{2\lambda}{c_0}\int_0^t\int_0^L \{|\theta_{xxt}|^2 + |\theta_{xtt}|^2\}\,\mathrm{d}x\,\mathrm{d}s < c\mu^2 \qquad \forall t \in [0, t_0[, \qquad (2.31)$$

for some $t_0 > 0$, $t_0 \le T_m$, $c = 6c_1/c_0 \ge 1$, where c_0, c_1 are defined by inequality (2.18), $\lambda = \min\{k^2/2\beta, \beta/16\}$. Let us define the functions

$$\mathcal{I}_1(t) = \mathfrak{M}(t; u, \theta) + \frac{2\lambda}{c_0}\int_0^t\int_0^L \{|\theta_{xx}|^2 + |\theta_{xt}|^2\}\,\mathrm{d}x\,\mathrm{d}\tau;$$

$$\mathcal{I}_2(t) = \mathfrak{M}(t; u_t, \theta_t) + \frac{2\lambda}{c_0}\int_0^t\int_0^L \{|\theta_{xxt}|^2 + |\theta_{xtt}|^2\}\,\mathrm{d}x\,\mathrm{d}\tau.$$

In this conditions there exists t_0 for which we have

$$\mathcal{G}_1(t) \le \frac{6c_1}{c_0}\varepsilon^2, \qquad \mathcal{G}_2(t) \le \frac{6c_1}{c_0}\mu^2 \qquad \forall t \in [0, t_0[. \tag{2.32}$$

Denoting by

$$\tau_1 = \sup\left\{\tau_1 > 0; \mathcal{G}_1(t) \le \frac{6c_1}{c_0}\varepsilon^2 \text{ in } [0, \tau_1[\right\},$$

$$t_2 = \sup\left\{\tau_2 > 0; \mathcal{G}_2(t) \le \frac{6c_1}{c_0}\mu_2 \text{ in } [0, \tau_0[\right\},$$

and by $t_3 =: \min\{t_1, t_2\}$. We only have two cases: (i) $t_3 = T_m$, (ii) $t_3 < T_m$. The first one implies that the solution is bounded and therefore $T_m = +\infty$. It remains only to consider the case (ii), which will be studied in our final theorem.

From Sobolev's embedding theorem and inequality (2.32) we get

$$|u_x(x,t)| \le c_0\varepsilon, \qquad |\theta(x,t)| \le c_0\varepsilon \qquad \forall (x,t) \in [0,L] \times [0, t_3[. \tag{2.33}$$

It is not difficult to see that there exists a positive constant c_1 such that

$$|\theta_x(x,t)| \le c_1\sqrt{\varepsilon}, \qquad \forall (x,t) \in [0,L] \times [0, t_3[. \tag{2.34}$$

In fact, we have

$$\int_0^L |\theta_{xx}|^2\,\mathrm{d}x = \int_0^L |\theta_{0,xx}|^2\,\mathrm{d}x + 2\int_0^t\int_0^L \theta_{xx}\theta_{xxt}\,\mathrm{d}x\,\mathrm{d}\tau$$

$$\le \varepsilon^2 + 2\left\{\int_0^t\int_0^L |\theta_{xx}|^2\,\mathrm{d}x\,\mathrm{d}\tau\right\}^{1/2}\left\{\int_0^t\int_0^L |\theta_{xxt}|^2\,\mathrm{d}x\,\mathrm{d}\tau\right\}^{1/2} \le c\varepsilon\mu. \tag{2.35}$$

For $\varepsilon < 1$ and $\mu > 1$. From the Gagliardo–Nirenberg inequality and (2.35) we get (2.34). Therefore, for $\delta > 0$ there exists $\varepsilon > 0$ for which (2.32), (2.33) implies

$$|\eta_i| < \delta, \quad i = 1, 2; \qquad |W_j| < \delta, \quad j = 1, \dots, 4. \tag{2.36}$$

From the Gagliardo–Nirenberg inequality and (2.32) we easily deduce that

$$|u_{xt}(x,t)| \le c_2\sqrt{\varepsilon}, \qquad |u_{tt}(x,t)| \le c_2\sqrt{\varepsilon}, \qquad |\theta_t(x,t)| \le c_2\sqrt{\varepsilon},$$
$$\forall (x,t) \in [0,L] \times [0, t_3], \tag{2.37}$$

for some $c_2 > 0$. Finally, using equation (2.6), (2.7) and inequalities (2.36), (2.37), we conclude that there exists $c_3 > 0$ satisfying

$$|\theta_{xx}(x,t)| \le c_3\sqrt{\varepsilon}; \qquad |u_{xx}(x,t)| \le c_2\sqrt{\varepsilon} \qquad \text{in } [0,L] \times [0, t_3[. \tag{2.38}$$

