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EXISTENCE AND MULTIPLICITY RESULTS FOR SEMICOERCIVE UNILATERAL PROBLEMS

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In this paper, we investigate a general class of variational inequalities. Existence and multiplicity results are obtained by using minimax principles for lower semi-continuous functions due to A. Szulkin.

1. INTRODUCTION

The aim of this paper is the study of problems in mechanics characterised by a general mechanics law which may be written in the form

$$0 \in \operatorname{grad} W(u) + \partial\Phi(u), \; u \in U,$$

that is, the law is the sum of a potential law and a superpotential law. W is the potential and Φ is a proper convex function and is the superpotential. We denote by $\partial\Phi$ the convex subdifferential of Φ. U is the space of all fields of possible displacements. In this paper it will be assumed that W can be written in the following form

$$W(u) = \langle u, Tu\rangle/2 + Cu,$$

where T is linear symmetric and C is $C^1(U, \mathbb{R})$.

As an example, we shall consider the following problem: let $T > 0$ and let $H^1(\Pi, \mathbb{R})$ be the Sobolev space obtained by completing the set of C^∞ real-valued T-periodic functions on $\Pi := \mathbb{R}/T\mathbb{Z}$ with the norm

$$\|u\| = \int_0^T |u|^2 + |\dot{u}|^2 \, dt.$$

Let K be the closed convex cone defined by

$$K := \{u \in H^1(\Pi, \mathbb{R}) : u(x) \geqslant 0 \text{ on } [0, T]\}.$$

We consider the following periodic unilateral problem

$$[\mathrm{P}_0] \qquad u \in K: \int_0^T \dot{u}.(\dot{v} - \dot{u})dt + \int_0^T \nabla_u V(t, u).(v - u)dt \geqslant 0, \quad \forall\, v \in K,$$

when V is α-positively homogeneous with respect to u and such that

(a) $\forall\, u \in \mathbb{R} \setminus \{0\}$: $\int_0^T V(t, u)dt > 0$,
(b) $\exists v \in \mathbb{R}^+$: $V(., v) < 0$ on a non zero measure subset.

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In this case,

$$W(u) := \frac{1}{2}\int_0^T |\dot{u}|^2\, dt + \int_0^T V(t, u)dt,$$

where the former term correspond to the kinetic energy and the latter to the potential energy. $\Phi(u) := I_K$ characterises the constraints on the displacement field.

We now detail the framework of our paper.

Let $\langle X, X^*\rangle$ be a dual system of real Hilbert spaces, let $T\colon X \to X^*$ be a symmetric bounded linear operator and let $C \in C^1(X, \mathbb{R})$. C is assumed to be β-positively homogeneous that is, $\langle C'(u), u\rangle = \beta C(u))$ and strongly continuous (that is C maps weakly converging sequences into converging sequences). Let $\Phi\colon X \to (-\infty, +\infty]$ be a proper convex functional. Φ is assumed to be α-positively homogeneous and weakly lower semicontinuous.

We are looking for *non trivial solutions*, that is, $x^* \notin \operatorname{Ker} T$, of the following variational inequality:

$$\text{[P]} \qquad x^* \in X : \langle v - x^*, Tx^* + C'(x^*)\rangle + \Phi(v) - \Phi(x^*) \geqslant 0, \quad \forall\, v \in X.$$

In case of bilateral problems (that is, $\Phi = 0$), the first result concerning this problem is due to Lassoued [**2**]. Recently, using a version of the Ljusternik-Schnirelman theory on C^1-manifolds due to Szulkin [**5**] Ben Naoum, Troestler and Willem obtained a general abstract existence and multiplicity theory for bilateral problems [**1**], where homogeneous second order differential equations were considered. For a basic work on critical point theory and its applications to bilateral problems we shall refer to [**3**]. In this paper, we use a version of minimax primciples for lower semicontinuous functions due to Szulkin, to get new results for the variational inequality [P] and related unilateral problems.

2. Existence result

Theorem 2.1. *If the following conditions hold true: $\alpha < \beta < 2$ and*

(1) *T is semicoercive, that is, there exists $c > 0$ such that*

$$\langle x, Tx\rangle \geqslant c.\, \|Px\|^2 \quad \textit{for each} \quad x \in X$$

with $P = I - Q$, where I denotes the identity mapping and Q denotes the orthogonal projection of X onto $\operatorname{Ker}(T)$.

