

# EXISTENCE AND MULTIPLICITY FOR ELLIPTIC PROBLEMS WITH QUADRATIC GROWTH IN THE GRADIENT

LOUIS JEANJEAN AND BOYAN SIRAKOV

ABSTRACT. We show that a class of divergence-form elliptic problems with quadratic growth in the gradient and non-coercive zero order terms are solvable, under essentially optimal hypotheses on the coefficients in the equation. In addition, we prove that the solutions are in general not unique. The case where the zero order term has the opposite sign was already intensively studied and the uniqueness is the rule.

## 1. INTRODUCTION

Boundary value problems for elliptic equations like

$$(1.1) \quad -\operatorname{div}(a(x, u, \nabla u)) = B(x, u, \nabla u) + f(x), \quad x \in \Omega \subset \mathbb{R}^N,$$

where  $-\operatorname{div}(a(x, \cdot, \nabla \cdot))$  is a Leray-Lions operator on some Sobolev space, have been one of the central problems in the theory of elliptic PDE in divergence form. This paper is a contribution to this study for the widely explored case when the nonlinear term  $B(x, u, \xi)$  has “natural growth” in the unknown function, that is, grows linearly in  $u$  and quadratically in  $\xi \in \mathbb{R}^N$ . The model case for our study is

$$(1.2) \quad a(x, u, \xi) = A(x)\xi, \quad B(x, u, \xi) = c_0(x)u + \mu(x)|\xi|^2,$$

where  $A$  is a positive bounded matrix,  $\mu \in L^\infty(\Omega)$ , and  $c_0, f$  belong to suitably chosen Lebesgue spaces.

This type of problems have generated a considerable literature. Let us mention here [10, 12, 15, 18, 20, 6, 7, 8, 3, 1, 2] as reference papers on this subject, most closely related to the problem we consider. In these works the existence, uniqueness or multiplicity of solutions of (1.1) is established under various conditions on  $a$ ,  $B$  and  $f$ , which will be discussed below.

Most of the works quoted above, when reduced to (1.2), assume that the coefficient  $c_0$  is nonpositive, that is, the equation is *coercive* or *proper*. The only exception to this rule is [2], in which the particular

---

2000 *Mathematics Subject Classification.* 35J25, 35J62.

*Key words and phrases.* Elliptic equation, natural growth, quadratic growth in the gradient, non-coercive, variational methods, sub- and super-solutions.

case  $c_0 = f \not\geq 0$  in the model problem (1.1)-(1.2) was mentioned; in the next section we will give a more detailed account on the results which appeared prior to this paper. Here we consider the general problem (2.1) below, with non-coercive dependence in the unknown function  $u$ . Specifically, we are going to see that, *when  $c_0$  is positive and sufficiently close to zero, the same type of existence result as in the case  $c_0 \leq 0$  can be obtained, but the bounded solutions are not unique.*

The paper is organized as follows. The next section contains our hypotheses and main results, and situates them with respect to previous works. A brief overview of the proofs is given in Section 3, while the proofs themselves can be found in Sections 4–7. We conclude with some final remarks in Section 8, where we discuss possible extensions and open problems.

## 2. MAIN RESULTS

In this section we state our main results. We study the equation

$$(2.1) \quad -\operatorname{div}(A(x)\nabla u) = H(x, u, \nabla u), \quad u \in H_0^1(\Omega),$$

where  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 3$  is a bounded domain in  $\mathbb{R}^N$ ,

$$(H1) \quad \left\{ \begin{array}{l} A \in L^\infty(\Omega)^{N \times N}, \Lambda I \geq A \geq \lambda I, \text{ for some } \Lambda \geq \lambda > 0, \text{ and} \\ |H(x, s, \xi)| \leq c_0(x)|s| + \mu|\xi|^2 + f(x), \\ \text{for some } \mu \in \mathbb{R}^+, c_0, f \in L^p(\Omega) \text{ with } p > \frac{N}{2}. \end{array} \right.$$

In the sequel we denote with  $C_N$  the optimal Sobolev constant, defined in (4.8) below. We have the following main existence result.

**Theorem 1.** *Assume that (H1) holds and*

$$(2.2) \quad \mu \|f\|_{L^{\frac{N}{2}}(\Omega)} < C_N.$$

*Then there exists a constant  $\bar{c} > 0$  depending on  $N, p, |\Omega|, \mu, \mu \|f\|_{L^p(\Omega)}$ , such that if*

$$\|c_0\|_{L^p(\Omega)} < \bar{c}$$

*then (2.1) admits a bounded solution.*

Next, we show that introducing a non-coercive zero order term in (2.1) induces *non-uniqueness* of the bounded solutions of this equation, in the extremal cases of the structural hypothesis (H1) above. In other words, we prove a multiplicity result for the equation

$$(2.3) \quad -\Delta u = c_0(x)u + \mu|\nabla u|^2 + f(x), \quad u \in H_0^1(\Omega),$$

where  $\mu \in \mathbb{R}$ ,  $c_0, f \in L^p(\Omega)$ .

**Theorem 2.** *Assume that*

$$\mu \neq 0 \quad \text{and} \quad c_0 \not\geq 0 \quad \text{in } \Omega.$$

*If*

$$(2.4) \quad \|[\mu f]^+\|_{L^{\frac{N}{2}}(\Omega)} < C_N,$$

*and*

$$\max\{\|c_0\|_{L^p(\Omega)}, \|[\mu f]^-\|_{L^p(\Omega)}\} < \bar{c},$$

*where  $\bar{c} > 0$  depends only on  $N$ ,  $p$ ,  $|\Omega|$ ,  $|\mu|$ ,  $\|[\mu f]^+\|_{L^p(\Omega)}$ , then (2.3) admits at least two bounded solutions.*

**Remark 1.** It is easy to check that the hypotheses in the above theorems are necessary, in the sense that (2.3) has no bounded solutions if  $c_0 = 0$  and  $\mu$  is large, or if  $c_0 = 0$  and  $f$  is large, or if  $\mu = 0$  and  $c_0$  is large; also if  $c_0 = 0$  or  $\mu = 0$  the solution given by Theorem 1 is unique. See for instance the last remarks in Section 3 of [24], pages 598-599 in that paper.

**Remark 2.** Note that in Theorem 2 there is no restriction on the sign of the source term  $f(x)$ .

**Remark 3.** A slightly more general version of Theorem 2 will be given in Section 7 (see also the remarks in Section 8).

Next we review the existence and uniqueness results which appeared prior to our work. Because of the very large literature we restrict ourselves to works which encompass the model case (1.2) (the reader may consult the references in the papers quoted below for various related problems).

We begin with references concerning Theorem 1. A weaker version of this result appeared already in Kazdan and Kramer [22], where equations in non-divergence form are studied. Later, in [10, 12], Boccardo, Murat and Puel showed that the sub- and super-solution method applies to general divergence-form equations with quadratic growth in the gradient, and proved existence of bounded solutions of such equations under a hypothesis of strict coercivity in  $u$ , that is,  $c_0(x) \leq -\alpha_0 < 0$  in (1.2). For results on strictly coercive equations we refer also to dall'Aglio, Giachetti and Puel [15]. Next, the equation (2.1) in the weakly coercive case ( $c_0 \equiv 0$ ) was studied by Maderna, Pagani and Salsa [23], and Ferone and Murat [17, 18]. In that case it turns out that existence can be proved only under a smallness hypothesis on the source term  $f$ , as in (2.2). Theorem 1 reduces to these results when  $c_0 = 0$ , and extends them to non-coercive zero-order terms.

Let us give some more context on coercive problems. Further existence results with weaker assumptions of regularity on the coefficients can be found in Grenon, Murat and Porretta [20]. Uniqueness results in natural spaces associated to the coercive problem were obtained by Barles and Murat [7], Barles, Blanc, Georgelin, and Kobylanski [6], Barles and Porretta [8]. We also refer to the recent works by Abdellaoui, dall’Aglia and Peral [3], and Abdel Hamid and Bidaut-Véron [1] for a deep study of (2.3) in the particular case  $c_0 = 0$ ,  $\mu = 1$ , and  $f \geq 0$ . They show that in this case the problem (2.3) has infinitely many solutions, of which only one is such that  $e^u - 1 \in H_0^1(\Omega)$ . For results on other classes of equations of type (1.1), with  $H$  being for instance in the form  $H(x, s, \xi) = \beta(s)|\xi|^2$  for some real function  $\beta$ , we refer to Boccardo, Gallouët, and Murat [9], as well as to [23], [3], [1, 2]. We note that in many of these papers equations involving quasilinear operators modeled on the  $p$ -Laplacian are also studied. Finally, the second author [24] recently obtained existence and uniqueness results for fully nonlinear equations in non-divergence form with quadratic dependence in the gradient, in which case the adapted weak notion of solution is the viscosity one (see [24] for references on these types of problems). The idea of our study originated from that paper.