Let us denote by

$$\nu = \sup_{|x| \le c\varepsilon} \{|\partial^\rho \eta_i(x)|, |\partial^\rho W_j(x)|; i = 1,2; j = 1, \ldots, 4, 1 \le \rho \le 4\},$$

where ∂^ρ stands for the partial derivative of order $|\rho|$. From (2.6) it follows

$$u_{xxx} = u_{ttx} - \alpha\theta_{xx} - \nabla\eta_1 \cdot (u_{xx}, \theta_x)u_{xx} - \eta_1 u_{xxx} - \nabla\eta_2 \cdot (u_{xx}, \theta_x)\theta_x - \eta_2\theta_{xx},$$

from this identity we find that

$$\|u_{xxx}\|^2_{L^2(0,L)} \le c_4\mu^2 \qquad \text{in } [0, t_3[.$$

Similarly we have

$$\|\theta_{xxx}\|^2_{L^2(0,L)} \le c_4\mu^2; \qquad \|\theta_{xxt}\|^2_{L^2(0,L)} \le c_4\mu^2 \qquad \text{in } [0, t_3[.$$

Finally, we get

$$\|\theta_{xt}\|^2_{L^2(0,L)} \le 3\varepsilon\mu \quad \text{in } [0, t_3[; \qquad |\theta_{xt}(x,t)|^2 \le c_5\sqrt{\varepsilon} \quad \text{in } [0, L] \times [0, t_3[. \tag{2.39}$$

LEMMA 2.3. Let us suppose that the initial data satisfies conditions (2.1), (2.2), (2.3), (2.4), (2.5), (2.27) and (2.28), then there exist positive constants C_i for which the following inequalities hold

$$\int_0^L \mathfrak{F}_t U_{xx}\, dx \le \frac{1}{2}\frac{d}{dt}\int_0^L \eta_1 |U_{xx}|^2\, dx + C_1(\sqrt{\varepsilon} + \delta)\mathfrak{M}(t; U, \psi). \tag{2.40}$$

$$\left|\int_0^L \left(x - \frac{L}{2}\right)\mathfrak{F}_t U_{xt}\, dx\right| \le C_2(\sqrt{\varepsilon} + \delta)\mathfrak{M}(t; U, \psi) + \frac{L\delta}{4}\{|U_{xt}(0,t)|^2 + |U_{xt}(L,t)|^2\} \tag{2.41}$$

$$\int_0^L \mathcal{G}_t \psi_t\, dx \le C_3(\delta + \sqrt[4]{\varepsilon})\mathfrak{M}(t; U, \psi). \tag{2.42}$$

Proof. First we consider the case $(U, \psi) = (u, \theta)$ and $\mathfrak{F} = F$. From (2.8) it follows that

$$\int_0^L \mathfrak{F}_t U_{xx}\, dx = \int_0^L [\eta_1 u_{xx}]_t u_{xx}\, dx + \int_0^L [\eta_2\theta_x]_t u_{xx}\, dx.$$

On the other hand

$$\int_0^L [\eta_1 u_{xx}]_t u_{xx}\, dx = \frac{1}{2}\int_0^L \nabla\eta_1 \cdot (u_{xt}, \theta_t)|u_{xx}|^2\, dx + \frac{1}{2}\frac{d}{dt}\int_0^L \eta_1|u_{xx}|^2\, dx.$$

$$\int_0^L [\eta_2\theta_x]_t u_{xx}\, dx = \int_0^L \nabla\eta_2 \cdot (u_{xt}, \theta_t)\theta_x u_{xx}\, dx + \int_0^L \eta_2\theta_{xt}u_{xx}\, dx.$$

We then have from (2.33), (2.36) and (2.38)

$$\int_0^L [\eta_1 u_{xx}]_t u_{xx}\,\mathrm{d}x \le c_1 \nu\sqrt{\varepsilon}\,\mathfrak{M}(t; u, \theta) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \eta_1 |u_{xx}|^2\,\mathrm{d}x.$$

$$\int_0^L [\eta_2 \theta_x]_t u_{xx}\,\mathrm{d}x \le c_1 \nu(\sqrt{\varepsilon} + \delta)\mathfrak{M}(t; u, \theta),$$

so from the last three inequalities relation (2.40) follows. Let us consider the case $(U, \psi) = (u_t, \theta_t)$ and $\mathfrak{F} = F_t$. So we have

$$\int_0^L \mathfrak{F}_t U_{xx}\,\mathrm{d}x = \int_0^L F_{tt} u_{xxt}\,\mathrm{d}x.$$