(2) $\dim \operatorname{Ker} T < +\infty$,

(3) $\exists z \in X : \langle C'(z), z\rangle + \Phi(z) < 0$,

(4) $C(u) > 0$, $\forall\, u \in \operatorname{Ker} T$, $u \neq 0$,

then problem [P] has a nontrivial solution.

PROOF: Let $J\colon X \to (-\infty, +\infty]$ be the functional defined by

$$J(u) := \frac{1}{2}\langle u, Tu\rangle + C(u) + \Phi(u).$$

Let $X = \overline{\bigcup_{n\in\mathbb{N}} X_n}$, where for each n, $X_n := \{x \in X\colon \|x\| \leqslant n\}$ is a weakly compact convex set in X. Since J is weakly lower semicontinuous, it reaches its minimum on each X_n, let us say at u_n.

We have $J(u_n) \leqslant J(v)$, for each $v \in X_n$.

Let $v \in X_n$, $tv + (1-t)u_n \in X_n$ for each $t \in [0, 1]$ and since Φ is convex, we get

$$\Phi(v) - \Phi(u_n) + \frac{1}{2}[\langle T(u_n + t(v-u_n)), u_n + t(v-u_n)\rangle - \langle Tu_n, u_n\rangle]/t$$
$$+ [C(u_n + t(v-u_n)) - C(u_n)]/t \geqslant 0, \text{ for all } v \in X_n.$$

Computing the limit as $t \to 0^+$ we get

$$\langle v - u_n, Tu_n + C'(u_n)\rangle + \Phi(v) - \Phi(u_n) \geqslant 0, \text{ for all } v \in X_n. \tag{2.1}$$

We show first that the sequence $\{u_n\}$ is bounded.

(a) If $0 < \beta < 2$. Suppose on the contrary that $\{u_n\}$ is unbounded. Passing possibly to a subsequence, we can suppose that $w - \lim_{n\to\infty} x_n = x^*$, where $x_n := u_n/\|u_n\|$.

Put $v = 0$ in (2.1); we obtain

$$\langle u_n, Tu_n\rangle + \beta.C(u_n) + \Phi(u_n) \leqslant 0$$

which implies:

$$\langle x_n, Tx_n\rangle + \beta.C(x_n).\|u_n\|^{\beta-2} + \Phi(x_n)\|u_n\|^{\alpha-2} \leqslant 0. \tag{2.2}$$

Taking the limit as $n \to +\infty$ in (2.2), we get

$$\langle x^*, Tx^*\rangle \leqslant \liminf\langle x_n, Tx_n\rangle \leqslant 0,$$

and since $\langle x^*, Tx^*\rangle \geqslant 0$ we obtain

$$\langle x^*, Tx^*\rangle = 0,$$

and thus $Tx^* = 0$ and $c.\liminf\|Px_n\|^2 \leqslant \liminf\langle x_n, Tx_n\rangle \leqslant 0$. Going if necessary to a subsequence we can assume that $Px_n \to 0$, $x_n \to x^* \in \operatorname{Ker} T$, since $\dim\operatorname{Ker} T < \infty$. Moreover $\|x^*\| = 1$.

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By positivity, $\langle u_n, Tu_n \rangle \geqslant 0$ for each $n \in \mathbb{N}$, and thus we have from (2.1)

$$\langle x, Tu_n \rangle + \langle x, C'(u_n) \rangle + \Phi(x) \geqslant \beta C(u_n) + \Phi(u_n), \text{ for each } x \in X_n. \tag{2.3}$$

Choosing $x = 0$, we obtain

$$\Phi(u_n) + \beta C(u_n) \leqslant 0.$$

Dividing by $\|u_n\|^\beta$,

$$\beta C(x_n) + \Phi(x_n) \|u_n\|^{\alpha-\beta} \leqslant 0$$

and taking the limit, we obtain

$$C(x^*) \leqslant 0,$$

and since $x^* \in \operatorname{Ker} T$, this is a contradiction to assumption (4).