As far as Theorem 2 is concerned, the fact that in problems with natural growth in the gradient the presence of a non-coercive zero-order term may lead to non-uniqueness of bounded solutions was observed only very recently in [24], for the equation (2.3) with  $f = 0$ . Subsequently the case when  $f \equiv c_0 \not\geq 0$  was considered in the work by Abdel Hamid and Bidaut-Véron [2] (their model equation is  $-\Delta_p u = |\nabla u|^p + \lambda f(x)(1 + u)^b$ ,  $b \geq p - 1$ ). Theorem 2 is valid for arbitrary source term  $f$ , which in particular may not be positive, and thus shows the multiplicity result is independent of the source term – as long as it has a small norm, of course, otherwise solutions may not exist.

To summarize, Theorem 1 is an essentially optimal, with respect to the coefficients, result on existence of bounded solutions of (2.1), for equations in divergence form with possibly non-coercive zero-order terms; while Theorem 2 shows uniqueness of bounded solutions is lost in the presence of non-coercive zero-order terms, at least in the model cases. We do not know whether a more general non-uniqueness result is valid (see Section 8).

In the next section we give more details on the underlying ideas in our approach, and discuss the difference between coercive and non-coercive problems.

**Notation.**

- (1) We denote by  $X$  the space  $H_0^1(\Omega)$  equipped with the Poincaré norm  $\|u\| := \int_{\Omega} |\nabla u|^2$ , and by  $X^{-1}$  its dual.
- (2) For  $v \in L^1(\Omega)$  we define  $v^+ = \max(v, 0)$  and  $v^- = \max(-v, 0)$ .
- (3) The norm  $(\int_{\Omega} |u|^p dx)^{1/p}$  in  $L^p(\Omega)$  is denoted by  $\|\cdot\|_p$ . We denote by  $p'$  the conjugate exponent of  $p$ , namely  $p' = (p-1)/p$ .
- (4) We denote by  $C, D > 0$  any positive constants which are not essential in the arguments and may vary from one line to another.

## 3. DISCUSSION AND GENERAL FRAME OF THE PROOFS

The aim of this section is, first, to provide some intuition on the hypotheses in our theorems and the differences they introduce with respect to previous works on problems with natural growth in the gradient, and second, to describe the ideas of the proofs of Theorem 1 and Theorem 2. To this goal, and in order to help the reader understand why the case  $c_0^+ \not\equiv 0$  is different from the cases  $c_0 \leq 0$  or  $c_0 \leq -\alpha_0 < 0$ , we present a variational interpretation of the model problem (2.3).

Let us assume, for the time being, that  $\mu > 0$  is a constant and  $c_0$  and  $f$  are smooth functions. Making the well-known change of unknown  $v = \frac{1}{\mu}(e^{\mu u} - 1)$  in (2.3) we observe that if a solution of

$$(3.1) \quad -\Delta v - [c_0(x) + \mu f(x)]v = c_0(x)g(v) + f(x), \quad v \in X,$$

where

$$(3.2) \quad g(s) = \begin{cases} \frac{1}{\mu}(1 + \mu s)\ln(1 + \mu s) - s & \text{if } s > -\frac{1}{\mu} \\ -s & \text{if } s \leq -\frac{1}{\mu}, \end{cases}$$

satisfies  $v > -\frac{1}{\mu}$ , then  $u = \frac{1}{\mu}\ln(1 + \mu v)$  is a solution of (2.3). In the next section we are going to see that this procedure can be made rigorous in general, and we will obtain a priori bounds on solutions of (3.1) which show that they indeed give solutions of (2.3), under the hypotheses of our theorems.

Equation (3.1) admits a variational formulation, in other words, its solutions in  $H_0^1(\Omega)$  can be represented as critical points of a functional defined on this space. Specifically, critical points of

$$I(v) := \frac{1}{2} \int_{\Omega} |\nabla v|^2 - [c_0(x) + \mu f(x)]v^2 dx - \int_{\Omega} c_0(x)G(v) dx - \int_{\Omega} f(x)v dx$$

on  $H_0^1(\Omega)$  are weak solutions of (3.1). Here  $G(s) = \int_0^s g(t) dt$ .

No such link between problems of type (2.1) and problems admitting a variational formulation has appeared in the earlier works on coercive equations with natural growth in the gradient [10, 12, 15, 23, 17, 18].

The fact that the general problem (2.1) does not have such a formulation surely explains this; however the validity of the results obtained in these papers can be explained, in a different light, by looking at the model problem (3.1).

First, if we assume that

$$c_0 \leq -\alpha_0 < 0$$

(as in [10, 12, 15]), it is easily seen that we have, independently of the size of  $f \in L^{\frac{N}{2}}(\Omega)$ ,

$$(3.3) \quad \lim_{\|v\| \rightarrow \infty} I(v) = +\infty,$$

or in other words  $I$  is *coercive*, from which the existence of a global minimum of  $I$  follows. Indeed, to prove (3.3) we observe that the second term in the definition of  $I$  dominates, for  $\|v\|$  large, the first and third terms, since (see Lemma 7)

$$(3.4) \quad \lim_{s \rightarrow \infty} \frac{G(s)}{s^2} = +\infty.$$

Next, if  $c_0 = 0$  (as in [23], [17], [18]), then  $I$  becomes

$$I(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \mu f(x)v^2 dx - \int_{\Omega} f(x)v dx$$

and it is easily seen that this functional is coercive if and only if (see Lemma 4)

$$(3.5) \quad \inf_{\|v\|_{L^2(\Omega)}=1} \int_{\Omega} |\nabla v|^2 - \mu f(x)v^2 dx > 0,$$

which in turn holds under the condition (2.2), discovered in [17].

On the other hand, in the case we are interested in

$$c_0^+ \not\equiv 0,$$

the geometry of  $I$  is completely different, now (3.4) implies

$$(3.6) \quad \inf_{v \in X} I(v) = \liminf_{\|v\| \rightarrow \infty} I(v) = -\infty,$$

and in particular no global minimum of  $I$  exists.

However, as we are going to see, it turns out that if  $c_0^+$  is appropriately small, the functional  $I$  takes strictly positive values on the boundary of some *large* ball  $B$  in  $H_0^1(\Omega)$ . In other words, we show that letting the coefficient  $c_0$  be slightly positive perturbs badly  $I$  at infinity (compare (3.3) to (3.6)) but keeps  $I$  “sufficiently large” on some large sphere. Hence, in view of  $I(0) = 0$ , it follows that  $I$  attains a *local* minimum in  $B$ , which is then a critical point of  $I$ .

The latter argument applies only to the extremal case (2.3) but yields existence for general equations as in Theorem 1, via the method of sub- and super-solutions. This method requires no variational structure at all, and applies to very general equations (see for instance [4, 11, 14]). Note that the method of sub- and super-solutions is particularly useful in searching for stable solutions, and a local minimum of a functional corresponds precisely to a stable solution.

Let us now explain why Theorem 2 is valid. The existence of a local minimum of  $I$  and (3.6) suggest that at least one more critical point (of saddle type) of  $I$  could be expected to exist. Proving this type of statement is the object of a large branch of the theory of variational methods in PDE, whose development started with the acclaimed work by Ambrosetti and Rabinowitz [5] on functionals which have “mountain-pass” geometry, that is, are positive on a small sphere and tend to  $-\infty$  at infinity. In our case we are able to prove that a second critical point of  $I$  exists by showing that Cerami sequences for  $I$  are bounded, from which classical arguments permit us to deduce the result. The boundedness of Cerami sequences is a significant difficulty and to overcome it we need to develop further some ideas introduced in [21].