Using the identities

$$F_{tt} = \eta_{1,tt} u_{xx} + 2\eta_{1,t} u_{xxt} + \eta_1 u_{xxtt} + \eta_{2,tt}\theta_x + 2\eta_{2,t}\theta_{xt} + \eta_2\theta_{xtt} \tag{2.43}$$

$$\eta_{1,tt} = (u_{xt}, \theta_t)\mathcal{H}_{\eta_1}(u_{xt}, \iota_t)^T + \nabla\eta_1 \cdot (u_{xtt}, \theta_{tt}), \qquad \eta_{1,t} = \nabla\eta_1 \cdot (u_{xt}, \theta_t), \tag{2.44}$$

we get that inequality (2.40) also holds in this case for an appropriate constant C_1. To prove (2.41) we only consider the case $(U, \psi, \mathfrak{F}) = (u_t, \theta_t, F_t)$. The other is simpler. So

$$\int_0^L \left(x - \frac{L}{2}\right)\mathfrak{F}_t U_{xt}\,\mathrm{d}x = \int_0^L \left(x - \frac{L}{2}\right)F_{tt} u_{xtt}\,\mathrm{d}x,$$

using (2.34), (2.36), (2.37) we then have from (2.43) and (2.44) that

$$-\int_0^L \left(x - \frac{L}{2}\right)F_{tt} u_{xtt}\,\mathrm{d}x \le C_2(\delta + \sqrt{\varepsilon})\mathfrak{M}(t; u_t, \theta_t) - \int_0^L \left(x - \frac{L}{2}\right)\eta_1 u_{xxtt} u_{xtt}\,\mathrm{d}x.$$

Integrating by parts and using (2.36) we obtain

$$-\int_0^L \left(x - \frac{L}{2}\right)\eta_1 u_{xxtt} u_{xtt}\,\mathrm{d}x \le \int_0^L \left(x - \frac{L}{2}\right)\nabla\eta_1 \cdot (u_{xx}, \theta_x)|u_{xtt}|^2\,\mathrm{d}x$$
$$+ \frac{L\delta}{4}\{|U_{xt}(0, t)|^2 + |U_{xt}(L, t)|^2\},$$

so (2.41) follows. Finally, to prove (2.42) we only prove for $(U, \psi) = (u_t, \theta_t)$ and $\mathfrak{F} = F_t$, $\mathcal{G} = G_t$. Then we have

$$\int_0^L \mathcal{G}_t \psi_t\,\mathrm{d}x = \int_0^L G_{tt}\theta_{tt}\,\mathrm{s}x,$$

where

$$G = W_1\theta_{xx} - W_2 u_{xt} + W_3 u_{xx} + W_4\theta_x.$$

We will prove that

$$\int_0^L \{W_1\theta_{xx}\}_{tt}\theta_{tt}\,\mathrm{d}x \le C_2(\delta + \sqrt{\varepsilon})\mathcal{M}(t; u_t, \theta_t).$$

The other terms are proved in a similar way.

$$\begin{aligned}\int_0^L \{W_1\theta_{xx}\}_{tt}\theta_{tt}\,\mathrm{d}x &= \int_0^L (u_{xt}, \theta_{xt}, \theta_t)\mathcal{H}_{W_1}(u_{xt}, \theta_{xt}, \theta_t)^T\theta_{xx}\theta_{tt}\,\mathrm{d}x\\ &\quad + \int_0^L \nabla W_1\cdot(u_{xtt}, \theta_{xtt}, \theta_{tt})\theta_{xx}\theta_{tt}\,\mathrm{d}x + \int_0^L W_1\theta_{xxtt}\theta_{tt}\,\mathrm{d}x\\ &\quad + 2\int_0^L \nabla W_1\cdot(u_{xt}, \theta_{xt}, \theta_t)\theta_{xxt}\theta_{tt}\,\mathrm{d}x\\ &\le cv\sqrt[4]{\varepsilon}\mathcal{M}(t; u_t, \theta_t) + \int_0^L W_1\theta_{xxtt}\theta_{tt}\,\mathrm{d}x.\end{aligned}$$

Since

$$\begin{aligned}\int_0^L W_1\theta_{xxtt}\theta_{tt}\,\mathrm{d}x &= -\int_0^L \nabla W_1\cdot(u_{xx}, \theta_{xx}, \theta_x)\theta_{xtt}\theta_{tt}\,\mathrm{d}x - \int_0^L W_1|\theta_{xtt}|^2\,\mathrm{d}x\\ &\le Cv(\sqrt[4]{\varepsilon} + \delta)\mathcal{M}(t; u_t, \theta_t).\end{aligned}$$