Thus the sequence $\{u_n\}$ is bounded. Without loss of generality, we can suppose that

$$u^* = w - \lim_{n \to \infty} u_n.$$

For $y \in X$, there exists $m \in \mathbb{N}$ such that $y \in X_n$ for all $n \geqslant m$. Hence $J(u_n) \leqslant J(y)$, for all $n \geqslant m$ and since J is weakly lower semicontinuous, we get

$$J(u^*) \leqslant J(y),$$

and therefore $J(u^*) = \min\limits_X J(y)$.

We have thus

$$\langle Tu^* + C'(u^*), v - u^* \rangle + \Phi(v) - \Phi(u^*) \geqslant 0, \qquad \forall\, y \in X.$$

With $v = 0$, we obtain $\beta C(u^*) + \Phi(u^*) \leqslant 0$, and thus by assumption (4)

$$u^* \notin \operatorname{Ker} T \setminus \{0\}.$$

Now,

$$J(u^*) \leqslant J(v), \quad \text{for all} \quad v \in X,$$

and thus

$$J(u^*) \leqslant C(v), \quad \text{for all} \quad v \in \operatorname{Ker} T.$$

By assumption (3), we get

$$J(u^*) \leqslant C(z) + \Phi(z) < 0,$$

and thus $u^* \neq 0$. ∎

Corollary 2.1. *Let K be a nonempty closed convex cone of X. If the following conditions hold true: $\alpha < \beta < 2$ and*

(1) *T is semicoercive.*
(2) $\dim \operatorname{Ker} T < +\infty$,
(3) $\exists z \in K \colon \langle C'(z), z\rangle < 0$,
(4) $C(u) > 0$, $\forall\, u \in K \cap \operatorname{Ker} T$, $u \neq 0$,

then there exists $u \in K \setminus \operatorname{Ker} T$ such that

$$\langle v - x^*, Tx^* + C'(x^*)\rangle \geqslant 0, \quad \forall\, v \in K.$$

3. Multiplicity result

We shall assume that $C \in C^1(X, \mathbb{R})$ is even, β-positively homogeneous and strongly continuous, and $\Phi\colon X \to (-\infty, +\infty]$ is even, α-positively homogeneous and strongly continuous.

In order to obtain a multiplicity result to prove that [P] has many pairs of solutions $(-x^*, x^*)$, we verify the assumptions of Theorem 4.4 in [4] due to Szulkin.

Let us recall some definitions.

Let X be a real Banach space and J a function on X satisfying: $J = f + g$, where $f \in C^1(X, \mathbb{R})$ and $g\colon X \to (-\infty, +\infty]$ is convex, proper and lower semicontinuous. We say that J satisfies the Palais-Smale condition in the sense of Szulkin (PS), if $\{u_n\}$ is a sequence such that $J(u_n) \to c \in \mathbb{R}$, $z_n \in f'(u_n) + \partial g(u_n)$ where $z_n \to 0$, then $\{u_n\}$ possesses a convergent subsequence.

Theorem 3.1. (Szulkin [4].) *Suppose that J is defined as above and satisfies (PS), $J(0) = 0$ and f, g are even. Assume also that*

(1) *there exists a subspace X_1 of X, of finite codimension, and numbers γ, $\rho > 0$ such that $J\,|_{\partial B_\rho \cap X_1} \geqslant \gamma$,*
(2) *there is a finite dimensional subspace X_2 of X, $\dim X_2 > \operatorname{codim} X_1$ such that $J(u) \to -\infty$ as $\|u\| \to \infty$, $u \in X_2$.*

Then J has at least $\dim X_2 - \operatorname{codim} X_1$ distinct pairs of nonzero critical points $(-x^*, x^*)$, that is $0 \in f'(x^*) + \partial g(x^*)$.

Corollary 3.1. (Szulkin [4].) *Suppose that the hypotheses of Theorem 3.1 are satisfied with (2) replaced by*

(2′) *for any positive integer k there is a k-dimensional subspace X_2 of X such that $J(u) \to -\infty$ as $\|u\| \to +\infty$.*

Then J has infinitely many distinct pairs of nonzero critical points.