It is important to note that the latter argument depends strongly on the variational structure of the PDE in consideration, which is the reason for which we are able to prove Theorem 2 only for the model equation (2.3). See Open Problem 1 in Section 8, and the remarks therein.

Here is an outline of the following sections. First, in Section 4 we give some preliminaries and study the relation between the problems (2.3) and (3.1). In Section 5 we establish several facts on the geometry of the functional  $I(v)$ , and show it admits a local minimum. The core of the multiplicity result is in Section 6, where we show that Cerami sequences for  $I$  are bounded. In Section 7 we finish the proof of Theorems 1 and 2. Section 8 contains some closing remarks and open problems.

#### 4. THE LINK BETWEEN PROBLEMS (2.3) AND (3.1)

We consider the problem

$$(4.1) \quad -\Delta v - [c_0(x) + \mu f(x)]v = c_0(x)g(v) + f(x), \quad v \in X,$$

where  $g$  is given by (3.2) and  $\mu > 0$ .

**Lemma 3.** *If  $v \in X$  is a solution of (4.1) which satisfies*

$$v > -1/\mu + \varepsilon \quad \text{on } \Omega, \quad \text{for some } \varepsilon > 0,$$

*then  $u = \frac{1}{\mu} \ln(1 + \mu v)$  is a solution of (2.3).*

*Proof.* The equation (4.1) can be rewritten, for  $v > -1/\mu$ ,

$$(4.2) \quad -\Delta v = \frac{c_0(x)}{\mu}(1 + \mu v)\ln(1 + \mu v) + (1 + \mu v)f(x).$$

Let  $v \in X$  be a solution of (4.2), we want to show that  $u = \frac{1}{\mu}\ln(1 + \mu v)$  is a solution of (2.3), that is, if  $\phi \in C_0^\infty(\Omega)$ , then

$$(4.3) \quad \int_{\Omega} \nabla u \nabla \phi - \mu |\nabla u|^2 \phi - c_0(x)u\phi \, dx = \int_{\Omega} f(x)\phi \, dx.$$

Let  $\psi = \frac{\phi}{1 + \mu v}$ . Clearly  $\psi \in X$  and thus it can be used to test (4.2). We get

$$(4.4) \quad \int_{\Omega} \nabla v \nabla \psi \, dx = \int_{\Omega} \frac{c_0(x)}{\mu} \ln(1 + \mu v) \phi \, dx + \int_{\Omega} f(x)\phi \, dx.$$

But

$$(4.5) \quad \int_{\Omega} \frac{c_0(x)}{\mu} \ln(1 + \mu v) \phi \, dx = \int_{\Omega} c_0(x)u\phi \, dx$$

and

$$\begin{aligned} \int_{\Omega} \nabla v \nabla \psi \, dx &= \int_{\Omega} \nabla \left( \frac{1}{\mu}(e^{\mu v} - 1) \right) \nabla \left( \frac{\phi}{1 + \mu v} \right) \, dx \\ &= \int_{\Omega} e^{\mu v} \nabla u \left( \frac{\nabla \phi}{1 + \mu v} - \frac{\mu \phi \nabla v}{(1 + \mu v)^2} \right) \, dx \\ &= \int_{\Omega} \nabla u \left( \nabla \phi - \frac{\mu \phi \nabla (e^{\mu v} - 1)}{(1 + \mu v)} \right) \, dx \\ &= \int_{\Omega} \nabla u (\nabla \phi - \mu \phi \nabla u) \, dx \\ (4.6) \quad &= \int_{\Omega} \nabla u \nabla \phi - \mu |\nabla u|^2 \phi \, dx. \end{aligned}$$

Combining (4.4), (4.5) and (4.6), we see that  $u$  satisfies (4.3).  $\square$

Next we recall the following standard fact.

**Lemma 4.** *Given  $h \in L^{N/2}(\Omega)$ , set*

$$E_h^2(u) = \int_{\Omega} |\nabla u|^2 - h(x)|u|^2 \, dx,$$

for  $u \in X$ . Then

$$\|h^+\|_{\frac{N}{2}} < C_N$$

implies that the quantity  $E_h(u)$  defines a norm on  $X$  which is equivalent to the standard norm, and

$$\lambda(h, \Omega) := \inf_{u \in X \setminus \{0\}} \frac{E_h(u)}{\|u\|^2} > 0.$$



This last property implies that the operator  $-\Delta - h$  satisfies the maximum principle in  $\Omega$ , that is, if  $-\Delta u - hu \geq 0$  in  $X^{-1}$  for some  $u \in X$ , then  $u^- \in X$  yields  $u^- \equiv 0$  in  $\Omega$ .

*Proof.* The first statement trivially follows from the Sobolev embedding and the fact that for any  $v \in X$ ,

$$(4.7) \quad \int_{\Omega} h(x)v^2 dx \leq \|h\|_{\frac{N}{2}} \|v\|_{2^*}^2 \leq \frac{1}{C_N} \|h\|_{\frac{N}{2}} \|\nabla v\|_2^2.$$

Here  $2^* = \frac{2N}{N-2}$  and

$$(4.8) \quad C_N = \inf\{\|\nabla v\|_2^2 : v \in X, \|v\|_{2^*}^2 = 1\} > 0$$

is the optimal constant in Sobolev's inequality. Note  $C_N$  depends only on  $N$ ; the exact value of  $C_N$  can be found in [25]. The maximum principle is obtained by multiplying  $-\Delta u - hu \geq 0$  by  $u^-$  and by integrating.  $\square$

**Definition.** In the rest of the paper we assume that the constant  $\bar{c} > 0$  in the main theorems is fixed so small that

$$(H2) \quad \|c_0 + \mu f^+\|_{\frac{N}{2}} < C_N.$$

We denote with  $\|\cdot\|$  the norm defined by  $E_{c_0+\mu f}(\cdot)$  which, by Lemma 4, is equivalent to the standard norm on  $X$ .

By (2.4), the validity of (H2) can be ensured by taking

$$(4.9) \quad \|c_0\|_{\frac{N}{2}} \leq \varepsilon_0/2, \quad \text{with} \quad \varepsilon_0 := C_N - \mu \|f^+\|_{\frac{N}{2}} > 0,$$

which occurs for  $\|c_0\|_p$  sufficiently small, since  $|\Omega| < \infty$ .

Now we recall the following global boundedness lemma, which is a consequence of results due to Stampacchia and Trudinger.

**Lemma 5.** Assume that  $A \in L^\infty(\Omega)^{N \times N}$ ,  $\Lambda I \geq A \geq \lambda I$ , for some  $\Lambda \geq \lambda > 0$ , and that  $c, f \in L^p(\Omega)$  for some  $p > \frac{N}{2}$ . Then if  $u \in X$  is a solution of

$$-\operatorname{div}(A(x)\nabla u) \leq (\geq) c(x)u + f(x)$$

then  $u$  is bounded above(below) and

$$\sup_{\Omega} u^+ (\sup_{\Omega} u^-) \leq C(\|u^+(u^-)\|_2 + \|f\|_p),$$

where  $C$  depends on  $N, p, \lambda, \Lambda, |\Omega|$ , and  $\|c\|_p$ .

*Proof.* This is a consequence of Theorem 4.1 in [26] combined with Remark 1 on page 289 in that paper. It can also be obtained by repeating the proof of Theorem 8.15 in [19] (which implies the same result for  $c \in L^\infty(\Omega)$ ), as remarked at the end of page 193 in that book.  $\square$

The next lemma shows that Lemma 3 can be applied, provided the function  $c_0$  is sufficiently small.

**Lemma 6.** *There exists a constant  $\bar{c} > 0$  depending on  $N, p, |\Omega|, \mu \|f^+\|_p$ , such that if*

$$\mu \|f^+\|_{\frac{N}{2}} < C_N, \quad \text{and} \quad \max\{\|c_0\|_p, \mu \|f^-\|_p\} < \bar{c},$$

*then any solution  $v$  of (4.1) satisfies  $v > -1/(2\mu)$  in  $\Omega$ .*

*Proof.* Since  $c_0 \geq 0$  on  $\Omega$  and  $g$  is nonnegative on  $\mathbb{R}$ , any solution of (4.1) satisfies

$$(4.10) \quad -\Delta v - [c_0(x) + \mu f^+(x)]v \geq -f^- \quad \text{on the set } \{v < 0\}.$$

We now use the global bound given in the previous lemma to infer that

$$(4.11) \quad \sup_{\Omega}(v^-) \leq C(\|v^-\|_2 + \|f^-\|_p),$$

for some constant  $C = C(N, p, |\Omega|, \|c_0 + \mu f^+\|_p)$ .