Therefore our result follows. ■

LEMMA 2.4. Under the same hypothesis as in Lemma 2.3, there exists positive constants C for which the following inequality holds

$$\begin{aligned}R(t; U, \psi) &\le \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(t; U\psi) + \frac{\beta L\delta}{24}\{|U_{xt}(0,t)|^2 + |U_{xt}(L,t)|^2\}\\ &\quad + C(\delta + \sqrt[4]{\varepsilon})\mathcal{M}(t; U, \psi),\end{aligned}$$

where

$$\begin{aligned}\mathcal{S}(t; U, \psi) &= \frac{1}{2}\left(\kappa_2 + \frac{10k}{\alpha}\right)\int_0^L \eta_1|U_{xx}|^2\,\mathrm{d}x - \frac{10k}{\alpha}\int_0^L \mathcal{F}U_{xx}\,\mathrm{d}x\\ &\quad - \alpha\kappa_2\int_0^L \mathcal{F}\psi_c\,\mathrm{d}x + \frac{\kappa_2}{2}\int_0^L |\mathcal{F}|^2\,\mathrm{d}x.\end{aligned}$$

Proof. Using Lemma 2.3 and recalling the definition of R our conclusion follows.

THEOREM 2.1. Let us take S, N, Q in C^3 satisfying (2.1)–(2.3) and with the same hypotheses as in Lemma 2.3, then there exists only one global solution of system (1.1)–(1.4) which decays exponentially as time goes to infinity.

Proof. We will suppose that S, N, Q are in C^4 and the initial data belongs to $H^4(0, L)$ satisfying the compatibility condition as in Theorem 5.1 of [1]. Our result will follow using the well known density arguments. From Lemmas 2.3 and 2.4 we get

$$\frac{\mathrm{d}}{\mathrm{d}t}K(t; U, \psi) \leq \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(t; U, \psi) - \min\left\{\frac{k^2}{2\beta}, \frac{\beta}{16}\right\}\mathfrak{M}(t; U, \psi), \tag{2.45}$$

provided ε and δ are small enough. Let us denote by

$$\mathcal{L}(t; U, \psi) = K(t; U, \psi) - \mathcal{S}(t; U, \psi).$$

From (2.18) we get

$$\frac{c_0}{2}\mathfrak{N}(t; U, \psi) \leq \mathcal{L}(t; U, \psi) \leq \frac{3}{2}c_1\mathfrak{N}(t; U, \psi), \tag{2.46}$$

provided ε and δ are small enough. Inequalities (2.45) and (2.46) imply

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t; U, \psi) \leq -c_4\mathfrak{N}(t; U, \psi)$$

for some positive constant c_4 which together with (2.46) yields

$$\frac{c_0}{2}\mathfrak{N}(t; U, \psi) \leq \mathcal{L}(t; U, \psi) \leq \mathcal{L}(0; U, \psi)\,\mathrm{e}^{-\gamma t} \leq \frac{3c_1}{2}\mathfrak{N}(0; U, \psi)\,\mathrm{e}^{-\gamma t}, \tag{2.47}$$

for a positive constant γ. On the other hand, for (2.45) and recalling the definition of $\mathcal{L}(t; U, \psi)$ we obtain

$$\mathcal{L}(t; U, \psi) + \lambda\int_0^t \mathfrak{M}(\tau; U, \psi)\,\mathrm{d}\tau \leq \mathcal{L}(0; U, \psi). \tag{2.48}$$

Where $\lambda = \min\{k^2/2\beta, \beta/16\}$. From (2.48), we get

$$\frac{2}{c_0}\lambda\int_0^t\int_0^L |\psi_{xx}|^2 + |\psi_{xt}|^2\,\mathrm{d}x\,\mathrm{d}\tau \leq \frac{2}{c_0}\mathcal{L}(0; U, \psi) \leq \frac{3c_1}{c_0}\mathfrak{N}(0; U, \psi). \tag{2.49}$$

From (2.29), (2.46), (2.48) and (2.49) we easily obtain

$$\mathcal{I}_1(t) < \frac{3c_1}{c_0}\varepsilon^2(\mathrm{e}^{-\gamma t} + 1), \qquad \mathcal{I}_2(t) < \frac{3c_1}{c_0}\mu^2(\mathrm{e}^{-\gamma t} + 1),$$

so, letting $t \to t_3$ we get

$$\mathcal{I}_1(t_3) \leq \frac{3c_1}{c_0}\varepsilon^2(\mathrm{e}^{-\gamma t_3} + 1) < \frac{6c_1}{c_0}\varepsilon^2, \tag{2.50}$$

$$\mathcal{I}_2(t_3) \leq \frac{3c_1}{c_0}\mu^2(\mathrm{e}^{-\gamma t_3} + 1) < \frac{6c_1}{c_0}\mu^2. \tag{2.51}$$

Since $t_3 = t_1$ or $t_3 = t_2$ inequality (2.50) or (2.51) is contradictory to the maximility of t_1 or t_2 respectively. Then it follows that $t_2 = T_m = \infty$ therefore the solution must be global and inequality (2.47) holds for any $t > 0$, thus the exponential decay follows. The proof is now complete. ■

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