From this theorem, we obtain the following

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THEOREM 3.2. *If $\alpha > 1$, $\beta > \max\{2, 2^\alpha - 1\}$ and*

(1) *T is semicoercive*

(2) $\dim \operatorname{Ker} T < +\infty$,

(3) *there exists a subspace X_n of X, such that $n := \dim X_n > \dim \operatorname{Ker} T$ and $C(y) < 0$, for all $y \in X_n$, $y \neq 0$,*

(4) $\Phi(u) > 0$, $\forall\, u \in (\operatorname{Ker} T) \setminus \{0\}$; $\Phi(u) \geqslant 0$, $\forall\, u \in X$.

Then there exist at least $n - \dim \operatorname{Ker} T$ distinct pairs of nontrivial solutions for problem [P].

PROOF: Let $f(x) := \langle x, Tx\rangle/2 + C(x)$, $g(x) := \Phi(x)$. Let $X_1 := (\operatorname{Ker} T)^{\perp}$, $X_2 := X_n$

(1) For every $x \in X_1$, we have

$$\langle x, Tx\rangle/2 + \Phi(x) + C(x) \geqslant c/2.\|x\|^2 - |C(x)|$$

and since Φ and C are continuous and positively homogeneous, there exist k, $k' > 0$ such that

$$\langle x, Tx\rangle/2 + \Phi(x) + \beta C(x) \geqslant c/2.\|x\|^2 - k'\|x\|^\beta .$$

It is always possible to choose ρ such that $\tau := c\rho^2/2 - k'\rho^\beta > 0$ and thus

$$J(x) \geqslant \tau, \quad \forall\, x \in \partial B_\rho \cap X_1.$$

(2) By assumption (3), there exists $\delta > 0$ such that

$$C(x) \leqslant -\delta\|x\|^\beta, \quad \text{for all} \quad x \in X_2.$$

We have

$$J(x) \leqslant \|T\|_* \|x\|^2 - \delta\|x\|^\beta + k\|x\|^\alpha$$

and thus

$$\lim_{\substack{\|x\| \to +\infty \\ x \in X_2}} J(x) = -\infty.$$

It remains to prove that J satisfies the (PS) condition. Let $u_n \in X$ be a sequence such that $J(u_n) \to c \in \mathbb{R}$, $z_n \in f'(u_n) + \partial g(u_n)$ where $z_n \to 0$; that is also (see [4] for more details)

$$\Phi(v) - \Phi(u_n) + \langle Tu_n + C'u_n, v - u_n\rangle \geqslant -\delta_n.\|v - u_n\|, \tag{3.1}$$

where $\delta_n \to 0$.

We claim that $\{u_n\}$ is bounded. Suppose that $\{u_n\}$ is unbounded. With $v = 2u_n$ in (3.1) we get

$$\langle u_n, Tu_n\rangle + \beta C(u_n) + (2^\alpha - 1)\Phi(u_n) \geqslant -\delta_n. \|u_n\|,$$

so that, for n large enough,

$$\beta J(u_n) - \{\langle u_n, Tu_n\rangle + \beta C(u_n) + (2^\alpha - 1)\Phi(u_n)\} \leqslant \beta(c+1) + \|u_n\|.$$

Thus

$$(\beta + 1 - 2^\alpha)\Phi(u_n) + (\beta/2 - 1)\langle u_n, Tu_n\rangle \leqslant \beta(c+1) + \|u_n\|. \tag{3.2}$$

By assumption (6) we have

$$(\beta/2 - 1)\langle u_n, Tu_n\rangle \leqslant \beta(c+1) + \|u_n\|.$$

Put $v_n := u_n/\|u_n\|$. We can suppose, by considering if necessary a subsequence, that $w - \lim\limits_{n\to\infty} v_n = v^*$.

We have

$$(\beta/2 - 1)\langle v_n, Tv_n\rangle \leqslant \beta(c+1)/\|u_n\|^2 + 1/\|u_n\|.$$

Taking the limit, we get

$$0 \leqslant \langle v^*, Tv^*\rangle \leqslant \liminf\langle v_n, Tv_n\rangle \leqslant 0,$$

and as in Theorem 2.1, going if necessary to a subsequence, we can assume that $\|v^*\| = 1$.

Since T is positive, from (3.2) we get also

$$(\beta + 1 - 2^\alpha)\Phi(u_n) \leqslant \beta(c+1) + \|u_n\|,$$

and thus

$$(\beta + 1 - 2^\alpha)\Phi(v_n) \leqslant \beta(c+1)/\|u_n\|^\alpha + 1/\|u_n\|^{\alpha-1}.$$

By taking the limit, we get $\Phi(v^*) \leqslant 0$, which is a contradiction to assumption (4).