Recall that if we assume  $\bar{c} > 0$  is small enough (H2) holds, and thus  $\|\cdot\|$  is equivalent to the standard norm on  $X$ . We multiply (4.1) by  $v^-$ , and integrate to get

$$\begin{aligned} \|v^-\|^2 &\leq \int_{\Omega} |\nabla v^-|^2 - [c_0(x) + \mu f(x)]|v^-|^2 dx \\ &\leq - \int_{\Omega} c_0(x)g(v)v^- dx - \int_{\Omega} f(x)v^- dx \\ &\leq \int_{\Omega} f^-(x)v^- dx \\ &\leq \|f^-\|_{\frac{N}{2}} \|v^-\|_{\frac{N}{N-2}} \leq C \|f^-\|_{\frac{N}{2}} \|v^-\|. \end{aligned}$$

Thus, in particular,

$$(4.12) \quad \|v^-\|_2 \leq C \|f^-\|_p.$$

Combining (4.11) and (4.12) we get  $\sup(v^-) \leq C \|f^-\|_p$ . This implies that  $\sup(v^-) < 1/(2\mu)$  if  $C \|f^-\|_p < 1/(2\mu)$ , that is, if  $\|\mu f^-\|_p \leq 1/(2C)$ . This finishes the proof.  $\square$

## 5. ON THE GEOMETRY OF THE FUNCTIONAL $I(v)$

We associate to (4.1) the functional  $I : X \rightarrow \mathbb{R}$  defined by

$$I(v) = \frac{1}{2} \|v\|^2 - \int_{\Omega} c_0(x)G(v) dx - \int_{\Omega} f(x)v dx.$$

Under our assumptions it is standard to show that  $I \in C^1(X, \mathbb{R})^1$ .

<sup>1</sup>Note that in Sections 5 and 6  $\mu$  can be an arbitrary function in  $L^\infty(\Omega)$

Recall  $G(s) = \int_0^s g(t)dt$  and define  $H(s) = \frac{1}{2}g(s)s - G(s)$ . In the following lemma we gather some simple and useful properties of  $g, G$  and  $H$ .

**Lemma 7.**

- (i) *The function  $g$  is continuous on  $\mathbb{R}$ ,  $g > 0$  on  $\mathbb{R} \setminus \{0\}$ ,  $G \geq 0$  on  $\mathbb{R}^+$  and  $G \leq 0$  on  $\mathbb{R}^-$ .*
- (ii) *For any  $r \in (1, 2)$  there exists  $C = C(r, \mu) > 0$  such that we have  $|g(s)| \leq C|s|^r$  for any  $s \in \mathbb{R}$ .*
- (iii) *We have  $g(s)/s \rightarrow 0$  as  $s \rightarrow 0$ .*
- (iv) *We have  $g(s)/s \rightarrow +\infty$  and  $G(s)/s^2 \rightarrow +\infty$  as  $s \rightarrow +\infty$ .*
- (v) *The function  $H$  satisfies  $H(s) \leq (s/t)H(t)$ , for  $0 \leq s \leq t$ .*
- (vi) *The function  $H$  is bounded on  $\mathbb{R}^-$ .*

*Proof.* We have  $g(0) = 0$  and, for  $s > -1/\mu$ ,  $g'(s) = \ln(1 + \mu s)$ . Thus  $g'(0) = 0$ ,  $g(s) > 0$  if  $s \neq 0$ . Now direct calculations show that

$$g(s) \leq \ln(1 + \mu s) s \quad \text{if } s \geq 0,$$

and  $g(s) \leq |s|$  if  $s \leq 0$ . Hence (i), (ii) and (iii) hold. By the definition of  $g$ , (iv) clearly holds. Also  $H(0) = 0$  and we get, for  $s \geq 0$ ,

$$H'(s) = \frac{1}{2}[g'(s)s - g(s)] = \frac{1}{2}\left[s - \frac{1}{\mu}\ln(1 + \mu s)\right].$$

Thus  $H''(s) = \frac{\mu s}{2(1 + \mu s)} \geq 0$  for  $s \geq 0$ . From the convexity of  $H$ , we deduce that, if  $0 < s \leq t$ ,

$$H(s) \leq \frac{s}{t}H(t) + \left(1 - \frac{s}{t}\right)H(0) = \frac{s}{t}H(t),$$

which proves (v). Finally, we trivially check that

$$H(s) = -G\left(-\frac{1}{\mu}\right) - \frac{1}{2\mu^2}$$

is constant for  $s \leq -1/\mu$ , which implies (vi). The lemma is proved.  $\square$

The next lemma concerns the geometrical structure of  $I$ . We are going to denote with  $B(0, \rho)$  the ball in  $X$  with center 0 and radius  $\rho$ .

**Lemma 8.** *Assume (H2). There exist constants  $\alpha = \alpha(N, |\Omega|, \mu) > 0$ ,  $\beta > 0$  and  $\rho > 0$  such that if  $0 < \|c_0\|_p \leq \alpha$  then*

- (i)  *$I(v) \geq \beta$  for  $\|u\| = \rho$ .*
- (ii)  *$\inf_{v \in B(0, \rho)} I(v) \leq 0$ , and  $\inf_{v \in B(0, \rho)} I(v) < 0$  if  $f \neq 0$ .*
- (iii) *There exists  $v_0 \in X$  such that  $\|v_0\| > \rho$  and  $I(v_0) \leq 0$ .*

*Proof.* Let  $r > 1$ , close to 1, satisfy  $(r+1)p' < \frac{2N}{N-2}$ . We can choose such  $r$  since  $p > \frac{N}{2}$ . By Lemma 7 we have

$$(5.1) \quad |G(s)| \leq C|s|^{r+1}, \quad \text{for all } s \in \mathbb{R}.$$

Using (5.1), we get, for any  $v \in X$ ,

$$(5.2) \quad \int_{\Omega} c_0(x)G(v)dx \leq C\|c_0\|_p\|v\|_{(r+1)p'}^{r+1} \leq C\|c_0\|_p\|v\|^{r+1},$$

where we used the Hölder and Sobolev inequalities. Also

$$\begin{aligned} \int_{\Omega} f(x)v(x)dx &\leq \|f\|_{\frac{N}{2}}\|v\|_{\frac{N}{N-2}} \leq D(\|f^+\|_{\frac{N}{2}} + \|f^-\|_{\frac{N}{2}})\|v\| \\ &\leq (D/\mu)(C_N + \mu\|f^-\|_{\frac{N}{2}})\|v\|, \end{aligned}$$

for some  $D = D(N, |\Omega|) > 0$ , by the hypotheses of Theorem 2. We then get, for any  $v \in X$ , because of (4.9),

$$(5.3) \quad I(v) \geq \frac{1}{2}\|v\|^2 - (D/\mu)(C_N + \mu\|f^-\|_{\frac{N}{2}})\|v\| - C\|c_0\|_p\|v\|^{r+1}.$$

We fix first  $\rho > 0$  sufficiently large so that if  $\|v\| = \rho$

$$\frac{1}{2}\|v\|^2 - (D/\mu)(C_N + \mu\|f^-\|_{\frac{N}{2}})\|v\| \geq \frac{1}{4}\rho,$$

and then  $\|c_0\|_p$  small enough to ensure that  $I(v) \geq \frac{1}{8}\rho$ , for any  $v \in X$  with  $\|v\| = \rho$ . This proves (i).

Next, note that  $I(0) = 0$ , so  $\inf_{v \in B(0, \rho)} I(v) \leq 0$ . If  $f \not\equiv 0$ , take a function  $v \in C_0^\infty(\Omega)$ , such that  $\int_{\Omega} f(x)v dx > 0$  and consider the map  $t \rightarrow I(tv)$  for  $t > 0$ . We have

$$(5.4) \quad \begin{aligned} I(tv) &= \frac{t^2}{2}\|v\|^2 - \int_{\Omega} c_0(x)G(tv) dx - t \int_{\Omega} f(x)v dx \\ &= t^2 \left[ \frac{1}{2}\|v\|^2 - \int_{\Omega} c_0(x) \frac{G(tv)}{t^2 v^2} v^2 dx - \frac{1}{t} \int_{\Omega} f(x)v dx \right]. \end{aligned}$$

By Lemma 7 we have  $G(s)/s^2 \rightarrow 0$  as  $s \rightarrow 0$ , thus

$$\int_{\Omega} c_0(x) \frac{G(tv)}{t^2 v^2} |v|^2 dx \rightarrow 0$$

as  $t \rightarrow 0$ , since  $v \in C_0^\infty(\Omega)$ . Then (5.4) implies  $I(tv) < 0$  for  $t > 0$  small enough. This proves (ii).

Finally, to prove (iii) we consider again the map  $t \rightarrow I(tv)$ ,  $t > 0$ , and take  $v \in C_0^\infty(\Omega)$  with  $v \geq 0$ ,  $c_0 v \not\equiv 0$ . Then since by Lemma 7  $G(s)/s^2 \rightarrow +\infty$  as  $s \rightarrow +\infty$ , we now have

$$\int_{\Omega} c_0(x) \frac{G(tv)}{t^2 v^2} |v|^2 dx \rightarrow +\infty,$$

so  $I(tv) \rightarrow -\infty$  as  $t \rightarrow +\infty$ . This of course implies (iii).  $\square$

In view of Lemma 8 it can be expected that for  $\|c_0\|_p$  sufficiently small  $I$  has two critical points, one of which is a local minimum, while the other is of saddle type.

**Lemma 9.** *Assume that  $\|c_0\|_p$  is sufficiently small to ensure that (H2) and Lemma 8 hold. Then the functional  $I$  possesses a critical point  $v \in B(0, \rho)$ , with  $I(v) \leq 0$ , which is a local minimum of  $I$ .*

*Proof.* By Lemma 8 (i) and (ii) there are  $\rho, \beta > 0$  such that

$$m := \inf_{v \in B(0, \rho)} I(v) \leq 0 \quad \text{and} \quad I(v) \geq \beta > 0 \quad \text{if} \quad \|v\| = \rho.$$

Let  $(v_n) \subset B(0, \rho) \subset X$  be a sequence such that  $I(v_n) \rightarrow m$ . Since  $(v_n) \subset X$  is bounded we have, up to a subsequence,  $v_n \rightharpoonup v$  weakly in  $X$ , for some  $v \in X$ . Now, by standard properties of the weak convergence and since  $f \in L^{N/2}(\Omega) \subset X^{-1}$ ,

$$\|v\|^2 \leq \liminf_{n \rightarrow \infty} \|v_n\|^2 \quad \text{and} \quad \int_{\Omega} f(x)v_n \, dx \rightarrow \int_{\Omega} f(x)v \, dx$$

as  $n \rightarrow \infty$ . Also, since  $v_n \rightarrow v$  in  $L^q(\Omega)$  for  $1 \leq q < \frac{2N}{N-2}$  and  $c_0 \in L^p(\Omega)$  we readily obtain, using (5.1), that

$$\int_{\Omega} c_0(x)G(v_n) \, dx \rightarrow \int_{\Omega} c_0(x)G(v) \, dx \quad \text{as} \quad n \rightarrow \infty.$$

We deduce that  $v \in B(0, \rho)$  and

$$I(v) \leq \liminf_{n \rightarrow \infty} I(v_n) = m = \inf_{v \in B(0, \rho)} I(v).$$

Thus  $v$  is a local minimum of  $I$  and, by standard arguments, a critical point of  $I$ .  $\square$

Now we define the mountain pass level

$$\hat{c} = \inf_{g \in \Gamma} \max_{t \in [0, 1]} I(g(t))$$

where

$$\Gamma = \{g \in C([0, 1], X) : g(0) = 0, g(1) = v_0\},$$

with  $v_0 \in X$  given by Lemma 8 (iii). We shall prove that  $I$  possesses a critical point at the mountain pass level, that is, there exists  $v \in X$  such that  $I(v) = \hat{c}$  and  $I'(v) = 0$ . Since  $\hat{c} > 0$  (by Lemma 8 (i)), this critical point must be different from the local minimum given by Lemma 9.

It is a standard fact that any  $C^1$ -functional having a mountain pass geometry admits a Cerami sequence at the mountain pass level (see for

instance [13, 16]). In other words, there exists a sequence  $(v_n) \subset X$  such that

$$I(v_n) \rightarrow \hat{c} \quad \text{and} \quad (1 + \|v_n\|)I'(v_n) \rightarrow 0.$$

If we manage to show that  $(v_n) \subset X$  admits a strongly convergent subsequence, its limit is the desired critical point. A first essential step in the proof of this fact is showing that  $(v_n)$  is bounded.

## 6. BOUNDEDNESS OF THE CERAMI SEQUENCES

The following lemma is the key point in the proof of Theorem 2.

**Lemma 10.** *Assume that  $\|c_0\|_p$  is sufficiently small to ensure (H2) and Lemma 8 hold. Then the Cerami sequences for  $I$  at any level  $d \in \mathbb{R}^+$  are bounded.*

*Proof.* Let  $(v_n) \subset X$  be a Cerami sequence for  $I$  at a level  $d \in \mathbb{R}^+$ . Assume for contradiction that  $\|v_n\| \rightarrow \infty$  and set

$$w_n = \frac{v_n}{\|v_n\|}.$$

Since  $(w_n) \subset X$  is bounded we have  $w_n \rightharpoonup w$  weakly in  $X$  and  $w_n \rightarrow w$  strongly in  $L^q(\Omega)$ , for  $1 \leq q < \frac{2N}{N-2}$  (up to a subsequence). We write  $w = w^+ - w^-$ . We shall distinguish the two cases  $c_0 w^+ \equiv 0$  and  $c_0 w^+ \not\equiv 0$ , and prove they are both impossible.

First we assume that  $c_0 w^+ = 0$ , and define the sequence  $(z_n) \subset X$  by  $z_n = t_n v_n$  with  $t_n \in [0, 1]$  satisfying

$$(6.1) \quad I(z_n) = \max_{t \in [0, 1]} I(tv_n)$$

(if  $t_n$  defined by (6.1) is not unique we choose its smallest possible value). Let us show that

$$(6.2) \quad \lim_{n \rightarrow \infty} I(z_n) = +\infty.$$

Seeking a contradiction we assume that for some  $M < \infty$

$$(6.3) \quad \liminf_{n \rightarrow \infty} I(z_n) \leq M,$$

and we define  $(k_n) \subset X$  by

$$k_n = \frac{\sqrt{4M}}{\|v_n\|} v_n = \sqrt{4M} w_n.$$

Then  $k_n \rightharpoonup k := \sqrt{4M} w$  weakly in  $X$  and  $k_n \rightarrow k$  strongly in  $L^q(\Omega)$  for any  $1 \leq q < \frac{2N}{N-2}$ . Thus, as in the proof of Lemma 9, we have

$$(6.4) \quad \int_{\Omega} c_0(x) G(k_n) dx \rightarrow \int_{\Omega} c_0(x) G(k) dx.$$

Now, recall that  $G(s) \leq 0$  for  $s \leq 0$ , see Lemma 7. Since we have assumed  $c_0(x) = 0$  if  $k(x) > 0$ , we obtain

$$(6.5) \quad \int_{\Omega} c_0(x)G(k) dx \leq 0.$$

Also, since  $f \in L^{N/2}(\Omega) \subset X^{-1}$

$$(6.6) \quad \left| \int_{\Omega} f(x)k_n dx \right| \leq \sqrt{4M} \|f\|_{X^{-1}} \|w_n\| \leq \sqrt{4M} \|f\|_{X^{-1}}.$$

Combining (6.4), (6.5) and (6.6) it follows that

$$(6.7) \quad \begin{aligned} I(k_n) &= 2M - \int_{\Omega} c_0(x)G(k_n) dx - \int_{\Omega} f(x)k_n dx \\ &\geq 2M - \sqrt{4M} \|f\|_{X^{-1}} + o(1). \end{aligned}$$

Thus, taking  $M > 0$  larger if necessary, we can assume that

$$(6.8) \quad I(k_n) \geq (3/2)M$$

for all sufficiently large  $n \in \mathbb{N}$ . Since  $k_n$  and  $z_n$  lay on the same ray in  $X$  for all  $n \in \mathbb{N}$ , we see by the definition of  $z_n$  that (6.8) contradicts (6.3) (note  $\sqrt{4M}/\|v_n\| < 1$  since  $\|v_n\| \rightarrow \infty$ ). Thus (6.2) holds.