Thus $\{u_n\}$ is bounded and by considering possibly a subsequence, we may suppose that u_n is weakly convergent. Let $u^* = w - \lim u_n$. Put $v = u^*$ in (3.1). We get

$$\langle Tu_n, u^* - u_n\rangle + \langle C'u_n, u^* - u_n\rangle + \Phi(u^*) - \Phi(u_n) \geqslant -\delta_n. \|u^* - u_n\|.$$

Taking the limit, we get

$$\varlimsup_{n\to\infty} \langle Tu_n, u_n - u^*\rangle \leqslant 0.$$

The orthogonal decomposition $\overline{X} \oplus \operatorname{Ker}(T)$ allows us to write $u_n =: \overline{u}_n + \widehat{u}_n$. Thus we have

$$\varlimsup_{n\to\infty} c. \|\overline{u}_n - \overline{u}^*\|^2 \leqslant \varlimsup_{n\to\infty} \langle T(\overline{u}_n - \overline{u}^*), \overline{u}_n - \overline{u}^*\rangle \leqslant 0,$$

and $\overline{u}_n$ is strongly convergent to $\overline{u}^*$. Since $\dim \operatorname{Ker} T < +\infty$, going if necessary to a subsequence, $\widehat{u}_n$ is strongly convergent to Q^* and the conclusion follows. ∎

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4. Examples

EXAMPLE 4.1. Let $T > 0$ and let $X := H^1(\Pi, \mathbb{R})$. Let K be the closed convex cone defined by $K := \{u \in H^1(\Pi, \mathbb{R}) : u(x) \geqslant 0 \text{ in } [0, T]\}$. We consider the periodic unilateral problem

$$(1) \qquad u \in K\colon \int_0^T \dot{u}.(\dot{v} - \dot{u})dt + \int_0^T \nabla_u V(t, u).(v - u)dt \geqslant 0, \ \forall\, v \in K.$$

We assume that:

(a) $\forall\, u \in \mathbb{R}$, $V(\cdot, u)$ is measurable and there exist $a, b \in L^1([0, T], \mathbb{R}_+)$ such that $\forall\, t \in [0, T]$, $\forall\, u \in \mathbb{R}$, $|u| = 1$, $|V(t, u)| \leqslant a(t)$ and $|\nabla_u V(t, u)| \leqslant b(t)$.

(b) for almost all $t \in \mathbb{R}$, $V(t, .) \in C^1$,

(c) $\forall\, u \in \mathbb{R} \setminus \{0\} : \int_0^T V(t, u)dt > 0$,

(d) $\exists \nu \in \mathbb{R}^+ : V(., v) < 0$, on a non zero measure subset,

(e) V is β-positively homogeneous ($\beta < 2$) with respect to u.

Let $T\colon X \to X^*$ and $C\colon X \to \mathbb{R}$ be defined by

$$\langle Tu, v\rangle := \int_0^T \dot{u}.(\dot{v} - \dot{u})dt, \ C(u) := \int_0^T V(t, u)dt.$$

We can prove that if V satisfies (a)–(e), then all assumptions of Corollary 2.1 are satisfied [1], so that (1) has at least one non-constant solution.

EXAMPLE 4.2. We consider the problem

$$(2) \qquad u \in X\colon \int_0^T \dot{u}.(\dot{v} - \dot{u})dt + \int_0^T \nabla_u V(t, u).(v - u)dt + \int_0^T g(t)\Big(|v|^3 - |u|^3\Big)dt \geqslant 0, \ \forall\, u \in X.$$

Let $T\colon X \to X^*$ and $C\colon X \to \mathbb{R}$ be defined as in Example 4.1 and put $\Phi(u) := \int_0^T g(t)\, |u|^3\, dt$. We assume that g is a positive ($g \neq 0$) bounded function.

We can prove that if V satisfies (a)–(d) and (e) with $\beta > 7$ and even, then all assumptions of Theorem 3.2 [1] are satisfied. Therefore (1) has infinitly many distinct pairs of non-constant solutions.

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