We remark that  $I(v_n) \rightarrow d$  and  $I(z_n) \rightarrow \infty$  imply that  $t_n \in (0, 1)$ . Hence by the definition of  $z_n$  we have that  $\langle I'(z_n), z_n \rangle = 0$ , for all  $n \in \mathbb{N}$ . Thus, with  $H$  defined as in Lemma 7,

$$(6.9) \quad \begin{aligned} I(z_n) &= I(z_n) - \frac{1}{2} \langle I'(z_n), z_n \rangle \\ &= \int_{\Omega} c_0(x)H(z_n) dx - \frac{1}{2} \int_{\Omega} f(x)z_n dx. \end{aligned}$$

Combining (6.2) and (6.9) we see that

$$(6.10) \quad \frac{1}{2} \int_{\Omega} f(x)z_n dx = -M(n) + \int_{\Omega} c_0(x)H(z_n) dx$$

where  $M(n)$  is a quantity such that  $M(n) \rightarrow +\infty$  as  $n \rightarrow \infty$ . In order to show that  $c_0 w^+ = 0$  does not occur we next prove that (6.10) is impossible.

Observe that, for  $n \in \mathbb{N}$  large enough,

$$(6.11) \quad \begin{aligned} d + 1 &\geq I(v_n) = I(v_n) - \frac{1}{2} \langle I'(v_n), v_n \rangle + o(1) \\ &= \int_{\Omega} c_0(x)H(v_n) dx - \frac{1}{2} \int_{\Omega} f(x)v_n dx + o(1) \end{aligned}$$

(note that  $\langle I'(v_n), v_n \rangle \rightarrow 0$ , since  $(v_n)$  is a Cerami sequence). Thus, for some  $D > 0$ ,

$$\int_{\Omega} c_0(x)H(v_n) dx \leq D + \frac{1}{2} \int_{\Omega} f(x)v_n dx = D + \frac{1}{2t_n} \int_{\Omega} f(x)z_n dx$$

or equivalently, using (6.10)

$$(6.12) \quad \int_{\Omega} c_0(x)H(v_n) dx \leq D - \frac{M(n)}{t_n} + \frac{1}{t_n} \int_{\Omega} c_0(x)H(z_n) dx.$$

Now we decompose  $\Omega$  into  $\Omega = \Omega_n^+ \cup \Omega_n^-$  with

$$\Omega_n^+ = \{x \in \Omega : z_n(x) \geq 0\} \quad \text{and} \quad \Omega_n^- = \Omega \setminus \Omega_n^+.$$

On  $\Omega_n^+$  we have, by Lemma 7 (v) and  $c_0 \geq 0$ , that

$$\int_{\Omega_n^+} c_0(x)H(z_n) dx \leq t_n \int_{\Omega_n^+} c_0(x)H(v_n) dx.$$

On  $\Omega_n^-$  we have, by Lemma 7 (vi) and  $|\Omega| < \infty$ , that for some  $D > 0$

$$\int_{\Omega_n^-} c_0(x)H(z_n) dx \leq D.$$

Then it follows from (6.12) that

$$\int_{\Omega_n^-} c_0(x)H(v_n) dx \leq D - \frac{M(n)}{t_n} + \frac{D}{t_n}.$$

Letting  $n \rightarrow \infty$  and using  $t_n \in [0, 1]$  we see that

$$\int_{\Omega_n^-} c_0(x)H(v_n) dx \rightarrow -\infty$$

which is impossible since, by Lemma 7 (vi),  $H$  is bounded on  $\mathbb{R}^-$  and  $|\Omega| < \infty$ . At this point we have shown that  $c_0 w^+ = 0$  is impossible.

We now assume that  $c_0 w^+ \neq 0$  and we show that this property also leads to a contradiction. Since  $(v_n) \subset X$  is a Cerami sequence we have  $\langle I'(v_n), v_n \rangle \rightarrow 0$ . Thus

$$\|v_n\|^2 - \int_{\Omega} c_0(x)g(v_n)v_n dx - \int_{\Omega} f(x)v_n dx \rightarrow 0.$$

Dividing by  $\|v_n\|^2$  we get

$$\|w_n\|^2 - \int_{\Omega} c_0(x) \frac{g(v_n)}{\|v_n\|} w_n dx \rightarrow 0,$$

and since  $\|w_n\| = 1$  we have

$$(6.13) \quad \int_{\Omega} c_0(x) \frac{g(v_n)}{\|v_n\|} w_n dx = \int_{\Omega} c_0(x) \frac{g(v_n)}{v_n} w_n^2 dx \rightarrow 1.$$

Let

$$\Omega^+ = \{x \in \Omega : c_0(x)w(x) > 0\} \neq \emptyset.$$

We also define

$$\Omega_n^+ = \{x \in \Omega : v_n(x) \geq 0\} \quad \text{and} \quad \Omega_n^- = \Omega \setminus \Omega_n^+.$$



Now since  $g(s)/s \rightarrow +\infty$  as  $s \rightarrow +\infty$  and  $w_n \rightarrow w > 0$  a.e. on  $\Omega^+$  it follows that

$$c_0 \frac{g(v_n)}{v_n} w_n^2 \rightarrow +\infty \quad \text{a.e. on } \Omega^+.$$

Thus, taking into account that  $|\Omega^+| > 0$ , we deduce that

$$(6.14) \quad \lim_{n \rightarrow \infty} \int_{\Omega^+} c_0(x) \frac{g(v_n)}{v_n} w_n^2 dx \rightarrow +\infty.$$

On the other hand we have

$$(6.15) \quad \begin{aligned} \int_{\Omega^+} c_0(x) \frac{g(v_n)}{v_n} w_n^2 dx &= \int_{\Omega} c_0(x) \frac{g(v_n)}{v_n} w_n^2 dx \\ &\quad - \int_{(\Omega \setminus \Omega^+) \cap \Omega_n^+} c_0(x) \frac{g(v_n)}{v_n} w_n^2 dx \\ &\quad - \int_{(\Omega \setminus \Omega^+) \cap \Omega_n^-} c_0(x) \frac{g(v_n)}{v_n} w_n^2 dx. \end{aligned}$$

But, for all  $n \in \mathbb{N}$ , since  $g$  is non negative,

$$(6.16) \quad \int_{(\Omega \setminus \Omega^+) \cap \Omega_n^+} c_0(x) \frac{g(v_n)}{v_n} w_n^2 dx \geq 0.$$

Also, since  $g(s)/s$  is bounded for  $s \leq 0$  we have, for some  $D > 0$ ,

$$(6.17) \quad \begin{aligned} \left| \int_{(\Omega \setminus \Omega^+) \cap \Omega_n^-} c_0(x) \frac{g(v_n)}{v_n} w_n^2 dx \right| &\leq D \int_{\Omega} c_0(x) w_n^2 dx \\ &\leq D \|c_0\|_{\frac{N}{2}} \|w_n\|^2 \leq D \|c_0\|_{\frac{N}{2}}. \end{aligned}$$

Now combining (6.13)-(6.17) we get a contradiction. This shows that  $c_0 w^+ \neq 0$  is impossible and ends the proof of the lemma.  $\square$

**Lemma 11.** *Under the hypotheses of Lemma 10 any Cerami sequence for  $I$  at a level  $d \in \mathbb{R}^+$  admits a strongly convergent subsequence.*

*Proof.* Let  $(v_n) \subset X$  be a Cerami sequence for  $I$  at a level  $d \in \mathbb{R}^+$ . Since by Lemma 10 this sequence is bounded, by passing to a subsequence we can assume that  $v_n \rightharpoonup v$  weakly in  $X$  and  $v_n \rightarrow v$  strongly in  $L^q(\Omega)$ , for each  $1 \leq q < \frac{2N}{N-2}$ . The condition  $I'(v_n) \rightarrow 0$  in  $X^{-1}$  means precisely that

$$-\Delta v_n - [c_0(x) + \mu f(x)]v_n - c_0(x)g(v_n) - f(x) \rightarrow 0 \quad \text{in } X^{-1}.$$

Because  $v_n \rightarrow v$  in  $L^q(\Omega)$ , for  $1 \leq q < \frac{2N}{N-2}$  and  $c_0 \in L^p(\Omega)$  for some  $p > \frac{N}{2}$  we readily have that  $c_0(x)g(v_n) \rightarrow c_0(x)g(v)$  in  $X^{-1}$ . Thus

$$(6.18) \quad -\Delta v_n - [c_0(x) + \mu f(x)]v_n \rightarrow c_0(x)g(v) + f(x) \quad \text{in } X^{-1}.$$

Now let  $L : X \rightarrow X^{-1}$  be defined by

$$(Lu)v = \int_{\Omega} \nabla u \nabla v - [c_0(x) + \mu f(x)]uv dx.$$

The operator  $L$  is invertible by (4.9), so we can deduce from (6.18) that  $v_n \rightarrow L^{-1}[c_0(x)g(v) + f(x)]$  in  $X$ . Consequently, by the uniqueness of the limit,  $v_n \rightarrow v$  in  $X$ .  $\square$

## 7. PROOFS OF THE MAIN THEOREMS

With the results from the previous section at hand, we are ready to prove Theorem 2. We assume that  $\bar{c} > 0$  is chosen sufficiently small to ensure that the conclusions of Lemmas 4–11 hold.

*Proof of Theorem 2.* Let first  $\mu > 0$ . By Lemma 9 we have the existence of a first critical point which is a local minimum of  $I$ , whereas by Lemmas 10 and 11 we obtain a second critical point at the mountain pass level  $\hat{c} > 0$ . So we obtain two different solutions of (4.1) in  $X$ . By Lemma 6 and Lemma 3 they give two different solutions of (2.3). These solutions are bounded, as a consequence of Lemma 13 below.

Next, if  $\mu < 0$  we replace  $u$  by  $-u$ , which is equivalent to replacing  $\mu$  by  $-\mu$  and  $f$  by  $-f$ . Theorem 2 is proved.  $\square$

Now consider the equation

$$(7.1) \quad -\operatorname{div}(A(x)\nabla u) = \mu \langle A(x)\nabla u, \nabla u \rangle + c_0(x)u + f(x),$$

and assume  $\Lambda I \geq A(x) \geq \lambda I$ , where  $\Lambda \geq \lambda > 0$ . We have just proved Theorem 2 for (7.1) with  $A(x) = I$ .

It is trivial to check that the change of unknown  $v = \frac{1}{\mu}(e^{\mu u} - 1)$  transforms (7.1) into

$$(7.2) \quad -\operatorname{div}(A(x)\nabla v) - [c_0(x) + \mu f(x)]v = c_0(x)g(v) + f(x).$$

This equation is variational and can be treated exactly like (4.1). Repeating the arguments from the previous sections we are led to the following result.

**Theorem 12.** *Assume that*

$$c_0 \not\equiv 0 \text{ in } \Omega \quad \text{and} \quad \mu \neq 0.$$

*If*

$$\|[\mu f]^+\|_{L^{\frac{N}{2}}(\Omega)} < \lambda C_N$$

*and*

$$\max\{\|c_0\|_{L^p(\Omega)}, \|[\mu f]^-\|_{L^p(\Omega)}\} < \bar{c},$$

*where  $\bar{c} > 0$  depends only on  $N, p, \lambda, \Lambda, |\Omega|, |\mu|, \|[\mu f]^+\|_{L^p(\Omega)}$ , then (7.1) admits at least two bounded solutions.*

The boundedness of the solutions obtained in this theorem (which contains Theorem 2 as a particular case) is a consequence of the following lemma.

**Lemma 13.** *Assume that  $\Lambda I \geq A(x) \geq \lambda I$  for some  $\Lambda \geq \lambda > 0$ ,  $\mu \in L^\infty(\Omega)$ , and that  $c_0$  and  $f$  belong to  $L^p(\Omega)$ , for some  $p > \frac{N}{2}$ . Then any solution  $v \in X$  of (7.2) belongs to  $L^\infty(\Omega)$ .*

*Proof.* Let  $v \in X$  be a solution of (7.2), which we recast as

$$-\operatorname{div}(A(x)\nabla v) = [c_0(x) + \mu f(x) + c_0(x)\frac{g(v)}{v}]v + f(x).$$

By our assumptions  $c_0$ ,  $f$ , and  $\mu f$  belong to  $L^p(\Omega)$ , for some  $p > N/2$ . We will be in position to apply Lemma 5 provided we show that the term  $c_0(x)\frac{g(v)}{v}$  has the same property. This is indeed the case because of the slow growth of  $g(s)/s$  as  $|s| \rightarrow \infty$  (recall Lemma 7). Specifically, for any  $r \in (0, 1)$  there exists a  $D > 0$  such that

$$\left| \frac{g(s)}{s} \right| \leq D|s|^r, \quad \text{for any } s \in \mathbb{R}.$$

Thus, since  $c_0 \in L^p(\Omega)$  for some  $p > \frac{N}{2}$ , and  $v$  is in some Lebesgue space ( $v \in L^{2N/(N-2)}(\Omega)$ ), by taking  $r > 0$  sufficiently small ( $r < \frac{4p-2N}{p(N-2)}$ ) and by using the Hölder inequality we see that  $c_0 g(v)/v \in L^{p_1}(\Omega)$ , for some  $p_1 \in (N/2, p)$ .

So  $v$  is bounded, by Lemma 5.  $\square$

We are now ready to prove Theorem 1. The idea is to use Theorem 12 in order to obtain a supersolution and a subsolution to (2.1) which can be proved to be ordered. Then we can obtain the existence of one solution to (2.1) by appealing to a theorem which states the existence of a solution between ordered sub- and super-solutions. Such results abound in the theory of elliptic PDE, see for instance [4], [14], and the references in these works. We are going to use Theorem 3.1 from [11], which is particularly adapted to our setting.

We recall, see Definition 3.1 of [11], that a function  $\underline{w} \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$  is a subsolution of (2.1) if

$$-\operatorname{div}(A(x)\nabla \underline{w}) \leq H(x, \underline{w}(x), \nabla \underline{w}(x)) \quad \text{in } \Omega \quad \text{and} \quad \underline{w} \leq 0 \quad \text{on } \partial\Omega.$$

Respectively, a function  $\bar{w} \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$  is a supersolution of (2.1) if

$$-\operatorname{div}(A(x)\nabla \bar{w}) \geq H(x, \bar{w}(x), \nabla \bar{w}(x)) \quad \text{in } \Omega \quad \text{and} \quad \bar{w} \geq 0 \quad \text{on } \partial\Omega.$$

The function  $H$  obviously satisfies the hypothesis (1.5) from [11], since

$$|H(x, s, \xi)| \leq (1 + \mu + |s|)(|c_0(x)| + |f(x)| + |\xi|^2).$$

*Proof of Theorem 1.* Observe that any solution  $\bar{u} \in X$  of the equation

$$(7.3) \quad -\operatorname{div}(A(x)\nabla u) = \frac{\mu}{\lambda} \langle A(x)\nabla u, \nabla u \rangle + c_0(x)u + f(x),$$

given by applying Theorem 12 to (7.3) in the particular case  $\mu > 0$  and  $f \geq 0$ , is such that  $\bar{u}$  is a *supersolution* to the original equation (2.1), thanks to (H1). In addition  $\bar{u} \geq 0$ , by the maximum principle (Lemma 4) which can be applied to (7.2). Similarly, it is easily checked that  $\underline{u} = -\bar{u}$  is a *subsolution* to (2.1), and of course  $\underline{u} \leq 0 \leq \bar{u}$ , so  $\underline{u}, \bar{u}$  are ordered. These functions are bounded, by Lemma 13. Thus Theorem 3.1 of [11] implies Theorem 1.  $\square$

## 8. FINAL REMARKS

The hypotheses we made on (1.1) in order to prove Theorem 2 can be generalized in various ways. For instance, if  $f \geq 0$  we see that any solution  $v$  of (4.1) satisfies  $v \geq 0$  on  $\Omega$ , provided  $\|c_0 + \mu f\|_{N/2} < C_N$  (by Lemma 4). Then we do not need any more Lemma 6 and, inspecting the proofs of the remaining lemmas, one can see that requiring that  $c_0$  belongs to  $L^p(\Omega)$  for some  $p > \frac{N}{2}$  and that  $f \in L^{\frac{N}{2}}(\Omega)$  suffices to get the conclusion of Theorem 2. In this case the solutions of (4.1) and thus of (2.3) which we obtain are not necessarily bounded. One may in general ask whether it is possible to consider coefficients  $c_0$  and  $f$  which are less regular, thus obtaining solutions with lower regularity, like for instance in [20].

Let us also make some remarks on the importance of the change of variables  $u = \frac{1}{\mu} \ln(1 + \mu v)$  which we used. If the operators  $\operatorname{div}(a)$  and  $B$  in (1.1) can be appropriately bounded above and below by quantities such that this change can be made in the corresponding “extremal” equations, leading to new equations for which our critical point method can be applied, then we obtain a subsolution and a supersolution for the initial problem, and hence a solution to this problem. This approach is in some sense alternative, as well as complementary, to the one used in many previous papers on coercive problems with natural growth. In these papers the idea was to mimic the change of variables in the initial problem, by testing the weak formulation of (1.1) with suitably chosen functions, which somehow take account of the change of unknown (see for instance Remark 2.10 in [18] for more details).

We stress that, in contrast with the existence result in Theorem 1 which is very general with respect to the structure of the equation, our multiplicity result, Theorem 12, depends on a strict link between the second order term and the gradient term. In other words, to obtain multiplicity we do need to be able to make the change of variables in the initial equation. It is certainly a very interesting open problem whether a multiplicity result can be proved for more general non-coercive problems with natural growth. It can be expected that topological methods,

in particular index theory, should permit us to deduce multiplicity of solutions for equations which do not have an equivalent formulation in terms of critical points of a functional. For instance, we state

**Open Problem 1.** Under appropriate smallness condition on  $c_0$  and  $f$  is it true that the equation

$$-\Delta u = c_0(x)u + \mu(x)|\nabla u|^2 + f(x), \quad u \in X,$$

has at least two bounded solutions, provided  $0 < \mu_1 \leq \mu(x) \leq \mu_2$  and  $\mu$  is not constant ?

*Acknowledgement.* We thank M.-F. Bidaut-Véron for a number of very useful remarks, which improved our presentation.

#### REFERENCES

- [1] H. ABDEL HAMID, M.F. BIDAUT-VÉRON, Correlation between two quasilinear elliptic problems with a source term involving the function or its gradient, *C. R. Acad. Sci. Paris Sér. I Math.*, **346**, (2008), 1251-1256.
- [2] H. ABDEL HAMID, M.F. BIDAUT-VÉRON, On the connection between two quasilinear elliptic problems with source terms of order 0 or 1 *Comm. Contemp. Math.*, **125**, (2010), 727-788.
- [3] B. ABDELLAOUI, A. DALL'AGLIO, I. PERAL, Some remarks on elliptic problems with critical growth in the gradient, *J. Diff. Equations*, **222**, (2006), 21-62.
- [4] H. AMANN, M.G. CRANDALL, On some existence theorems for semi-linear elliptic equations, *Indiana Univ. Math. J.* **27**(5), (1978), 779-790.
- [5] A. AMBROSETTI, P.H. RABINOWITZ, Dual variational methods in critical point theory and applications, *J. Functional Analysis*, **14**, (1973), 349-381.
- [6] G. BARLES, A.P. BLANC, C. GEORGELIN, M. KOBYLANSKI, Remarks on the maximum principle for nonlinear elliptic PDE with quadratic growth conditions, *Ann. Scuola Norm. Sup. Pisa*, **28**, (1999), 381-404.
- [7] G. BARLES, F. MURAT, Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions, *Arch. Rational. Mech Anal.*, **133**, (1995), 77-101.
- [8] G. BARLES, A. PORETTA, Uniqueness for unbounded solutions to stationary viscous Hamilton-Jacobi equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **5**, (2006), 107-136.
- [9] L. BOCCARDO, T. GALLOUËT, F. MURAT, A unified presentation of two existence results for problems with natural growth, in *Progress in partial differential equations: the Metz surveys*, 2 (1992), 127-137, Pitman Res. Notes Math. Ser., 296, Longman Sci. Tech., Harlow, 1993.
- [10] L. BOCCARDO, F. MURAT, J.P. PUEL, Résultats d'existence pour certains problèmes elliptiques quasilineaires, *Ann. Scuola Norm. Sup. Pisa*, **11**, (1984), 213-235.
- [11] L. BOCCARDO, F. MURAT, J.P. PUEL, Quelques propriétés des opérateurs elliptiques quasi-linéaires, *C. R. Acad. Sci. Paris Sér. I Math.* **307**(14), (1988), 749-752.

- [12] L. BOCCARDO, F. MURAT, J.P. PUEL,  $L^\infty$  estimate for some nonlinear elliptic partial differential equations and application to an existence result, *SIAM J. Math. Anal.* **23**(2), (1992), 326–333.
- [13] G. CERAMI, Un criterio di esistenza per i punti critici su varietà illimitate, *Rend. Istit. Lombardo, Accademia di Scienze e Lettere (A)*, **112**, (1978), 332–336.
- [14] C. DE COSTER, M. HENRARD, Existence and localization of solution for second order elliptic BVP in presence of lower and upper solutions without any order, *J. Diff. Equa.*, **145**, (1998), 420–452.
- [15] A. DALL’AGLIO, D. GIACHETTI, J.P. PUEL, Nonlinear elliptic equations with natural growth in general domains, *Ann. Mat. Pura Appl.*, **181** (2002), 407–426.
- [16] I. EKELAND, Convexity methods in Hamiltonian mechanics. *Springer-Verlag, Berlin*, (1990).
- [17] V. FERONE, F. MURAT, Quasilinear problems having quadratic growth in the gradient: an existence result when the source term is small, *Équations aux dérivées partielles et applications, Gauthier-Villars, Éd. Sci. Méd. Elsevier, Paris*, (1998), 497–515.
- [18] V. FERONE, F. MURAT, Nonlinear problems having quadratic growth in the gradient: an existence result when the source term is small, *Non. Anal. TMA*, **42**, (2000), 1309–1326.
- [19] D. GILBARG, N.S. TRUDINGER, Elliptic Partial Differential Equations of Second Order, 2nd edition, Springer Verlag, (1983).
- [20] N. GRENON, F. MURAT, A. PORRETTA, Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms, *C. R. Math. Acad. Sci. Paris* **342**(1), (2006), 23–28.
- [21] L. JEANJEAN, On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on  $\mathbb{R}^N$ , *Proc. Roy. Soc. Edinburgh*, **129A**, (1999), 787–809.
- [22] J.L. KAZDAN, R.J. KRAMER, Invariant criteria for existence of solutions to second-order quasilinear elliptic equations, *Comm. Pure Appl. Math.*, **31**, (1978), 619–645.
- [23] C. MADERNA, C. PAGANI, S. SALSA, Quasilinear elliptic equations with quadratic growth in the gradient, *J. Diff. Eq.*, **97**(1) (1992), 54–70.
- [24] B. SIRAKOV, Solvability of uniformly elliptic fully nonlinear PDE, *Arch. Rat. Mech. Anal.*, **195**, (2010), 579–607.
- [25] G. TALENTI, Best constant in Sobolev’s inequality, *Ann. Mat. Pura Apl.* **110** (1976), 353–372.
- [26] N.S. TRUDINGER, Linear elliptic operators with measurable coefficients, *Ann. Scu. Nor. Sup. Pisa*, **27**, (1973), 265–308.

LOUIS JEANJEAN  
LABORATOIRE DE MATHÉMATIQUES (UMR 6623)  
UNIVERSITÉ DE FRANCHE-COMTÉ  
16, ROUTE DE GRAY 25030 BESANÇON CEDEX, FRANCE  
*E-mail address:* louis.jeanjean@univ-fcomte.fr

BOYAN SIRAKOV  
PONTIFÍCIA UNIVERSIDADE CATÓLICA DO RIO DE JANEIRO  
DEPARTAMENTO DE MATEMÁTICA,  
RUA MARQUÊS DE SÃO VICENTE 225, GÁVEA  
RIO DE JANEIRO - RJ, CEP 22451-900,  
BRAZIL  
*E-mail address:* bsirakov@mat.puc-rio.br